INFLUENCE OF SURFACE ROUGHNESS TO SOLUTIONS OF THE BOUSSINESQ EQUATIONS WITH ROBIN BOUNDARY CONDITION

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Abstract. Every 'real existing' domain Ω is covered by microscopic asperities. We replace Ω by a sequence $(\Omega_k)_{k\in\mathbb{N}}$ of domains with rough boundaries where $\Omega_k \to \Omega$. Consider weak solutions $(u_k, \theta_k)\,, k \in \mathbb{N}$, of the Boussinesq equations in $[0, T[\times\Omega_k$ where the temperature θ_k satisfies a Robin boundary condition. Passing to the limit we will observe an additional weight factor in the Robin boundary condition which reflects the rugosity of the boundaries $(\partial\Omega_k)_{k\in\mathbb{N}}$. Motivated by this observation, fix $\Lambda \in L^\infty(\partial\Omega)\,, \Lambda \geq 1$. We will construct domains $(\Omega_k)_{k\in\mathbb{N}}$ and weak solutions $(u_k, \theta_k)_{k\in\mathbb{N}}$ of the Boussinesq equations in $[0, T[\times\Omega_k$ with $\frac{\partial\theta_k}{\partial N} + (\theta_k - \zeta) = 0$ on $]0, T[\times\partial\Omega_k$ converging to a weak solution (u, θ) in $[0, T[\times\Omega$ with $\frac{\partial\theta}{\partial N} + \Lambda(\theta - \zeta) = 0$ on $]0, T[\times\partial\Omega$; here ζ describes the external temperature. This result is essentially based on the construction of some special gradient Young measures.

1. Introduction and main results

In this paper we consider the Boussinesq equations

$$\begin{aligned} u_t - \Delta u + u\cdot\nabla u + \nabla p &= \theta g && \text{in }]0,T[\times\Omega\,,\\ \operatorname{div} u &= 0 && \text{in }]0,T[\times\Omega\,,\\ \theta_t - \Delta\theta + u\cdot\nabla\theta &= 0 && \text{in }]0,T[\times\Omega\,,\\ u &= u_0 && \text{at } t=0\,,\\ \theta &= \theta_0 && \text{at } t=0\,, \end{aligned} \tag{1.1}$$

in a domain $\Omega \subseteq \mathbb{R}^n\,, n \in \{2, 3\}$, and a finite time interval $[0, T[$. The unknowns in (1.1) are u, θ, p, where u denotes the velocity of the fluid, θ denotes the difference of the temperature to a fixed reference temperature and p denotes the pressure. The following data are given: $u_0\,, \theta_0$ are the initial values and g the gravitational force. To simplify the notation we have set the density, kinematic viscosity and thermal conductivity to 1. The Boussinesq equations constitute a model of motion of a viscous, incompressible buoyancy-driven fluid flow coupled with heat convection; for further information we refer to [20, 27]. Many researchers have investigated the Boussinesq system, see e.g. [4, 9, 12, 17, 18, 19, 22] and papers cited there. In this paper we supplement the Boussinesq system (1.1) with the *no slip boundary condition*

$$u = 0 \quad \text{on }]0,T[\times\partial\Omega \tag{1.2}$$

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and the *Robin boundary condition*

$$\frac{\partial \theta}{\partial N} + \Lambda(\theta - \zeta) = 0 \quad \text{on }]0,T[\times\partial\Omega \tag{1.3}$$

where ζ denotes the exterior temperature on $\partial\Omega$ and where $\Lambda : \partial\Omega \to]0,\infty[$ is a positive, scalar function describing the ratio of heat transfer to the difference $\theta - \zeta$.

The effect of surface roughness to weak solutions of the Navier-Strokes equations was observed in several papers. The common idea is that the 'physical' domain $\Omega \subseteq \mathbb{R}^3$ is covered by microscopic asperities and is therefore replaced by a sequence $(\Omega_k)_{k\in\mathbb{N}}$ of domains with rough boundaries where $\Omega_k \to \Omega$. It follows from the pioneering work [10] that for periodically, smooth oscillating Ω_k the complete slip boundary condition on $\partial\Omega_k$ transforms into the no slip boundary condition on $\partial\Omega$ if there is 'enough boundary rugosity'. Introducing a Young measure which describes the character of oscillations of $(\partial\Omega_k)_{k\in\mathbb{N}}$ the authors in [6] removed the periodicity assumption on Ω_k. For further results and approaches we refer to [3, 5, 7, 8, 11].

In the present paper we want to investigate the influence of surface roughness to weak solutions of the Boussinesq equations where the temperature satisfies a Robin boundary condition. Consider $n \in \{2, 3\}$, and

$$\Omega = \{\, (x', x_n) \in \mathbb{R}^n;\ x_n > 0;\ x' \in \mathbb{R}^{n-1}\,\}. \tag{1.4}$$

We replace the 'ideal' domain Ω by the sequence $(\Omega_k)_{k\in\mathbb{N}}$ of 'domains with rough boundaries' defined by

$$\Omega_k = \{\, (x', x_n) \in \mathbb{R}^{n-1};\ x_n > -h_k(x');\ x' \in \mathbb{R}^{n-1}\}\,, \quad k \in \mathbb{N}\,, \tag{1.5}$$

where $(h_k)_{k\in\mathbb{N}}$ are *admissible functions*, i.e. non-negative functions $h_k : \mathbb{R}^{n-1} \to \mathbb{R}_0^+$, $k \in \mathbb{N}$, which are equi-Lipschitz continuous and $h_k \to 0$ uniformly on all compact subsets of $\mathbb{R}^{n-1}$. By definition, the equi-Lipschitz continuity means that there is a constant $L > 0$ such that

$$\frac{|h_k(x') - h_k(y')|}{|x' - y'|} \le L \quad \forall x', y' \in \mathbb{R}^{n-1}, x' \neq y' \quad \forall k \in \mathbb{N}.$$

It follows that $(\nabla h_k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^{n-1})$. By Theorem 4.1 we can assume (after a not relabelled subsequence) that $(\nabla h_k)_{k\in\mathbb{N}}$ generates a Young measure $\nu = (\nu_{x'})_{x'\in\mathbb{R}^{n-1}}$.

Consider weak solutions (u_k, θ_k), $k \in \mathbb{N}$, of the Boussinesq equations (1.1) in $[0,T[\times\Omega_k$ satisfying the energy inequalities (1.11), (1.12) below and the following boundary conditions:

$$u_k = 0\,, \quad \frac{\partial \theta_k}{\partial N} + (\theta_k - \zeta) = 0 \quad \text{on }]0,T[\times\partial\Omega_k. \tag{1.6}$$

Let (u, θ) be a weak limit of $(u_k, \theta_k)_{k\in\mathbb{N}}$ in $[0,T[\times\Omega$. In Theorem 1.2 we will show that (u, θ) is a weak solution of (1.1) in $[0,T[\times\Omega$ and

$$u = 0\,, \quad \frac{\partial \theta}{\partial N} + \Lambda(\theta - \zeta) = 0 \quad \text{on }]0,T[\times\partial\Omega \tag{1.7}$$

where $\Lambda : \partial\Omega \to [1,\infty[$ describes an additional heat transfer coefficient to the exterior which is due to the rugosity of the boundaries. This function can be computed using the Young measure ν, see (1.16).

Motivated by this result, fix $\Lambda \in L^\infty(\partial\Omega)\,,\Lambda \geq 1$. In Theorem 1.3 we will construct domains $(\Omega_k)_{k\in\mathbb{N}}$ as in (1.5) and weak solutions $(u_k,\theta_k)_{k\in\mathbb{N}}$ of (1.1) in $[0,T[\times\Omega_k$ with (1.6) converging to a weak solution (u,θ) of (1.1) in $[0,T[\times\Omega$ with (1.7). This result makes essentially use of the Young measures constructed in Theorem 4.3.

We need the following spaces of test functions:

$$C_0^\infty([0,T[;C_{0,\sigma}^\infty(\Omega)) := \{\, w|_{[0,T[\times\Omega};\ w\in C_0^\infty(]-1,T[\times\Omega)\,;\ \mathrm{div} w = 0\,\}\,,$$
$$C_0^\infty([0,T[\times\overline{\Omega}) := \{\, \phi|_{[0,T[\times\overline{\Omega}}\,;\ \phi\in C_0^\infty(]-1,T[\times\mathbb{R}^n)\,\}.$$

Motivated by the concept of a weak solution of the Navier-Stokes equations in the sense of Leray-Hopf we arrive at the following

Definition 1.1. Let $\Omega\subseteq\mathbb{R}^n\,,n\in\{2,3\}$, be a uniform Lipschitz domain, let $0<T<\infty, g\in L^\infty(]0,T[\times\Omega)$, let $\zeta\in L^2(0,T;L^2(\partial\Omega))$, $\Lambda\in L^\infty(\partial\Omega)$ with $\Lambda(x)\geq 0$ for almost all $x\in\partial\Omega$. Further assume $u_0\,,\theta_0\in L^2(\Omega)$.
(i) A pair

$$\begin{aligned} &u\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))\,,\\ &\theta\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)) \end{aligned} \tag{1.8}$$

is called a *weak solution* of the Boussinesq equations (1.1) with no slip boundary condition (1.2) and Robin boundary condition (1.3) if

$$\begin{aligned} -\int_0^T \langle u,w_t\rangle_\Omega\,dt + \int_0^T\langle\nabla u,\nabla w\rangle_\Omega\,dt + \int_0^T\langle u\cdot\nabla u,w\rangle_\Omega\,dt\\ = \int_0^T\langle\theta g,w\rangle_\Omega\,dt + \langle u_0,w(0)\rangle_\Omega \end{aligned} \tag{1.9}$$

for all $w\in C_0^\infty([0,T[;C_{0,\sigma}^\infty(\Omega))$ and

$$\begin{aligned} -\int_0^T \langle \theta,\phi_t\rangle_\Omega\,dt + \int_0^T\langle\nabla\theta,\nabla\phi\rangle_\Omega\,dt + \int_0^T\langle u\cdot\nabla\theta,\phi\rangle_\Omega\,dt\\ = -\int_0^T\langle\Lambda(\theta-\zeta),\phi\rangle_{\partial\Omega}\,dt + \langle\theta_0,\phi(0)\rangle_\Omega \end{aligned} \tag{1.10}$$

for all $\phi\in C_0^\infty([0,T[\times\overline{\Omega})$. In the identities above $\langle\cdot,\cdot\rangle_\Omega$ denotes the usual L^2-scalar product in Ω.
(ii) Consider a weak solution of (1.1) with (1.2), (1.3). Then (u,θ) satisfies the *energy inequalities* if

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t\|\nabla u\|_2^2\,d\tau \leq \frac{1}{2}\|u_0\|_2^2 + \int_0^t\langle\theta g,u\rangle_\Omega\,d\tau\,, \tag{1.11}$$

$$\begin{aligned} &\frac{1}{2}\|\theta(t)\|_2^2 + \int_0^t\|\nabla\theta\|_2^2\,d\tau + \int_0^t\|\sqrt{\Lambda}\,\theta\|_{2,\partial\Omega}^2\,d\tau\\ &\quad\leq \frac{1}{2}\|\theta_0\|_2^2 + \int_0^t\langle\Lambda\zeta,\theta\rangle_{\partial\Omega}\,d\tau \end{aligned} \tag{1.12}$$

are satisfied for almost all $t\in[0,T[$.

Consider a weak solution (u,θ) as above. After a redefinition on a null set of $[0,T[$ we have that $u:[0,T[\to L^2_\sigma(\Omega)$ and $\theta:[0,T[\to L^2(\Omega)$ are weakly continuous functions and $u(0)=Pu_0\,,\theta(0)=\theta_0$ where $P:L^2(\Omega)\to L^2_\sigma(\Omega)$

denotes the usual Helmholtz projection, see [23, IV, Lemma 2.4.2]. Further, there exists a distribution p, called an associated pressure, such that

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = \theta g$$

holds in the sense of distributions in $]0, T[\times\Omega$, see [23, V.1.7]. Our first main result reads as follows.

