OPTIMAL INITIAL VALUE CONDITIONS FOR THE EXISTENCE OF STRONG SOLUTIONS OF THE BOUSSINESQ EQUATIONS

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ABSTRACT. Consider the instationary Boussinesq equations in a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$ with initial values $u_0 \in L^2_\sigma(\Omega)$, $\theta_0 \in L^2(\Omega)$ and gravitational force g. We call (u, θ) strong solution if (u, θ) is a weak solution and additionally Serrin's condition $u \in L^s(0, T; L^q(\Omega))$ holds where $1 < s, q < \infty$ satisfy $\frac{2}{s} + \frac{3}{q} = 1$. In this paper we show that $\int_0^\infty \|e^{-tA}u_0\|_q^s \, dt < \infty$ is necessary and sufficient for the existence of such a strong solution (u, θ) in a sufficiently small interval $[0, T[\,, 0 < T \leq \infty$. Furthermore we show that strong solutions are uniquely determined and that they are smooth if the data are smooth. The crucial point is the fact that we have required no additional integrability condition for θ in the definition of a strong solution (u, θ).

1. INTRODUCTION AND MAIN RESULTS

Let $\Omega \subseteq \mathbb{R}^3$ be a domain, and let $[0, T[\,, 0 < T \leq \infty$, be a time interval. Then we consider the Boussinesq equations

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p &= \theta g && \text{in }]0, T[\times\Omega\,, \\ \operatorname{div} u &= 0 && \text{in }]0, T[\times\Omega\,, \\ \theta_t - \Delta\theta + u \cdot \nabla\theta &= 0 && \text{in }]0, T[\times\Omega\,, \\ u = 0\,, \quad \theta &= 0 && \text{on }]0, T[\times\partial\Omega\,, \\ u = u_0\,, \quad \theta &= \theta_0 && \text{at } t = 0\,, \end{aligned} \tag{1.1}$$

where u denotes the velocity of the fluid, θ the difference of the temperature to a fixed reference temperature and p denotes the pressure. Further, $u_0\,, \theta_0$ are the initial values. For mathematical completeness we allow a time dependent gravitational force $g = g(t, x)$. However, in most applications the gravitational force is a constant vector field in time. To simplify the notation we have set the density, kinematic viscosity and thermal conductivity to 1. The Boussinesq equations constitute a model of motion of a viscous, incompressible buoyancy-driven fluid flow coupled with heat convection. For further information about the Boussinesq system we refer to [17, 22]. The Boussinesq equations have been investigated by many researchers, see e.g. [1, 2, 3, 11, 12, 14, 16, 18, 21] and papers cited there.

We need the following space of test functions:

$$C_0^\infty([0, T[; C_{0,\sigma}^\infty(\Omega)) := \{\, w|_{[0,T[\times\Omega} \,; w \in C_0^\infty(]-1, T[\times\Omega) \,; \operatorname{div} w = 0 \,\}.$$

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Motivated by the concept of a weak solution of the instationary Navier-Stokes equations (in the sense of Leray-Hopf) we arrive at the following

Definition 1.1. Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, let $0 < T \leq \infty$, let $g :]0,T[\times\Omega \to \mathbb{R}^3$ be a measurable vector field. Further assume $u_0 \in L^2_\sigma(\Omega)$ and $\theta_0 \in L^2(\Omega)$. A pair

$$u \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2_{\text{loc}}([0,T[; W^{1,2}_{0,\sigma}(\Omega)), \tag{1.2}$$

$$\theta \in L^\infty(0,T; L^2(\Omega)) \cap L^2_{\text{loc}}([0,T[; H^1_0(\Omega)), \tag{1.3}$$

with $\theta g \in L^1_{\text{loc}}([0,T[; L^2(\Omega))$ is called a weak solution of the Boussinesq system (1.1) if the following identities are satisfied for all $w \in C_0^\infty([0,T[; C^\infty_{0,\sigma}(\Omega))$ and all $\phi \in C_0^\infty([0,T[\times\Omega)$:

$$- \langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} + \langle u\cdot\nabla u, w\rangle_{\Omega,T} = \langle \theta g, w\rangle_{\Omega,T} + \langle u_0, w\rangle_\Omega ,$$

$$- \langle \theta, \phi_t\rangle_{\Omega,T} + \langle \nabla\theta, \nabla\phi\rangle_{\Omega,T} + \langle u\cdot\nabla\theta, \phi\rangle_{\Omega,T} = \langle \theta_0, \phi(0)\rangle_\Omega .$$

In the identities above $\langle\cdot,\cdot\rangle_\Omega$, $\langle\cdot,\rangle_{\Omega,T}$ denotes the usual L^2-scalar product in Ω and in $]0,T[\times\Omega$, respectively.

Given a weak solution (u,θ) of (1.1) we may assume, after a possible redefinition on a set of Lebesgue measure 0, that $u : [0,T[\to L^2_\sigma(\Omega)$ and $\theta : [0,T[\to L^2(\Omega)$ are both weakly continuous functions and the initial values u_0, θ_0 are attained in the following sense:

$$\lim_{t\searrow 0}\langle u(t), w\rangle_\Omega = \langle u_0, w\rangle_\Omega , \quad \lim_{t\searrow 0}\langle \theta(t), \phi\rangle_\Omega = \langle \theta_0, \phi\rangle_\Omega$$

for all $w \in L^2_\sigma(\Omega)$ and all $\phi \in L^2(\Omega)$. If $g \in L^\infty(]0,T[\times\Omega)$ it can be proved using the Faedo-Galerkin method that there exists a weak solution (u,θ) of (1.1) in $[0,T[\times\Omega$. We can show this analogously as in [16, Theorem 1] where the corresponding result is proven in the case of mixed Dirichlet/Neumann boundary conditions. Moreover, there exists a distribution p, called an associated pressure, such that

$$u_t - \Delta u + u\cdot\nabla u + \nabla p = \theta g$$

holds in the sense of distributions in $]0,T[\times\Omega$, see [19, V.1.7]. For exponents s, q with $1 < q,\ s < \infty$ we define the *Serrin number* by

$$\mathcal{S}(s,q) := \frac{2}{s} + \frac{3}{q}.$$

Up to now, uniqueness and regularity of a weak solution u of the three-dimensional instationary Navier-Stokes equations is an unsolved problem. However, it is known that uniqueness and regularity holds if additionally Serrin's condition $u \in L^s(0,T;L^q(\Omega))$ holds where $1 < s, q < \infty$ with $\mathcal{S}(s,q) = 1$. Since the Navier-Stokes equations can reduced to the Boussinesq equations we give the following definition.

Definition 1.2. Consider data as in Definition 1.1. We say that (u,θ) is a *strong solution* of (1.1) if (u,θ) is a weak solution of (1.1) and if there are exponents $1 < s, q < \infty$ with $\mathcal{S}(s,q) = 1$ such that $u \in L^s(0,T;L^q(\Omega))$.

The present paper deals with optimal initial value conditions, uniqueness and regularity of strong solutions as defined above. The crucial point in this analysis is the fact that we have required no additional integrability condition for θ.

Our first main result is a sufficient criterion for the existence of a strong solution of (1.1). We denote by $\Delta = \Delta_2\,, A = A_2$ the Laplace and Stokes operator, respectively. For further information about these operators we refer to the preliminaries.

Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^{2,1}$, let $0 < T \leq \infty$. Consider $1 < s, q < \infty$ with $\mathcal{S}(s,q) = 1$. Let $1 < s_1, s_2, q_1 < \infty$ be defined by*

$$\frac{1}{s_1} = \frac{1}{2} - \frac{1}{s}\,, \quad \frac{1}{q_1} = \frac{1}{2} - \frac{1}{q}\,, \quad \frac{1}{s_2} = \frac{1}{2} + \frac{1}{s}.$$

Consider $g \in L^{s_2}(0,T;L^q(\Omega)) \cap L^{\mu}(0,T;L^p(\Omega))$ where $1 < \mu, p < \infty$ satisfy $\mathcal{S}(\mu,p) = \frac{3}{2}$ and $2 + \frac{3}{q} > \frac{3}{q_1} + \frac{3}{p}$. Further assume $u_0 \in L^2_\sigma(\Omega)\,, \theta_0 \in L^2(\Omega)$. Then there exists a constant $\epsilon_* = \epsilon_*(\Omega, p, q) > 0$ with the following property: If the conditions

$$\int_0^T \|e^{-tA}u_0\|_q^s\,dt \leq \epsilon_*\,, \tag{1.4}$$

$$\int_0^T \|e^{t\Delta}\theta_0\|_{q_1}^{s_1}\,dt \leq \epsilon_*\,, \tag{1.5}$$

$$\|g\|_{p,\mu;T} \leq \epsilon_*\,, \tag{1.6}$$

are satisfied, then there exists a strong solution (u,θ) of the Boussinesq equations (1.1). After a possible redefinition on a null set, $u : [0,T[\to L^2_\sigma(\Omega)$ and $\theta : [0,T[\to L^2(\Omega)$ are strongly continuous and the energy equalities

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2\,d\tau = \frac{1}{2}\|u_0\|_2^2 + \int_0^t \langle \theta g, u\rangle_\Omega\,d\tau\,, \tag{1.7}$$

$$\frac{1}{2}\|\theta(t)\|_2^2 + \int_0^t \|\nabla \theta(\tau)\|_2^2\,d\tau = \frac{1}{2}\|\theta_0\|_2^2 \tag{1.8}$$

are satisfied for all $t \in [0,T[$. Moreover (u,θ) is the only strong solution of (1.1) in $[0,T[\times\Omega$.

Remark. It follows from (2.2), (2.4) below that $e^{-tA}u_0 \in L^q(\Omega)$ for a.a. $t > 0$ and consequently the left hand side of (1.4) is well defined. It is easy to see that for all $q > 3$ there exists $p > 2$ satisfying $2 + \frac{3}{q} > \frac{3}{q_1} + \frac{3}{p}$. Therefore, the requirements on μ, p can be fulfilled for all possible exponents s, q. In the case $3 < q < 9$ it is possible to choose $p = q_1$.

For a proof of this theorem we refer to Section 4. The idea is to construct (u,θ) as a solution in $L^s(0,T;L^q_\sigma(\Omega)) \times L^{s_1}(0,T;L^{q_1}(\Omega))$ of a suitable nonlinear system, see (3.14) below. The estimates needed to solve this system with the help of Banach's fixed point theorem are presented in Lemma 3.2. The final step is to prove that (u,θ) fulfils (1.2), (1.3) and is therefore a strong solution of (1.1).

In [5, 6] the authors proved that (1.9) below is the optimal initial value condition on $u_0 \in L^2_\sigma(\Omega)$ to get a strong solution $u \in L^s(0,T;L^q(\Omega))\,, 0 < T \leq \infty$, of the instationary Navier-Stokes equations in a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$ where $\mathcal{S}(s,q) = 1$. In the following theorem we show that this condition also characterizes the class of initial values $u_0 \in L^2_\sigma(\Omega)\,, \theta_0 \in L^2(\Omega)$ that allow a strong solution of the Boussinesq equations (1.1) in a sufficiently small interval $[0,T[\,, 0 < T \le \infty$. Especially no additional integrability condition is required for θ_0.

Theorem 1.4. *Consider Ω, T, g and exponents s, q, s_1, s_2, q_1 as in Theorem 1.3. Further assume $u_0 \in L^2_\sigma(\Omega)\,, \theta_0 \in L^2(\Omega)$. Then the condition*

$$\int_0^\infty \|e^{-tA}u_0\|_q^s \, dt < \infty \tag{1.9}$$

is necessary and sufficient for the existence of $0 < T' \le T$ and a strong solution (u,θ) with $u \in L^s(0,T';L^q(\Omega))$ of the Boussinesq equations (1.1).

It is known (see [16, Theorem 3]) that weak solutions of the Boussinesq equations are uniquely determined if $u \in L^s(0,T;L^q(\Omega))\,, \theta \in L^s(0,T;L^q(\Omega))$ where $\mathcal{S}(s,q) = 1$. The following uniqueness theorem for (1.1) needs no additional integrability condition for θ.

Theorem 1.5. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^{2,1}$, let $0 < T \le \infty$, and $1 < s, q < \infty$ with $\mathcal{S}(s,q) = 1$. Consider $g \in L^\mu_{loc}([0,T[;L^p(\Omega))$ where $1 < \mu, p < \infty$ satisfy $\mathcal{S}(\mu,p) = \frac{3}{2}$ and $2 + \frac{3}{q} > \frac{3}{q_1} + \frac{3}{p}$. Assume that (u_1,θ_1) and (u_2,θ_2) are weak solutions of (1.1) such that additionally $u_1\,, u_2 \in L^s_{loc}([0,T[;L^q(\Omega))$. Then $u_1(t) = u_2(t)$ and $\theta_1(t) = \theta_2(t)$ for almost all $t \in [0,T[$.*

The next theorem states that strong solutions (u,θ) of (1.1) are smooth if the data are sufficiently smooth. A proof can be found in Section 5. In this theorem we allow domains which are not necessarily bounded.

Theorem 1.6. *Let $\Omega \subseteq \mathbb{R}^3$ be a uniform C^2-domain (which is not necessarily bounded) such that Ω is also a C^∞-domain. Let $0 < T \le \infty$ and $g \in C_0^\infty(\overline{]0,T[\times\Omega})$. Consider $u_0 \in W^{1,2}_{0,\sigma}(\Omega)\,, \theta_0 \in H^1_0(\Omega)$. Let (u,θ) be a strong solution of (1.1) in $[0,T[\times\Omega$. Then, after redefinition on a null set of $]0,T[\times\Omega$,*

$$u \in C^\infty_{loc}(\overline{]\epsilon,T'[\times\Omega})\,, \quad \theta \in C^\infty_{loc}(\overline{]\epsilon,T'[\times\Omega}) \tag{1.10}$$

for all ϵ, T' with $0 < \epsilon < T' < T$. There exists an associated pressure p of u satisfying

$$p \in C^\infty_{loc}(\overline{]\epsilon,T'[\times\Omega}) \tag{1.11}$$

for all ϵ, T' with $0 < \epsilon < T' < T$.

The paper is organized as follows: In Section 2 we present some preliminaries. Section 3 deals with the construction of a fixed point needed for the proof of Theorem 1.3. In the following section we will prove Theorems 1.3-1.5. Finally, Section 1.6 is dedicated to the proof of Theorem 1.6.

2. Preliminaries

Given a domain $\Omega \subseteq \mathbb{R}^n\,, n \in \mathbb{N}$, and $1 \le q \le \infty\,, k \in \mathbb{N}$, we need the usual Lebesgue and Sobolev spaces, $L^q(\Omega)\,, W^{k,q}(\Omega)$ with norm $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and $\|\cdot\|_{W^{k,q}(\Omega)}$, respectively. For two measurable functions f, g with $f \cdot g \in L^1(\Omega)$, where $f \cdot g$ means the usual scalar product of vector or matrix fields, we set $\langle f, g\rangle_\Omega := \int_\Omega f(x)\cdot g(x)\,dx$. Note that the same symbol $L^q(\Omega)$

etc. will be used for spaces of scalar-, vector- or matrix-valued functions. Let $C^m(\Omega)\,, m = 0, 1, \ldots, \infty$, denote the space of functions for which all partial derivatives of order $|\alpha| \le m$ ($|\alpha| < \infty$ when $m = \infty$) exist and are continuous. As usual, $C_0^m(\Omega)$ is the set of all functions from $C^m(\Omega)$ with compact support in Ω. Further $C_{0,\sigma}^\infty(\Omega) := \{\, v \in C_0^\infty(\Omega);\, \operatorname{div} v = 0 \,\}$. For $1 < q < \infty$ we define $L^q_\sigma(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_q}$ and $W^{1,2}_{0,\sigma}(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{W^{1,2}}}$. For $1 \le q \le \infty$ let q' be the dual exponent such that $\frac{1}{q} + \frac{1}{q'} = 1$. It is well known that $L^q_\sigma(\Omega)' \cong L^{q'}_\sigma(\Omega)\,, 1 < q < \infty$, using the standard pairing $\langle \cdot, \cdot \rangle_\Omega$.

