On regularity of weak solutions to the instationary Navier-Stokes system - a review on recent results

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Consider a weak instationary solution u of the Navier-Stokes equations in a domain $\Omega \subset \mathbb{R}^3$, i.e., u solves the Navier-Stokes system in the sense of distributions and

$$u \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;W_0^{1,2}(\Omega)\big).$$

Since the pioneering work of J. Leray 1933/34 it is an open problem whether weak solutions are unique and smooth. The main step - to nowadays knowledge - is to show that the given weak solution is a strong one in the sense of J. Serrin, i.e., $u \in L^s\big(0,T;L^q(\Omega)\big)$ where $s > 2$, $q > 3$ and $\frac{2}{s} + \frac{3}{q} = 1$. This review reports on recent progress in this important problem, considering this issue locally on an initial interval $[0,T')$, $T' < T$, i.e., the problem of optimal initial values $u(0)$, globally on $[0,T)$, and from a one-sided point of view $u \in L^s\big(T'-\varepsilon,T';L^q(\Omega)\big)$ or $u \in L^s\big(T',T'+\varepsilon;L^q(\Omega)\big)$. Further topics deal with the energy (in-)equality, uniqueness of weak solutions and blow-up phenomena.

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1 Introduction

The Navier-Stokes system is the most classical model to describe the flow of a viscous incompressible fluid, the so-called Newtonian fluids. Despite of about 80 years of mathematical analysis, since the seminal paper of J. Leray [36] on the existence of global weak solutions in the whole space $\mathbb{R}^3$ and corresponding results of E. Hopf [29] for domains, basic questions on uniqueness and regularity of weak solutions are still open. These fundamental problems are also of importance for the general theory of partial differential equations and brought Clay Mathematics Institute in 2000 to classify the issue of regularity as one of the seven Millennium Prize Problem, see C. Fefferman [22].

Given a domain $\Omega \subset \mathbb{R}^3$ and a time interval $[0,T)$, $0 < T \le \infty$, let there be given an external force $f : \Omega \times (0,T) \to \mathbb{R}^3$ and an initial value $u_0 : \Omega \to \mathbb{R}^3$.

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Then we are looking for a velocity field u and an associated scalar pressure function p such that

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p = f, \qquad & \operatorname{div} u = 0 \text{ in } \Omega \times (0,T), \\ u(0) = u_0, \qquad & u = 0 \text{ on } \partial\Omega \times (0,T). \end{aligned} \tag{1.1}$$

For simplicity the coefficient of viscosity $\nu > 0$ has been set to $\nu = 1$. The nonlinear transport term $u \cdot \nabla u$ is defined by $\sum_{j=1}^3 u_j \partial_j u$ and can also be written in the form $u \cdot \nabla u = \operatorname{div}(u \otimes u)$ since u is solenoidal; recall that $u \otimes v = (u_i v_j)_{i,j=1}^3$ and $\operatorname{div}\big((F_{ij})_{i,j}\big) = \big(\sum_{i=1}^3 \partial_i F_{ij}\big)_{j=1}^3$ for a matrix field $F = (F_{ij})$.

In this article we use standard notation for Lebesgue, Sobolev and Bochner spaces spaces, i.e. $\big(L^q(\Omega) = L^q, \|\cdot\|_q\big)$, $\big(W^{k,q}(\Omega) = W^{k,q}, \|\cdot\|_{W^{k,q}}\big)$, and $\big(L^s(0,T); L^q(\Omega) = L^s(L^q)); \|\cdot\|_{q,s;T} = \|\cdot\|_{L^s(L^q)}\big)$, $1 \le s, q \le \infty$, respectively. We do not differ between spaces of scalar-, vector- and matrix-valued functions. The index σ will denote a subspace of solenoidal vector fields, the subscript 0 a subspace of functions with vanishing trace. Duality products of functions on Ω and $\Omega \times (0,T)$ will be denoted by $\langle \cdot, \cdot \rangle_\Omega$ and $\langle \cdot, \cdot \rangle_{\Omega,T}$, respectively.

To consider (1.1) as an abstract nonlinear evolution problem we introduce the Helmholtz projection on $L^q(\Omega)$,

$$P_q : L^q(\Omega) \to L^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q},$$

where $1 < q < \infty$ and $C^\infty_{0,\sigma}(\Omega) = \{u \in C_0^\infty(\Omega) : \operatorname{div} u = 0\}$. We recall that P_q is a well-defined bounded projection for bounded and exterior C^1-domains and defines an algebraic and topological decomposition

$$L^q(\Omega) = L^q_\sigma(\Omega) \oplus G_q(\Omega)$$

with $L^q_\sigma(\Omega) = \mathcal{R}(P_q)$, the range of P_q, and $G_q(\Omega) = \{\nabla p \in L^q(\Omega) : p \in L^q_{\text{loc}}(\overline{\Omega})\} = \mathcal{N}(P_q)$, the kernel of P_q; for details see [24], [45], [46].

Then we define the Stokes operator $A_q = -P_q \Delta$ on $L^q_\sigma(\Omega)$, $1 < q < \infty$, by

$$\begin{aligned} \mathcal{D}(A_q) &= W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega), \\ A_q : \mathcal{D}(A_q) &\subset L^q_\sigma(\Omega) \to L^q_\sigma(\Omega), \quad u \mapsto A_q u = -P_q \Delta u. \end{aligned}$$

It is well-known, see e.g. [14], that $-A_q$ generates a bounded analytic semigroup $\{e^{-tA_q}; t \ge 0\}$ for bounded and exterior domains of class $C^{1,1}$. Since A_q coincides with A_r on $\mathcal{D}(A_q) \cap \mathcal{D}(A_r)$, $1 < r, q < \infty$, we simply write A; by analogy, since $P_q u = P_r u$ for $u \in C^\infty_{0,\sigma}(\Omega)$, we simply write P. Note that in general non-smooth or general unbounded domains P and A may fail to exist, see [3], [38]. However, for $q = 2$ and any domain $\Omega \subset \mathbb{R}^3$, Hilbert space methods can be used to define P_2 and $A_2 = -P_2 \Delta$ with the properties mentioned above.

