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Approximation with Diversified B-Splines

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Abstract

When approximating functions defined on some domain $\Omega \subset \mathbb{R}^d$, standard tensor product splines reveal sub-optimal behavior, in particular, if Ω is non-convex. As an alternative, we suggest a natural diversification strategy for the B-spline basis $\{B_i\}_i$. It is grounded on employing a separate copy $B_{i,\gamma}$ of B_i for every connected component γ of its support $\operatorname{supp} B_i \cap \Omega$. In the bivariate case, which is important for applications, this process enhances the spline space to a crucial extent. Concretely, we prove that the error in uniform tensor product spline approximation of a function $f : \mathbb{R}^2 \supset \Omega \to \mathbb{R}$ can be bounded in terms of the pure partial derivatives of f, where the constant depends neither on the shape of Ω nor on the knot grid. An example shows that a similar result cannot hold true for higher dimensions, even if the domain is convex and has a smooth boundary.

1 Introduction

Spline approximation is a fundamental issue in theory and applications like reverse engineering [VMC97] or simulation [CHB09, Höl03, HRW01]. However, our current knowledge on the subject is leaving some important questions unanswered when it comes to approximation of multivariate functions defined on subsets $\Omega \subset \mathbb{R}^d$. Open issues include the appropriate choice of the spline space itself and the dependence of constants in error estimates on the shape of Ω or the chosen knot sequence.

Let $\mathscr{S}_n(T, \Omega)$ denote the span of tensor product B-splines of coordinate order $n = (n_1, \ldots, n_d)$ with knots $T = (T_1, \ldots, T_d)$ restricted to Ω. In the fundamental work [DDS80], it is shown that

$$\min_{s \in \mathscr{S}_n(T,\Omega)} \|f - s\|_{\Omega,L^p} \le C \sum_{i=1}^{d} h_i^{n_i} \|\partial_i^{n_i} f\|_{\Omega,L^p} \tag{1}$$

for some $C > 0$, where $h = (h_1, \ldots, h_d)$ is the maximal spacing of knots. In [MR09] and [Rei12], this result is elaborated for the special cases of interpolation and approximation with polynomials, respectively. The estimate suggests that the pure partial derivatives of f alone should be sufficient to bound the error, and that a fine knot sequence in a distinct coordinate direction should be sufficient to compensate for large derivatives in that direction. Unfortunately, such a conclusion may not be drawn imprudently from the results in [DDS80], in particular for the following reasons: First, the domain Ω

is assumed to be coordinate-wise convex, what is a severe restriction of the range of applicability. Second, Ω has to live up to a series of technical assumptions which may be hard to verify in a concrete setting. Third, and perhaps most impedingly, a detailed analysis of statements and arguments reveals a hidden dependence of the number C on the *aspect ratio*

$$\varrho := \max_{i,j} \frac{h_i}{h_j}$$

of the knot grid, even in the case of uniform splines. Thus, the before mentioned compensation of a large value of $\|\partial_i^{n_i} f\|_{\Omega,L^p}$ by an exclusive refinement of the knot sequence T_i is questioned, as decreasing h_i alone is increasing ϱ. A similar dependence of C on the aspect ratio can also be observed in other approaches to the topic, like [HRW01] or [MR08]. So it is plausible, but by no means evident, that this phenomenon is not an artifact of insufficient proof techniques, but a matter of fact.

In the second section of this paper, we present examples in two and three variables which actually prove that C cannot be independent of ϱ. While the 2d example exploits the non-convexity of the domain, the 3d case gets along with a domain which is strictly convex and has a perfectly smooth boundary. The special structure of the 3d counterexample might be useful to identify a subclass of domains where (1) is valid with uniform C. However, this topic is not addressed here.

Instead, in the third section, we propose a remedy to the problems observed in the 2d case. It is based on the observation that the spline space $\mathscr{S}_n(T, \Omega)$ is not rich enough to deal adequately with non-convexity. Let S_i denote the support of the B-spline B_i. Then its relevant part $S_i \cap \Omega$ might consist of several connected components. In the standard setting, the single B-spline B_i is overcharged by possibly conflicting demands coming from simultaneous error minimization on all these components. So it is a natural approach to use a separate copy $B_{i,\gamma}$ of B_i for each connected component γ of $S_i \cap \Omega$. This process, called *diversification*, provides a significant amount of extra flexibility. Our main theorem on approximation with diversified B-splines states that (1) holds true for a broad class of domains $\Omega \subset \mathbb{R}^2$ with a constant C which is independent of the aspect ratio and the shape of Ω.

For the proof of our main theorem, another new concept, called *condensation*, is introduced to address the notorious problem of lacking stability of the basis when working on domains with boundary. Condensation is replacing a given knot sequence by a finer one without changing the span of B-splines on the considered domain. Thus, the size of the support and the knot spacing can be made comparable, what facilitates the construction of stable quasi-interpolants.

To focus on the essence of ideas, we confine ourselves to the case of uniform knot sequences and to error measurements with respect to the sup-norm, i.e., $p = \infty$. Arbitrary knot sequences and exponents p can be dealt with in a similar fashion, but require a significantly increased complexity of notations and arguments.

2 Issues in one, two, and three variables

In this section, we elaborate on some phenomena occurring in spline approximation on domains of different dimension.

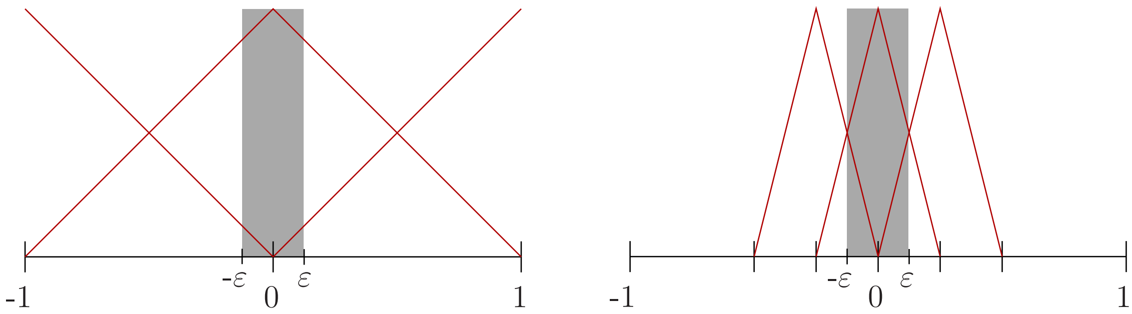

Figure 1: Cardinal B-splines and condensed variant on domain $\omega = (-\varepsilon, \varepsilon)$.

2.1 Univariate case

In principle, the univariate case is understood in any detail. However, the following thoughts are not only a preparation for the bivariate theory, to be developed in the next section, but might also be useful to understand situations, where a given knot sequence $T = h\mathbb{Z} + \tau$ is not well adapted to the given domain $\omega = (\underline{\omega}, \overline{\omega})$ in the sense that $|\omega| := \overline{\omega} - \underline{\omega}$ is much smaller than the knot spacing h. Here, the B-spline basis looses its uniform stability, as can exemplarily be seen already in the linear case $n = 2$. For $h = 1$ and $\tau = 0$, let $b_i, i \in \mathbb{Z}$, denote the corresponding B-splines with support $[i, i+2]$. When approximating on the domain $\omega := (-\varepsilon, \varepsilon), \varepsilon < 1$, then only the functions b_{-1}, b_0, b_1 are active, all others vanish identically. The lower bound $\operatorname{cond}_\infty\{b_{-1}, b_0, b_1\} \geq 1/\varepsilon$ on the condition number of this basis follows already from

$$\|b_{-1}\|_{\omega,\infty} = \|b_1\|_{\omega,\infty} = \varepsilon, \quad \|b_0\|_{\omega,\infty} = 1.$$

That is, the basis is ill-conditioned for small ε. This observation is not in conflict with the famous result on the uniform stability of B-spline bases [dB76]. As is easily overseen, it applies only in special situations, e.g., for splines defined on the whole real line. It is fairly obvious how to choose a knot sequence which is better adapted to the case considered above. For instance, let $T' := 2\varepsilon\mathbb{Z}$, then the corresponding B-splines b'_{-1}, b'_0, b'_1 are active on ω. They span exactly the same space of functions, $\operatorname{span}\{b_{-1}, b_0, b_1\} = \operatorname{span}\{b'_{-1}, b'_0, b'_1\}$, but now we have $\operatorname{cond}_\infty\{b'_{-1}, b'_0, b'_1\} = 2$, independent of ε. The situation is illustrated in Figure 1.