Given a Banach space X, $1 \le p \le \infty$, and an interval $]0, T[$ we denote by $L^p(0, T; X)$ the space of (equivalence classes of) strongly measurable functions $f :]0, T[\to X$ such that $\|f\|_p := \left(\int_0^T \|f(t)\|_X^p\, dt\right)^{\frac{1}{p}} < \infty$ if $1 \le p < \infty$ and $\|f\|_\infty := \operatorname{ess\,sup}_{t \in]0,T[} \|f(t)\|_X$ if $p = \infty$. Moreover

$$\begin{aligned} L^p_{\mathrm{loc}}([0, T[; X) := \{\, u : [0, T[\to X \text{ strongly measurable}, \\ u \in L^p(0, T'; X) \text{ for all } 0 < T' < T \}. \end{aligned}$$

If $X = L^q(\Omega)$, $1 \le q \le \infty$, the norm in $L^p(0, T; L^q(\Omega))$ will be denoted by $\|f\|_{q,p;T}$.

Fix a bounded domain $\Omega \subseteq \mathbb{R}^3$ with $\partial\Omega \in C^{2,1}$ and $1 < q < \infty$. Let $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$, be the *Helmholtz projection* and let Δ_q denote the *Laplace operator* with domain $\mathcal{D}(\Delta_q) := W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega)$. We introduce the *Stokes operator* by

$$\mathcal{D}(A_q) = L^q_\sigma(\Omega) \cap W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega), \quad A_q u := -P_q \Delta_q u\,, \quad u \in \mathcal{D}(A_q).$$

The Stokes operator is *consistent* in the sense that for $1 < q, r < \infty$

$$A_q u = A_r u \quad \forall u \in \mathcal{D}(A_q) \cap \mathcal{D}(A_r). \tag{2.1}$$

Throughout this paper we will write $A = A_2$. For $\alpha \in [0, 1]$ the fractional power $A_q^\alpha : \mathcal{D}(A_q^\alpha) \to L^q_\sigma(\Omega)$ with dense domain $\mathcal{D}(A_q^\alpha) \subseteq L^q_\sigma(\Omega)$ and range $\mathcal{R}(A_q^\alpha) = L^q_\sigma(\Omega)$ is a well defined, bijective, closed operator (see [8, 9]). In particular the inverse operator $(A_q^\alpha)^{-1} := A_q^{-\alpha}$ is a bounded operator on $L^q_\sigma(\Omega)$. Further $(A_q^\alpha)' \cong A_{q'}^\alpha$ for the adjoint operator. The space $\mathcal{D}(A_q^\alpha)$ equipped with the graph norm $\|u\|_{\mathcal{D}(A_q^\alpha)} := \|u\|_q + \|A_q^\alpha u\|_q$ which is equivalent to $\|A_q^\alpha u\|_q$ is a Banach space. Analogous properties hold for fractional powers $(-\Delta_q)^\alpha : \mathcal{D}((-\Delta_q)^\alpha) \subseteq L^q(\Omega) \to L^q(\Omega)$ of $-\Delta_q$.

It is well known that $-A_q$ generates a uniformly bounded analytic semigroup $\{\, e^{-tA_q}; t \ge 0 \,\}$ on $L^q_\sigma(\Omega)$ and that Δ_q generates a bounded analytic semigroup $\{\, e^{t\Delta_q}; t \ge 0\}$ on $L^q(\Omega)$. The decay estimates

$$\|A_q^\alpha e^{-tA_q}\|_q \le c\, t^{-\alpha}, \quad t > 0\,, \tag{2.2}$$

$$\|(-\Delta_q)^\alpha e^{t\Delta_q}\|_q \le c\, t^{-\alpha} \quad t > 0\,, \tag{2.3}$$

are satisfied where $\alpha \ge 0\,, q > 1$, and $c = c(\Omega, q, \alpha) > 0$. There holds

$$\|u\|_\gamma \le c \|A_q^\alpha u\|_q \qquad \forall u \in \mathcal{D}(A_q^\alpha)\,, \tag{2.4}$$

$$\|\phi\|_\gamma \le c \|(-\Delta_q)^\alpha \phi\|_q \quad \forall \phi \in \mathcal{D}((-\Delta_q)^\alpha) \tag{2.5}$$

with a constant $c = c(\Omega, q, \alpha) > 0$ where $0 \le \alpha \le 1\,, 1 < q < \infty$ with $2\alpha + \frac{3}{\gamma} = \frac{3}{q}$. Furthermore

$$\mathcal{D}(A_q^{1/2}) = W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega)\,, \quad \|\nabla u\|_q \le c\|A_q^{1/2} u\|_q\,, u \in \mathcal{D}(A_q^{1/2})\,, \tag{2.6}$$

$$\mathcal{D}((-\Delta_q)^{1/2}) = W_0^{1,q}(\Omega)\,, \quad \|\nabla u\|_q \le c\|(-\Delta_q)^{1/2} u\|_q\,, u \in \mathcal{D}((-\Delta_q)^{1/2}) \tag{2.7}$$

for all $1 < q < \infty$ with a constant $c = c(\Omega, q) > 0$. If $q = 2$ it is possible to choose $c = 1$ in (2.2), (2.3), (2.6), (2.7). We refer to [7, 8, 9, 10] for the results above and further properties.

To proceed we formulate integral equations which characterize weak solutions of the Boussinesq system (1.1).

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, let $0 < T \le \infty$, let $g :]0, T[\times\Omega \to \mathbb{R}^3$ be a measurable vector field. Further assume $u_0 \in L^2_\sigma(\Omega)$ and $\theta_0 \in L^2(\Omega)$. Then (u, θ) satisfying (1.2), (1.3) and $\theta g \in L^1_{loc}([0, T[; L^2(\Omega))$ is a weak solution of (1.1) if and only if the integral equations*

$$\begin{aligned} u(t) = e^{-tA}u_0 &+ \int_0^t e^{-(t-\tau)A} P\big(\theta(\tau)g(\tau)\big)\, d\tau \\ &- A^{1/2}\int_0^t e^{-(t-\tau)A} A^{-1/2} P\mathrm{div}\big(u(\tau)\otimes u(\tau)\big)\, d\tau\,, \end{aligned} \tag{2.8}$$

$$\theta(t) = e^{t\Delta}\theta_0 - (-\Delta)^{1/2}\int_0^t e^{(t-\tau)\Delta}(-\Delta)^{-1/2}\mathrm{div}\big(\theta(\tau)u(\tau)\big)\, d\tau \tag{2.9}$$

are satisfied for almost all $t \in [0, T[$.

Proof. The representation formula (2.8) follows from [19, Chapter IV, Section 2.4] with $f := \theta g \in L^1_{\mathrm{loc}}([0, T[; L^2(\Omega))$. To prove (2.9) we replace $-A$ by Δ and use the same argumentation as in the proof of (2.8). □

3. Construction of a suitable fixed point

The proof of Theorem 1.3 is based on the construction of a solution of the system (3.14) below. To solve this system with the help of Banach's fixed point theorem we need the estimates presented in Lemma 3.2. In the following lemma we use the boundedness of Ω to show that the assumption $p > \frac{3}{2}$ in [4, Lemma 3.1] can be removed.

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^{2,1}$ and $p > 1\,, F \in L^p(\Omega)$. Choose $r, \sigma \ge 0$ with*

$$2\sigma + \frac{3}{r} = \frac{3}{p}\,, \quad 0 \le \sigma \le \frac{1}{2}. \tag{3.1}$$

There exists a unique element in $L^r_\sigma(\Omega)$ denoted by $A_r^{-1/2-\sigma} P_r \mathrm{div} F \in L^r_\sigma(\Omega)$ with

$$\langle A_r^{-1/2-\sigma} P_r \mathrm{div} F, A_{r'}^{1/2+\sigma} w\rangle_\Omega = -\langle F, \nabla w\rangle_\Omega \tag{3.2}$$

for all $w \in \mathcal{D}(A_{r'}^{1/2+\sigma})$. There holds

$$\|A_r^{-1/2-\sigma} P_r \mathrm{div} F\|_r \le c\|F\|_p \tag{3.3}$$

with a constant $c = c(\Omega, p, r) > 0$.