Theorem 1.2. *Let $n \in \{2, 3\}$, let $\Omega\,, (\Omega_k)_{k\in\mathbb{N}}$ be as in (1.4), (1.5), and $0 < T < \infty$. Further let $g \in L^\infty(]0, T[\times\mathbb{R}^n)\,, \zeta \in L^2(0, T; H^1(\mathbb{R}^n))$, and $u_0\,, \theta_0 \in L^2(\mathbb{R}^n)$. Assume that $(\nabla h_k)_{k\in\mathbb{N}}$ generates the Young measure $\nu = (\nu_{x'})_{x'\in\mathbb{R}^{n-1}}$. For every $k \in \mathbb{N}$ let (u_k, θ_k) be a weak solution of the Boussinesq equations (1.1) in $[0, T[\times\Omega_k$ (with data $g, \zeta|_{]0,T[\times\Omega_k}$ and $u_0, \theta_0|_{\Omega_k}$) satisfying the energy inequalities (1.11), (1.12) (where Ω is replaced by Ω_k) and*

$$u_k = 0\,, \quad \frac{\partial\theta_k}{\partial N} + (\theta_k - \zeta) = 0 \quad \text{on }]0, T[\times\partial\Omega_k. \tag{1.13}$$

Consider $u\,, \theta \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ such that

$$\begin{aligned} u_k &\underset{k\to\infty}{\rightharpoonup^*} u \quad \text{in } L^\infty(0, T; L^2(\Omega))\,, \quad u_k \underset{k\to\infty}{\rightharpoonup} u \quad \text{in } L^2(0, T; H^1(\Omega))\,, \\ \theta_k &\underset{k\to\infty}{\rightharpoonup^*} \theta \quad \text{in } L^\infty(0, T; L^2(\Omega))\,, \quad \theta_k \underset{k\to\infty}{\rightharpoonup} \theta \quad \text{in } L^2(0, T; H^1(\Omega)). \end{aligned} \tag{1.14}$$

Then (u, θ) is a weak solution of (1.1) in $[0, T[\times\Omega$ (with data $g, \zeta|_{]0,T[\times\Omega}$ and $u_0, \theta_0|_\Omega$) satisfying

$$u = 0\,, \quad \frac{\partial\theta}{\partial N} + \Lambda(\theta - \zeta) = 0 \quad \text{on }]0, T[\times\partial\Omega \tag{1.15}$$

where the function $\Lambda : \partial\Omega \to \mathbb{R}$ fulfils

$$\Lambda(x', 0) = \int_{\mathbb{R}^{n-1}} \sqrt{1 + |\lambda|^2}\, d\nu_{x'}(\lambda) \tag{1.16}$$

for almost all $x = (x', 0) \in \partial\Omega$ with $x' \in \mathbb{R}^{n-1}$.

We proceed with the following theorem.

Theorem 1.3. *Let $n \in \{2, 3\}$, let $\Omega := \{\, (x', x_n) \in \mathbb{R}^n;\ x_n > 0\,, x' \in \mathbb{R}^{n-1}\,\}$, and $0 < T < \infty$. Further assume $g \in L^\infty(]0, T[\times\mathbb{R}^n), \zeta \in L^2(0, T; H^1(\mathbb{R}^n))$, and u_0, $\theta_0 \in L^2(\mathbb{R}^n)$. Let $\Lambda : \partial\Omega \to [1, \infty[$ be a bounded, measurable function. If $n = 3$ we need additionally $g \in L^4(0, T; L^2(\mathbb{R}^n))$. Then there exist admissible functions $(h_k)_{k\in\mathbb{N}}$ and domains $(\Omega_k)_{k\in\mathbb{N}}$ defined as in (1.5) such that the following properties are fulfilled:*

- *There exist weak solutions $(u_k, \theta_k)\,, k \in \mathbb{N}$, of (1.1) in $[0, T[\times\Omega_k$ (with data $g, \zeta|_{]0,T[\times\Omega_k}$ and $u_0, \theta_0|_{\Omega_k}$) satisfying*

$$u_k = 0\,, \quad \frac{\partial\theta_k}{\partial N} + (\theta_k - \zeta) = 0 \quad \text{on }]0, T[\times\partial\Omega_k. \tag{1.17}$$

- *There exists a weak solution (u, θ) of (1.1) in $[0, T[\times\Omega$ (with data $g, \zeta|_{]0,T[\times\Omega}$ and $u_0, \theta_0|_\Omega$) satisfying*

$$u = 0\,, \quad \frac{\partial\theta}{\partial N} + \Lambda(\theta - \zeta) = 0 \quad \text{on }]0, T[\times\partial\Omega. \tag{1.18}$$

- *We have*

$$u_k \underset{k\to\infty}{\rightharpoonup^*} u \quad in\ L^\infty(0,T;L^2(\Omega))\,, \quad u_k \underset{k\to\infty}{\rightharpoonup} u \quad in\ L^2(0,T;H^1(\Omega))\,,$$
$$\theta_k \underset{k\to\infty}{\rightharpoonup^*} \theta \quad in\ L^\infty(0,T;L^2(\Omega))\,, \quad \theta_k \underset{k\to\infty}{\rightharpoonup} \theta \quad in\ L^2(0,T;H^1(\Omega)).$$

In the two-dimensional case we can use (see Theorem 3.1 (iii)) the uniqueness of weak solutions of the Boussinesq equations (1.1) with (1.2), (1.3) to obtain the following 'stronger version' of Theorem 1.3.

Theorem 1.4. *Let* $\Omega := \{ (x', x_2) \in \mathbb{R}^2;\ x_2 > 0\,, x' \in \mathbb{R} \}$*, let* $0 < T < \infty$*, let* $g \in L^\infty(]0,T[\times\mathbb{R}^2)\,, \zeta \in L^2(0,T;H^1(\mathbb{R}^2))$*, let* $u_0\,, \theta_0 \in L^2(\mathbb{R}^2)$*. Consider a bounded, measurable function* $\Lambda : \partial\Omega \to [1,\infty[$*. Let* (u,θ) *be the unique weak solution of (1.1) in* $[0,T[\times\Omega$ *(with data* $g, \zeta|_{]0,T[\times\Omega}$ *and* $u_0, \theta_0|_\Omega$*) satisfying*

$$u = 0\,, \quad \frac{\partial\theta}{\partial N} + \Lambda(\theta - \zeta) = 0 \quad on\ ]0,T[\times\partial\Omega.$$

Then there exist admissible functions $(h_k)_{k\in\mathbb{N}}$ *and domains* $(\Omega_k)_{k\in\mathbb{N}}$ *defined as in (1.5) such that if* $(u_k,\theta_k)\,, k \in \mathbb{N}$*, denotes the unique weak solution of (1.1) in* $[0,T[\times\Omega_k$ *(with data* $g, \zeta|_{]0,T[\times\Omega_k}$ *and* $u_0, \theta_0|_{\Omega_k}$*) satisfying*

$$u_k = 0\,, \quad \frac{\partial\theta_k}{\partial N} + (\theta_k - \zeta) = 0 \quad on\ ]0,T[\times\partial\Omega_k$$

then

$$u_k \underset{k\to\infty}{\rightharpoonup^*} u \quad in\ L^\infty(0,T;L^2(\Omega))\,, \quad u_k \underset{k\to\infty}{\rightharpoonup} u \quad in\ L^2(0,T;H^1(\Omega))\,,$$
$$\theta_k \underset{k\to\infty}{\rightharpoonup^*} \theta \quad in\ L^\infty(0,T;L^2(\Omega))\,, \quad \theta_k \underset{k\to\infty}{\rightharpoonup} \theta \quad in\ L^2(0,T;H^1(\Omega)).$$

The paper is organized as follows. In Section 2 we present some preliminaries. The following section investigates existence and uniqueness of weak solutions of (1.1) with (1.2), (1.3). In Section 4 we present the needed results from the theory of Young measures and construct some 'special gradient Young measures'. After some preparation in Section 5 we prove Theorem 1.2. Finally, Section 7 is dedicated to the proof of Theorem 1.3.

2. Preliminaries

Let $\Omega \subseteq \mathbb{R}^n\,, n \in \mathbb{N}$, be an open set, let $1 \le p \le \infty\,, k \in \mathbb{N}$. We denote by $L^p(\Omega), W^{k,p}(\Omega), W_0^{k,p}(\Omega)$ the usual Lebesgue and Sobolev spaces with norm $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}(\Omega)}$, respectively. We set $H^k(\Omega) := W^{k,2}(\Omega)$ and $H_0^k(\Omega) := W_0^{k,2}(\Omega)$. Furthermore $H^{-1}(\Omega) := H_0^1(\Omega)'$. For $s \in \mathbb{R}^+ \setminus \mathbb{N}$ let $W^{s,2}(\Omega)$ denote the usual Sobolev-Slobodeckij space, see [25, Definition II.3.1]. Looking at [25, Satz II.5.3, Satz II.5.4 and Satz II.7.9] we get that for a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ and $0 \le s_2 < s_1 \le 1$ the imbedding

$$W^{s_1,2}(\Omega) \hookrightarrow W^{s_2,2}(\Omega) \tag{2.1}$$

is compact. For two measurable functions f, g with the property $f\cdot g \in L^1(\Omega)$ where $f \cdot g$ means the usual scalar product of scalar, vector or matrix fields, we set $\langle f, g\rangle_\Omega := \int_\Omega f(x)\cdot g(x)\,dx$. Note that (in general) the symbol $L^p(\Omega)$ etc. will be used for spaces of scalar, vector or matrix-valued functions. By $v \otimes v = (v_i v_j)_{i,j=1}^n$ we denote the usual tensor product of $v \in \mathbb{R}^n$. Let $C^m(\Omega)\,, m = 0, 1, \ldots, \infty$, denote the usual space of functions for which all partial derivatives of finite order $|\alpha| \le m$ exist and are continuous and let $C^m(\overline{\Omega}) := \{\phi|_{\overline{\Omega}}; \phi \in C^m(\mathbb{R}^n)\}$. As usual, $C_0^m(\Omega)$ is the set of all functions from $C^m(\Omega)$ with compact support in Ω and let $C_0^\infty(]0,T[\times\Omega)$ denote the space of smooth function with compact support in $]0,T[\times\Omega$. Further we introduce $C_{0,\sigma}^\infty(\Omega) := \{\, v \in C_0^\infty(\Omega); \operatorname{div} v = 0\,\}$ as the space of smooth solenoidal vector fields. For $1 < q < \infty$ we define the spaces $L_\sigma^q(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_q}$ and $W_{0,\sigma}^{1,2}(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{W^{1,2}}}$. For $1 \le q \le \infty$ let q' be the dual exponent such that $\frac{1}{q} + \frac{1}{q'} = 1$. It is well known that $L_\sigma^q(\Omega)' \cong L_\sigma^{q'}(\Omega)\,, 1 < q < \infty$, using the standard pairing $\langle\cdot,\cdot\rangle_\Omega$. Given a Banach space X and an interval $]0,T[\,, 0 < T < \infty$, we denote by $L^p(0,T;X)$ the Banach space of (equivalence classes) of strongly measurable functions $f :]0,T[\to X$ such that $\|f\|_p := \left(\int_0^T \|f(t)\|_X^p \, dt\right)^{\frac{1}{p}} < \infty$ if $1 \le p < \infty$ and $\|f\|_\infty := \operatorname{ess\,sup}_{t\in]0,T[} \|f(t)\|_X$, if $p = \infty$. If $X = L^q(\Omega)\,, 1 \le q \le \infty$, the norm in the space $L^p(0,T;L^q(\Omega))$ is denoted by $\|\cdot\|_{q,p;\Omega;T}$.

Let $\Omega \subseteq \mathbb{R}^n\,, n \ge 2$, be a uniform Lipschitz domain (see f. ex. [1, 4.9].) Let dS denote the surface measure on $\partial\Omega$. The space $L^2(\partial\Omega)$ should denote the usual Lebesgue space on $\partial\Omega$ with scalar product $\langle\cdot,\cdot\rangle_{\partial\Omega}$. If Ω is bounded and $0 < s < 1$ we define the trace space $W^{s,2}(\partial\Omega)$ as in [23, I.3.6]. Further (see [25, Satz II.8.7]) for $\frac{1}{2} < s \le 1$ there exists a continuous, linear trace operator $T : W^{s,2}(\Omega) \to W^{s-\frac{1}{2},2}(\partial\Omega)$ with the property $T\phi = \phi|_{\partial\Omega}$ for $\phi \in C^1(\overline{\Omega})$. When there is no possibility of confusion we will identify $\phi \in W^{s,2}(\Omega)$ with its trace and just write ϕ instead of $\phi|_{\partial\Omega} := T\phi$.