Using the Helmholtz projection P and the Stokes operator $A = -P\Delta$ we get rid of the pressure term ∇p in (1.1) $\big(P(\nabla p) = 0\big)$ and rewrite (1.1) as an abstract nonlinear evolution equation

$$u_t + Au + P(u \cdot \nabla u) = Pf \text{ in } (0,T), \quad u(0) = u_0 \tag{1.2}$$

in $L^q_\sigma(\Omega)$. Here we assume the condition $u_0 \in L^q_\sigma(\Omega)$ leading to $P_q u_0 = u_0$. Now we are in the position to introduce several notions of solutions u to (1.1).

Definition 1.1 *(Definition of several notions of solutions)*

1. *A weak solution u (in the sense of Leray-Hopf) is a solution in the sense of distributions, i.e.*

$$-\langle u, w_t\rangle_{\Omega,T}+\langle\nabla u,\nabla w\rangle_{\Omega,T}+\langle u\cdot\nabla u, w\rangle_{\Omega,T} = \langle f, w\rangle_{\Omega,T}+\langle u_0, w(0)\rangle_\Omega \quad (1.3)$$

for all test functions $w \in C_0^\infty\big([0,T); C_{0,\sigma}^\infty(\Omega)\big)$, lying in the Leray-Hopf class

$$u \in \mathcal{LH}_T = L^\infty\big(0,T; L^2_\sigma(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0,T); H^1_0(\Omega)\big) \quad (1.4)$$

and satisfying the energy inequality (EI)

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2 \,\mathrm{d}\tau \le \frac{1}{2}\|u_0\|_2^2 + \int_0^t \langle f, u\rangle_\Omega \,\mathrm{d}\tau \quad (1.5)$$

for all $t \in [0,T)$. Redefining u on a subset of $(0,T)$ of Lebesgue measure equal to 0 we may assume that

$$u : [0,T) \to L^2_\sigma(\Omega) \text{ is weakly continuous,} \quad (1.6)$$

or $u \in C^0_w\big([0,T); L^2_\sigma(\Omega)\big)$ for short.

2. *A weak solution $u \in \mathcal{LH}_T$ is called a strong solution (in the sense of Serrin) if there are exponents s, q such that*

$$u \in L^s\big(0,T; L^q(\Omega)\big), \quad s > 2,\ q > 3, \quad \frac{2}{s} + \frac{3}{q} = 1. \quad (1.7)$$

Under the assumption $\frac{2}{s} + \frac{3}{q} = 1$ the space $L^s\big(0,T; L^q(\Omega)\big)$ is called a Serrin class.

3. *A mild solution $u \in C^0([0,T); L^q(\Omega))$, $1 < q < \infty$, is a solution of the nonlinear integral equation*

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}\big(f(\tau) - \operatorname{div}(u\otimes u)(\tau)\big)\,\mathrm{d}\tau. \quad (1.8)$$

which originates from the variation of constants formula applied to the evolution problem (1.2).

4. *Classical solutions have weak or even classical derivatives with respect to time and space such that u_t, ∇u, $\nabla^2 u \in L^q(\Omega)$, $q > 1$. A classical solution satisfying $u \in C^\infty\big(\Omega \times (0,T)\big)$ when $f \in C^\infty\big(\Omega\times(0,T)\big)$ and even $u \in C^\infty\big(\overline{\Omega}\times[0,T)\big)$ when $f \in C^\infty\big(\overline{\Omega}\times[0,T)\big)$, $u_0 \in C^\infty(\overline{\Omega})$ and $\partial\Omega \in C^\infty$, is called a smooth solution.*

5. *Strong solutions (in the sense of maximal regularity) are solutions of the evolution equation (1.2) with $f \in L^s\big(0,T; L^q(\Omega)\big)$ and adequate initial value u_0 such that e.g.*

$$u_t, \nabla^2 u \in L^s\big(0,T; L^q(\Omega)\big) \text{ for some } 1 < s,\ q < \infty. \quad (1.9)$$

For a more precise definition of maximal regularity of the linear Stokes system and its properties we refer to Section 3.

6. *Very weak solutions are solutions u in Serrin's class $L^s\big(0,T;L^q(\Omega)\big)$ as in* (1.7) *with no differentiability (except for the weak divergence* $\operatorname{div} u = 0$*). To be more precise, for any test function $w \in C_0^\infty\big([0,T); C^2_{0,\sigma}(\overline{\Omega})\big)$ (i.e. $w(t) \in C^2(\overline{\Omega})$, $w(t)_{|\partial\Omega} = 0$, $\operatorname{div} w(t) = 0$, $\operatorname{supp} w(t) \subset \overline{\Omega}$ compact)*

$$\begin{aligned} -\langle u, w_t\rangle_{\Omega,T} - \langle u, \Delta w\rangle_{\Omega,T} - \langle u \otimes u, \nabla w\rangle_{\Omega,T} \\ = \langle f, w\rangle_{\Omega,T} + \langle u_0, w(0)\rangle_\Omega \\ \operatorname{div} u(t) = 0 \ \text{ in } \Omega, \quad u(t)\cdot N = 0 \ \text{ on } \partial\Omega \ \text{ for a.a. } t \in (0,T). \end{aligned} \tag{1.10}$$

Here $N = N(x)$ denotes the outward normal at $x \in \partial\Omega$, and the tangential component $u - (u\cdot N)N$ vanishes on $\partial\Omega$ due to the first equation of (1.10).