Proof. From $2 \cdot \frac{1}{2} + \frac{3}{r} \geq \frac{3}{p}$ it follows $2 \cdot \frac{1}{2} + \frac{3}{p'} \geq \frac{3}{r'}$. Sobolev's imbedding theorem yields the continuous imbedding

$$W^{1,r'}(\Omega) \hookrightarrow L^{p'}(\Omega). \tag{3.4}$$

(If $r' \geq 3$ we use the boundedness of Ω.) Fix $w \in \mathcal{D}(A_{r'})$. Using (2.6) and (3.4) we see $w \in \mathcal{D}(A_{p'}^{1/2})$. From the consistence of the Stokes operator (see (2.1)) it follows $A_{p'}^{1/2} w = A_{r'}^{1/2} w$. Further, since $A_{r'}^{1/2} w \in \mathcal{D}(A_{r'}^{1/2})$ we get from (2.6), (2.4)

$$\begin{aligned} |\langle F, \nabla w\rangle_\Omega| &\leq \|F\|_p \|\nabla w\|_{p'} \\ &\leq c\|F\|_p \|A_{p'}^{1/2} w\|_{p'} \\ &= c\|F\|_p \|A_{r'}^{1/2} w\|_{p'} \\ &\leq c\|F\|_p \|A_{r'}^{1/2+\sigma} w\|_{r'} \end{aligned}$$

with a constant $c = c(\Omega, p, r) > 0$. The rest of the proof can be finished as in [4, Lemma 3.1]. There are no problems occurring although we allow $1 < p < \infty$. □

In the same way $(-\Delta_r)^{-1/2-\sigma}\mathrm{div}F$ is well defined by

$$\langle (-\Delta_r)^{-1/2-\sigma}\mathrm{div}F, (-\Delta_{r'})^{1/2+\sigma}\phi\rangle_\Omega = -\langle F, \nabla\phi\rangle_\Omega$$

for all $\phi \in \mathcal{D}((-\Delta_{r'})^{1/2+\sigma})$. We proceed with the lemma below.

Lemma 3.2. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^{2,1}$ and $0 < T \leq \infty$. Consider $1 < s, q < \infty$ with $\mathcal{S}(s,q) = 1$. Let $1 < s_1, q_1 < \infty$ be defined by*

$$\frac{1}{s_1} = \frac{1}{2} - \frac{1}{s}, \quad \frac{1}{q_1} = \frac{1}{2} - \frac{1}{q}.$$

Consider $g \in L^\mu(0,T; L^p(\Omega))$ where $1 < \mu, p < \infty$ satisfy $\mathcal{S}(\mu,p) = \frac{3}{2}$ and $2 + \frac{3}{q} > \frac{3}{q_1} + \frac{3}{p}$. Define $\alpha := \frac{1}{2} + \frac{3}{2q}$ and the Banach spaces

$$X := L^s(0,T; L^q_\sigma(\Omega)), \quad Y := L^{s_1}(0,T; L^{q_1}(\Omega)).$$

(i) *Define the bilinear form $\mathcal{F}_1 : X \times X \to X$ by*

$$\mathcal{F}_1(u,v)(t) := -A_q^\alpha \int_0^t e^{-(t-\tau)A_q} A_q^{-\alpha} P_q \mathrm{div}\big(u(\tau) \otimes v(\tau)\big)\, d\tau$$

for a.a. $t \in [0,T[$.

Then

$$\|\mathcal{F}_1(u,v)\|_X \leq c\|u \otimes v\|_{\frac{q}{2},\frac{s}{2};T} \leq c\|u\|_X \|v\|_X \tag{3.5}$$

for all $u, v \in X$ where $c = c(\Omega, q) > 0$ is a constant.

(ii) *Define the bilinear form $\mathcal{F}_2 : X \times Y \to Y$ by*

$$\mathcal{F}_2(u,\theta)(t) := -(-\Delta_{q_1})^\alpha \int_0^t e^{(t-\tau)\Delta_{q_1}} (-\Delta_{q_1})^{-\alpha} \mathrm{div}\big(\theta(\tau) u(\tau)\big)\, d\tau$$

for a.a. $t \in [0,T[$.

Then

$$\|\mathcal{F}_2(u,\theta)\|_Y \leq c\|\theta u\|_{2,2;T} \leq c\|u\|_X \|\theta\|_Y \tag{3.6}$$

for all $u \in X$, $\theta \in Y$ with $c = c(\Omega, q) > 0$.

(iii) *Define the linear map* $\mathcal{L} : Y \to X$ *by*

$$\big(\mathcal{L}\theta\big)(t) := \int_0^t e^{-(t-\tau)A} P\big(\theta(\tau)g(\tau)\big)\,d\tau \quad \textit{for a.a. } t \in [0,T[.$$

Then

$$\|\mathcal{L}\theta\|_X \leq c\|g\|_{p,\mu;T}\|\theta\|_Y \tag{3.7}$$

for all $\theta \in Y$ *with* $c = c(\Omega, p, q) > 0$.

Proof. Fix $u\,, v \in L^s(0,T; L^q_\sigma(\Omega))$ and $\theta \in L^{s_1}(0,T; L^{q_1}(\Omega))$.

Proof of (i). We use (2.2), (3.3) and get

$$\begin{aligned}\|\mathcal{F}(u,v)(t)\|_q &\leq c\int_0^t |t-\tau|^{-\alpha}\|A_q^{-\alpha}P_q\mathrm{div}\big(u(\tau)\otimes v(\tau)\big)\|_q\,d\tau \\ &\leq c\int_0^T |t-\tau|^{-\alpha}\|u(\tau)\otimes v(\tau)\|_{\frac{q}{2}}\,d\tau\end{aligned}$$

for almost all $t \in [0,T[$ with $c = c(\Omega, q) > 0$. Apply the Hardy-Littlewood inequality (see [20, Ch. V, 1.2]) with $(1-\alpha) + \frac{1}{s} = \frac{1}{s/2}$ to obtain

$$\|\mathcal{F}(u,v)\|_{q,s;T} \leq c\|u\otimes v\|_{\frac{q}{2},\frac{s}{2};T} \leq c\|u\|_{q,s;T}\|v\|_{q,s;T} \tag{3.8}$$

with $c = c(\Omega, q) > 0$.

Proof of (ii). There holds $2\cdot\frac{3}{2q} + \frac{3}{q_1} = \frac{3}{2}$. By an analogous version of (3.3) for $-\Delta_{q_1}$ we get

$$\|(-\Delta_{q_1})^{-\alpha}\mathrm{div}\big(\theta(t)u(t)\big)\|_{q_1} \leq c(\Omega,q)\|\theta(t)u(t)\|_2 \tag{3.9}$$

for a.a. $t \in [0,T[$. It follows from (2.3), (3.9)

$$\begin{aligned}\|(\mathcal{F}_2(u,\theta))(t)\|_{q_1} &\leq c\int_0^t |t-\tau|^{-\alpha}\|(-\Delta_{q_1})^{-\alpha}\mathrm{div}\big(\theta(\tau)u(\tau)\,\big)\|_{q_1}\,d\tau \\ &\leq c\int_0^T |t-\tau|^{-\alpha}\|\theta(\tau)u(\tau)\|_2\,d\tau\end{aligned}$$

for a.a. $t \in [0,T[$ with $c = c(\Omega,q) > 0$. The Hardy-Littlewood inequality with $(1-\alpha) + \frac{1}{s_1} = \frac{1}{2}$, combined with Hölder's inequality, yields

$$\|\mathcal{F}_2(u,\theta)\|_{q_1,s_1;T} \leq c\|\theta u\|_{2,2;T} \leq c(\Omega,q)\|u\|_{q,s;T}\|\theta\|_{q_1,s_1;T}. \tag{3.10}$$

Proof of (iii). Choose $0 \leq \sigma < 1$ such that $2\sigma + \frac{3}{q} = \frac{3}{q_1} + \frac{3}{p}$. Define $1 < p_*, \mu_* < \infty$ by

$$\frac{1}{p_*} = \frac{1}{q_1} + \frac{1}{p}\,, \quad \frac{1}{\mu_*} = \frac{1}{s_1} + \frac{1}{\mu}. \tag{3.11}$$

Using (2.2), (2.4) yields

$$\begin{aligned}\|(\mathcal{L}\theta)(t)\|_q &\leq c\left\|A^\sigma_{p_*}\int_0^t e^{-(t-\tau)A_{p_*}}P_{p_*}\big(\theta(\tau)g(\tau)\big)\,d\tau\right\|_{p_*} \\ &\leq c\int_0^T |t-\tau|^{-\sigma}\|\theta(\tau)g(\tau)\|_{p_*}\,d\tau\end{aligned} \tag{3.12}$$

for a.a. $t \in [0,T[$ with $c = c(\Omega,p,q) > 0$. Since

$$(1-\sigma) + \frac{1}{s} = \frac{1}{\mu_*} = \frac{1}{s_1} + \frac{1}{\mu}$$

we can apply the Hardy-Littlewood estimate to (3.12) and get

$$\|\mathcal{L}\theta\|_{q,s;T} \leq c\|\theta g\|_{p_*,\mu_*;T} \leq c(\Omega,p,q)\|g\|_{p,\mu;T}\|\theta\|_{q_1,s_1;T}. \qquad \square$$

Now we have all ingredients at hand to construct a solution of (3.14).