Further if $(X_n)_{n\in\mathbb{N}}$ is a sequence of Banach spaces we will write that a sequence $v_n \in X_n, n \in \mathbb{N}$, is bounded in $(X_n)_{n\in\mathbb{N}}$ if there is a constant $M > 0$ such that $\|v_n\|_{X_n} \le M$ for all $n \in \mathbb{N}$.

3. Existence and uniqueness of weak solutions

The following theorem is needed for the construction of the weak solutions stated in Theorems 1.3, 1.4. It is not known whether weak solutions of (1.1) are uniquely determined if $n = 3$ since the corresponding problem for the Navier-Stokes equations is not solved.

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^n\,, n \in \{2,3\}$, be a uniform Lipschitz domain, let $0 < T < \infty, g \in L^\infty(]0,T[\times\Omega)$, let $\zeta \in L^2(0,T;L^2(\partial\Omega))\,, \Lambda \in L^\infty(\partial\Omega)$ with $\Lambda(x) \ge 0$ for almost all $x \in \partial\Omega$. Further assume $u_0\,, \theta_0 \in L^2(\Omega)$. Then the following statements are satisfied:*

(i) *There exists a weak solution of the Boussinesq equations of (1.1) with boundary conditions (1.2), (1.3).*
(ii) *If $n = 3$ and if additionally $g \in L^4(0,T;L^2(\Omega))$ holds then there exists a weak solution of (1.1) with (1.2), (1.3) satisfying the energy inequalities (1.11), (1.12) for a.a. $t \in [0,T[$.*
(iii) *If $n = 2$ there exists exactly one weak solution (u,θ) of the Boussinesq equations (1.1) with (1.2), (1.3). After a redefinition on a null set of $[0,T[$ we have that $u : [0,T[\to L_\sigma^2(\Omega)\,, \theta : [0,T[\to L^2(\Omega)$ are strongly continuous and the energy equalities are satisfied, i.e. (1.11), (1.12) hold as equality for all $t \in [0,T[$.*

The most parts of the proof of this theorem are based on well known arguments. Therefore we will only give a sketch of proof and focus on the parts of the proof which are not standard, i.e. (3.4) and statement (ii).

Sketch of proof. Choose linearly independent vectors $w_k \in C^\infty_{0,\sigma}(\Omega), k \in \mathbb{N}$, such that $\operatorname{span}\{w_k; k \in \mathbb{N}\}$ is dense in $W^{1,2}_{0,\sigma}(\Omega)$ and $\langle w_i, w_j\rangle = \delta_{i,j}$ for $i, j \in \mathbb{N}$. Further, choose linearly independent vectors $\psi_k \in C^\infty_0(\overline{\Omega})\,, k \in \mathbb{N}$, such that $\operatorname{span}\{\psi_k; k \in \mathbb{N}\}$ is dense in $H^1(\Omega)$ and $\langle \psi_i, \psi_j\rangle_\Omega = \delta_{i,j}$ for $i, j \in \mathbb{N}$.

Fix $k \in \mathbb{N}$. Using the well known Galerkin procedure and existence theory for ordinary differential equations we can prove (see [24, Chapter 3, Theorem 3.1], respectively [19, Theorem 1]) that there exist unique functions $u_k(t) := \sum_{i=1}^k \alpha_{i,k}(t) w_i$ and $\theta_k(t) := \sum_{i=1}^k \beta_{i,k}(t)\psi_i$ which satisfy the finite dimensional **Galerkin approximation system**

$$\frac{d}{dt}\langle u_k, w_i\rangle_\Omega + \langle \nabla u_k, \nabla w_i\rangle_\Omega + \langle u_k \cdot \nabla u_k, w_i\rangle_\Omega = \langle \theta_k g, w_i\rangle_\Omega \tag{Gal1}$$

$$\text{for a.a. } t \in]0, T[\,, \quad \forall i = 1, \dots, k\,,$$

$$\frac{d}{dt}\langle \theta_k, \psi_l\rangle_\Omega + \langle \nabla\theta_k, \nabla\psi_l\rangle_\Omega + \langle u_k \cdot \nabla\theta_k, \psi_l\rangle_\Omega + \langle \Lambda(\theta_k - \zeta), \psi_l\rangle_{\partial\Omega} = 0 \tag{Gal2}$$

$$\text{for a.a. } t \in]0, T[\,, \quad \forall l = 1, \dots, k\,,$$

$$u_k(0) = \sum_{i=1}^k \langle u_0, w_i\rangle_\Omega\, w_i\,, \tag{Gal3}$$

$$\theta_k(0) = \sum_{i=1}^k \langle \theta_0, \psi_i\rangle_\Omega\, \psi_i. \tag{Gal4}$$

Further, there exists a (not relabelled) subsequence $(u_k, \theta_k)_{k\in\mathbb{N}}$ and (u, θ) such that

$$u_k \underset{k\to\infty}{\rightharpoonup^*} u \quad \text{in } L^\infty(0, T; L^2(\Omega))\,, \quad u_k \underset{k\to\infty}{\rightharpoonup} u \quad \text{in } L^2(0, T; W^{1,2}_{0,\sigma}(\Omega))\,, \tag{3.1}$$

$$\theta_k \underset{k\to\infty}{\rightharpoonup^*} \theta \quad \text{in } L^\infty(0, T; L^2(\Omega))\,, \quad \theta_{m_k} \underset{k\to\infty}{\rightharpoonup} \theta \quad \text{in } L^2(0, T; H^1(\Omega)). \tag{3.2}$$

Analogously as in [24, Chapter 3, Theorem 3.1] it follows that for $0 < \gamma < \frac{1}{4}$ the sequence $\left(\int_\mathbb{R} |\tau|^{2\gamma} \|\widehat{\theta_k}(\tau)\|^2_{2,\Omega}\, d\tau\right)_{k\in\mathbb{N}}$ is bounded where $\widehat{\theta_k} : \mathbb{R} \to L^2(\Omega)$ denotes the Fourier transform of $1_{[0,T[}\theta_k$. Fix a bounded Lipschitz domain $G \subseteq \Omega$ and $\frac{1}{2} < s < 1$. Consequently, (see 2.1)) the imbeddings

$$H^1(G) \underset{\text{compact}}{\hookrightarrow} W^{s,2}(G) \underset{\text{continuous}}{\hookrightarrow} L^2(G)\,, \tag{3.3}$$

and Theorem [24, Chapter 3, Theorem 2.2] imply that $(\theta_k)_{k\in\mathbb{N}}$ contains a strongly convergent subsequence in $L^2(0, T; W^{s,2}(G))$. Due to the continuous trace operator $W^{s,2}(\Omega) \to L^2(\partial\Omega))$ it follows

$$\theta_k \underset{k\to\infty}{\to} \theta \quad \text{strongly in } L^2(0, T; L^2(G)) \cap L^2(0, T; L^2(\partial G)). \tag{3.4}$$

A similar argumentation as above (see also [24, III, (3.41)]) shows

$$u_k \underset{k\to\infty}{\to} u \quad \text{strongly in } L^2(0, T; L^2(G)). \tag{3.5}$$

Using (Gal1)- (Gal4) and (3.1), (3.2), (3.4), (3.5) we can prove that (u, θ) is a weak solution of (1.1) with (1.2), (1.3).

Proof of (ii). The crucial point in the proof of (1.11) is to show

$$\lim_{k\to\infty} \int_0^t \langle \theta_k u_k, g\rangle_\Omega \, d\tau = \int_0^t \langle \theta u, g\rangle_\Omega \, d\tau \tag{3.6}$$

for a.a. $t \in [0, T[$. Sobolev's imbedding theorem implies that $(\theta_k u_k)_{k\in\mathbb{N}}$ is bounded in $L^{4/3}(0, T; L^2(\Omega))$. Looking at (3.4), (3.5) it follows $\theta_k u_k \rightharpoonup \theta u$ in $L^{4/3}(0, T; L^2(\Omega))$. Now we make use of the assumption $g \in L^4(0, T; L^2(\Omega))$ to obtain (3.6). From (Gal1), (Gal3) we get

$$\frac{1}{2}\|u_k(t)\|_2^2 + \int_0^t \|\nabla u_k\|_2^2 \, d\tau = \frac{1}{2}\|u_0\|_2^2 + \int_0^t \langle \theta_k g, u_k\rangle_\Omega \, d\tau \tag{3.7}$$

for all $k \in \mathbb{N}$ and $t \in [0, T[$. By (3.6), (3.7) and (3.1), (3.5) it follows that (1.11) is satisfied for almost all $t \in [0, T[$. Using (3.1), (3.2), (3.4) we can show in a standard way that (1.12) is fulfilled for a.a. $t \in [0, T[$.

Proof of (iii). This can be shown using the ideas presented in [24, Chapter 3, Section 3.3] in a standard way. □

4. Young measures

Our main tool to analyse the boundary behaviour of weak solutions of the Boussinesq equations is the theory of Young measures. Theorem 4.1 deals with the existence of Young measures generated by bounded sequences in $L^\infty(\Omega)$. Afterwards, we will construct some 'special gradient Young measures' which will be needed for the existence of the domains with rough boundaries stated in Theorem 1.3 and Theorem 1.4.

Consider $n, d \in \mathbb{N}$ and a measurable set $\Omega \subseteq \mathbb{R}^n$. A *Carathéodory function* is a function $\psi : \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that for almost all $x \in \Omega$ the function $\psi(x, \cdot) : \mathbb{R}^d \to \mathbb{R}$ is a continuous function and for all $\lambda \in \mathbb{R}^d$ the function $\psi(\cdot, \lambda) : \Omega \to \mathbb{R}$ is measurable. We denote by $\mathcal{B}(\mathbb{R}^d)$ the σ-algebra of Borel sets on $\mathbb{R}^d$, define $C_0(\mathbb{R}^d) := \{\, \phi \in C(\mathbb{R}^d); \lim_{\lambda\to\infty} \phi(\lambda) = 0 \,\}$. A Young measure $\nu = (\nu_x)_{x\in\Omega}$ is a family of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that for all $\phi \in C_0(\mathbb{R}^d)$ the function

$$\Omega \to \mathbb{R}, \quad x \mapsto \int_{\mathbb{R}^d} \phi(\lambda)\, d\nu_x(\lambda)$$

is measurable. We will use the notation $\nu = (\nu_x)_{x\in\Omega}$. Consider measurable function $z_k : \Omega \to \mathbb{R}^d, k \in \mathbb{N}$. We say that $(z_k)_{k\in\mathbb{N}}$ generates the Young measure $\nu = (\nu_x)_{x\in\Omega}$ if for all $\phi \in C_0(\mathbb{R}^d)$ we have

$$\phi(z_k) \underset{k\to\infty}{\overset{*}{\rightharpoonup}} \overline{\phi} \quad \text{in } L^\infty(\Omega) \tag{4.1}$$

where

$$\overline{\phi}(x) := \int_{\mathbb{R}^d} \phi(\lambda)\, d\nu_x(\lambda) \quad \text{for a.a. } x \in \Omega. \tag{4.2}$$

We proceed with the following theorem which plays a crucial role in this paper. For a proof we refer to [2, Section 2]. We remark that $\psi(\cdot, z_{m_k}(\cdot))$ is an abbreviation for the function $x \mapsto \psi(x, z_{m_k}(x))\,, x \in A$.