Let us recall several important results on these different notions of solutions.

Remark 1.2 1. The existence of weak solutions is known for arbitrary domains $\Omega \subset \mathbb{R}^3$, initial values in $u_0 \in L^2_\sigma(\Omega)$ and forces $f \in L^1\big(0,T;L^2(\Omega)\big)$. More generally, f may be assumed to be a functional of the form $f = \operatorname{div} F$, $F \in L^2\big(0,T;L^2(\Omega)\big)$ so that in (1.3)

$$\langle f, w\rangle_{\Omega,T} := -\langle F, \nabla w\rangle_{\Omega,T}.$$

2. Weak solutions can be constructed by several methods, e.g. by Galerkin's approximation method, approximation by time discretization or by approximation of the term $u\cdot\nabla u$ by $J_k u \cdot \nabla u$ via Yosida operators $J_k = \big(I + \frac{1}{k}A^{1/2}\big)^{-1}$.

3. Weak solutions constructed as in 2. satisfy the energy inequality (EI) in (1.5). The reason for the inequality rather than an equality is the use of approximation techniques and the lower semi-continuity of the norm $\|\cdot\|_2$ with respect to weak convergence. When Ω is bounded, the weak solutions constructed as in 2. even satisfy the *strong energy inequality* (SEI)

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^{t} \|\nabla u\|_2^2 \,\mathrm{d}\tau \le \frac{1}{2}\|u(t_0)\|_2^2 + \int_{t_0}^{t} \langle f, u\rangle_\Omega \,\mathrm{d}\tau \tag{1.11}$$

for *almost all* $t_0 \in [0,T)$ (including $t_0 = 0$) and for all $t \in (t_0, T)$. In the case of an exterior domain maximal regularity estimates yielding $L^s(L^q)$-estimates of the associated pressure p are needed to construct a weak solution satisfying (SEI), see [41].

When a weak solution satisfies (1.11) for a specific t_0, we will say that u satisfies $(EI)_{t_0}$. Hence

$$u \text{ satisfies } (SEI) \quad \Leftrightarrow \quad u \text{ satisfies } (EI)_{t_0} \text{ for a.a. } t_0 \in [0,T].$$

It is an open problem whether (EI) for a weak solutions implies (SEI). Moreover, it is even open whether every weak solution satisfying (1.3) and (1.4) automatically satisfies the energy inequality (EI). Note that weak solutions in our definition are assumed to satisfy $(EI) = (EI)_0$.

4. One of the fundamental problems on weak solutions is the question of uniqueness. A classical theorem, the so-called *Serrin uniqueness theorem*, states that a weak solution u lying in a Serrin class $L^s\big(0,T;L^q(\Omega)\big)$, see (1.7), is unique in the class of all Leray-Hopf type weak solutions. Note that by definition a Leray-Hopf type weak solution satisfies (EI) and that a weak solution in a Serrin class even satisfies the energy equality (EE), see 5. below. Uniqueness for a weak solution in the limit class $L^\infty\big(0,T;L^3(\Omega)\big)$ was proved by Kozono-Sohr [35] and in [19], see also Theorem 2.14 below.

5. Less restrictive assumptions than in 4. are needed to prove that a weak solution u satisfies the energy equality (EE), i.e.

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2 \,\mathrm{d}\tau = \frac{1}{2}\|u_0\|_2^2 + \int_0^t \langle f, u\rangle_\Omega \,\mathrm{d}\tau \tag{1.12}$$

for all $t \in (0,T)$. A classical theorem requires the condition

$$u \in L^4\big(0,T;L^4(\Omega)\big) \tag{1.13}$$

or equivalently $u\otimes u \in L^2\big(0,T;L^2(\Omega)\big)$. In this case, the term $u\cdot\nabla u = \operatorname{div}(u\otimes u)$ has the same properties as the external force $\operatorname{div} F$ with $F \in L^2\big(0,T;L^2(\Omega)\big)$. In particular, u may be used as test function w in (1.3) leading to the integrability $(u\cdot\nabla u)u \in L^1\big(0,T;L^1(\Omega)\big)$ and the fact that $\langle u\cdot\nabla u, u\rangle_\Omega(t) = 0$ for a.a. $t\in(0,T)$. Note that in the general three-dimensional case a weak solution $u \in \mathcal{LH}_T$ is not an admissible test function in (1.3).

Assumptions different to (1.13) were discussed by Shinbrot [44]: if $u \in L^r\big(0,T;L^q(\Omega)\big)$ where $\frac{2}{r}+\frac{3}{q} \le 1+\frac{1}{q}$, $q\ge 4$, then u satisfies (EE). Actually, Shinbrot's condition together with the Leray-Hopf integrability $u \in L^\infty\big(0,T;L^2(\Omega)\big)$ implies by Hölder's inequality that $u\in L^4\big(0,T;L^4(\Omega)\big)$. A similar argument can be applied when $u \in L^r\big(0,T;L^q(\Omega)\big)$, where $\frac{2}{r}+\frac{3}{q} \le 1+\frac{1}{r}$, $r\ge 4$, together with the fact that $u\in L^2\big(0,T;L^6(\Omega)\big)$. Definitely weaker assumptions than (1.13) will be discussed in §2.3.