Theorem 3.3. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C^{2,1}$, let $0 < T \leq \infty$. Consider $1 < s, q < \infty$ with $\mathcal{S}(s,q) = 1$. Let $1 < s_1, q_1 < \infty$ be defined by*

$$\frac{1}{s_1} = \frac{1}{2} - \frac{1}{s}, \quad \frac{1}{q_1} = \frac{1}{2} - \frac{1}{q}.$$

Consider $g \in L^\mu(0,T;L^p(\Omega))$ where $1 < \mu, p < \infty$ satisfy $\mathcal{S}(\mu,p) = \frac{3}{2}$ and $2 + \frac{3}{q} > \frac{3}{q_1} + \frac{3}{p}$. Then there exists a constant $\epsilon_* = \epsilon_*(\Omega,q,p) > 0$ with the following property: If $E_1 \in L^s(0,T;L^q_\sigma(\Omega))$, $E_2 \in L^{s_1}(0,T;L^{q_1}(\Omega))$ fulfil

$$\|E_1\|_{q,s;T} + \|E_2\|_{q_1,s_1;T} + \|g\|_{p,\mu;T} \leq \epsilon_* \tag{3.13}$$

then there exists $u \in L^s(0,T;L^q_\sigma(\Omega))$, $\theta \in L^{s_1}(0,T;L^{q_1}(\Omega))$ satisfying

$$\begin{aligned} u &= E_1 + \mathcal{F}_1(u,u) + \mathcal{L}\theta\,, \\ \theta &= E_2 + \mathcal{F}_2(u,\theta) \end{aligned} \tag{3.14}$$

and

$$\|u\|_{q,s;T} + \|\theta\|_{q_1,s_1;T} \leq 4(\|E_1\|_{q,s;T} + \|E_2\|_{q_1,s_1;T}).$$

Proof. Let X, Y, let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{L}$ be defined as in Lemma 3.2, and let $c = c(\Omega,p,q) > 0$ be a constant such that the estimates (3.5), (3.6), (3.7) are fulfilled. We endow $X \times Y$ with the norm $\|(u,\theta)\|_{X\times Y} := \|u\|_X + \|\theta\|_Y$ and obtain that $X \times Y$ is a Banach space. Introduce the nonlinear map

$$T : X \times Y \to X \times Y\,, \quad T(u,\theta) := \big(E_1 + \mathcal{F}_1(u,u) + \mathcal{L}\theta, E_2 + \mathcal{F}_2(u,\theta)\big).$$

Define $\epsilon_* := \frac{1}{32c}$. Therefore

$$\|\mathcal{L}\theta\|_X \leq c\|g\|_{p,\mu;T}\|\theta\|_Y \leq \frac{1}{2}\|\theta\|_Y$$

for all $\theta \in Y$. Introduce $M := \|E_1\|_X + \|E_2\|_Y$. With no loss of generality assume $M > 0$. Since $4cM < 1$ we can define R as the smallest positive root of the polynomial $cx^2 - \frac{1}{2}x + M$, i.e.

$$R = \frac{\frac{1}{2} - \sqrt{\frac{1}{4} - 4cM}}{2c} = \frac{2M}{\frac{1}{2} + \sqrt{\frac{1}{4} - 4cM}}.$$

We define the closed ball $\mathcal{B} := \{(u,\theta) \in X \times Y; \|(u,\theta)\|_{X\times Y} \leq R\}$. Hence there holds

$$\|T(u,\theta)\|_{X\times Y} \leq c\|u\|_X(\|u\|_X + \|\theta\|_Y) + \frac{1}{2}\|\theta\|_Y + M \leq cR^2 + \frac{1}{2}R + M = R.$$

Thus $T(\mathcal{B}) \subseteq \mathcal{B}$. We obtain

$$\begin{aligned}
&\|T(u,\theta) - T(\tilde{u},\tilde{\theta})\|_{X\times Y} \\
&\quad = \big(\mathcal{F}_1(u, u-\tilde{u}) + \mathcal{F}_1(u-\tilde{u},\tilde{u}) + \mathcal{L}(\theta-\tilde{\theta})\,, \mathcal{F}_2(u,\theta-\tilde{\theta}) + \mathcal{F}_2(u-\tilde{u},\tilde{\theta})\big) \\
&\quad \le c(\|u\|_X + \|\tilde{u}\|_X + \|\tilde{\theta}\|_Y)\|u-\tilde{u}\|_X + c\|u\|_X\|\theta-\tilde{\theta}\|_Y + \frac{1}{2}\|\theta-\tilde{\theta}\|_Y \\
&\quad \le 2cR\|u-\tilde{u}\|_X + cR\|\theta-\tilde{\theta}\|_Y + \frac{1}{2}\|\theta-\tilde{\theta}\|_Y \\
&\quad \le (2cR + \frac{1}{2})\|(u,\theta) - (\tilde{u},\tilde{\theta})\|_{X\times Y}
\end{aligned}$$

for all $(u,\theta)\,,(\tilde{u},\tilde{\theta}) \in \mathcal{B}$. We get from $R < 4M$ and (3.13) that

$$2cR + \frac{1}{2} < 8cM + \frac{1}{2} < 1.$$

Altogether $T : \mathcal{B} \to \mathcal{B}$ is a strict contraction. By Banach's fixed point theorem there exists $(u,\theta) \in \mathcal{B}$ with $T(u,\theta) = (u,\theta)$. Especially

$$\|u\|_X + \|\theta\|_Y \le R < 4M. \qquad \square$$

4. Proof of Theorems 1.3, 1.4 and 1.5

Proof of Theorem 1.3. Step 1. Define

$$E_1(t) := e^{-tA}u_0\,, \quad E_2(t) := e^{t\Delta}\theta_0 \qquad \text{for } t \in [0,T[. \tag{4.1}$$

We use [19, IV, Theorems 2.3.1 and 2.4.1] to get $E_1 \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2_{\text{loc}}([0,T[;W^{1,2}_{0,\sigma}(\Omega))$ and that E_1 is a weak solution to the (linear) Stokes system with initial value u_0 and external force 0. Analogously it follows $E_2 \in L^\infty(0,T;L^2(\Omega)) \cap L^2_{\text{loc}}([0,T[;H^1_0(\Omega))$ and that E_2 is a weak solution to the heat equation with initial value θ_0 and external force 0.

Let $\mathcal{F}_1\,,\mathcal{F}_2\,,\mathcal{L}$ be defined as in Lemma 3.2 and let $\epsilon_* = \epsilon_*(\Omega,p,q) > 0$ be the constant constructed in Theorem 3.3. Thus, if

$$\|E_1\|_{q,s;T} + \|E_2\|_{q_1,s_1;T} + \|g\|_{p,\mu;T} \le \epsilon_* \tag{4.2}$$

then there exists $u \in L^s(0,T;L^q_\sigma(\Omega))\,,\theta \in L^{s_1}(0,T;L^{q_1}(\Omega))$ satisfying

$$\begin{aligned} u &= E_1 + \mathcal{F}_1(u,u) + \mathcal{L}\theta\,, \\ \theta &= E_2 + \mathcal{F}_2(u,\theta) \end{aligned} \tag{4.3}$$

and

$$\|u\|_{q,s;T} + \|\theta\|_{q_1,s_1;T} \le 4(\|E_1\|_{q,s;T} + \|E_2\|_{q_1,s_1;T}). \tag{4.4}$$

In the following assume that $u \in L^s(0,T;L^q_\sigma(\Omega))\,,\theta \in L^{s_1}(0,T;L^{q_1}(\Omega))$ satisfy (4.3), (4.4). We will show that, after a possible reduction of ϵ_* (see the discussion following (4.8)), that (u,θ) is a weak solution of (1.1), in particular (u,θ) fulfils (1.2), (1.3).