Theorem 4.1. *Let $n, d \in \mathbb{N}$, let $\Omega \subseteq \mathbb{R}^n$ be a measurable set, let $z_k : \Omega \to \mathbb{R}^d$, $k \in \mathbb{N}$, be measurable functions, assume that $(z_k)_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. Then there exist a subsequence $(z_{m_k})_{k\in\mathbb{N}}$ and a Young measure $\nu = (\nu_x)_{x\in\Omega}$ such that the following properties are satisfied:*

(i) *The sequence $(z_{m_k})_{k\in\mathbb{N}}$ generates ν.*
(ii) *There is a compact subset $K \subseteq \mathbb{R}^d$ with $\operatorname{supp}(\nu_x) \subseteq K$ for a.a. $x \in \Omega$.*
(iii) *Let $A \subseteq \Omega$ be a measurable set, let $\psi : A \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function with the property that $\big(\psi(\cdot, z_{m_k}(\cdot))\big)_{k\in\mathbb{N}}$ is a weakly convergent sequence in $L^1(A)$. Then $\psi(x,\cdot)$ is integrable with respect to ν_x for a.a. $x \in A$. Define*

$$\overline{\psi}(x) := \int_{\mathbb{R}^d} \psi(x,\lambda)\, d\nu_x(\lambda) \qquad \text{for a.a. } x \in A.$$

Then $\overline{\psi} \in L^1(A)$ and

$$\psi(\cdot, z_{m_k}(\cdot)) \underset{k\to\infty}{\rightharpoonup} \overline{\psi} \qquad \text{in } L^1(A).$$

The rest of this section is dedicated to the proof of Theorem 4.3 below. First, we consider the 'homogeneous case' of this theorem.

Lemma 4.2. *Let $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded Lipschitz domain, let $z \in \mathbb{R}^n$. Define the (homogeneous) Young measure $\nu = (\nu_x)_{x\in\Omega}$ by*

$$\nu_x := \frac{1}{2}\big(\delta_z + \delta_{-z}\big)\,, \quad x \in \Omega.$$

Then there exist non-negative functions $u_k : \Omega \to \mathbb{R}_0^+$, $k \in \mathbb{N}$, fulfilling the following properties:

(i) *The sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,\infty}(\Omega)$ with $u_k|_{\partial\Omega} = 0$ and $(\nabla u_k)_{k\in\mathbb{N}}$ generates ν.*
(ii) *There holds $u_k \to 0$ strongly in $L^\infty(\Omega)$ as $k \to \infty$ and $\|\nabla u_k\|_\infty \le \frac{3}{2}|z|$ for all $k \in \mathbb{N}$.*

Proof. **Step 1.** Define for $x \in [0,1[$

$$\chi(x) := \begin{cases} 1\,, & x \in [0,\frac{1}{2}]\,,\\ -1\,, & x \in]\frac{1}{2},1[\,, \end{cases}$$

and extend χ by periodicity to all of $\mathbb{R}$. Introduce

$$h : \mathbb{R}^n \to \mathbb{R}\,, \quad h(x) := 1 + \int_0^{x\cdot z} \chi(s)\, ds.$$

Then $\nabla h(x) = \chi(x\cdot z)z$ for almost all $x \in \mathbb{R}^n$. Define

$$w_k(x) := \frac{1}{k}h(kx) = \frac{1}{k} + \frac{1}{k}\int_0^{kx\cdot z} \chi(s)\, ds\,, \quad x \in \mathbb{R}^n.$$

Consequently $\nabla w_k(x) = \chi(kx\cdot z)z$ for almost all $x \in \mathbb{R}^n$. There holds $\|w_k\|_\infty \le \frac{2}{k}$, $k \in \mathbb{N}$, and therefore $w_k \to 0$ strongly in $L^\infty(\Omega)$ as $k \to \infty$. Moreover $w_k \in W^{1,\infty}(\Omega)$ and $\|\nabla w_k\|_\infty = |z|$. In the following we will show that $(\nabla w_k)_{k\in\mathbb{N}}$ generates ν. For $\phi \in C_0(\mathbb{R}^n)$ we introduce

$$f : \mathbb{R}^n \to \mathbb{R}\,, \quad f(x) := \phi(\chi(x\cdot z)z).$$

Define $f_k : \mathbb{R}^n \to \mathbb{R}\,, f_k(x) := f(kx)$ for every $k \in \mathbb{N}$. By the Riemann-Lebesgue Lemma, see [13, Theorem 2.6], we get

$$f_k \underset{k\to\infty}{\overset{*}{\rightharpoonup}} \overline{f} \quad \text{in } L^\infty(\Omega) \tag{4.3}$$

where $\overline{f} : \Omega \to \mathbb{R}$ denotes the constant function with $\overline{f}(x) := \frac{1}{2}(\phi(z)+\phi(-z))$ for every $x \in \Omega$. The definition of ν implies

$$\int_{\mathbb{R}^n} \phi(\lambda)\, d\nu_x(\lambda) = \frac{1}{2}(\phi(z) + \phi(-z)) \tag{4.4}$$

for all $x \in \Omega$. Combining (4.3), (4.4) yields

$$\phi(\nabla w_k) \underset{k\to\infty}{\overset{*}{\rightharpoonup}} \Big(x \mapsto \int_{\mathbb{R}^n} \phi(\lambda)\, d\nu_x(\lambda)\Big) \quad \text{in } L^\infty(\Omega)$$

for a.a. $x \in \Omega$. Therefore $(\nabla w_k)_{k\in\mathbb{N}}$ generates ν.

Step 2. We have to modify the sequence $(w_k)_{k\in\mathbb{N}}$ such that additionally $w_k|_{\partial\Omega} = 0$. We follow the ideas presented in [21, Lemma 8.3]. Let $\eta_k \in C^1(\overline{\Omega})$ be a sequence of cut-off functions with $0 \le \eta_k \le 1$ such that:

1. $\eta_k(x) = 0$ if $x \in \partial\Omega$.
2. $\eta_k(x) = 1$ if $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > \frac{1}{k}$.
3. We have $|\nabla\eta_k(x)| \le ck$ for all $x \in \Omega\,, k \in \mathbb{N}$, with $c = c(\Omega) > 0$.

For $j, k \in \mathbb{N}$ define

$$w_{j,k}(x) := \eta_k(x) w_j(x) \quad \text{for a.a. } x \in \Omega\,, \tag{4.5}$$

so that $w_{j,k}|_{\partial\Omega} = 0$ and $\nabla w_{j,k}(x) = \eta_k(x)\nabla w_j(x) + w_j(x)\nabla\eta_k(x)$ for a.a. $x \in \Omega$. Choose a strictly increasing sequence $(j_k)_{k\in\mathbb{N}}$ of natural numbers $j_k \in \mathbb{N}\,, k \in \mathbb{N}$, such that $\|w_{j_k,k}\|_\infty \le \frac{1}{k^3}\,, k \in \mathbb{N}$, and define $u_k := w_{j_k,\,k}$ for $k \in \mathbb{N}$. Then

$$\|\nabla u_k\|_\infty \le \|\nabla w_{j_k}\|_\infty + \|w_{j_k}\|_\infty \|\nabla\eta_k\|_\infty \le \|\nabla w_{j_k}\|_\infty + c\frac{1}{k^2}. \tag{4.6}$$

By (4.5), (4.6) we get that $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,\infty}(\Omega)$ and $u_k \to 0$ strongly in $L^\infty(\Omega)$ as $k \to \infty$. From

$$\big|\{x \in \Omega; \nabla u_k(x) \neq \nabla w_{j_k}(x)\}\big| \le \big|\{x \in \Omega; \operatorname{dist}(x, \partial\Omega) \le \frac{1}{k}\}\big| \underset{k\to\infty}{\to} 0$$

and [21, Lemma 6.3] it follows that $(\nabla u_k)_{k\in\mathbb{N}}$ generates ν. Due to (4.6) we can choose a subsequence $(l_k)_{k\in\mathbb{N}}$ of $(j_k)_{k\in\mathbb{N}}$ with $\|\nabla u_{l_k}\|_\infty \le \frac{3}{2}|z|$ for all $k \in \mathbb{N}$. Altogether, $(u_{l_k})_{k\in\mathbb{N}}$ satisfies (i), (ii). □

Now we have all ingredients at hand to prove the following

Theorem 4.3. *Let $\Omega \subseteq \mathbb{R}^n\,, n \in \mathbb{N}$, be an arbitrary open set, let $z : \Omega \to \mathbb{R}^n$ be a bounded, measurable function. Define the Young measure $\nu = (\nu_x)_{x\in\Omega}$ by*

$$\nu_x := \frac{1}{2}\big(\delta_{z(x)} + \delta_{-z(x)}\big)\,, \quad x \in \Omega. \tag{4.7}$$

Then there exist non-negative functions $u_k : \Omega \to \mathbb{R}_0^+\,, k \in \mathbb{N}$, such that $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,\infty}(\Omega)$, we have that $(\nabla u_k)_{k\in\mathbb{N}}$ generates ν and $u_k \to 0$ strongly in $L^\infty(\Omega)$ as $k \to \infty$.

Proof. Choose a sequence $(h_i)_{i\in\mathbb{N}} \subseteq C_0^\infty(\Omega)$ which is dense in $L^1(\Omega)$ and a sequence $(\phi_j)_{j\in\mathbb{N}} \subseteq C_0(\mathbb{R}^n)$ which is dense in $C_0(\mathbb{R}^n)$. Define for $j \in \mathbb{N}$

$$\overline{\phi_j}(x) := \int_{\mathbb{R}^n} \phi_j(\lambda)\, d\nu_x(\lambda) = \frac{1}{2}\big(\, \phi_j(z(x)) + \phi_j(-z(x))\,\big) \qquad \text{for a.a. } x \in \Omega. \tag{4.8}$$

Step 1. Define the set

$$A := \bigcap_{j\in\mathbb{N}} \Big\{\, x \in \Omega; \lim_{r\searrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\overline{\phi_j}(y) - \overline{\phi_j}(x)|\, dy = 0 \,\Big\}.$$

It is easily verified that $A \subseteq \Omega$ is measurable. By Lebesgue's differentiation theorem we have $|\Omega \setminus A| = 0$. For $k \in \mathbb{N}$ define

$$\begin{aligned} \mathcal{O}_k := \Big\{\, & B(x,r) \subseteq \Omega \text{ such that } x \in A\,, 0 < r < \frac{1}{k} \text{ satisfy} \\ & \frac{1}{|B(x,r)|} \int_{B(x,r)} |\overline{\phi_j}(y) - \overline{\phi_j}(x)|\, dy \le \frac{1}{k} \text{ for all } j = 1, \dots, k \,\Big\}. \end{aligned} \tag{4.9}$$

By Vitali's covering theorem (see [15, Theorem 1.150 and Remark 1.151]) there exist for every $k \in \mathbb{N}$ a countable set $I_k \subseteq \mathbb{N}$ and pairwise disjoint $B(x_{k,l}, r_{k,l}) \in \mathcal{O}_k\,, l \in I_k$, such that

$$\big| A \setminus \bigcup_{l\in I_k} B(x_{k,l}, r_{k,l}) \big| = 0.$$

To simplify the notation assume $I_k = \mathbb{N}\,, k \in \mathbb{N}$. Fix $j \in \mathbb{N}$ and $h \in C_0^\infty(\Omega)$. Without loss of generality assume $\operatorname{supp}(h) \neq \varnothing$. Define $\delta := \operatorname{dist}(\operatorname{supp}(h), \mathbb{R}^n \setminus \Omega)$ if $\Omega \neq \mathbb{R}^n$ and $\delta := 1$ if $\Omega = \mathbb{R}^n$. Introduce

$$\begin{aligned} D &:= \{x \in \Omega; \operatorname{dist}(\operatorname{supp}(h), x) < \frac{\delta}{2}\}\,, \\ M_k &:= \{\, l \in \mathbb{N}; \operatorname{supp}(h) \cap B(x_{k,l}, r_{k,l}) \neq \varnothing \,\}\,, \quad k \in \mathbb{N}. \end{aligned}$$

For all $k \in \mathbb{N}$ with $k > \max\{\frac{4}{\delta}, j\}$ it follows with (4.9)

$$\begin{aligned} &\Big| \int_\Omega h(x)\overline{\phi_j}(x)\, dx - \sum_{l\in\mathbb{N}} \overline{\phi_j}(x_{k,l}) \int_{B(x_{k,l},r_{k,l})} h(x)\, dx \Big| \\ &\le \|h\|_\infty \sum_{l\in M_k} \int_{B(x_{k,l},r_{k,l})} |\overline{\phi_j}(x) - \overline{\phi_j}(x_{k,l})|\, dx \\ &\le \|h\|_\infty \sum_{l\in M_k} \frac{1}{k} |B(x_{k,l}, r_{k,l})| \\ &\le \frac{1}{k} \|h\|_\infty |D|. \end{aligned}$$

Passing to the limit in the inequality above yields

$$\int_\Omega h(x)\overline{\phi_j}(x)\, dx = \lim_{k\to\infty} \sum_{l\in\mathbb{N}} \overline{\phi_j}(x_{k,l}) \int_{B(x_{k,l},r_{k,l})} h(x)\, dx \tag{4.10}$$

for all $h \in C_0^\infty(\Omega)\,, j \in \mathbb{N}$. For $k, l \in \mathbb{N}$ define the homogeneous Young measure

$$\nu^{k,l}(x) := \nu_{x_{k,l}} \qquad \text{for all } x \in B(x_{k,l}, r_{k,l}).$$

Step 2. In this step we will define functions $u_k : \Omega \to \mathbb{R}_0^+ , k \in \mathbb{N}$, with $u_k \to 0$ strongly in $L^\infty(\Omega)$ as $k \to \infty$ and $\|\nabla u_k\|_\infty \le \frac{3}{2}\|z\|_\infty , k \in \mathbb{N}$, in such a way that

$$\lim_{k\to\infty} \sum_{l\in\mathbb{N}} \int_{B(x_{k,l},r_{k,l})} h_i(x)\phi_j(\nabla u_k(x))\,dx = \lim_{k\to\infty} \sum_{l\in\mathbb{N}} \overline{\phi_j}(x_{k,l}) \int_{B(x_{k,l},r_{k,l})} h_i(x)\,dx \tag{4.11}$$

for all $i, j \in \mathbb{N}$.