The process of replacing a given knot sequence $T = h\mathbb{Z} + \tau$ by another one $T^\omega = h^\omega\mathbb{Z} + \tau^\omega$ with spacing $h^\omega \leq |\omega|$ under the condition that knots within the domain ω remain unchanged is called *condensation*. Concretely, we distinguish three cases to define that process:

- If $|\omega| > h$, then there are at least two knots of T in ω and nothing is changed, i.e., $T^\omega := T$.
- If $|\omega| \leq h$, then knot spacing is reduced to $h^\omega := |\omega|$. To define the new shift τ^ω, let i denote the largest integer satisfying $hi + \tau < \overline{\omega}$.
  - If $hi + \tau \in \omega$, then $\tau^\omega := i(h - h^\omega) + \tau$. In this way, the single knot of T contained in ω is retained in T^ω.
  - If $hi + \tau \notin \omega$, then $\tau^\omega := \underline{\omega} - h^\omega i$. In this way, the boundary points of ω become knots of T^ω.

The B-splines with respect to T^ω are denoted by $b_i^\omega, i \in \mathbb{Z}$, and called *condensed B-splines*. Let $I_\omega := \{i \in \mathbb{Z} : \operatorname{supp} b_i \cap \omega \neq \emptyset\}$ denote the set of indices of active B-splines with knots T, then the above definition guarantees that the same indices yield active B-splines with knots T^ω, i.e., $I_\omega = \{i \in \mathbb{Z} : \operatorname{supp} b_i^\omega \cap \omega \neq \emptyset\}$. The supports satisfy

$$\operatorname{supp} b_i^\omega \subset \operatorname{supp} b_i, \quad i \in I_\omega.$$

Further, by construction, the spanned spline spaces coincide,

$$\operatorname{span}\{b_i \chi(\omega) : i \in I_\omega\} = \operatorname{span}\{b_i^\omega \chi(\omega) : i \in I_\omega\}.$$

Here, multiplication by the characteristic function $\chi(\omega)$ of the domain is used to dismiss irrelevant differences outside ω. For later reference, we briefly recall the representation of polynomials in terms of uniform splines. For $T = \mathbb{Z}$ and any polynomial $p \in \mathbb{P}_n$ of order n, the coefficient s_i in the representation $p = \sum_i b_i s_i$ can be determined as linear combination of the values of p at the points $i + 1/2, \ldots, i + n - 1/2$ and certain weights $\alpha_1^n, \ldots, \alpha_n^n$, i.e., $s_i = \sum_{m=1}^n \alpha_m^n p(i+m-1/2)$. By invariance of B-splines with respect to affine transformation of the argument, the same weights can be used for any uniform knot sequence, and in particular for the condensed sequence T^ω: Let $\mu_m^\omega := h^\omega(m-1/2) + \tau^\omega$ denote the midpoints of knots in T^ω, then

$$p(x) = \sum_{i \in I_\omega} b_i(x) \sum_{m=1}^n \alpha_m^n p(\mu_{i+m}^\omega), \quad x \in \omega.$$

Partition of unity corresponds to the fact that the weights α_m^n sum up to 1,

$$\sum_{m=1}^n \alpha_m^n = 1. \tag{2}$$

2.2 Bivariate case

Already in the bivariate case, spline approximation is much more subtle. As an example, consider

$$\Omega := \big\{(x_1, x_2) \in \mathbb{R}^3 : -1 < x_2 < |x_1| < 1\big\}$$

and knots $T_1 := h_1\mathbb{Z} + 1, T_2 := h_2\mathbb{Z}$, where $h_1 := 2$ and $h_2 < 1$, see Figure 2. In this case the aspect ratio $\varrho = 2/h_2$ becomes large for small h_2. For symmetry reasons, the best approximation of the function

$$f(x_1, x_2) := \begin{cases} x_2 & x_2 \geq 0, x_1 \geq 0 \\ -x_2 & x_2 \geq 0, x_1 \leq 0 \\ 0 & x_2 \leq 0 \end{cases}$$

with splines of order $n = (1, 1)$ is $s = 0$. So the maximal error is attained on the topmost support of B-splines, shaded in red in the figure, and we obtain $\inf_s \|f - s\|_{\Omega,\infty} \geq \|f\|_{\Omega,\infty} = 1$. It is easy to verify that the right hand side in (1) is Ch_2. Hence, the constant $C \geq \varrho/2$ in (1) increases unboundedly with the aspect ratio. This is due to the fact that there are supports of B-splines whose intersection with Ω consists of several connected components. In the next section, we will devise a remedy to this problem. Essentially, multiple copies of B-splines, always one for each connected component of its support, are employed to generate a spline space which is rich enough to account for the non-convexity of the domain.

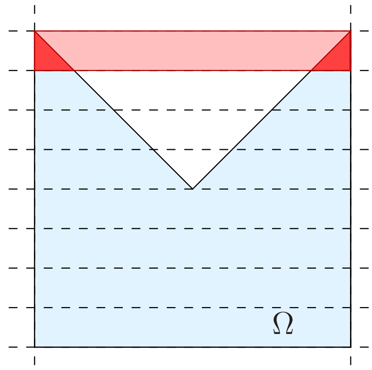

Figure 2: Non-convex domain and knot sequence with large aspect ratio.

2.3 Trivariate case

In the trivariate case, and equally in more variables, even convex domains with perfectly smooth boundary may reveal a dependence of the constant C in (1) on the mesh ratio ϱ. This statement is now substantiated by the following example: Let

$$\Omega := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1 - x_2)^2 + (x_1 + x_2)^4 + (1 - x_3)^2 < 1\right\},$$

see Figure 3. Given $h > 0$, the spline space to be considered has knots

$$T_1 := T_2 := h\mathbb{Z}, \quad T_3 := h^5\mathbb{Z}$$

so that the aspect ratio is $\varrho = h^{-4}$. The coordinate orders are $n = (n_1, n_2, n_3)$. We set $m := n_1 + n_2 - 1$ and assume $n_1 \geq 2$. The family of functions

$$f_h(x_1, x_2, x_3) := \left(m x_1 x_2^{m-1} - (n_1 - 1)x_2^m\right) \exp(-x_3/h^4)$$

is to be approximated. Skipping the technical details, we note that there exists a constant c_1 such that

$$|x_1 - x_2| \leq c_1\sqrt{x_3}, \quad |x_1| \leq c_1\sqrt[4]{x_3}, \quad |x_2| \leq c_1\sqrt[4]{x_3}$$

for all points $(x_1, x_2, x_3) \in \Omega$. These bounds can be used to show that there exist constants c_2, c_3 with

$$\begin{aligned}
&\|D_1^{n_1} f_h\|_{\Omega,\infty} = 0\\
&\|D_2^{n_2} f_h\|_{\Omega,\infty} \leq \|c_2 x_3^{n_1/4} \exp(-x_3/h^4)\|_{\Omega,\infty} \leq c_3 h^{n_1}\\
&\|D_3^{n_3} f_h\|_{\Omega,\infty} \leq \|c_2 x_3^{m/4} \exp(-x_3/h^4) h^{-4n_3}\|_{\Omega,\infty} \leq c_3 h^{m-4n_3}.
\end{aligned}$$

That is, the right hand side in (1) is bounded from above by $Cc_3h^m(h+h^{n_3})$. To estimate the left hand side, we consider points of the form $x_h(t) = (t, t, h^4)$ with $h \leq 1/2$ and $t \in (0, h/2)$, which lie in Ω. Any trivariate spline s with knots T_1, T_2, T_3 evaluated at $x_h(t)$ is a polynomial of order m in t. Its maximal deviation from the prescribed values $f_h(x_h(t)) = t^m n_2/e$, which form a polynomial of order $m+1$ in t, is bounded from below by $c_4 h^m$ for some positive constant c_4. This implies $\|f - s\|_{\Omega,\infty} \geq c_4 h^m$ for the overall deviation of any spline s. Comparison of the two estimates yields $C \geq c_4/(c_3(h + h^4))$, showing that the constant C becomes arbitrarily large for small h.