6. The main open problem for weak solutions is the question of regularity, see [22]: is every weak solution u (and an associated pressure p) of class C^∞ in space and time provided that f, u and $\partial\Omega$ are of class C^∞? The classical result requires that u lies in Serrin's class $L^s\big(0,T;L^q(\Omega)\big)$ as in (1.7); the proof of the implication $u \in L^s\big(0,T;L^q(\Omega)\big) \Rightarrow u,p \in C^\infty$ is based on mathematical induction, cf. [47, Ch. V, Theorems 1.8.1 and 1.8.2].

7. A weak solution u of (1.1) is defined on the half-open interval $[0,T)$ as a solution of a partial differential equation. However, a careful check of the proof of existence and of the definition shows that e.g. under the assumption $F\in L^2\big(0,T;L^2(\Omega)\big)$ the solution u can be extended to $t=T$ and considered as a function $u\in C_w^0\big([0,T];L^2_\sigma(\Omega)\big)$ and that the energy inequality (and energy equality, if possible) do hold for all t up to $t=T$.

There are numerous results on *conditional regularity*, i.e., *a posteriori* conditions on a given weak solution u to guarantee its regularity. Most of these criteria are of Serrin type controlling ∇u, $\omega = \operatorname{rot} u$ or various components of u, ∇u, ω; other conditions work with the deformation tensor $\frac{1}{2}\big(\nabla u + (\nabla u)^T\big)$ or the pressure p; concerning a recent review on results using the vorticity we refer to [2]. Rather than trying to summarize and describe these results the focus of this review is on a recent approach to use an optimal initial value condition on u_0 and on function values $u(t)$ of u for all or almost all t. These and further related results, including also for unbounded domains, can be found in the articles [8], [9], [10], [11], [12], [13], [15], [16], [17], [18], [19], [20], [21] and [33].

This article is organized as follows. In Section 2 we present the main results, postponing the proofs to Section 3 - except for some short proofs. §2.1 starts with the discussion of the optimal initial value condition on u_0 to guarantee the existence of a local strong solution. The main Theorem 2.1 is the basis for most of the following results and uses a Besov space characterization to be introduced in the beginning. Next, §2.2 deals with further regularity criteria which either in some sense are beyond Serrin's condition or need the kinetic energy function. Then we discuss conditions to guarantee the energy equality and the uniqueness of weak solutions in §2.3. Section 3 contains all longer proofs, starting with some preliminaries on the Stokes operator in §3.1. The proof of Theorem 2.1 can be found in §3.2, all other proofs in §3.3. Note that we consider mainly the case of bounded domains, shifting further results on unbounded domains mainly to the remarks.

2 Main Results

2.1 Optimal Initial Values

To describe an optimal condition on initial values $u_0 \in L^2_\sigma(\Omega)$ to allow for a local in time strong solution $u \in L^s\big(0,T;L^q(\Omega)\big)$ of Serrin type of the Navier-Stokes system (1.1) it is natural to require that the solution

$$u(t) = E_0(t) := e^{-tA}u_0$$

of the corresponding linear Stokes system with vanishing external force has the property $E_0 \in L^s\big(0,T;L^q(\Omega)\big)$. Actually, this condition which is well-known for the case of the whole space $\Omega = \mathbb{R}^3$ yields also a necessary and sufficient condition for smooth bounded and exterior domains, see [15], [17], [18] and [9], [33], respectively. The integrability condition on E_0, say

$$\int_0^\infty \|e^{-\tau A}u_0\|_q^s \,\mathrm{d}\tau < \infty, \tag{2.1}$$

can be considered in terms of Besov spaces. Starting with the classical Besov space $B^{2/s}_{q',s'}(\Omega)$ for a domain $\Omega \subset \mathbb{R}^3$ (cf. [50, Ch. 4]) where q', s' are the conjugate exponents to q, s, respectively, and $\frac{2}{s} + \frac{3}{q} = 1$, solenoidal subspaces

$$\mathbb{B}^{2/s}_{q',s'}(\Omega) = B^{2/s}_{q',s'}(\Omega) \cap L^{q'}_\sigma(\Omega) = \big\{v \in B^{2/s}_{q',s'}(\Omega) :\ \operatorname{div} v = 0,\ N\cdot v\big|_{\partial\Omega} = 0\big\}$$

were defined in [1]. Actually, $\mathbb{B}^{2/s}_{q',s'}(\Omega)$ coincides with the real interpolation space $\big(L^{q'}_\sigma(\Omega), \mathcal{D}(A_{q'})\big)_{1/s,s'} \subset L^{q'}_\sigma(\Omega)$ ([1, Prop. 3.4, (3.18)]) yielding an optimal space of initial values u_0 such that $E_0(t) = e^{-tA}u_0$ satisfies $(E_0)_t,\ AE_0 \in L^{s'}\big(0,T; L^{q'}_\sigma(\Omega)\big)$, i.e., E_0 is a classical strong solution of the homogeneous Stokes problem with initial value u_0.

Here we do need the dual space

$$\mathbb{B}^{-2/s}_{q,s}(\Omega) := \big(\mathbb{B}^{2/s}_{q',s'}(\Omega)\big)^*. \tag{2.2}$$