Fix $k, l \in \mathbb{N}$. In the following u_k will be defined on $B(x_{k,l}, r_{k,l})$. Use Lemma 4.2 to obtain functions $z_m^{k,l} : B(x_{k,l}, r_{k,l}) \to \mathbb{R}_0^+ , m \in \mathbb{N}$, such that:

(i) $(z_m^{k,l})_{m\in\mathbb{N}}$ is bounded in $W^{1,\infty}(B(x_{k,l}, r_{k,l}))$, we have $z_m^{k,l} \to 0$ strongly in $L^\infty(\Omega)$ as $m \to \infty$ and $z_m^{k,l}|_{\partial B(x_{k,l},r_{k,l})} = 0$, $m \in \mathbb{N}$.

(ii) $(\nabla z_m^{k,l})_{m\in\mathbb{N}}$ generates $\nu^{k,l}$ and $\|\nabla z_m^{k,l}\|_\infty \le \frac{3}{2}\|z\|_\infty$ for all $m \in \mathbb{N}$.

Thus

$$\lim_{m\to\infty} \int_\Omega h(x)\phi(\nabla z_m^{k,l}(x))\,dx = \overline{\phi}(x_{k,l}) \int_\Omega h(x)\,dx \tag{4.12}$$

for all $h \in L^1(\Omega)$ and $\phi \in C_0(\mathbb{R}^n)$. Choose $m = m(k, l) \in \mathbb{N}$ (see (4.12)) such that $\|z_{m(k,l)}^{k,l}\|_\infty \le \frac{1}{k}$ and that

$$\Big|\int_{B(x_{k,l},r_{k,l})} h_i(x)\phi_j(\nabla z_{m(k,l)}^{k,l}(x))\,dx - \overline{\phi_j}(x_{k,l}) \int_{B(x_{k,l},r_{k,l})} h_i(x)\,dx\Big| \le \frac{1}{2^l k} \tag{4.13}$$

is satisfied for all $i, j = 1, \ldots, k$. Now we define

$$u_k(x) := \begin{cases} z_{m(k,l)}^{k,l}(x)\,, & \text{if } x \in B(x_{k,l}, r_{k,l})\,, \\ 0\,, & \text{if } x \in N_k. \end{cases} \tag{4.14}$$

Fix $k \in \mathbb{N}$. Since $z_{m(k,l)}^{k,l}|_{\partial B(x_{k,l},r_{k,l})} = 0$ for all $l \in \mathbb{N}$ we obtain that the weak gradient ∇u_k exists as function $\nabla u_k \in L^\infty_{\text{loc}}(\Omega)$. For fixed $i, j \in \mathbb{N}$ we sum over $l \in \mathbb{N}$ in (4.13), let $k \to \infty$ in this sum and use (4.14) to prove (4.11). Moreover by construction $\|u_k\|_\infty \le \frac{1}{k}$ and $\|\nabla u_k\|_\infty \le \frac{3}{2}\|z\|_\infty$ for $k \in \mathbb{N}$.

Step 3. Fix $i, j \in \mathbb{N}$. Then

$$\begin{aligned} \lim_{k\to\infty} \int_\Omega h_i(x)\phi_j(\nabla u_k(x))\,dx &= \lim_{k\to\infty} \sum_{l\in\mathbb{N}} \int_{B(x_{k,l},r_{k,l})} h_i(x)\phi_j(\nabla u_k(x))\,dx \\ &= \lim_{k\to\infty} \sum_{l\in\mathbb{N}} \overline{\phi_j}(x_{k,l}) \int_{B(x_{k,l},r_{k,l})} h_i(x)\,dx \\ &= \int_\Omega h_i(x)\overline{\phi_j}(x)\,dx. \end{aligned}$$

To obtain the identity above we have used (4), (4.11) and (4.10). By density of $(h_i)_{i\in\mathbb{N}}$ in $L^1(\Omega)$ and of $(\phi_j)_{j\in\mathbb{N}}$ in $C_0(\mathbb{R}^n)$ it follows that $(\nabla u_k)_{k\in\mathbb{N}}$ generates ν. □

5. Preparation of the proof of Theorem 1.2

5.1. **Decomposition of the pressure.** Let $G \subseteq \mathbb{R}^n , n \ge 2$, be a bounded Lipschitz domain. For $u \in L^2(G)$ we write $\text{div}u = 0$ or $\Delta u = 0$ if these identities are satisfied in the sense of distributions in G. If $u \in L^2(G)$

satisfies $\Delta u = 0$, we can apply Weyl's Lemma to get (after a redefinition on a null set) that u is smooth, i.e. $u \in C^\infty(G)$. Define

$$\begin{aligned} \Delta W_0^{2,2}(G) &:= \{\, \Delta p; p \in W_0^{2,2}(G) \,\}\,, \\ L_0^2(G) &:= \{\, p \in L^2(G); \int_G p\,dx = 0 \,\}. \end{aligned}$$

A major point in the proof of Theorem 1.2 is to prove identity (6.6) below. This proof is based on the construction of a 'local pressure' introduced by J. Wolf. We proceed with the following theorem which is a variant of [26, Theorem 2.6].

Theorem 5.1. *Let $G \subseteq \mathbb{R}^n\,, n \geq 2$, be a bounded C^2-domain, let $1 < \gamma < \infty$, $0 < T < \infty$, let $u_0 \in L^2(G)$, let $Q_1 \in L^\gamma(0,T;L^2(G))$, and let $Q_2 \in L^\gamma(0,T;L^2(G))$. Consider $u \in L^\infty(0,T;L^2(G))$ with $\mathrm{div} u(t) = 0$ for a.a. $t \in]0,T[$ and*

$$-\int_0^T \langle u, w_t\rangle_G\,dt + \int_0^T \langle Q_1, \nabla w\rangle_G\,dt + \int_0^T \langle Q_2, w\rangle_G\,dt - \langle u_0, w(0)\rangle_G = 0 \tag{5.1}$$

for all $w \in C_0^\infty([0,T[;C_{0,\sigma}^\infty(G))$. Then there exist unique functions $p_r \in L^\gamma(0,T;L^2(G))$, $p_h \in L^\infty(0,T;L_0^2(G))$ with $p_r(t) \in \Delta W_0^{2,2}(G)\,, \Delta_x p_h(t) = 0$ for a.a. $t \in]0,T[$ such that

$$\begin{aligned} &-\int_0^T \langle u + \nabla_x p_h, w_t\rangle_G\,dt + \int_0^T \langle Q_1, \nabla w\rangle_G\,dt + \int_0^T \langle Q_2, w\rangle_G\,dt \\ &= \langle u_0, w(0)\rangle_G + \int_0^T \langle p_r, \mathrm{div} w\rangle_G\,dt \end{aligned} \tag{5.2}$$

for all $w \in C_0^\infty([0,T[;C_0^\infty(G))$. We have

$$\|p_r\|_{2,\gamma;G;T} \leq c(\|Q_1\|_{2,\gamma;G;T} + \|Q_2\|_{2,\gamma;G;T})\,, \tag{5.3}$$

$$\|p_h\|_{2,\infty;G;T} \leq c(\|u\|_{2,\infty;G;T} + \|Q_1\|_{2,\gamma;G;T} + \|Q_2\|_{2,\gamma;G;T}) \tag{5.4}$$

with a constant $c = c(G,\gamma,T) > 0$.

Proof. **Step 1.** Define $u(0) := u_0$. For $w \in C_{0,\sigma}^\infty(G)\,, \eta \in C_0^\infty([0,T[)$ we have

$$\begin{aligned} &-\int_0^T \langle u(t), w\rangle_G\,\eta'(t)\,dt - \langle u_0, w(0)\rangle_G\,\eta(0) \\ &= -\int_0^T \big(\,\langle Q_1(t), \nabla w\rangle_G + \langle Q_2(t), w\rangle_G\big)\eta(t)\,dt. \end{aligned} \tag{5.5}$$

Identity (5.5) implies that there exists a null set $N = N(w)$ such that

$$\langle u(t), w\rangle_G - \langle u_0, w\rangle_G = -\int_0^t \big(\langle Q_1(t), \nabla w\rangle_G + \langle Q_2(t), w\rangle_G\,\big)\,dt \tag{5.6}$$

for all $t \in [0,T[\setminus N$. Using $u \in L^\infty(0,T;L^2(G))$ and a separability argument u can be redefined on a null set of $[0,T[$ such that (5.6) holds for all $t \in [0,T[$ and all $w \in C_{0,\sigma}^\infty(G)$. For $t \in [0,T[$ define $\widetilde{Q_1}(t) := \int_0^t Q_1(s)\,ds$ and $\widetilde{Q_2}(t) :=$

$\int_0^t Q_2(s)\,ds$. We employ Fubini's Theorem and [23, II, Lemma 2.2.2] to get for each fixed $t \in [0,T[$ a unique $p(t) \in L^2_0(G)$ such that

$$\langle u(t) - u_0, w\rangle_G + \langle \widetilde{Q_1}(t), \nabla w\rangle_G + \langle \widetilde{Q_2}(t), w\rangle_G = \langle p(t), \mathrm{div} w\rangle_G \tag{5.7}$$

for all $w \in W^{1,2}_0(G)$. Estimate [23, II, (2.2.6)] yields

$$\|p(t)\|_2 \le c(G,\gamma)(\|u(t) - u_0\|_2 + \|\widetilde{Q}_1(t)\|_2 + \|\widetilde{Q}_2(t)\|_2) \tag{5.8}$$

for all $t \in [0,T[$. Using (5.8) we can show that $[0,T[\to L^2(G)\,, t \mapsto p(t)$, is Bochner measurable. Furthermore

$$\|p\|_{2,\infty;G;T} \le c(G,\gamma,T)(\|u\|_{2,\infty;G;T} + \|Q_1\|_{2,\gamma;G;T} + \|Q_2\|_{2,\gamma;G;T}). \tag{5.9}$$

Step 2. From [26, Corollary 2.5] we get unique $\widetilde{p_r}(t) \in \Delta W^{2,2}_0(G)\,, p_h(t) \in L^2(G)$ with $\Delta_x p_h(t) = 0$ for all $t \in [0,T[$ such that

$$p(t) = \widetilde{p_r}(t) + p_h(t)\,, \quad \|\widetilde{p_r}(t)\|_2 + \|p_h(t)\|_2 \le c\|p(t)\|_2 \tag{5.10}$$

with $c = c(G,\gamma,T)$ for a.a. $t \in [0,T[$. Since $p(t) \in L^2_0(G)$ we get $p_h(t) \in L^2_0(G)\,, t \in [0,T[$. Combining (5.9), (5.10) yields $p_h \in L^\infty(0,T;L^2(G))$. From (5.8), (5.10) it follows $\widetilde{p_r}(0) = 0$. For fixed $w \in C^\infty_0([0,T[\times G)$ we insert $w(t)$ in (5.7) and integrate with respect to $t \in [0,T[$ to obtain

$$\begin{aligned}\int_0^T \langle u(t) - u_0, w\rangle_G\,dt + \int_0^T \langle \widetilde{Q_1}(t), \nabla w\rangle_G\,dt + \int_0^T \langle \widetilde{Q_2}(t), w\rangle_G\,dt \\ = \int_0^T \langle \widetilde{p_r}(t), \mathrm{div} w\rangle_G\,dt - \int_0^T \langle \nabla_x p_h(t), w\rangle_G\,dt.\end{aligned} \tag{5.11}$$