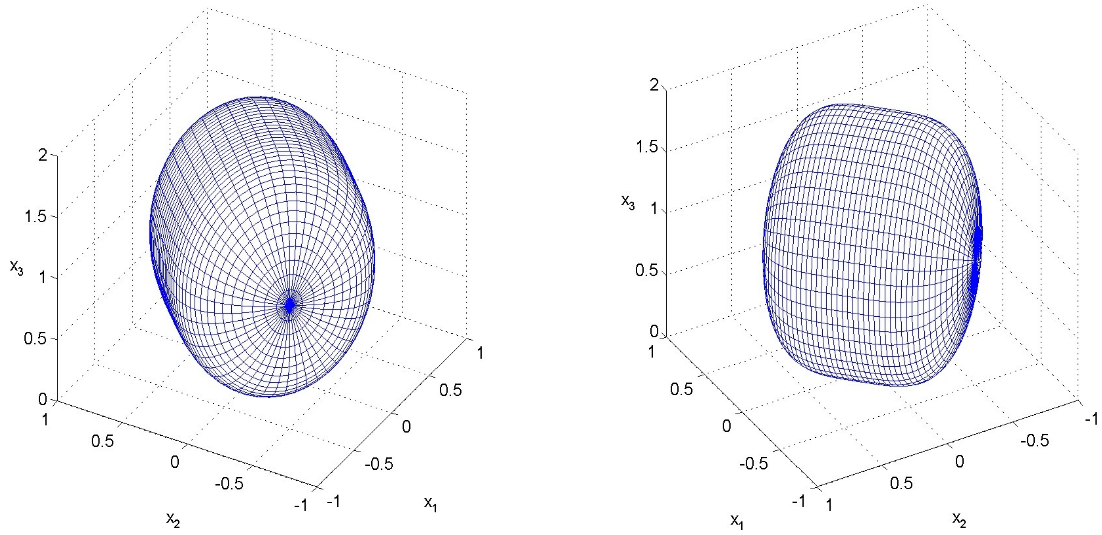

Figure 3: Convex domain Ω for the trivariate example.

3 Bivariate approximation

In this section, we focus on the bivariate case. Applications include the reconstruction of surface patches by tensor product splines in the context of reverse engineering. We show that diversification yields a significant improvement of approximation properties so that, beyond its theoretical implications, this technique is also recommended for practical use.

After introducing some notation, our main theorem is presented in Section 3.2. The remaining part of the paper is demanded by the proof.

3.1 Notation

A two-dimensional *interval* in $\mathbb{R}^2$ is understood as the cartesian product of two intervals in $\mathbb{R}$. The *size* of a bounded interval I is defined as the vector of side-lengths and denoted by $|I| := \sup\{x-y : x, y \in I\}$, where the supremum is understood component-wise. Let $\Omega, M \subset \mathbb{R}^2$ be two subsets of $\mathbb{R}^2$. The set of connected components of $M \cap \Omega$ is denoted by $\mathcal{C}_\Omega(M)$. The *bounding box* of M, i.e., the smallest interval containing it, is denoted by $\mathcal{B}(M) = \mathcal{B}_1(M) \times \mathcal{B}_2(M)$. If M is connected, the *pruned bounding box* $\mathcal{B}_\Omega(M)$ of M is defined as the element of $\mathcal{C}_\Omega(\mathcal{B}(M))$ containing M. The *size* of M is understood as the size of its bounding box, i.e., $|M| := |\mathcal{B}(M)|$. Given a vector $h \in [0,\infty]^2$ and a second set $\Omega \subset \mathbb{R}^2$ containing M, the *h-neighborhood* of M is defined by

$$\mathcal{N}(M,h) := \big\{x \in \mathbb{R}^2 : |\{x\} \cup M| \leq |M| + h\big\}.$$

The connected component in $\mathcal{C}_\Omega(\mathcal{N}(M,h))$ containing M is called the *local h-neighborhood* and denoted by $\mathcal{N}_\Omega(M,h)$. We note that $\mathcal{N}(M,h)$ is always an interval. Further, $\mathcal{N}(M,0) = \mathcal{B}(M)$ and $\mathcal{N}_\Omega(M,0) = \mathcal{B}_\Omega(M)$

Throughout, and without further notice, the index $\sigma \in \{1,2\}$ is addressing the two coordinate directions. Components of two-dimensional objects are tagged by the prepending subscript σ. For instance, $x = (x_1, x_2) \in \mathbb{R}^2$ or $x_j = (x_{1,j}, x_{2,j})$ for a sequence $\{x_j\}_j$ in $\mathbb{R}^2$.

To define a spline space, let $T = (T_1, T_2)$ be a pair of uniform knot sequences with *grid width* $h \in \mathbb{R}^2_{>0}$. Concretely, $T_\sigma := h_\sigma \mathbb{Z} + \tau_\sigma$ for some shift τ_σ. The individual knots and their midpoints are denoted by

$$t_k := (h_1 k_1 + \tau_1, h_2 k_2 + \tau_2), \quad \mu_k := (t_{k-(1,1)} + t_k)/2, \tag{3}$$

respectively. For example, according to our conventions, the two components of t_k are labeled $t_{1,k}, t_{2,k}$. The $h/2$-neighborhood of μ_k is the *grid cell* $\Gamma_k := \mathcal{N}(\{\mu_k\}, h/2)$. The vector of coordinate orders is denoted by $n \in \mathbb{N}^2$, and we define

$$\bar{n} := \max(n_1, n_2).$$

Multiplication of univariate B-splines b_{σ,i_σ} with knots T_σ yields the *tensor product B-splines*

$$B_i(x) := b_{1,i_1}(x_1) b_{2,i_2}(x_2), \quad x \in \mathbb{R}^2,\ i \in \mathbb{Z}^2.$$

The support of B_i is given by $S_i := \operatorname{supp} B_i = \mathcal{B}(\{t_i\} \cup \{t_{i+n}\})$.

3.2 Main result

In this subsection, we introduce the space of diversified B-splines and present the main result of our work. We start with the definition of the class of sets to which it applies.

Definition 1. *Let $\Phi = [a, \varphi]$ be a pair consisting of the real number $a > 0$ and a continuous function $\varphi : X \to \mathbb{R}_{>0}$ defined on the interval $X := [-a, a]$. With $X^\delta := [-a + \delta, a - \delta]$ the sub-interval with margin $\delta \in \mathbb{R}$, let*

$$\Phi^\delta := \{x \in X^\delta \times \mathbb{R} : \delta < x_2 < \varphi(x_1)\}.$$

An axis-aligned isometry in $\mathbb{R}^2$ is a composition of a translation and a rotation by an integer multiple of $\pi/2$.

A subset $\Omega \subset \mathbb{R}^2$ is called a graph domain with parameter $h_0 \in \mathbb{R}_{>0}$ *if there exists an index set $R \subset \mathbb{N}$, axis-aligned isometries Σ_r, and pairs $\Phi_r = [a_r, \varphi_r]$ as above with $a_r > h_0$ and $\min_{X_r} \varphi_r > h_0$ such that*

$$\Omega = \bigcup_{r \in R} \Sigma_r(\Phi_r^\delta), \quad 0 \le \delta \le h_0.$$

The value of h_0 guarantees a least amount of overlap of the sets $\Sigma_r(\Phi_r^0)$ so that any sufficiently small subset of Ω is contained in one of them. More precisely, the following argument will be used repeatedly, and without further notice: Given Ω as above, let $M \subset \Omega$ be an arbitrary subset of size $|M| \le 2h_0$. There exists a point $x \in M$ satisfying $M \subset \mathcal{N}_\Omega(\{x\}, h_0)$ and an index $r \in R$ such that $x \in \Sigma_r(\Phi_r^{h_0})$. Hence, $M \subset \Sigma_r(\Phi_r^0)$. Since our constructions disregard translations and are completely congeneric for the different coordinate directions and orientations, we may assume that Σ_r is the identity. All other cases could be treated in an analogous manner. Further, the value of the index $r \in R$ is irrelevant. Hence, to avoid excessive notation, we drop the index r and write $\Phi = [a, \varphi]$ instead of $\Phi_r = [a_r, \varphi_r]$ when examining M. In particular, it is $M \subset \Phi^0$.