By elementary properties of real interpolation and the duality theorem [50, Theorem 1.11.2]

$$\mathbb{B}^{-2/s}_{q,s}(\Omega) = \Big(\big(\mathcal{D}(A_{q'}), L^{q'}_\sigma(\Omega)\big)_{1/s',s'}\Big)^* = \big(\mathcal{D}(A_{q'})^*, L^q_\sigma(\Omega)\big)_{1/s',s}$$

since $\big(L^{q'}_\sigma(\Omega)\big)^* = L^q_\sigma(\Omega)$. Hence

$$\begin{aligned}\|u_0\|_{\mathbb{B}^{-2/s}_{q,s}} &\sim \|u_0\|_{\left(\mathcal{D}(A_{q'})^*, L^q_\sigma(\Omega)\right)_{1/s',s}} \sim \|A^{-1}u_0\|_{\left(L^q_\sigma(\Omega), \mathcal{D}(A_q)\right)_{1-1/s,s}} \\ &\sim \|A^{-1}u_0\|_q + \Bigg(\int_0^\infty \|Ae^{-\tau A}(A^{-1}u_0)\|_q^s \,\mathrm{d}\tau\Bigg)^{1/s}\end{aligned} \tag{2.3}$$

where the second equivalence of norms uses the identity $\langle A^{-1}u_0, A\varphi\rangle_\Omega = \langle u_0, \varphi\rangle_\Omega$ for $\varphi \in \mathcal{D}(A_{q'})$ and the equivalence $\|\varphi\|_{W^{2,q'}} \sim \|A_{q'}\varphi\|_{q'}$ (for bounded Ω). The norm on the right-hand side of (2.3) is the norm of $A^{-1}u_0$ in $\mathbb{B}^{2/s}_{q,s}(\Omega)$, and, by [50, Theorem 1.14.5], the interval of integration $(0,\infty)$ may be replaced by any interval $(0,\delta)$, $0 < \delta \le \infty$, yielding an equivalent norm on $\mathbb{B}^{-2/s}_{q,s}(\Omega)$. Finally, for a bounded domain, the term $\|A^{-1}u_0\|_q$ in (2.3) may be omitted.

Given $\delta \in (0,\infty]$ we denote the space $\mathbb{B}^{-2/s}_{q,s}(\Omega)$ also by

$$\mathcal{B}^{q,s}_\delta(\Omega) = \Bigg\{u_0 : \|u_0\|_{\mathcal{B}^{q,s}_\delta} := \Bigg(\int_0^\delta \|e^{-\tau A}u_0\|_q^s \,\mathrm{d}\tau\Bigg)^{1/s} < \infty\Bigg\}. \tag{2.4}$$

Recall that $\mathcal{B}^{q,s}_\delta(\Omega) \subset \mathcal{D}(A_{q'})^*$ is a reflexive, separable Banach space and that all norms $\|\cdot\|_{\mathcal{B}^{q,s}_\delta(\Omega)}$, $\delta > 0$, are equivalent. To be more precise, there exists $c(\delta) > 0$ such that

$$c(\delta)\|\cdot\|_{\mathcal{B}^{q,s}_\infty(\Omega)} \le \|\cdot\|_{\mathcal{B}^{q,s}_\delta(\Omega)} \le \|\cdot\|_{\mathcal{B}^{q,s}_\infty(\Omega)}.$$

For $\mathcal{B}^{q,s}_\infty(\Omega)$ we also simply write $\mathcal{B}^{q,s}(\Omega)$.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^{1,1}$, let $0 < T \le \infty$, $2 < s < \infty$, $3 < q < \infty$ with $\frac{2}{s} + \frac{3}{q} = 1$ be given, and consider the Navier-Stokes system with initial value $u_0 \in L^2_\sigma(\Omega)$ and an external force $f = \operatorname{div} F$ where $F \in L^2\big(0,T; L^2(\Omega)\big) \cap L^{s/2}\big(0,T; L^{q/2}(\Omega)\big)$.*

(i) *There exists an absolute constant* $\varepsilon_* = \varepsilon_*(q,\Omega) > 0$ *with the following property: If*

$$\|u_0\|_{\mathcal{B}_T^{q,s}} + \|F\|_{q/2,s/2;T} \le \varepsilon_*, \tag{2.5}$$

then the Navier-Stokes system (1.1) *has a unique strong solution* $u \in L^s\big(0,T;L^q(\Omega)\big)$.

(ii) *The condition* $u_0 \in \mathcal{B}_\infty^{q,s}(\Omega)$ *is sufficient and necessary for the existence and uniqueness of a local in time strong solution* $u \in L^s\big(0,T';L^q(\Omega)\big)$ *of* (1.1) *for some* $0 < T' \le T$.

Let us recall further results and extensions of Theorem 2.1.

Remark 2.2 1. A classical result, see e.g. [31], states that an initial value $u_0 \in \mathcal{D}(A_2)$ allows for a local strong solution. The condition $u_0 \in \mathcal{D}(A_2^{1/4})$ is due to Fujita and Kato [23] for a smooth bounded domain. Since this result is based on L^2-theory, it can be generalized to arbitrary bounded and unbounded domains, see [47, Ch. V, Theorem 4.2.2]. Fabes, Jones and Rivière [7] as well as Miyakawa [40] proved that the condition $u_0 \in L^r(\Omega)$, $r > 3$, yields a local strong solution.

2. The condition $u_0 \in \mathcal{D}(A_2^{1/4})$ has the important property of being *scaling invariant*. To explain this notion recall that with a solution u of (1.1) also $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$, $\lambda > 0$, is a solution of (1.1) with the same Reynolds number (here $Re = \frac{1}{\nu} = 1$) on the time interval $(0, \lambda^{-2}T)$ and domain $\lambda^{-1}\Omega$. Note that

$$\|u_\lambda\|_{L^s(L^q)} = \lambda^{1-(\frac{2}{s}+\frac{3}{q})}\|u\|_{L^s(L^q)}.$$

Hence $\|u_\lambda\|_{L^s(L^q)} = \|u\|_{L^s(L^q)}$ if and only if $L^s(L^q)$ is a Serrin class; in this case, the space $L^s(L^q)$ (or its norm) is called scaling invariant.

The corresponding condition for the initial value is related to $(u_0)_\lambda(x) = \lambda u_0(\lambda x)$. Scaling invariant initial value conditions are $u_0 \in \mathcal{D}(A^{1/4})$ and $u_0 \in L^3_\sigma(\Omega)$. The latter case was considered by Kato [30] and Giga [27].