Step 3. Fix $\tau \in]0,T[$. Given $\phi \in C^\infty_0(G)$ insert $w := \nabla\phi$ in (5.7). Consider $t \in]0,T-\tau[$ such that $\mathrm{div} u(t) = 0\,, \mathrm{div} u(t+\tau) = 0$. We make use of $\Delta_x p_h(t)\,, \Delta_x p_h(t+\tau) = 0$ and obtain

$$\begin{aligned}\langle \widetilde{p_r}(t+\tau) - \widetilde{p_r}(t), \Delta\phi\rangle_G &= \langle \widetilde{Q_1}(t+\tau) - \widetilde{Q_1}(t), \nabla^2\phi\rangle_G \\ &\quad + \langle \widetilde{Q_2}(t+\tau) - \widetilde{Q_2}(t), \nabla\phi\rangle_G.\end{aligned}$$

Since $\widetilde{p_r}(t) \in \Delta W^{2,2}_0(G)$ we obtain from [26, (2.1), (2.2)] that

$$\|\widetilde{p_r}(t+\tau) - \widetilde{p_r}(t)\|_2^\gamma \le c\sum_{i=1}^2 \Big\|\int_t^{t+\tau} Q_i(s)\,ds\Big\|_2^\gamma \tag{5.12}$$

with a constant $c = c(G,\gamma,T)$ independent of t,τ. Especially, (5.12) is satisfied for a.a. $t \in]0,T-\tau[$. Fix $0 < T' < T$. We get

$$\begin{aligned}\int_0^{T'} \|\widetilde{p_r}(t+\tau)-\widetilde{p_r}(t)\|_2^\gamma\,dt &\le c\sum_{i=1}^2 \int_0^{T'} \Big(\int_t^{t+\tau} \|Q_i(s)\|_2\,ds\Big)^\gamma\,dt\\ &\le c\sum_{i=1}^2 \int_0^{T'} \int_t^{t+\tau} \|Q_i(s)\|_2^\gamma\,ds\,\tau^{\gamma/\gamma'}\,dt\\ &= c\tau^{\gamma/\gamma'}\sum_{i=1}^2 \int_0^{T}\int_0^{T'} 1_{[t,t+\tau]}(s)\|Q_i(s)\|_2^\gamma\,dt\,ds\\ &\le c\tau^{\gamma}\sum_{i=1}^2 \int_0^{T} \|Q_i(s)\|_2^\gamma\,ds\end{aligned} \tag{5.13}$$

for all $\tau \in]0,T-T'[$ with a constant $c = c(\gamma,G,T) > 0$ independent of T', τ. Estimate (5.13) yields $\widetilde{p_r} \in W^{1,\gamma}(0,T;L^2(G))$ and

$$\|\partial_t\widetilde{p_r}\|_{2,\gamma;T} \le c\,(\,\|Q_1\|_{2,\gamma;T} + \|Q_2\|_{2,\gamma;T}\,).$$

Step 4. Define $p_r := \partial_t\widetilde{p_r} \in L^\gamma(0,T;L^2(G))$. For $w \in C_0^\infty([0,T[\times G)$ insert w_t instead of w in (5.11) and integrate by parts to get (5.2). In this argument $\mathrm{div} u = 0\,, w(T) = 0$ and $\widetilde{p_r}(0) = 0$ were used. The uniqueness follows from (5.3), (5.4). □

5.2. **Two auxiliary lemmata.**

Lemma 5.2. *Let $\Omega\,,(\Omega_k)_{k\in\mathbb{N}}$ be as in (1.4), (1.5), let $0 < T < \infty$, and $K \subseteq \mathbb{R}^{n-1}$ be compact.*

(1) *Let $(v_k)_{k\in\mathbb{N}}$ be a bounded sequence in $H^1(\Omega_k)$. Then*

$$\lim_{k\to\infty}\int_K |v_k(x',-h_k(x')) - v_k(x',0)|\,dx' = 0. \tag{5.14}$$

(2) *If $(\theta_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^2(0,T;H^1(\Omega_k))$ then*

$$\lim_{k\to\infty}\int_{[0,T[\times K} |\theta_k(t,x',-h_k(x')) - \theta_k(t,x',0)|\,d(x',t) = 0. \tag{5.15}$$

Proof. For $v \in C_0^\infty(\overline{\Omega_k})$ and $k \in \mathbb{N}$ there holds

$$\int_K |v(x',-h_k(x')) - v(x',0)|\,dx' = \int_K \Big|\int_0^{-h_k(x')} \frac{\partial v}{\partial x_n}(x',\tau)\,d\tau\Big|\,dx'. \tag{5.16}$$

By a density argument, identity (5.16) holds true for $v_k \in H^1(\Omega_k)$. We deduce

$$\begin{aligned}&\int_K |v_k(x',-k_k(x')) - v_k(x',0)|\,dx'\\ &\le \Big(\int_K\int_{-h_k(x')}^0 |\frac{\partial v_k}{\partial x_n}(x',\tau)|^2\,d\tau\,dx'\Big)^{1/2}\Big(\int_K\int_{-h_k(x')}^0 1\,d\tau\,dx'\Big)^{1/2} \qquad (5.17)\\ &\le |K|^{1/2}\big(\sup_{x'\in K} h_k(x')\big)^{1/2}\|v_k\|_{H^1(\Omega_k)}\end{aligned}$$

for all $k \in \mathbb{N}$. Using $h_k(x') \to 0$ as $k \to \infty$ uniformly for $x' \in K$, we get (5.14). We employ $u_k(\cdot, t) \in H^1(\Omega_k)$ for a.a. $t \in [0, T[$ and integrate (5.17) with respect to $t \in [0, T[$ to show (5.15). □

Lemma 5.3. *Let Ω, $(\Omega_k)_{k\in\mathbb{N}}$ be as in (1.4), (1.5), let $0 < T < \infty$. Consider a sequence $(u_k)_{k\in\mathbb{N}}$ which is bounded in $L^{4/3}(0, T; L^2(\Omega_k))$, consider $u \in L^{4/3}(0, T; L^2(\Omega))$ with $u_k \rightharpoonup u$ as $k \to \infty$ in $L^{4/3}(0, T; L^2(\Omega))$. Then*

$$\lim_{k\to\infty} \int_0^T \langle u_k, w\rangle_{\Omega_k}\, dt = \int_0^T \langle u\,, w\rangle_\Omega\, dt \tag{5.18}$$

for all $w \in C_0^0([0, T[\times\mathbb{R}^n)$.

***Proof*.** Choose a ball $B \subseteq \mathbb{R}^n$ such that $\mathrm{supp}(w) \subseteq [0, T[\times B$. Define $B_k := (\Omega_k \setminus \Omega) \cap B\,, k \in \mathbb{N}$. Then

$$\int_0^T \langle u_k, w\rangle_{\Omega_k}\, dt = \int_0^T \langle u_k, w\rangle_\Omega\, dt + \int_0^T \langle u_k, w\rangle_{B_k}\, dt. \tag{5.19}$$

We obtain with Hölder's inequality

$$\begin{aligned}\int_0^T |\langle u_k, w\rangle_{B_k}|\, dt &\le \|u_k\|_{L^{4/3}(0,T;L^2(B_k))}\|w\|_{L^4(0,T;L^2(B_k))}\\ &\le |B_k|^{1/2}\, T^{1/4}\, \|u_k\|_{L^{4/3}(0,T;L^2(B_k))} \sup_{(t,x)\in[0,T[\times\mathbb{R}^n} |w(t, x)|\end{aligned}$$

for all $k \in \mathbb{N}$. Since $|B_k| \to 0$ as $k \to \infty$, we get $\lim_{k\to\infty} \int_0^T \langle u_k, w\rangle_{B_k}\, dt = 0$. Consequently from (5.19) we conclude that (5.18) is fulfilled. □

6. Proof of Theorem 1.2

Since the sequence $(h_k)_{k\in\mathbb{N}}$ is equi-Lipschitz continuous the norm of the trace operator on $H^1(\Omega_k)$ can be chosen independently of $k \in \mathbb{N}$, i.e. there is a constant $c > 0$ such that

$$\|\phi\|_{H^1(\Omega_k)} \le c\|\phi\|_{L^2(\partial\Omega_k)} \tag{6.1}$$

for all $k \in \mathbb{N}$ and all $\phi \in H^1(\Omega_k)$. Since $(u_k, \theta_k)_{k\in\mathbb{N}}$ satisfies (1.12) in $[0, T[\times\Omega_k$ we get with Hölder's inequality, Young's inequality and (6.1) that

$$\begin{aligned}&\frac{1}{4}\|\theta_k\|^2_{2,\infty;\Omega_k;T} + \frac{1}{2}\|\nabla\theta_k\|^2_{2,2;\Omega_k;T} + \|\theta_k\|^2_{2,2;\partial\Omega_k;T}\\ &\quad\le \|\theta_0\|^2_{2,\Omega_k} + c\|\zeta\|^2_{2,1;\partial\Omega_k;T} + c\|\zeta\|^2_{2,2;\partial\Omega_k;T}\\ &\quad\le c\big(\|\theta_0\|^2_{2,\mathbb{R}^n} + \|\zeta\|^2_{L^2(0,T;H^1(\mathbb{R}^n))}\big)\end{aligned}$$

for all $k \in \mathbb{N}$ with a constant $c > 0$ independent of k. Thus

$$(\theta_k)_{k\in\mathbb{N}} \text{ is bounded in } L^\infty(0, T; L^2(\Omega_k)) \cap L^2(0, T; H^1(\Omega_k)). \tag{6.2}$$

Furthermore, since (u_k, θ_k) fulfils (1.11) in $[0, T[\times\Omega_k$ we get with (6.2)

$$(u_k)_{k\in\mathbb{N}} \text{ is bounded in } L^\infty(0, T; L^2(\Omega_k)) \cap L^2(0, T; H^1(\Omega_k)). \tag{6.3}$$

The proof of Theorem 1.2 is based on Lemma 6.1, 6.2 and 6.3 below.