Restricting the span of the B-splines $B_i, i \in \mathbb{Z}^2$, to Ω yields the spline space

$$\mathscr{S}_n(T,\Omega) := \Big\{ \big(\sum_{i\in\mathbb{Z}^2} B_i s_i\big)_{|\Omega} : s_i \in \mathbb{R} \Big\},$$

as typically considered in the literature. However, this space is by no means an evident choice. Instead, one might consider the space $\mathscr{S}'_n(T,\Omega)$ of all functions which coincide with polynomials of order n on grid cells, and which are C^{n_1-2} in x_1-direction and C^{n_2-2} in x_2-direction. In general, $\mathscr{S}'_n(T,\Omega)$ is larger than $\mathscr{S}_n(T,\Omega)$ so that the problem discussed in the preceding section might disappear. As it is not obvious how to construct $\mathscr{S}'_n(T,\Omega)$, we propose a different approach, which yields an intermediate spline space $\mathscr{S}^*_n(T,\Omega)$ with the desired properties. It is a subspace[1] of $\mathscr{S}^*_n(T,\Omega)$, and, in general, larger than $\mathscr{S}_n(T,\Omega)$.

Given some B-spline B_i, the idea is to use it not only once, but possibly several copies of it – always one for each connected component of its support in Ω. More precisely, we introduce the index set

$$J := \big\{(i,\gamma) : i \in \mathbb{Z}^2, \gamma \in \mathcal{C}_\Omega(S_i)\big\}$$

and define the *diversified B-splines*

$$B_j := B_i \chi(\gamma), \quad j = (i,\gamma) \in J,$$

with support $S_j := \operatorname{supp} B_j = \operatorname{supp} \chi(\gamma)$, see Figure 4(b). We note that the similar expressions B_i and B_j refer to different objects, relatable by the type of the subscript: Subscripts $i \in \mathbb{Z}^2$ indicate standard B-splines, while subscripts $j = (i,\gamma) \in J$ indicate their diversified descendants. The indices i and j will be used in a consistent way to simplify reading. The diversified B-splines span the space

$$\mathscr{S}^*_n(T,\Omega) := \Big\{ \sum_{j\in J} B_j s_j : s_j \in \mathbb{R} \Big\}.$$

Clearly, $\mathscr{S}_n(T,\Omega) \subset \mathscr{S}^*_n(T,\Omega) \subset \mathscr{S}'_n(T,\Omega)$.

Let us briefly reconsider the second example given in the introduction. The large error $\min_{s\in\mathscr{S}_n(T,\Omega)} \|f-s\|_{\infty,\Omega} = h_1/2 = 1$ is caused by the B-splines in the upper half-plane, for which the intersection of their supports with Ω consists of two connected components. When split into two separate copies, the error drops down to $\min_{s\in\mathscr{S}^*_n(T,\Omega)} \|f-s\|_{\infty,\Omega} = h/2$, and (1) is valid with $C = 1/2$.

To formulate our main result, we define the *anisotropic Sobolev space* $W^n_\infty(\Omega)$ of order $n \in \mathbb{N}^2$ as the space of essentially bounded functions $f : \Omega \to \mathbb{R}$ whose partial derivatives $\partial_1^{n_1} f, \partial_2^{n_2} f$ are also essentially bounded.

Theorem 1. *Let Ω be a graph domain with parameter h_0, and let $n \in \mathbb{N}^2$. There exists a constant C depending only on n such that*

$$\inf_{s\in\mathscr{S}^*_n(T,\Omega)} \|f-s\|_{\Omega,\infty} \le C\,(h_1^{n_1}\|\partial_1^{n_1} f\|_{\Omega,\infty} + h_2^{n_2}\|\partial_2^{n_2} f\|_{\Omega,\infty})$$

for any $f \in W^n_\infty(\Omega)$ and any uniform knot sequence T with grid width $h \le h_0/(\bar{n}+1)$.

[1] We conjecture $\mathscr{S}'_n(T,\Omega) = \mathscr{S}^*_n(T,\Omega)$, but this is irrelevant in this context.

The rest of this section is devoted to the proof of the theorem. The main problem with establishing an error bound is the lack of stability of the basis $B_j, j \in J$. It is not related to the existence of copies, but to the potentially small size of supports. Below, we apply the principle of condensation, as introduced above in the univariate case, in a specific way to the bivariate setting. The existence of a universal set of knots is fundamental for any known analysis of univariate or tensor product spline spaces. Condensation abandons that paradigm by assigning individual knots to each basis function. However, the special construction presented below is preserving just as much of the conventional structure to permit the use of suitably modified standard quasi interpolants.

3.3 Condensation

In this subsection, we apply condensation to diversified B-splines. Given the index $i \in \mathbb{Z}^2$, we define the unbounded intervals

$$W_{1,i} := \mathcal{N}(S_i, (\infty, 0)), \quad W_{2,i} := \mathcal{N}(S_i, (0, \infty))$$

as extensions of the support S_i of B_i in x_1- and x_2-direction, respectively. For $j = (i, \gamma) \in J$, let

$$W_{1,j} := \mathcal{N}_\Omega(S_j, (\infty, 0)), \quad W_{2,j} := \mathcal{N}_\Omega(S_j, (0, \infty))$$

denote the connected components of $W_{\sigma,i}$ which contain the support S_j of B_j. These definitions are illustrated in figures 4(c) and 4(d).

The intervals $\omega_{\sigma,j} := \mathcal{B}_\sigma(W_{\sigma,j})$, which characterize the extent of the bounding box of $W_{\sigma,j}$ in x_σ-direction, are used for condensation, as described in the preceding section: We define the *condensed diversified B-splines (or briefly cdB-splines)* B_j^* by

$$B_j^*(x) := b_{1,i_1}^{\omega_{1,j}}(x_1) b_{2,i_2}^{\omega_{2,j}}(x_2) \chi(\gamma), \quad j = (i, \gamma) \in J.$$

Their support $S_j^* := \operatorname{supp} B_j^*$ coincides with that of the associated original B-spline. We have $S_j^* = S_j \subset S_i$, and in particular

$$|S_j^*| = |S_j| \leq \bar{n}h < h_0.$$

The condensed knot sequences corresponding to B_j^* are denoted by

$$T_j^* = (T_{1,j}^*, T_{2,j}^*), \quad T_{\sigma,j}^* := h_{\sigma,j}^* \mathbb{Z} + \tau_{\sigma,j}^* := T_\sigma^{\omega_{\sigma,j}}.$$

Analogous to (3), the individual knots and their midpoints are given by

$$t_{j,k}^* = (h_{1,j}^* k_1 + \tau_1^*, h_{2,j}^* k_2 + \tau_2^*), \quad \mu_{j,k}^* = (t_{j,k-(1,1)} + t_{j,k})/2,$$

respectively. The supports of cdB-splines are illustrated in figures 4(e) and 4(f).