Step 2. We obtain

$$\|\theta u\|_{2,2;T} \le \|\theta\|_{q_1,s_1;T}\|u\|_{q,s;T} < \infty.$$

Consequently $\theta u \in L^2(0,T;L^2(\Omega))$. By construction

$$\theta(t) = e^{t\Delta}\theta_0 - (-\Delta)^{1/2}\int_0^t e^{(t-\tau)\Delta}(-\Delta)^{-1/2}\text{div}\big(\theta(\tau)u(\tau)\big)\,d\tau$$

for a.a. $t \in [0,T[$. Thus, from [19, Chapter IV, Lemma 2.4.2] (with A replaced by $-\Delta$) it follows that θ fulfils (1.3).

Step 3. Define

$$E(t) := E_1(t) + \mathcal{L}(\theta)(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A} P\big(\theta(\tau)g(\tau)\big)\, d\tau\,, \tag{4.5}$$

$$\tilde{u}(t) := -\int_0^t A_q^{\alpha} e^{-(t-\tau)A_q} A_q^{-\alpha} P_q \mathrm{div}\big((\tilde{u}+E)\otimes(\tilde{u}+E)\big)(\tau)\, d\tau \tag{4.6}$$

for a.a. $t \in [0,T[$. Thus $u = \tilde{u} + E$. From $g \in L^{s_2}(0,T;L^q(\Omega))$ we get that $\theta g \in L^1(0,T;L^2(\Omega))$. Therefore [19, IV, (2.3.2)] implies $\nabla E \in L^2(0,T;L^2(\Omega))$. Let $1 < \gamma < \infty$ be defined by $\frac{1}{\gamma} = \frac{1}{2} + \frac{1}{q}$. In the following we use the consistence of the Stokes operator and duality arguments to rewrite (4.6). We apply [4, (3.11)] (which is also true for a smooth bounded domain) with $r_1 = q/2\,, r_2 = q$ and $F := u \otimes u \in L^{\frac{s}{2}}(0,T;L^{\frac{q}{2}}(\Omega))$ to (4.6) and obtain

$$\tilde{u}(t) = -\int_0^t A_{q/2}^{1/2} e^{-(t-\tau)A_{q/2}} A_{q/2}^{-1/2} P_{q/2} \mathrm{div}\big((\tilde{u}+E)\otimes(\tilde{u}+E)\big)(\tau)\, d\tau$$

for a.a. $t \in [0,T[$. Using (2.1) and $F \in L^{s_2}_{\mathrm{loc}}([0,T[;L^{\gamma}(\Omega))$ yields

$$\tilde{u}(t) = -\int_0^t A_{\gamma}^{1/2} e^{-(t-\tau)A_{\gamma}} A_{\gamma}^{-1/2} P_{\gamma} \mathrm{div}\big((\tilde{u}+E)\otimes(\tilde{u}+E)\big)(\tau)\, d\tau \tag{4.7}$$

for a.a. $t \in [0,T[$. Let $J_n := (I + \frac{1}{n}A_{\gamma}^{1/2})^{-1}\,, n \in \mathbb{N}$, be the Yosida approximation of I in $L^{\gamma}_{\sigma}(\Omega)$, so that $\tilde{u} = J_n\tilde{u} + \frac{1}{n}A_{\gamma}^{1/2}J_n\tilde{u}$. For further properties of J_n we refer to [19, II, Section 3.4]. Applying Yosida's smoothing procedure in combination with the consistence of the Stokes operator in the same way as in [6] that leads from (2.46) to (2.48) in this paper shows that

$$\|A_{\gamma}e^{1/2}J_n\tilde{u}\|_{2,2;T} \le c_1\|u\|_{q,s;T}\big(\|A_{\gamma}^{1/2}J_n\tilde{u}\|_{2,2;T} + \|\nabla E\|_{2,2;T}\big) \tag{4.8}$$

for all $n \in \mathbb{N}$ with a fixed constant $c_1 = c_1(\Omega,q) > 0$. Replacing ϵ_* by $\min\{\epsilon_*, \frac{1}{8c_1}\}$ it follows from (4.4) that

$$c_1\|u\|_{q,s;T} \le 4c_1(\|E_1\|_{q,s;T} + \|E_2\|_{q_1,s_1;T}) \le \frac{1}{2}.$$

By construction $\epsilon_* = \epsilon_*(\Omega,p,q) > 0$. We can apply the absorption principle to (4.8) and get

$$\|A_{\gamma}^{1/2}J_n\tilde{u}\|_{2,2;T} \le c\|u\|_{q,s;T}\|\nabla E\|_{2,2;T} \tag{4.9}$$

with a constant $c = c(\Omega,q) > 0$ independent of $n \in \mathbb{N}$. By a functional analytic argument (see [19, II. (3.1.8), (3.1.9)]) in combination with the consistence of the Stokes operator it follows $\tilde{u}(t) \in \mathcal{D}(A^{1/2})$ for a.a. $t \in [0,T[$ and $A^{1/2}\tilde{u} \in L^2(0,T;L^2(\Omega))$. Therefore $\nabla\tilde{u} \in L^2(0,T;L^2(\Omega))$.

Step 4. Since $\nabla u \in L^2(0,T;L^2(\Omega))$ we can write

$$\tilde{u}(t) = -\int_0^t e^{-(t-\tau)A_{\gamma}} P_{\gamma}(u \cdot \nabla u)(\tau)\, d\tau$$

for a.a. $t \in [0, T[$. The same argumentation as in [6, page 640] shows that (4) implies $u \otimes u \in L^2(0, T; L^2(\Omega))$. We get

$$\tilde{u}(t) = -\int_0^t A^{1/2} e^{-(t-\tau)A} A^{-1/2} P \mathrm{div}(u \otimes u)(\tau)\, d\tau$$

for a.a. $t \in [0, T[$. Therefore, $\tilde{u}$ can be considered as a weak solution of the (linear) Stokes system with initial value 0 and external force $f = -\mathrm{div}(u \otimes u)$ where $u \otimes u \in L^2(0, T; L^2(\Omega))$. Then linear theory (see [19, IV, Theorems 2.3.1 and 2.4.1]) implies that $\tilde{u}$ satisfies (1.2).

Step 5. We have proven that (u, θ) satisfies (1.2), (1.3). Lemma 2.1 yields that (u, θ) is a strong solution of (1.1). We obtain (see [19, IV, Theorem 2.3.1]), after a possible redefinition on a null set, that $u : [0, T[\to L^2_\sigma(\Omega)$ and $\theta : [0, T[\to L^2(\Omega)$ are strongly continuous. The uniqueness of a strong solution follows from Theorem 1.5. The proof is complete. □

Proof of Theorem 1.4. Let $\epsilon_* = \epsilon_*(\Omega, p, q) > 0$ be the constant constructed in Theorem 1.3. Let E_1, E_2 be defined as in (4.1). Since $E_2 \in L^\infty(0, T; L^2(\Omega)), \nabla E_2 \in L^2(0, T; L^2(\Omega))$ and $\mathcal{S}(s_1, q_1) = \frac{3}{2}$ it follows by interpolation and the continuous imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ that $E_2 \in L^{s_1}(0, T; L^{q_1}(\Omega))$.

First, assume (1.9). Due to $\|E_2\|_{q_1, s_1; T} + \|g\|_{p, \mu; T} < \infty$ we can choose $0 < T' \le T$ such that (1.4), (1.5), (1.6) are fulfilled where T is replaced by T'. Consequently, by Theorem 1.3 there exists a strong solution (u, θ) with $u \in L^s(0, T'; L^q(\Omega))$ of (1.1).