Lemma 6.1. *The weak limit (u, θ) in (1.14) satisfies (1.9) for all $w \in C_0^\infty([0, T[; C^\infty_{0,\sigma}(\Omega))$.*

Proof. By interpolation and Sobolev's imbedding theorem we get with (6.3)

$$\int_0^T \|u_k\otimes u_k\|_{2,\Omega}^{4/3}\,dt \le c\int_0^T \|u_k\|_{2,\Omega}^{2/3}\|u_k\|_{H^1(\Omega)}^2\,dt \le c \tag{6.4}$$

with a constant $c>0$ independent of $k\in\mathbb{N}$. Therefore we find a matrix field in $L^{4/3}(0,T;L^2(\Omega))$, denoted by $\overline{u\otimes u}$, such that (along a not relabelled subsequence)

$$u_k\otimes u_k \underset{k\to\infty}{\rightharpoonup} \overline{u\otimes u} \quad \text{in } L^{4/3}(0,T;L^2(\Omega)). \tag{6.5}$$

The main step in the proof of this lemma is to prove the following assertion.
Assertion. There holds (along a not relabelled subsequence)

$$\lim_{k\to\infty}\int_0^T \langle u_k\otimes u_k,\nabla w\rangle_\Omega\,dt = \int_0^T \langle u\otimes u,\nabla w\rangle_\Omega\,dt \tag{6.6}$$

for all $w\in C_0^\infty([0,T[;C_{0,\sigma}^\infty(\Omega))$.
Proof of (6.6). Fix $w\in C_0^\infty([0,T[;C_{0,\sigma}^\infty(\Omega))$. Choose smooth, bounded domains Ω_1,Ω_2 with $\overline{\Omega_1}\subseteq\Omega_2$ and $\mathrm{supp}(w(t,\cdot))\subseteq\Omega_1$ for all $t\in[0,T[$. Since $(u_k,\theta_k)_{k\in\mathbb{N}}$ is a weak solution of (1.1) in $[0,T[\times\Omega_k$ we see that (5.1) is fulfilled with $Q_1:=\nabla u_k - u_k\otimes u_k$ and $Q_2:=-\theta_k g$ on $G:=\Omega_2$. Therefore, there exist unique

$$\begin{aligned} &p_{r,k}\in L^{4/3}(0,T;L^2(\Omega_2))\,,\quad p_{r,k}(t)\in\Delta W_0^{2,2}(\Omega_2),\ \text{a.a. } t\in[0,T[\,,\\ &p_{h,k}\in L^\infty(0,T;L_0^2(\Omega_2))\,,\quad \Delta_x p_{h,k}(t)=0,\ \text{a.a. } t\in[0,T[\,,\end{aligned}$$

such that

$$\begin{aligned}-\int_0^T \langle u_k+\nabla_x p_{h,k},w\rangle_{\Omega_2}\eta'(t)dt = \int_0^T \big(&\langle u_k\otimes u_k,\nabla w\rangle_{\Omega_2} - \langle\nabla u_k,\nabla w\rangle_{\Omega_2}\\ &+\langle\theta_k g,w\rangle_{\Omega_2}+\langle p_{r,k},\mathrm{div} w\rangle_{\Omega_2}\big)\eta(t)\,dt\end{aligned} \tag{6.7}$$

for all $w\in C_0^\infty(\Omega_2)\,,\eta\in C_0^\infty(]0,T[)$. Since the constant c in (5.3) and (5.4) is independent of $k\in\mathbb{N}$ it follows that $(p_{h,k})_{k\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2(\Omega_2))$ and that $(p_{r,k})_{k\in\mathbb{N}}$ is bounded in $L^{4/3}(0,T;L^2(\Omega_2))$. Hence (along a not relabelled subsequence)

$$p_{h,k} \underset{k\to\infty}{\rightharpoonup^*} p_h \text{ in } L^\infty(0,T;L^2(\Omega_2))\,, \tag{6.8}$$

$$p_{r,k} \underset{k\to\infty}{\rightharpoonup} p_r \text{ in } L^{4/3}(0,T;L^2(\Omega_2)). \tag{6.9}$$

By (6.8) we conclude $\Delta_x p_h(t)=0$ for a.a. $t\in[0,T[$. Since $\overline{\Omega_1}\subseteq\Omega_2$ we get from (6.8) and the estimates for harmonic functions in [14, Theorem 2.2.7] that $(p_{h,k})_{k\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H^2(\Omega_1))$. Therefore

$$p_{h,k} \underset{k\to\infty}{\rightharpoonup} p_h \quad \text{in } L^2(0,T;H^2(\Omega_1)). \tag{6.10}$$

We use the imbedding $L^2(\Omega_1)\hookrightarrow H^{-1}(\Omega_1)$ to identify $u_k(t)+\nabla_x p_{h,k}(t)$ for a.a. $t\in[0,T[$ with the functional $w\mapsto\langle u_k(t)+\nabla_x p_{h,k}(t),w\rangle_{\Omega_1}\,,w\in H_0^1(\Omega_1)$.

Thus, we obtain from (6.7) with (6.2), (6.3), (6.6), (6.9)

$$\begin{aligned}\int_0^T \|\frac{d}{dt}(u_k + \nabla_x p_{h,k})\|_{H^{-1}(\Omega_1)}^{4/3}\, dt \\ \leq c \int_0^T \Big(\|u_k \otimes u_k\|_2^{4/3} + \|\nabla u_k\|_2^{4/3} + \|\theta_k g\|_2^{4/3} + \|p_{r,k}\|_2^{4/3}\Big)\, dt \\ \leq c\end{aligned} \tag{6.11}$$

with a constant $c > 0$ independent of $k \in \mathbb{N}$. Consider the imbedding scheme

$$H^1(\Omega_1) \underset{\text{compact}}{\hookrightarrow} L^2(\Omega_1) \underset{\text{continuous}}{\hookrightarrow} H^{-1}(\Omega_1). \tag{6.12}$$

We get with (6.3), (6.11), (6.12) and the Aubin-Lions compactness theorem (see [24, Theorem 3.2.2]) that (after a not relabelled subsequence)

$$u_k + \nabla_x p_{h,k} \underset{k\to\infty}{\to} u + \nabla_x p_h \quad \text{strongly in } L^2(0,T;L^2(\Omega_1)). \tag{6.13}$$

To proceed we need the following fact: Consider $\psi \in L^2(0,T;H^1(\Omega_2))$ with $\Delta\psi(t) = 0$ for a.a. $t \in [0,T[$. From Weyl's lemma and [14, Theorem 2.2.7] it follows $\psi \in L^2(0,T;H^2(\Omega_1))$. A short computation shows

$$\int_0^T \langle \nabla_x\psi \otimes \nabla_x\psi, \nabla w\rangle_{\Omega_1}\, dt = -\sum_{i,j=1}^n \int_0^T \frac{1}{2}\langle (\partial_i\psi)^2, \partial_j w_j\rangle_{\Omega_1}\, dt = 0. \tag{6.14}$$

We employ (1.14), (6.5), (6.10), (6.13), (6.14) to obtain

$$\begin{aligned}&\int_0^T \langle \overline{u\otimes u}, \nabla w\rangle_{\Omega_1}\, dt \\ &= \lim_{k\to\infty}\int_0^T \langle u_k \otimes u_k, \nabla w\rangle_{\Omega_1}\, dt \\ &= \lim_{k\to\infty}\int_0^T \langle (u_k + \nabla_x p_{h,k}) \otimes u_k, \nabla w\rangle_{\Omega_1}\, dt \\ &\quad - \lim_{k\to\infty}\int_0^T \langle \nabla_x p_{h,k} \otimes (u_k + \nabla_x p_{h,k}), \nabla w\rangle_{\Omega_1}\, dt \\ &= \int_0^T \langle (u + \nabla_x p_h) \otimes u, \nabla w\rangle_{\Omega_1} - \int_0^T \langle \nabla_x p_h \otimes (u + \nabla_x p_h), \nabla w\rangle_{\Omega_1}\, dt \\ &= \int_0^T \langle u \otimes u, \nabla w\rangle_{\Omega_1}\, dt.\end{aligned}$$

Now the proof of Lemma 6.1 can be finished. Fix $w \in C_0^\infty([0,T[;C_{0,\sigma}^\infty(\Omega))$. Since (u_k,θ_k) is a weak solution of (1.1) in $[0,T[\times\Omega_k$ and w has compact support in $[0,T[\times\Omega$ we have that (1.9) is fulfilled where (u,θ) is replaced by (u_k,θ_k). Passing to the limit in this identity and using (1.14), (6.6) we get that (u,θ) fulfils (1.9) .

□

Lemma 6.2. *(u,θ) satisfies (1.10) for all $\phi \in C_0^\infty([0,T[\times\mathbb{R}^n)$ with Λ defined by (1.16).*

***Proof*.** **Step 1.** In this step we show that for all bounded Lipschitz domains $G \subseteq \Omega$ there holds

$$\theta_k \underset{k\to\infty}{\to} \theta \quad \text{strongly in } L^2(0,T;L^2(G)) \cap L^2(0,T;L^2(\partial G))\,. \tag{6.15}$$

Proof of (6.15). Fix $\frac{1}{2} < s < 1$ and a bounded Lipschitz domain $G \subseteq \Omega$. From (1.10) it follows

$$-\int_0^T \langle \theta_k, \phi\rangle_G \eta'(t)\,dt = \int_0^T \Big(\langle \theta_k u_k, \nabla\phi\rangle_G - \langle \nabla\theta_k, \nabla\phi\rangle_G\Big)\eta(t)\,dt \tag{6.16}$$

for all $\phi \in C_0^\infty(G)$ and $\eta \in C_0^\infty(]0,T[)$. We use the continuous imbedding $L^2(G) \hookrightarrow H^{-1}(G)$ to identify $\theta_k(t)$ for a.a. $t \in [0,T[$ with the functional $\phi \mapsto \langle \theta_k(t), \phi\rangle_G\,, \phi \in H_0^1(G)$. Thus, we obtain from (6.16) with (6.2), (6.3)

$$\int_0^T \|\frac{d}{dt}\theta_k\|_{H^{-1}(G)}^{4/3}\,dt \le c\int_0^T \Big(\|\theta_k u_k\|_2^{4/3} + \|\nabla\theta_k\|_2^{4/3}\Big)\,dt \le c \tag{6.17}$$

for all $k \in \mathbb{N}$ with a constant $c > 0$ independent of k. Consider the imbedding scheme (see (2.1))

$$H^1(G) \underset{\text{compact}}{\hookrightarrow} W^{s,2}(G) \underset{\text{continuous}}{\hookrightarrow} H^{-1}(G). \tag{6.18}$$

From (6.2), (6.17), (6.18) and [24, Theorem 3.2.2] we get the existence of a subsequence $(\theta_{m_k})_{k\in\mathbb{N}}$ which is strongly convergent in $L^2(0,T;W^{s,2}(G))$. Since $\theta_k \rightharpoonup \theta$ as $k \to \infty$ in $L^2(0,T;L^2(G))$ it is possible to choose $m_k = k\,, k \in \mathbb{N}$. Looking at the continuous operator $W^{s,2}(G) \to L^2(\partial G)$ we see that (6.15) holds. □

Step 2. In the following we want to show that

$$\lim_{k\to\infty}\int_0^T \langle \theta_k, \phi\rangle_{\partial\Omega_k}\,dt = \int_0^T \langle \Lambda\theta, \phi\rangle_{\partial\Omega}\,dt \tag{6.19}$$

for all $\phi \in C_0^\infty([0,T[\times\mathbb{R}^n)$ where Λ is defined by (1.16).

Proof of (6.19). Choose an open ball $B := B_r(0) \subseteq \mathbb{R}^{n-1}$ with $0 < r < \infty$ such that

$$\operatorname{supp}(\phi) \subseteq \{\,(t,x',x_n) \in [0,T[\times\mathbb{R}^{n-1}\times\mathbb{R};\ x' \in B\,\}.$$

Define

$$s_k(x') := \sqrt{1 + |\nabla h_k(x')|^2} \quad \text{for a.a. } x' \in B \text{ and all } k \in \mathbb{N}\,,$$

and $Q := [0,T[\times B$. We get

$$\begin{aligned}
&\Big|\int_0^T \langle \theta_k, \phi\rangle_{\partial\Omega_k}\,dt - \int_0^T \langle \Lambda\theta, \phi\rangle_{\partial\Omega}\,dt\Big| \\
&\quad\le \int_Q |\,(\theta_k\phi)(t,x',-h_k(x')) - (\theta_k\phi)(t,x',0)\,|\,s_k(x')\,d(x',t) \\
&\quad\quad+ \int_Q |\,(\theta_k\phi)(t,x',0) - (\theta\phi)(t,x',0)\,|\,s_k(x')\,d(x',t) \\
&\quad\quad+ \Big|\int_Q (\theta\phi)(t,x',0)\Big(s_k(x') - \int_{\mathbb{R}^{n-1}}\sqrt{1+|\lambda|^2}\,d\nu_{x'}(\lambda)\Big)\,d(x',t)\Big|
\end{aligned} \tag{6.20}$$

for all $k \in \mathbb{N}$. Introduce $G := \{(x',x_n) \in \mathbb{R}^n; 0 < x_n < 1\,, x' \in B\}$. By (5.15), the boundedness of $(s_k)_{k\in\mathbb{N}}$ in $L^\infty(B)$, the uniform convergence of $\phi(t,x',-h_k(x'))$ for $(t,x')\in Q$, and (6.15) we obtain that the first two terms on the right hand side of (6.20) tend to zero as $k\to\infty$.