3.4 Local knot structure

In this subsection, we consider the structure of knot sequences in a vicinity of a grid cell. Defining the index set

$$L := \{(k, \gamma) : k \in \mathbb{Z}^2, \gamma \in \mathcal{C}_\Omega(\Gamma_k)\},$$

the *pruned grid cell* with index $\ell = (k,\gamma) \in L$ is just $\Gamma_\ell := \gamma$. Clearly, $\Omega = \bigcup_{\ell\in L} \Gamma_\ell$. Given $\ell \in L$, let

$$J_\ell := \{j \in J : S_j^* \cap \Gamma_\ell \neq \emptyset\}, \quad \Gamma_\ell^* := \bigcup_{j\in J_\ell} S_j^*, \tag{4}$$

denote the set of indices of cdB-splines which are active on Γ_ℓ and the union of the corresponding supports, respectively. It is $|\Gamma_\ell^*| \leq (2\bar{n}-1)h \leq 2h_0$. Hence, recalling the argument below Definition 1, let $\Gamma_\ell^* \subset \Phi^0$. For any $j \in J_\ell$, the interval $\omega_{2,j}$ satisfies $|\omega_{2,j}| \geq h_0 \geq h_2$. Hence, condensation in x_2-direction is effectless, i.e., $T_{2,j}^* = T_2$. By contrast, knot sequences in x_1-direction may be modified. However, the process is exactly the same for all diversified B-splines with equal index i_2. To show this, let

$$J_\ell^{i_2} := \{j' = (i',\gamma') \in J_\ell : i_2' = i_2\}$$

denote the set of all indices in J_ℓ with second component i_2. For any two indices $j, j' \in J_\ell^{i_2}$, both S_j^* and $S_{j'}^*$ contain Γ_ℓ. Hence, they lie in the same connected component $W' \in \mathcal{C}_\Omega(W_{1,i})$ of $W_{1,i}$. Let $\omega' := \mathcal{B}_1(W')$ denote the first component of its bounding box. Then all B-splines $b_{1,j}, j \in J_\ell^{i_2}$, are condensed with respect that interval. We write $b_{1,j} := b_{1,i_1}^{\omega_{1,j}} = b_{1,i_1}^{\omega'}$ and $T_{1,j}^* := T_1^{\omega_{1,j}} = T_1^{\omega'}$ and note that the knot sequences $T_{1,j}$ depend only on the component i_2 of the index $j = (i,\gamma) \in J_\ell^{i_2}$. Together, we have shown that locally any spline $s \in \mathscr{S}_n^*(T,\Omega)$ can be written in the form

$$s(x) = \sum_{j\in J_\ell} s_j B_j^*(x) = \sum_{i_2\in\mathbb{Z}} b_{2,i_2}(x_2) \sum_{j\in J_\ell^{i_2}} s_j b_{1,j}(x_1), \quad x \in \Gamma_\ell. \tag{5}$$

3.5 Representation of polynomials

Now, we consider the representation of polynomials $p \in \mathbb{P}_n$ of coordinate order $n \in \mathbb{N}^2$ in terms of cdB-splines. Recalling the discussion of the univariate case in the preceding section, we define the linear functional $P_j : \mathbb{P}_n \to \mathbb{R}$ by

$$P_j p := \sum_{m_1=1}^{n_1} \sum_{m_2=1}^{n_2} \alpha_{m_1}^{n_1} \alpha_{m_2}^{n_2} p(\mu_{1,j,i+m}^*, \mu_{2,j,i+m}^*), \quad j = (i,\gamma).$$

We claim that

$$p(x) = \sum_{j\in J} B_j^*(x) P_j p, \quad x \in \Omega. \tag{6}$$

To show this, it suffices to consider any monomial $p^d(x) := x_1^{d_1} x_2^{d_2}$ of coordinate degree $d < n$, and $x \in \Gamma_\ell$ for an arbitrary index $\ell \in L$. Without loss of generality, we assume that ℓ is such that (5) is applicable and find

$$\sum_{j\in J} B_j^*(x) P_j p^d = \sum_{i_2\in\mathbb{Z}} b_{2,i_2}(x_2) \sum_{j\in J_\ell^{i_2}} b_{1,j}(x_1) \sum_{m_1=1}^{n_1} \sum_{m_2=1}^{n_2} \alpha_{m_1}^{n_1} \alpha_{m_2}^{n_2} (\mu_{1,j,i+m}^*)^{d_1} (\mu_{2,j,i+m}^*)^{d_2}.$$

Since $(\mu_{2,j,i+m}^*)^{d_2} = \mu_{2,i+m}^{d_2}$ is independent of j, we obtain

$$\sum_{j\in J} B_j^*(x) P_j p^d = \sum_{i_2\in\mathbb{Z}} b_{2,i_2}(x_2) \sum_{m_2=1}^{n_2} \alpha_{m_2}^{n_2} \mu_{2,i+m}^{d_2} \Big(\sum_{j\in J_\ell^{i_2}} b_{1,j}(x_1) \sum_{m_1=1}^{n_1} \alpha_{m_1}^{n_1} (\mu_{1,j,i+m}^*)^{d_1} \Big).$$

The parenthesized expression is equal to $x_1^{d_1}$. Hence,

$$\sum_{j\in J} B_j^*(x)P_j p^d = x_1^{d_1} \sum_{i_2\in\mathbb{Z}} b_{2,i_2}(x_2) \sum_{m_2=1}^{n_2} \alpha_{m_2}^{n_2} \mu_{2,i+m}^{d_2} = x_1^{d_1} x_2^{d_2}.$$

This verifies $\sum_{j\in J_\ell} B_j^* P_j p^d = p^d$ on Γ_ℓ, as requested.

In particular, if $p(x) \equiv 1$, we use (2) to establish partition of unity,

$$1 = \sum_{j\in J_\ell} B_j^*(x)P_j p = \sum_{j\in J_\ell} B_j^*(x) \sum_{m_1=1}^{n_1} \sum_{m_2=1}^{n_2} \alpha_{m_1}^{n_1}\alpha_{m_2}^{n_2} = \sum_{j\in J_\ell} B_j^*(x), \quad x \in \Gamma_\ell.$$

Since cdB-splines are non-negative, this implies the local convex hull property,

$$\Big\| \sum_{j\in J_\ell} s_j B_j^* \Big\|_{\Gamma_\ell,\infty} \le \sup_{j\in J_\ell} |s_j|. \tag{7}$$

Let $S_j' := \mathcal{B}(\{t_{j,i}^*\} \cup \{t_{j,i+n}^*\})$ denote the support of the unrestricted condensed B-spline corresponding to B_j, see figures 4(e) and 4(f). Then all evaluation points lie in this set, $\mu_{j,i+m} \in S_j'$. Hence, the operator P_j is bounded by

$$\|P_j\| := \sup_{p\in\mathbb{P}_n} \frac{|P_j p|}{\|p\|_{S_j'}} \le \sum_{m_1=1}^{n_1} \sum_{m_2=1}^{n_2} |\alpha_{m_1}^{n_1}\alpha_{m_2}^{n_2}| =: c_n', \tag{8}$$

where the constant c_n' depends only on n.

3.6 Quasi-interpolation

The functionals P_j cannot be used directly for the definition of a quasi-interpolant since the evaluation points $\mu_{j,i+m}^*$ may lie outside of the domain. Rather, we have to prefix a local approximation process. To this end, we proceed in three steps: First, for any $j \in J$, we denote the local h_j^*-neighborhood of S_j^* by $S_j^+ := \mathcal{N}_\Omega(S_j^*, h_j^*)$ and claim existence of an interval $H_j^* \subset S_j^+$ of size $|H_j^*| = h_j^*$, see figures 4(g) through 4(j). Since $|S_j^+| \le (\bar n + 2)h_j^* \le 2h_0$, it is $S_j^+ \subset \Phi^0$. For $j = (i,\gamma)$, the interval used for condensation in x_1-direction is given by $\omega_{1,j} = \mathcal{B}_1(W_{1,j}) = \{x_1 \in \mathbb{R} : (x_1, t_{2,i}) \in W_{1,j}\}$. Its length is bounded from below by $|h_{1,j}^*|$ so that we can choose an interval $\omega' \subset \omega_{1,j}$ of length $|h_{1,j}^*|$ which contains some point $x = (x_1, t_{2,i}) \in S_j^*$ on the lower bound of S_j^*. Hence, $H_j^* := \omega' \times [t_{2,i-(1,1)}, t_{2,i}]$ has the desired properties.