For the proof of the converse direction consider $0 < T' < T$ and a strong solution (u, θ) with $u \in L^s(0, T'; L^q(\Omega))$ of (1.1). From (1.3) it follows by interpolation $\theta \in L^{s_1}(0, T'; L^{q_1}(\Omega))$. By (3.7) we get $\mathcal{L}\theta \in L^s(0, T'; L^q(\Omega))$. Using (2.8) we obtain

$$e^{-tA} u_0 = u(t) - (\mathcal{L}\theta)(t) - \tilde{u}(t) \tag{4.10}$$

for almost all $t \in [0, T'[$ where

$$\tilde{u}(t) := -\int_0^t A^{1/2} e^{-(t-\tau)A} A^{-1/2} P \mathrm{div}\big(u(\tau) \otimes u(\tau)\big)\, d\tau$$

for a.a. $t \in [0, T'[$. From [4, (3.11)] (which holds true for a bounded domain) it follows

$$\tilde{u}(t) = -\int_0^t A_q^\alpha e^{-(t-\tau)A_q} A_q^{-\alpha} P_q \mathrm{div}\big(u(\tau) \otimes u(\tau)\big)\, d\tau$$

for a.a. $t \in [0, T'[$ where $\alpha := \frac{1}{2} + \frac{3}{2q}$. Consequently, (3.5) yields $\tilde{u} \in L^s(0, T'; L^q(\Omega))$. Altogether, from (4.10) we get $e^{-tA} u_0 \in L^s(0, T'; L^q(\Omega))$. From (2.2), (2.4) and $\mathcal{S}(s, q) = 1$ we get

$$\int_{T'}^\infty \|e^{-tA} u_0\|_q^s\, dt \le c \int_{T'}^\infty t^{-\frac{3}{2}s(\frac{1}{2} - \frac{1}{q})} \|u_0\|_2^s\, dt < \infty.$$

Altogether (1.9) holds. □

Proof of Theorem 1.5. Throughout this proof we will use the results of Lemma 3.2 without referring back to them. Especially let s_1, q_1, μ_*, p_* be defined as in (3.2), (3.11). Further we introduce for $0 < T' \le T \le \infty$ and $f \in L^s(0,T; L^q(\Omega))$ the notation $\|f\|_{q,s;T',T} := \left(\int_{T'}^{T} \|f(t)\|_q^s \, dt\right)^{\frac{1}{s}}$.

Define

$$T_{max} := \sup\{\, T' \in [0,T[;\ u_1(t) = u_2(t)\,, \theta_1(t) = \theta_2(t) \quad \text{for a.a. } t \in [0,T'[\,\}.$$

Suppose by contradiction $T_{max} < T$. Consider any $T' \in [T_{max}, T[$. By Lemma 2.1 there holds

$$\begin{aligned} u_1 - u_2 &= \mathcal{F}_1(u_1, u_1 - u_2) + \mathcal{F}_1(u_1 - u_2, u_2) + \mathcal{L}(\theta_1 - \theta_2)\,, \\ \theta_1 - \theta_2 &= \mathcal{F}_2(u_1 - u_2, \theta_1) + \mathcal{F}_2(u_2, \theta_1 - \theta_2). \end{aligned}$$

Thus

$$\begin{aligned} &\|u_1 - u_2\|_{q,s;T'} \\ &\quad \le c\big(\, \|u_1 \otimes (u_1 - u_2)\|_{\frac{q}{2},\frac{s}{2};T'} + \|(u_1 - u_2) \otimes u_2\|_{\frac{q}{2},\frac{s}{2};T'} \\ &\qquad + \|(\theta_1 - \theta_2) g\|_{p_*,\mu_*;T'} \,\big) \\ &\quad = c\big(\, \|u_1 \otimes (u_1 - u_2)\|_{\frac{q}{2},\frac{s}{2};T_{max},T'} + \|(u_1 - u_2) \otimes u_2\|_{\frac{q}{2},\frac{s}{2};T_{max},T'} \\ &\qquad + \|(\theta_1 - \theta_2) g\|_{p_*,\mu_*;T_{max},T'} \,\big) \\ &\quad \le c\big(\, \|u_1\|_{q,s;T_{max},T'} + \|u_2\|_{q,s;T_{max},T'} \,\big) \|u_1 - u_2\|_{q,s;T_{max},T'} \\ &\qquad + c\|g\|_{p,\mu;T_{max},T'} \|\theta_1 - \theta_2\|_{q_1,s_1;T_{max},T'} \end{aligned} \tag{4.11}$$

with $c = c(\Omega, p, q) > 0$. The difference $\|\theta_1 - \theta_2\|_{q_1,s_1;T'}$ can be estimated analogously. We get

$$\begin{aligned} \|\theta_1 - \theta_2\|_{q_1,s_1;T'} \le c\big(\, &\|u_1 - u_2\|_{q,s;T_{\max},T'} \|\theta_1\|_{q_1,s_1;T_{\max},T'} \\ &+ \|u_2\|_{q,s;T_{\max},T'} \|\theta_1 - \theta_2\|_{q_1,s_1;T_{\max},T'} \,\big). \end{aligned}$$

Altogether

$$\begin{aligned} &\|u_1 - u_2\|_{q,s;T'} + \|\theta_1 - \theta_2\|_{q_1,s_1;T'} \\ &\le c_1\big(\|u_1\|_{q,s;T_{max},T'} + \|u_2\|_{q,s;T_{max},T'} + \|\theta_1\|_{q_1,s_1;T_{max},T'}\big) \|u_1 - u_2\|_{q,s;T_{max},T'} \\ &\quad + c_1\big(\|g\|_{p,\mu;T_{max},T'} + \|u_2\|_{q,s;T_{max},T'}\big) \|\theta_1 - \theta_2\|_{q_1,s_1;T_{max},T'} \end{aligned} \tag{4.12}$$

with a fixed constant $c_1 = c_1(\Omega, p, q) > 0$. Now $T' > T_{max}$ can be chosen such that the two conditions

$$\begin{aligned} &c_1\big(\|u_1\|_{q,s;T_{max},T'} + \|u_2\|_{q,s;T_{max},T'} + \|\theta\|_{q_1,s_1;T_{max},T'}\big) \le \frac{1}{4}\,, \\ &c_1\big(\|g\|_{p,\mu;T_{max},T'} + \|u_2\|_{q,s;T_{max},T'}\big) \le \frac{1}{4} \end{aligned} \tag{4.13}$$

are satisfied. From (4.12), (4.13) we get

$$\|u_1 - u_2\|_{q,s;T'} + \|\theta_1 - \theta_2\|_{q_1,s_1;T'} = 0.$$

Since $T' > T_{max}$ this is a contradiction to the definition of T_{max}. Therefore $T_{max} = T$. □

5. Regularity of a strong solution

The goal of this section is to show the smoothness of a strong solution (u,θ) of (1.1) if the data are sufficiently smooth. The definition of a strong solution (u,θ) of (1.1) is 'asymmetric' in the sense that we require $u \in L^s(0,T;L^q(\Omega))\,,\frac{2}{s}+\frac{3}{q}=1$ but no additional condition on θ. Therefore, the crucial point in the first step in the proof of Theorem 1.6 is to show $u\cdot\nabla\theta \in L^2_{\text{loc}}([0,T[;L^2(\Omega))$.

Theorem 5.1. *Let $\Omega\subseteq\mathbb{R}^3$ be a uniform C^2-domain, let $0<T\le\infty$. Consider $1<s,q<\infty$ with $\mathcal{S}(s,q)=1$ and $g\in L^8_{loc}([0,T[;L^4(\Omega))$. Consider $u_0\in W^{1,2}_{0,\sigma}(\Omega)\,,\theta_0\in H^1_0(\Omega)$. Let (u,θ) be a weak solution of (1.1) with $u\in L^s_{loc}([0,T[;L^q(\Omega))$. Then the following statements are satisfied:*

(i) *u fulfils*

$$u\in L^\infty_{loc}([0,T[;W^{1,2}_{0,\sigma}(\Omega))\cap L^2_{loc}([0,T[;H^2(\Omega))\,,\quad u_t\in L^2_{loc}([0,T[;L^2_\sigma(\Omega)) \tag{5.1}$$

and

$$u\cdot\nabla u\in L^2_{loc}([0,T[;L^2(\Omega)). \tag{5.2}$$

Moreover, there exists an associated pressure p of u with

$$p\in L^2_{loc}([0,T[;L^2_{loc}(\overline{\Omega}))\,,\quad \nabla p\in L^2_{loc}([0,T[;L^2(\Omega)). \tag{5.3}$$