To show that the third term in (6.20) converges to zero introduce the Carathéodory function

$$\psi(x',\lambda) := \sqrt{1+|\lambda|^2}\,, \quad x'\in B\,,\lambda\in\mathbb{R}^{n-1}. \tag{6.21}$$

By assumption $(\nabla h_k)_{k\in\mathbb{N}}$ generates ν. Combining Theorem 4.1 with the uniqueness statement in [21, Lemma 6.3] it follows that properties (ii), (iii) stated in Theorem 4.1 are fulfilled with $m_k = k\,, k\in\mathbb{N}$. Therefore

$$s_k \underset{k\to\infty}{\rightharpoonup} \Big(x'\mapsto \int_{\mathbb{R}^{n-1}} \sqrt{1+|\lambda|^2}\,d\nu_{x'}(\lambda)\Big) \tag{6.22}$$

in $L^1(B)$ and even in $L^2(B)$ due to the the boundedness of B. Especially we get for a.a. $t\in[0,T[$

$$\begin{aligned}&\lim_{k\to\infty}\int_B (\theta\phi)(t,x',0)\,s_k(x')\,dx'\\ &=\int_B (\theta\phi)(t,x',0)\int_{\mathbb{R}^{n-1}}\sqrt{1+|\lambda|^2}\,d\nu_{x'}(\lambda)\,dx'.\end{aligned} \tag{6.23}$$

Due to the boundedness of $(s_k)_{k\in\mathbb{N}}$ in $L^\infty(B)$ and (6.23) we conclude with Lebesgue's dominated convergence theorem that the third term on the right hand side of (6.20) tends to zero as $k\to\infty$. Altogether (6.19) holds. □

Analogously we can prove

$$\lim_{k\to\infty}\int_0^T \langle\zeta,\phi\rangle_{\partial\Omega_k}\,dt = \int_0^T \langle\Lambda\zeta,\phi\rangle_{\partial\Omega} \quad \text{for all } \phi\in C_0^\infty([0,T[\times\mathbb{R}^n). \tag{6.24}$$

Step 3. Since $(u_k,\theta_k)_{k\in\mathbb{N}}$ is a weak solution of (1.1) in $[0,T[\times\Omega_k$ there holds

$$\begin{aligned}&-\int_0^T \langle\theta_k,\phi_t\rangle_{\Omega_k}\,dt + \int_0^T \langle\nabla\theta_k,\nabla\phi\rangle_{\Omega_k}\,dt + \int_0^T \langle u_k\cdot\nabla\theta_k,\phi\rangle_{\Omega_k}\,dt\\ &\quad = -\int_0^T \langle\theta_k-\zeta,\phi\rangle_{\partial\Omega_k}\,dt + \langle\theta_0,\phi(0)\rangle_{\Omega_k}\end{aligned} \tag{6.25}$$

for all $\phi\in C_0^\infty([0,T[\times\mathbb{R}^n))$. From [1, Theorem 5.8], $n\in\{2,3\}$, and the equi-Lipschitz continuity of $(h_k)_{k\in\mathbb{N}}$ we get

$$\begin{aligned}\|\theta_k(t)u_k(t)\|_{2,\Omega_k} &\le \|\theta_k(t)\|_{4,\Omega_k}\|u_k(t)\|_{4,\Omega_k}\\ &\le c\|\theta_k(t)\|_{2,\Omega_k}^{1/4}\|\theta_k(t)\|_{H^1(\Omega_k)}^{3/4}\|u_k(t)\|_{2,\Omega_k}^{1/4}\|u_k(t)\|_{H^1(\Omega_k)}^{3/4}\\ &\le c\|\theta_k\|_{2,\infty;\Omega_k;T}^{1/4}\|u_k\|_{2,\infty;\Omega_k;T}^{1/4}\big(\|\theta_k(t)\|_{H^1(\Omega_k)}^{3/2}+\|u_k(t)\|_{H^1(\Omega_k)}^{3/2}\big)\end{aligned}$$

for a.a. $t\in]0,T[$ and all $k\in\mathbb{N}$ with a constant $c>0$ independent of k. From (6.2), (6.3) it follows that $(\theta_k u_k)_{k\in\mathbb{N}}$ is bounded in $L^{4/3}(0,T;L^2(\Omega_k))$. From (1.14), (6.15) we get

$$\theta_k u_k \underset{k\to\infty}{\rightharpoonup} \theta u \quad \text{in } L^{4/3}(0,T;L^2(\Omega)). \tag{6.26}$$

Integration by parts in space and employing (5.18), (6.26) implies

$$\begin{aligned}\lim_{k\to\infty}\int_0^T \langle u_k\cdot\nabla\theta_k,\phi\rangle_{\Omega_k}\,dt &= -\lim_{k\to\infty}\int_0^T \langle \theta_k u_k,\nabla\phi\rangle_{\Omega_k}\,dt\\ &= -\int_0^T \langle \theta u,\nabla\phi\rangle_{\Omega}\,dt\\ &= \int_0^T \langle u\cdot\nabla\theta,\phi\rangle_{\Omega}\,dt\end{aligned}\tag{6.27}$$

for all $\phi\in C_0^\infty([0,T[\times\mathbb{R}^n)$. Passing to the limit in (6.25) and making use of (1.14), (6.27) and of (6.19), (6.24) yield (1.10). □

Lemma 6.3. *We have $u(t)\in W^{1,2}_{0,\sigma}(\Omega)$ for almost all $t\in[0,T[$.*

Proof. First we prove the following

Assertion. *Let $(v_k)_{k\in\mathbb{N}}$ be a bounded sequence in $H^1_0(\Omega_k)$, let $v\in H^1(\Omega)$ such that $v_k\rightharpoonup v$ as $k\to\infty$ in $H^1(\Omega)$. Then $v\in H^1_0(\Omega)$.*

To prove the assertion consider an open ball $B\subseteq\mathbb{R}^{n-1}$ and define $G:=\{(x',x_n)\in\mathbb{R}^n;\,0<x_n<1\,,x'\in B\}$. Since the operator $H^1(G)\to L^2(\partial G)$ is compact it follows $v_k\to v$ as $k\to\infty$ strongly in $L^2(\partial G)$. Therefore

$$\lim_{k\to\infty}\int_B |v(x',0)-v_k(x',0)|\,dx'=0.\tag{6.28}$$

We use (5.14), (6.28) and $v_k|_{\partial\Omega_k}=0$ to obtain

$$\int_B |v(x',0)|\,dx' \le \underbrace{\int_B |v(x',0)-v_k(x',0)|\,dx' + \int_B |v_k(x',0)-v_k(x',-h_k(x))|\,dx'}_{\to 0 \text{ as } k\to\infty}.$$

Thus $\int_{\mathbb{R}^{n-1}}|v(x',0)|\,dx'=0$ and consequently $v|_{\partial\Omega}=0$. □

To prove Lemma 6.3 we define for $k\in\mathbb{N}$ and $\delta>0$ with $\delta<T-\delta$

$$u_k^\delta(t):=(\widetilde{u_k}*\rho_\delta)(t)=\int_{\mathbb{R}}\widetilde{u_k}(t-\tau)\rho_\delta(\tau)\,d\tau\,,\quad t\in[\delta,T-\delta]\,,$$

where $(\rho_\delta)_{\delta>0}$ is a smooth Dirac sequence with suitable compact support and $\widetilde{u_k}(\tau):=1_{[0,T[}(\tau)u_k(\tau)$. Then the sequence $(u_k^\delta(t))_{k\in\mathbb{N}}$ with $t\in[\delta,T-\delta]$ has the properties of the sequence $(v_k)_{k\in\mathbb{N}}$ of the assertion above. Hence $u^\delta(t)\in H^1_0(\Omega)$ for all $t\in[\delta,T-\delta]$. Since $u^\delta(t)\to u(t)$ for $\delta\searrow 0$ strongly in $H^1(\Omega)$ for a.a. $t\in[0,T[$ we get $u(t)\in H^1_0(\Omega)$ for a.a. $t\in[0,T[$.

From $u_k(t)\in L^2_\sigma(\Omega_k)$ for all $k\in\mathbb{N}$ and a.a. $t\in[0,T[$ and (1.14) it follows $\mathrm{div}u(t)=0$ for a.a. $t\in[0,T[$. Further, from [16, Chapter III, Section 4.3] we get $W^{1,2}_{0,\sigma}(\Omega)=\{v\in H^1(\Omega);\mathrm{div}v=0\,,v|_{\partial\Omega}=0\}$. Altogether $u(t)\in W^{1,2}_{0,\sigma}(\Omega)$ for a.a. $t\in[0,T[$. □

Proof of Theorem 1.2. Combine Lemma 6.1 and Lemma 6.2 to get that (u,θ) satisfies (1.9), (1.10) for all test function ϕ and w as in Definition 1.1. From Lemma 6.3 we obtain $u(t)\in W^{1,2}_{0,\sigma}(\Omega)$ for a.a. $t\in[0,T[$. □

7. Proof of Theorem 1.3

Step 1. Choose $w \in \mathbb{R}^{n-1}$ with $|w| = 1$. Introduce

$$z : \mathbb{R}^{n-1} \to \mathbb{R}\,, \quad z(x') := \sqrt{\big(\Lambda(x',0)\big)^2 - 1}\; w$$

Consequently,

$$\Lambda(x',0) = \sqrt{1 + |z(x')|^2} \tag{7.1}$$

for all $x = (x',0) \in \partial\Omega$ with $x' \in \mathbb{R}^{n-1}$. Define the Young measure $\nu = (\nu_{x'})_{x' \in \mathbb{R}^{n-1}}$ by

$$\nu_{x'} := \frac{1}{2}\big(\delta_{z(x')} + \delta_{-z(x')}\big)\,, \quad x' \in \mathbb{R}^{n-1}.$$

By Theorem 4.3 there exist non-negative functions $h_k : \mathbb{R}^{n-1} \to \mathbb{R}_0^+\,, k \in \mathbb{N}$, such that $(h_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,\infty}(\mathbb{R}^{n-1})$, we have that $(\nabla h_k)_{k\in\mathbb{N}}$ generates ν and $h_k \to 0$ strongly in $L^\infty(\mathbb{R}^{n-1})$ as $k \to \infty$. For every $k \in \mathbb{N}$ the function h_k can be redefined on a null set of $\mathbb{R}^{n-1}$ such that $h_k : \mathbb{R}^{n-1} \to \mathbb{R}_0^+$ is Lipschitz continuous. Altogether, the sequence $(h_k)_{k\in\mathbb{N}}$ is admissible. Further, the definition of ν and (7.1) imply that

$$\Lambda(x',0) = \int_{\mathbb{R}^{n-1}} \sqrt{1 + |\lambda|^2}\, d\nu_{x'}(\lambda) \quad \text{for a.a. } x' \in \mathbb{R}^{n-1}. \tag{7.2}$$

Step 2. Define

$$\Omega_k := \{\, (x', x_n) \in \mathbb{R}^n; x_n > -h_k(x'); x' \in \mathbb{R}^{n-1} \,\}\,, \quad k \in \mathbb{N}.$$

Making use of Theorem 3.1 we obtain for every $k \in \mathbb{N}$ a weak solution (u_k, θ_k) of the Boussinesq equations (1.1) in $[0,T[\times\Omega_k$ (with data $g, \zeta|_{]0,T[\times\Omega_k}$ and $u_0, \theta_0|_{\Omega_k}$) satisfying the energy inequalities (1.11), (1.12) (where Ω is replaced by Ω_k) and the boundary conditions (1.17).

Therefore we can choose $u\,, \theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and a subsequence $(u_{m_k}, \theta_{m_k})_{k\in\mathbb{N}}$ such that

$$u_{m_k} \underset{k\to\infty}{\rightharpoonup^*} u \quad \text{in } L^\infty(0,T;L^2(\Omega))\,, \quad u_{m_k} \underset{k\to\infty}{\rightharpoonup} u \quad \text{in } L^2(0,T;H^1(\Omega))\,,$$
$$\theta_{m_k} \underset{k\to\infty}{\rightharpoonup^*} \theta \quad \text{in } L^\infty(0,T;L^2(\Omega))\,, \quad \theta_{m_k} \underset{k\to\infty}{\rightharpoonup} \theta \quad \text{in } L^2(0,T;H^1(\Omega)).$$

Theorem 1.2 in combination with (1.16), (7.2) implies that (u,θ) is a weak solution of (1.1) with boundary conditions (1.18). □

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