Second, we define the linear operator $A_j : W_\infty^n(H_j^*) \to \mathbb{P}_n$ by

$$A_j f := \operatorname{argmin}_{p\in\mathbb{P}_n} \|f - p\|_{H_j^*,2}.$$

That is, A_j is mapping the function f to the polynomial which is best approximating on the interval H_j^* with respect to the L^2-norm. The neighborhood $H_j^+ := \mathcal{N}(H_j^*, \bar n h)$ is chosen large enough to ensure $S_j' \subset H_j^+$. Hence,

$$\|A_j\| := \sup_f \frac{\|A_j f\|_{S_j',\infty}}{\|f\|_{H_j^*,\infty}} \le \sup_f \frac{\|A_j f\|_{H_j^+,\infty}}{\|f\|_{H_j^*,\infty}}.$$

The rightmost expression is invariant under shifts and scalings in $\mathbb{R}^2$. Hence, to determine an upper bound on it, we may assume $H_j^* = H' := [0,1]^2$, and hence $H_j^+ = H'' := [-\bar{n}, \bar{n}+1]^2$, without loss of generality. For polynomials in $\mathbb{P}_n$, the 2-norm on H' and the sup-norm on H'' are equivalent. So there exists a constant c_n'' depending only on n such that $\|A_j f\|_{H'',\infty} \le c_n'' \|A_j f\|_{H',2}$, and we obtain

$$\|A_j\| \le c_n'' \sup_f \frac{\|A_j f\|_{H',2}}{\|f\|_{H',\infty}} \le c_n'' \sup_f \frac{2\|f\|_{H',2}}{\|f\|_{H',\infty}} \le 2c_n''. \tag{9}$$

Third, we define the *quasi interpolant* $Q : W_\infty^n(\Omega) \to \mathscr{S}_n^*(T,\Omega)$ by

$$Qf := \sum_{j\in J} B_j^* Q_j f, \quad Q_j := P_j A_j.$$

Let $p \in \mathbb{P}_n$. Then, by (6),

$$Qp = \sum_{j\in J} B_j^* P_j A_j p = \sum_{j\in J} B_j^* P_j p = p, \tag{10}$$

showing that Q is reproducing polynomials. Further, using (8) and (9), we see that the functionals Q_j are uniformly bounded by

$$\|Q_j\| := \sup_f \frac{|Q_j f|}{\|f\|_{H_j^*,\infty}} \le \|P_j\| \cdot \|A_j\| \le 2c_n' c_n'' =: c_n, \quad j \in J.$$

Hence, by(7), the spline Qf is bounded on Γ_ℓ by

$$\|Qf\|_{\Gamma_\ell,\infty} \le \max_{j\in J_\ell} |Q_j f| \le \max_{j\in J_\ell} \|Q_j\| \cdot \|f\|_{H_j^*,\infty} \le c_n \max_{j\in J_\ell} \|f\|_{H_j^*,\infty}. \tag{11}$$

3.7 Error estimate

The final step of the proof of Theorem 1 is based on a specific variant of the Bramble-Hilbert Lemma established in [Rei12]. For $f \in W_\infty^n(\Omega)$, we consider the deviation of the spline approximant $s := Qf$ on an arbitrary grid cell $\Gamma_\ell, \ell \in L$. Let

$$\Gamma_\ell^+ := \mathcal{B}_\Omega\Big(\bigcup_{j\in J_\ell} S_j^+\Big).$$

This is a restricted interval of size $|\Gamma_\ell^+| \le (2\bar{n}+1)h \le 2h_0$ containing all parts of the domain with potential influence on the approximant s on Γ_ℓ. In particular, $H_j^* \subset \Gamma_\ell^+$ for all $j \in J_\ell$. Let $\Gamma_\ell^+ \in \Phi^0$. Then this set can be written in the form

$$\Gamma_\ell^+ = \big\{x \in \omega : x_2 < \varphi(x_1)\big\}$$

for a certain interval $\omega = \omega_1 \times \omega_2 \subset \mathbb{R}^2$. With $\omega_2 = [a,b]$, it is either $b \le \min\varphi$ or $a \ge h$. In the first case, $\Gamma_\ell^+ = \omega$ is a complete interval and thus ready for further use. In the second case, there is enough space left below the bottom of Γ_ℓ^+ to attach an interval of height h. We set

$$\tilde{\varphi}(x_1) := \min(\varphi(x_1), b), \quad \tilde{a} := \begin{cases} a & \text{if } b \le \min\varphi \\ a-h & \text{else} \end{cases}$$

and define

$$\Gamma := \big\{x \in \omega_1 \times \mathbb{R} : \tilde{a} \le x_2 < \tilde{\varphi}(x_1)\big\}.$$

It is $\Gamma_\ell^+ \subset \Gamma \subset \Phi^0$ and $|\Gamma| \le (2\bar{n}+2)h$. Further, the values of the function $\tilde{\varphi}$ bounding Γ satisfy

$$\frac{\max \tilde{\varphi} - \tilde{a}}{\min \tilde{\varphi} - \tilde{a}} \le 2\bar{n} + 2.$$

According to [Rei12], Theorem 2.4, there exists a polynomial $p \in \mathbb{P}_n$ such that the error $\Delta := f - p$ satisfies

$$\|\Delta\|_{\Gamma,\infty} \le c_n^* \big(h_1^{n_1} \|\partial_1^{n_1} f\|_{\Omega,\infty} + h_2^{n_2} \|\partial_2^{n_2} f\|_{\Omega,\infty}\big),$$

where the constant c_n^* depends only on n. Eventually, we use reproduction of polynomials according to (10) and the bound (11) to obtain

$$\begin{aligned}\|f - Qf\|_{\Gamma_\ell,\infty} &\le \|\Delta\|_{\Gamma_\ell,\infty} + \|Q\Delta\|_{\Gamma_\ell,\infty} \le \|\Delta\|_{\Gamma_\ell,\infty} + c_n \max_{j \in J_\ell} \|\Delta\|_{H_j^*,\infty} \\ &\le (1 + c_n)\|\Delta\|_{\Gamma_\ell,\infty} \le (1 + c_n)c_n^* \big(h_1^{n_1} \|\partial_1^{n_1} f\|_{\Omega,\infty} + h_2^{n_2} \|\partial_2^{n_2} f\|_{\Omega,\infty}\big).\end{aligned}$$

Since $\ell \in L$ was chosen arbitrarily, the claim of Theorem 1 follows with $C := (1 + c_n)c_n^*$.

4 Conclusion

Theory developed in this paper clarifies the following issues:

- In the bivariate case, diversification of standard B-splines is the key to constructing spline spaces with optimal approximation properties in the sense that the error is bounded in terms of pure partial derivatives with a constant depending only on the order of the spline space.

- In three of more variables, diversification is reasonable and recommended for applications. However, equally strong results as in the 2d case cannot be expected, not even for convex domains with smooth boundary.

- Condensation is a powerful tool for the construction of bounded quasi-interpolants.

Future research will focus on a generalization of the ideas presented here to arbitrary knot sequences and to error estimates with respect to Sobolev norms. Further, the dependence of constants on the shape of domains in higher dimensions shall be explored.