The condition $u_0 \in L^3_\sigma(\Omega)$ can be weakened to assumptions in Lorentz spaces $L^{3,s}_\sigma(\Omega)$ when $q \le s < \infty$, see [48]. Here we mention the continuous embeddings

$$\mathcal{D}(A_2^{1/4}) \subset L^3_\sigma(\Omega) \subset L^{3,s}_\sigma(\Omega) \subset \mathcal{B}_\infty^{q,s}(\Omega) \tag{2.6}$$

where each space is scaling invariant; for the latter embedding which holds when $q \le s < \infty$ we refer to [1, (0.16)]. Replacing $\mathcal{B}_\infty^{q,s}(\Omega)$ by $\mathcal{B}_\delta^{q,s}(\Omega)$, the family of spaces $\mathcal{B}_\delta^{q,s}(\Omega)$ is scaling invariant in the sense that δ must be changed to $\lambda^{-2}\delta$ (and Ω to $\lambda^{-1}\Omega$) .

3. For a smooth exterior domain $\Omega \subset \mathbb{R}^3$ similar results were obtained in [9], [33]. It is also shown that the assumption $F \in L^{s/2}\big(L^{q/2}\big)$ (with $\frac{2}{s/2} + \frac{3}{q/2} = 2$, cf. Theorem 2.1) can be generalized to $F \in L^{s_*}(L^{q_*})$ with $\frac{2}{s_*} + \frac{3}{q_*} = 2$. Then the condition $\int_0^\infty \|e^{-\tau A}u_0\|_q^s \,\mathrm{d}\tau < \infty$ is necessary and sufficient for the existence of a local strong solution $u \in L^s(L^q)$, $\frac{2}{s} + \frac{3}{q} = 1$.

4. For a general bounded or unbounded domain $\Omega \subset \mathbb{R}^3$ only L^2-theory for the Stokes operator and the Helmholtz projection is available. In this case Theorem 2.1 (i) holds with the exponents $q = 4$, $s = 8$, cf. [17, Sect. 4].

5. The largest space of initial values to yield solutions in scaling invariant function spaces was found by Koch and Tataru [32] for the whole space case $\mathbb{R}^3$. Assume that $u_0 \in BMO^{-1}$, i.e., u_0 can be written in the form $u_0 = \operatorname{div} f$ with some $f \in BMO$. Then there exists a local solution $u \in L^2_{\mathrm{loc}}\big(\mathbb{R}^3 \times [0,\infty)\big)$ such that the scaling invariant norm

$$\sup_{x\in\mathbb{R}^3, R>0} \frac{1}{|B_R(x)|} \int_0^{R^2} \int_{B_R(x)} |u|^2 \,\mathrm{d}y\,\mathrm{d}\tau$$

is finite; here $B_R(x) \subset \mathbb{R}^3$ denotes the ball with center x and radius R.

For simplicity let $F = 0$ in the following. It suggests itself to use Theorem 2.1 (i) not only at $t_0 = 0$, but at all or almost all $t_0 \in [0, T)$ to show that a weak solution is a strong one. In view of (2.5) we need more information on the space $\mathcal{B}^{q,s}(\Omega)$ and on functions $u : [0,T) \to \mathcal{B}^{q,s}_\delta(\Omega)$. Although the constant ε_* in (2.5) cannot be determined precisely, we will fix some $\varepsilon_* > 0$ in the following so that Theorem 2.1 (i) can be applied.

Definition 2.3 *Let $0 \neq v \in L^2_\sigma(\Omega)$. Then*

$$\delta(v) := \begin{cases} 0 & \text{if } v \notin \mathcal{B}^{q,s}(\Omega) \\ \delta \in (0,\infty) & \text{if } v \in \mathcal{B}^{q,s}(\Omega) \text{ and } \|v\|_{\mathcal{B}^{q,s}_\delta} = \varepsilon_* < \|v\|_{\mathcal{B}^{q,s}_\infty} \\ \infty & \text{if } v \in \mathcal{B}^{q,s}(\Omega) \text{ and } \|v\|_{\mathcal{B}^{q,s}_\infty} \le \varepsilon_* \end{cases}$$

With an abuse of notation we set $\|v\|_{\mathcal{B}^{q,s}} = \infty$ when $v \notin \mathcal{B}^{q,s}(\Omega)$.

For a weak solution $u \in \mathcal{LH}_T$ of (1.1) we simply write $\delta(t)$ for $\delta(u(t))$.

The second case in Definition 2.3 is well defined: Assume that $v \in \mathcal{B}^{q,s}(\Omega)$ and $\|v\|_{\mathcal{B}^{q,s}} > \varepsilon_*$. Then the function $V : (0,\infty) \to (0,\infty)$, $V(t) := \|v\|_{\mathcal{B}^{q,s}_t}$ is a strictly increasing continuous function with range $(0, \|u\|_{\mathcal{B}^{q,s}_\infty}) \supset (0, \varepsilon_*]$. Hence there exists a unique $\delta \in (0,\infty)$ such that $V(\delta) = \varepsilon_*$. The strict monotonicity of V is based on the fact that if $\|e^{-\tau A}v\|_q = 0$ then $0 = \langle e^{-\tau A}v, v\rangle = \|e^{-\tau A/2}v\|_2^2$ and consequently $\|e^{-\tau A/2^k}v\|_2 = 0$ for all $k \in \mathbb{N}$; in the limit $k \to \infty$ we get $v = 0$ due to the strong continuity of the Stokes semigroup.