(ii) *θ fulfils*

$$\theta\in L^\infty_{loc}([0,T[;H^1_0(\Omega))\cap L^2_{loc}([0,T[;H^2(\Omega))\,,\quad \theta_t\in L^2_{loc}([0,T[;L^2(\Omega)) \tag{5.4}$$

and

$$u\cdot\nabla\theta\in L^2_{loc}([0,T[;L^2(\Omega)). \tag{5.5}$$

Proof. **Step 1.** Replacing T by $T'\,,0<T'<T$, we may assume with no loss of generality that $g\in L^8(0,T;L^4(\Omega))\,,u\in L^s(0,T;L^q(\Omega))$ and $u\in L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))\,,\theta\in L^2(0,T;H^1_0(\Omega))$. From (1.3) it follows by interpolation $\theta\in L^{8/3}(0,T;L^4(\Omega))$. Thus $\theta g\in L^2(0,T;L^2(\Omega))$. We consider u as a strong solution of the instationary Navier-Stokes equations with initial value u_0 and external force $f:=\theta g$. From [19, Chapter V, Theorem 1.8.1] we obtain (5.1), (5.2) and an associated pressure p of u satisfying (5.3).
Step 2. Fix $2<\gamma<4$. Define $\alpha:=\frac{1}{\gamma}$. Then $\frac14<\alpha<\frac12$. Choose $3<r<6$, such that $2\alpha+\frac{3}{r}=\frac{3}{2}$. We get with Sobolev's imbedding theorem and [19, Chapter III. (2.2.8), (2.4.18)] that

$$\begin{aligned}\|u(t)\|_\infty &\le c\|u(t)\|_{W^{1,r}(\Omega)}\\ &\le c\big(\|A^{1/2+\alpha}u(t)\|_2+\|u(t)\|_2\big)\\ &\le c\big(\|Au(t)\|_2^{2\alpha}\|A^{1/2}u(t)\|_2^{1-2\alpha}+\|u(t)\|_2\big)\end{aligned} \tag{5.6}$$

for almost all $t\in[0,T[$ with a constant $c>0$ independent of t. Integrating (5.6) yields

$$\|u\|_{\infty,\gamma;T}\le c\big(\|Au\|^{2\alpha}_{2,2;T}\|A^{1/2}u\|^{1-2\alpha}_{2,\infty;T}+\|u\|_{2,\gamma;T}\big)<\infty \tag{5.7}$$

for all $2 \leq \gamma < 4$. We know (see (2.9)) that

$$\theta(t) = E_2(t) + \tilde{\theta}(t) := e^{t\Delta}\theta_0 - (-\Delta)^{1/2} \int_0^t e^{(t-\tau)\Delta}(-\Delta)^{-1/2}\mathrm{div}\big(\theta(\tau)u(\tau)\big)\, d\tau \tag{5.8}$$

for a.a. $t \in [0, T[$. There holds

$$\|(-\Delta)^{1/2}e^{t\Delta}\theta_0\|_2 = \|e^{t\Delta}(-\Delta)^{1/2}\theta_0\|_2 \leq \|(-\Delta)^{1/2}\theta_0\|_2$$

for all $t > 0$. Thus

$$E_2 \in L^\infty(0, T; H_0^1(\Omega)). \tag{5.9}$$

We use [19, Chapter IV, Lemma 1.5.3] (also true for $(-\Delta)$) and obtain:

$$\|(-\Delta)e^{t\Delta}\theta_0\|_{2,2;T} = \|(-\Delta)^{1/2}e^{t\Delta}(-\Delta)^{1/2}\theta_0\|_{2,2;T} \leq \|(-\Delta)^{1/2}\theta_0\|_2.$$

By the maximal regularity of $-\Delta$ (c.f. [19, IV, Lemma 1.6.2] for $-\Delta$) it follows that E_2 fulfils (5.4) where θ is replaced by E_2.

By (5.7) and $\nabla\theta \in L^2(0, T; L^2(\Omega))$ it follows $u \cdot \nabla\theta \in L^1(0, T; L^2(\Omega))$. Consequently, since $\mathrm{div}\big((\theta u)(t)\big) = (u \cdot \nabla\theta)(t)$ for a.a. $t \in [0, T[$ in the sense of distributions in Ω, we obtain

$$\tilde{\theta}(t) = -\int_0^t e^{(t-\tau)\Delta}\big(u \cdot \nabla\theta\big)(\tau)\, d\tau \tag{5.10}$$

for a.a. $t \in [0, T[$. The proof of $u \cdot \nabla\theta \in L^2(0, T; L^2(\Omega))$ is based on the following

Assertion. *Consider $2 \leq \gamma_1 < \infty$ and assume $\nabla\theta \in L^{\gamma_1}(0, T; L^2(\Omega))$. Let $2 < \gamma_2 < \infty$ be defined by $\frac{1}{4} + \frac{1}{\gamma_2} = \frac{1}{\gamma_1}$. Then*

$$\nabla\theta \in L^\gamma(0, T; L^2(\Omega)) \tag{5.11}$$

for all $2 \leq \gamma < \gamma_2$.

Proof of the assertion. We get

$$\begin{aligned}\|(-\Delta)^{1/2}\tilde{\theta}(t)\|_2 &= \Big\|\int_0^t (-\Delta)^{1/2}e^{(t-\tau)\Delta}(u \cdot \nabla\theta)(\tau)\, d\tau\Big\|_2 \\ &\leq \int_0^T |t-\tau|^{-1/2}\|\big(u \cdot \nabla\theta\big)(\tau)\|_2\, d\tau\end{aligned}$$

for a.a. $t \in [0, T[$. Define $1 < \gamma_3 < \infty$ by $\frac{1}{\gamma_3} = \frac{1}{4} + \frac{1}{\gamma_1}$. Consequently, the estimate $\|u \cdot \nabla\theta\|_2 \leq \|u\|_\infty\|\nabla\theta\|_2$, Hölder's inequality and (5.7) imply $u \cdot \nabla\theta \in L^\gamma(0, T; L^2(\Omega))$ for all $1 \leq \gamma < \gamma_3$. Thus, the Hardy-Littlewood inequality with $\frac{1}{2} + \frac{1}{\gamma_2} = \frac{1}{\gamma_3}$ implies $(-\Delta)^{1/2}\tilde{\theta} \in L^\gamma(0, T; L^2(\Omega))$ for all $1 \leq \gamma < \gamma_2$. From (5.9) it follows $(-\Delta)^{1/2}E_2 \in L^\gamma(0, T; L^2(\Omega))\,, 1 \leq \gamma < \gamma_2$. Altogether, see (5.8), we get $(-\Delta)^{1/2}\theta \in L^\gamma(0, T; L^2(\Omega))$ for all $1 \leq \gamma < \gamma_2$ and consequently, (see (2.7)) the proof of the assertion is finished. □

We use an iterative procedure to show $u \cdot \nabla\theta \in L^2(0, T; L^2(\Omega))$. First, define $\gamma_1 := 2$. Consequently, we get $\nabla\theta \in L^\gamma(0, T; L^2(\Omega))$ for all $2 \leq \gamma < 4$. In the next step, the requirements of the assertion are fulfilled for all $2 \leq \gamma_1 < 4$. Thus $\nabla\theta \in L^\gamma(0, T; L^2(\Omega))$ for all $2 \leq \gamma < \infty$. Especially $u \cdot \nabla\theta \in L^2(0, T, L^2(\Omega))$.

Step 3. Since $u \cdot \nabla\theta \in L^2(0, T; L^2(\Omega))$ we obtain from [19, IV, Theorem 2.5.2] (with A replaced by $-\Delta$) that $\tilde{\theta}$ fulfils (5.4), (5.5). Thus $\theta = \tilde{\theta} + E_2$ satisfies (5.4), (5.5). The proof of this theorem is completed. □

Looking at Theorem 5.1 we see that u and θ obtain the same degree of regularity. Now, well known methods developed for the Navier-Stokes equations can be used to show the smoothness of (u, θ).

Proof of Theorem 1.6. From (2.8), (2.9) in combination with (5.2), (5.5) we get

$$\begin{aligned} u(t) &= e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}P(\theta g)\,d\tau - \int_0^t e^{-(t-\tau)A}P(u\cdot\nabla u)\,d\tau \\ \theta(t) &= e^{t\Delta}\theta_0 - \int_0^t e^{(t-\tau)\Delta}(u\cdot\nabla\theta)\,d\tau \end{aligned} \tag{5.12}$$

for almost all $t \in [0, T[$. By the maximal regularity of A and $-\Delta$ it follows

$$\begin{aligned} u_t + Au &= P(\theta g) - P(u\cdot\nabla u) \quad &&\text{in } L^2_{\text{loc}}([0,T[; L^2_\sigma(\Omega))\,, \\ \theta_t - \Delta\theta &= -u\cdot\nabla\theta &&\text{in } L^2_{\text{loc}}([0,T[; L^2(\Omega)). \end{aligned} \tag{5.13}$$

Further, let p be an associated pressure of u satisfying (5.3). A careful inspection of the proof of [19, Theorem V, 1.8.2] shows that $u\,, \theta\,, p$ can be redefined on a null set of $]0, T[\times\Omega$ such that (1.10), (1.11) are fulfilled. Every step in this proof to increase the regularity of u can be used analogously to increase the regularity of θ.

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