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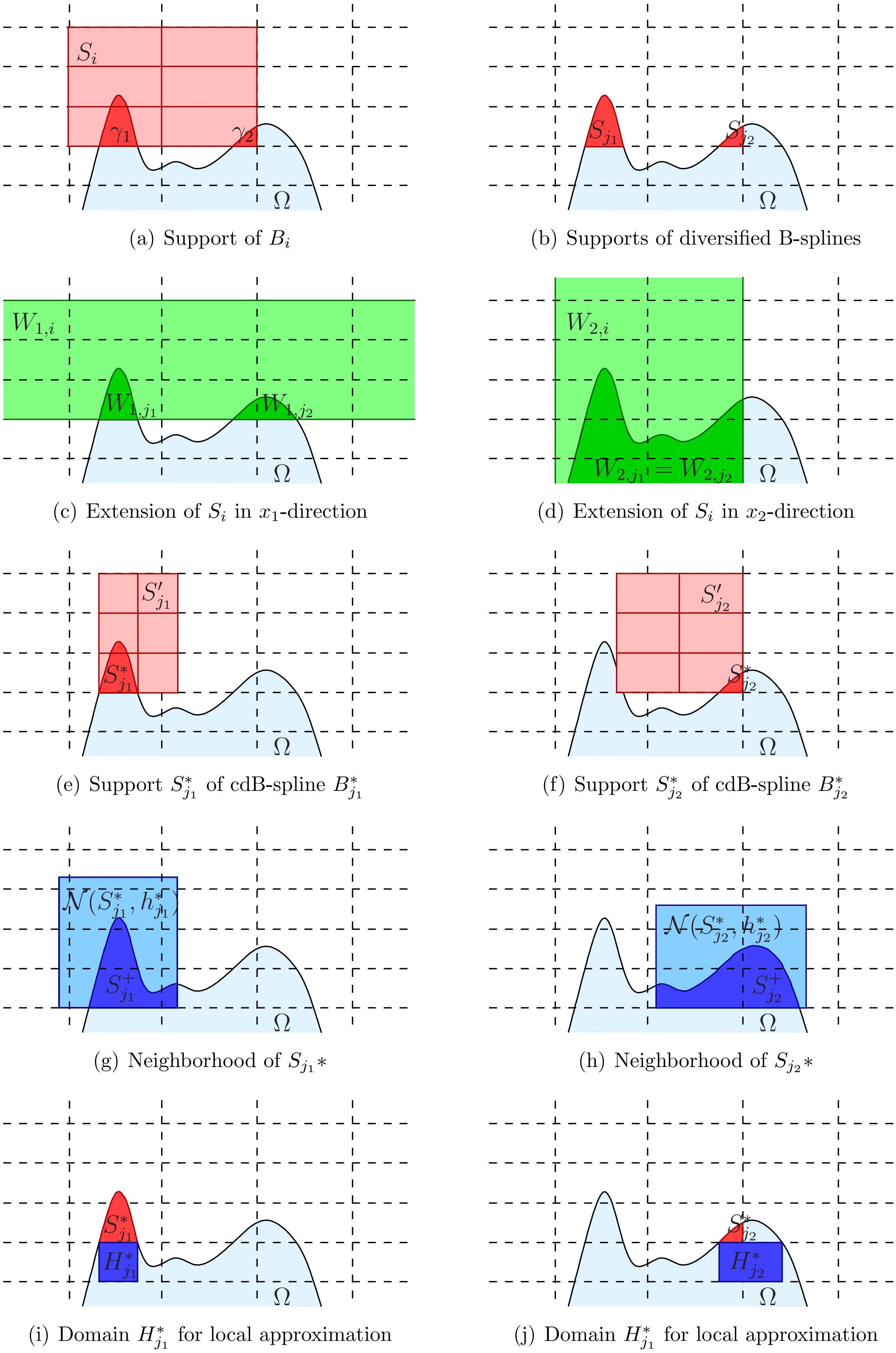


(a) Support of B_i

(b) Supports of diversified B-splines

(c) Extension of S_i in x_1-direction

(d) Extension of S_i in x_2-direction

(e) Support $S^*_{j_1}$ of cdB-spline $B^*_{j_1}$

(f) Support $S^*_{j_2}$ of cdB-spline $B^*_{j_2}$

(g) Neighborhood of $S_{j_1}*$

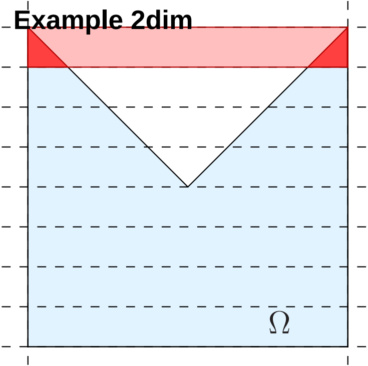
(h) Neighborhood of $S_{j_2}*$

(i) Domain $H^*_{j_1}$ for local approximation

(j) Domain $H^*_{j_1}$ for local approximation

Figure 4: Some sets used in the proof

Example 2dim



Example 3dim, (1)
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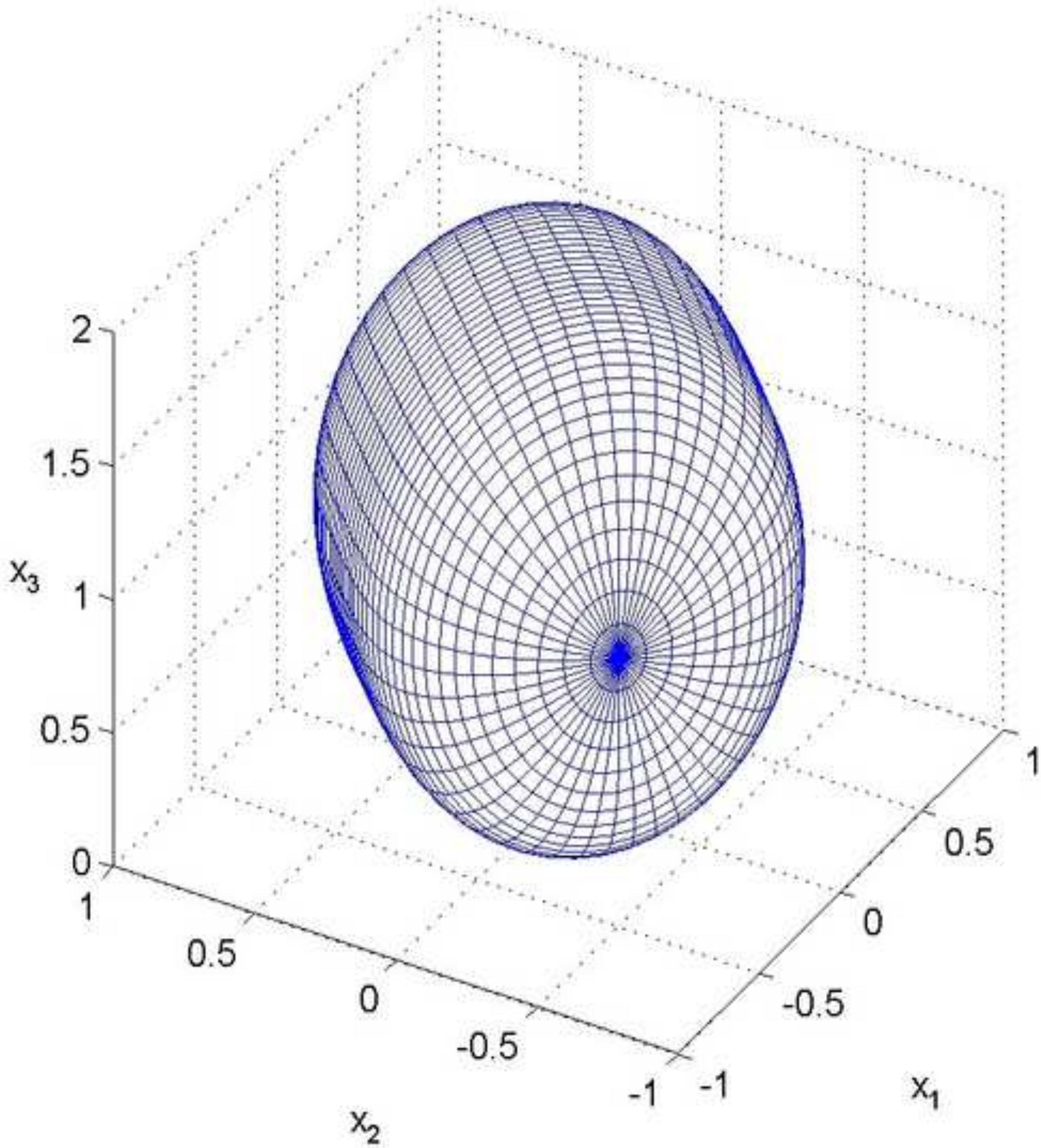