Lemma 2.4 *Let $u \in \mathcal{LH}_T$ be a Leray-Hopf type weak solution of (1.1), and $\delta > 0$.*

(i) If additionally $u \in L^\infty\big([0,T); \mathcal{B}^{q,s}_\delta(\Omega)\big)$, then $u \in C^0_w\big([0,T]; \mathcal{B}^{q,s}_\delta(\Omega)\big)$. In particular, $\|u(t)\|_{\mathcal{B}^{q,s}_\delta}$ is a lower semi-continuous function of $t \in [0,T]$, and

$$\|u\|_{L^\infty(0,T;\mathcal{B}^{q,s}_\delta)} = \sup_{t\in[0,T]} \|u(t)\|_{\mathcal{B}^{q,s}_\delta}. \tag{2.7}$$

(ii) If $u \in \mathcal{LH}_T$ is not necessarily contained in $L^\infty\big([0,T); \mathcal{B}^{q,s}_\delta(\Omega)\big)$, then the function $[0,T] \to [0,\infty]$, $t \mapsto \|u(t)\|_{\mathcal{B}^{q,s}_\delta}$ is lower semi-continuous.

(iii) The function $\delta(\cdot)$ from Definition 2.3 is upper semi-continuous.

For the problem of local regularity we need the following terminology:

- u is *left-sided* $L^s(L^q)$*-regular at* $t_0 \in (0,T]$ if there exists $\varepsilon = \varepsilon(t_0) \in (0,t_0)$ such that $u \in L^s((t_0-\varepsilon,t_0);L^q(\Omega))$
- u is *right-sided* $L^s(L^q)$*-regular at* $t_0 \in [0,T)$ if there exists $\varepsilon = \varepsilon(t_0) \in (0,T-t_0)$ such that $u \in L^s((t_0,t_0+\varepsilon);L^q(\Omega))$
- u is $L^s(L^q)$*-regular at* $t_0 \in (0,T)$ if u is left- as well as right-sided regular at t_0.

Theorem 2.5 *Let $u \in \mathcal{LH}_T$ be a weak solution of* (1.1) *in a bounded smooth domain $\Omega \subset \mathbb{R}^3$ satisfying the strong energy inequality (SEI), and let $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s}+\frac{3}{q}=1$.*

(i) Let u satisfy $u(t) \in \mathcal{B}^{q,s}(\Omega)$ for all $t \in [0,T)$. Given $t_1 \in (0,T)$ assume that

$$\delta(t) \geq t_1 - t \tag{2.8}$$

for a.a. t in a left-sided neighborhood of t_1. Then u is left-sided $L^s(L^q)$-regular at t_1.

(ii) Given $t_1 \in (0,T)$ assume that $u(t_1) \in \mathcal{B}^{q,s}(\Omega)$ and that u satisfies $(EI)_{t_1}$, i.e., the strong energy inequality in t_1. Then u is right-sided $L^s(L^q)$-regular at t_1.

(iii) Let u satisfy $u(t) \in \mathcal{B}^{q,s}(\Omega)$ for all $t \in [0,T)$ and condition (2.8) *at $t_1 \in (0,T)$. Then u is regular at t_1. If condition* (2.8) *is satisfied at* every *$t_1 \in (0,T)$, then $u \in L^s_{\mathrm{loc}}\big([0,T);L^q(\Omega)\big)$.*

(iv) Let u satisfy $u(t) \in \mathcal{B}^{q,s}(\Omega)$ for all $t \in [0,T)$. Assume that at $t_1 \in (0,T)$

$$\lim_{t \nearrow t_1} \|u(t)\|_{\mathcal{B}^{q,s}_{t_1-t}} = 0. \tag{2.9}$$

Then u is $L^s(L^q)$-regular at t_1. If condition (2.9) *is satisfied at* every *$t_1 \in (0,T)$, then $u \in L^s_{\mathrm{loc}}\big([0,T);L^q(\Omega)\big)$.*

Remark 2.6 The explicit lower bound $\delta(t) \geq t_1 - t$ in (2.8) leads to the question whether also the condition $\delta(t) \geq \alpha(t_1-t)$ with $\alpha \in (0,1)$ will suffice to get the same result. For simplicity let $T = t_1$. In this case, starting with $t = t_0 < T$ we get the sequence of instants (t_j) defined by $t_j = T - (1-\alpha)^j(T-t_0)$ converging to T; hence $u \in L^s\big(t_0,t_j;L^q(\Omega)\big)$. However, there is no uniform bound of $\|u\|_{q,s;(t_0,t_j)}$ as $j \to \infty$. To be more precise and to apply Serrin's uniqueness theorem in each step, each t_j can be replaced by $t'_j < t_j$ such that still $t'_j \to T$ as $j \to \infty$.

Corollary 2.7 *Let u be a weak solution of the Navier-Stokes system as in Theorem 2.5.*

(i) Assume that $u \in L^s_{\mathrm{loc}}\big([0,t_1);L^q_\sigma(\Omega)\big)$, but $u \notin L^s\big(0,t_1;L^q_\sigma(\Omega)\big)$. Then there exists an $\epsilon_* > 0$ such that for all $\alpha \in (\frac{1}{4},\frac{1}{2}]$

$$\|A^\alpha u(t)\|_2 > \epsilon_*(t_1-t)^{1/4-\alpha} \quad \text{for a.a. } t \in (0,t_1). \tag{2.10}$$

(ii) Assume that u satisfies $(EI)_{t_1}$ *(this condition is e.g. satisfied when* t_1 *is a left-sided regular point of u), but is not right-sided regular at* t_1. *Then* $u(t_1) \notin \mathcal{B}^{q,s}(\Omega)$ *and for each* $\delta > 0$ *and* $\alpha \in \left[\frac{1}{4}, \frac{1}{2}\right]$

$$\|u(t)\|_3,\ \|A^\alpha u(t)\|_2,\ \|u(t)\|_{\mathcal{B}^{q,s}_\delta} \to \infty \quad \text{as} \quad t \searrow t_1.$$

2.2 Regularity Criteria Beyond Serrin's Condition and Energy Criteria

The regularity criteria of Theorem 2.5 have the disadvantage that the norm of $u(t)$ in $\mathcal{B}^{q,s}_{t_1-t}(\Omega)$ cannot be controlled directly for the solution u. However, there are many easy applications of Theorem 2.5 yielding more concrete conditions.