Example 3dim, (2)
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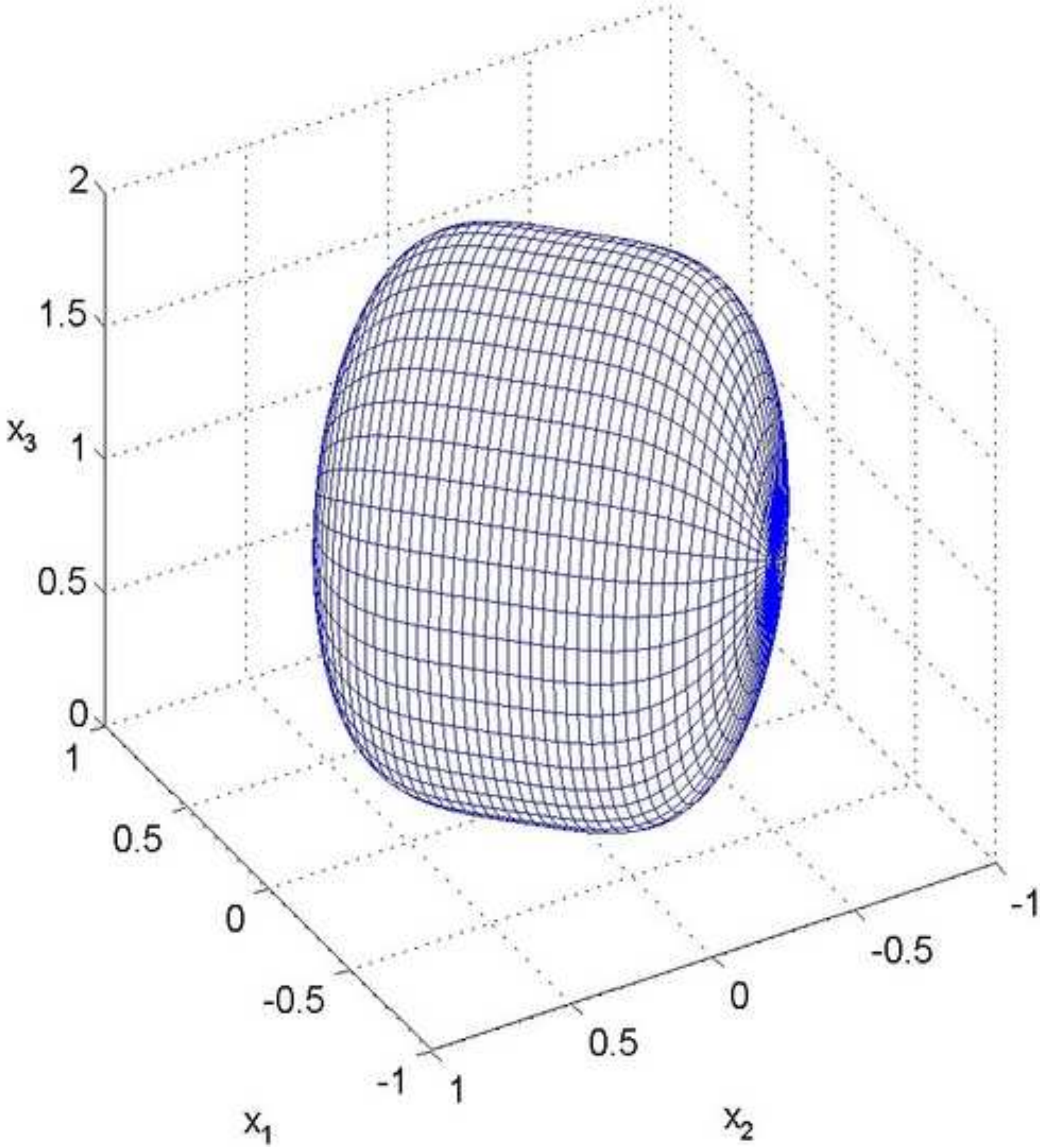
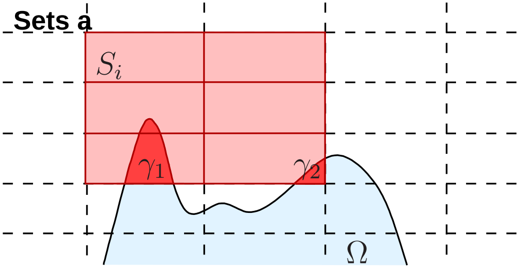
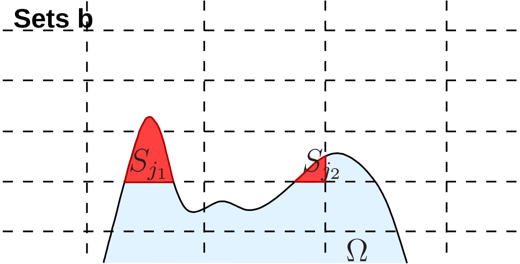


Sets a



Sets b



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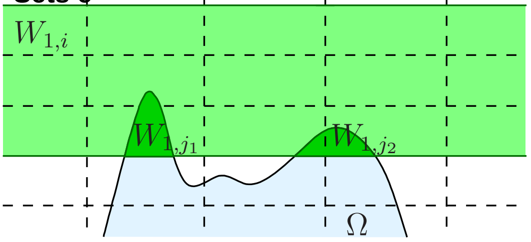

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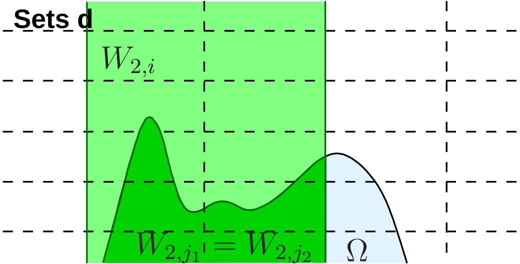

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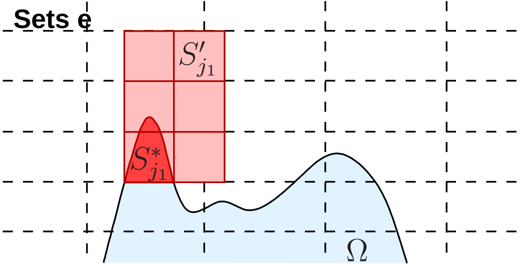

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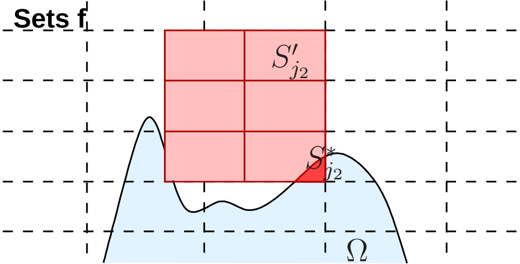
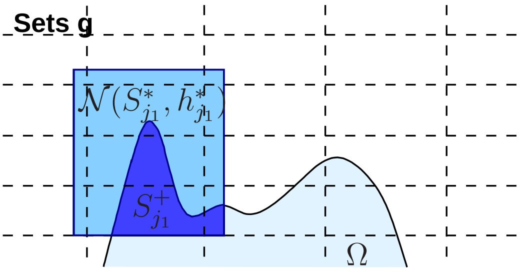
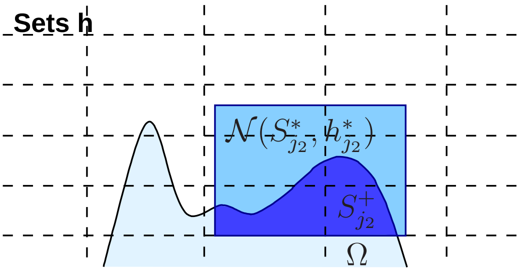
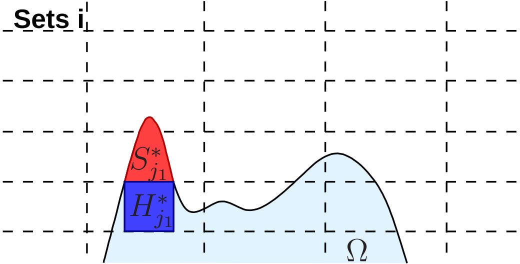
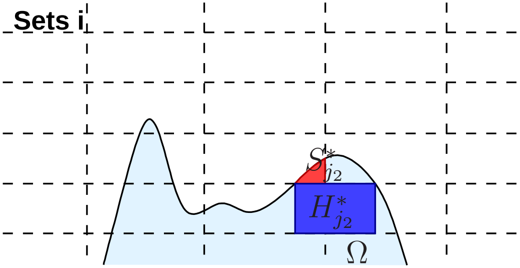



Sets g
$\mathcal{N}(S_{j_1}^*, h_{j_1}^*)$
$S_{j_1}^+$
Ω


Sets h
$\mathcal{N}(S^*_{j_2}, h^*_{j_2})$
$S^+_{j_2}$
Ω

Sets i





Sets i
$S^*_{j_2}$
$H^*_{j_2}$
Ω