Corollary 2.8 *Let* $u \in \mathcal{LH}_T$ *be a weak solution of* (1.1) *on a bounded smooth domain* $\Omega \subset \mathbb{R}^3$ *satisfying (SEI).*

(i) If there exists $\delta \in (0, \infty]$ *such that* $u \in C^0\big([0,T); \mathcal{B}^{q,s}_\delta(\Omega)\big)$, *then u is a strong solution on* $[0,T)$.

(ii) If there exists $\delta \in (0, \infty]$ *such* $\|u\|_{L^\infty(0,T;\mathcal{B}^{q,s}_\delta)} \le \varepsilon_*$, *then u is a strong solution on* $[0,T)$.

(iii) In (i) and (ii) the space $\mathcal{B}^{q,s}_\delta(\Omega)$ *can be replaced by any of the spaces* $\mathcal{D}(A^{1/4})$, $L^3_\sigma(\Omega)$ *and* $L^{3,s}_\sigma(\Omega)$ $(s \ge q > 3)$.

Proof (i) If $u \in C^0\big([0,T']; \mathcal{B}^{q,s}_\sigma(\Omega)\big)$, $0 < T' < T$, then u is uniformly continuous in $t \in [0,T']$ with values in $\mathcal{B}^{q,s}_\delta(\Omega)$. Given ε_* from Theorem 2.1 the uniform continuity in t allows to find $\delta' \in (0,\delta]$ such that $\|u(t)\|_{\mathcal{B}^{q,s}_{\delta'}} \le \varepsilon_*$ for all $t \in [0,T']$. Then a compactness argument on $[0,T']$ implies that $u \in L^s\big([0,T']; L^q(\Omega)\big)$.

(ii) From Lemma 2.4 we know that $\|u(t)\|_{\mathcal{B}^{q,s}_\delta} \le \|u\|_{L^\infty(0,T;\mathcal{B}^{q,s}_\delta)} \le \varepsilon_*$ for all $t \in [0,T)$. Now a compactness argument as in (ii) completes the proof.

(iii) The embeddings (2.6) and (i), (ii) immediately prove (iii). ∎

Note that $L^\infty\big(0,T;\mathcal{B}^{q,s}_\delta(\Omega)\big) = L^\infty\big(0,T;\mathcal{B}^{q,s}_\infty(\Omega)\big)$ and $C^0\big([0,T];\mathcal{B}^{q,s}_\delta(\Omega)\big) = C^0\big([0,T];\mathcal{B}^{q,s}_\infty(\Omega)\big)$, $\delta \in (0,\infty]$, with equivalent norms.

The next criteria are based on Theorem 2.1, however, in a certain sense work beyond Serrin's condition at the expense of a further smallness assumption.

Theorem 2.9 *Let* $u \in \mathcal{LH}_T$ *be a weak solution of* (1.1) *as in Corollary 2.8. Further let the exponents q and r, s satisfy* $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$ *and* $1 \le r \le s$.

(i) Assume that

$$\liminf_{\delta \to 0} \frac{1}{\delta^{1-r/s}} \int_{t_1-\delta}^{t_1} \|u(\tau)\|_q^r \,\mathrm{d}\tau = 0. \tag{2.11}$$

Then u is regular at t_1, *i.e.,* $u \in L^s\big(t_1 - \varepsilon, t_1 + \varepsilon; L^q(\Omega)\big)$ *for some* $\varepsilon > 0$.

(ii) Assume that for $0 \le t_0 < t_1 < T$ and some $t_1 < T' \le T$

$$\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} (T' - \tau)^{r/s} \, \|u(\tau)\|_q^r \, \mathrm{d}\tau \le \tilde{\varepsilon}_*. \tag{2.12}$$

Then u is regular at t_1. Here $\tilde{\varepsilon}_* > 0$ is a constant related to ε_* in (2.5).

An easy consequence of Theorem 2.9 (i) is the well-known fact that a weak solution $u \in \mathcal{LH}_T$ is regular almost everywhere (even everywhere in $(0,T)$ except for a possible set $S \subset (0,T)$ of vanishing Hausdorff measure $\mathcal{H}^{1/2}(S) = 0$). Actually, since $u \in L^2\big(0,T; L^6(\Omega)\big) \subset L^1\big(0,T; L^6(\Omega)\big)$, Lebesgue's differentiation theorem implies that $\frac{1}{\delta}\int_{t_1-\delta}^{t_1} \|u\|_6^1 \, \mathrm{d}\tau \to \|u(t_1)\|_6$ t-a.e as $\delta to 0$. Hence the term in (2.11) (with $q = 6$, $r = 1$ and $s = 4$) vanishes t-a.e.

Finally, we describe a regularity criterion based on the *kinetic energy*

$$E_k(t) = \frac{1}{2}\|u(t)\|_2^2. \tag{2.13}$$

Theorem 2.10 *Let $u \in \mathcal{LH}_T$ be a weak solution of* (1.1) *satisfying (SEI) as in Corollary 2.8. Assume that at $t_1 \in (0,T)$ for $\alpha \in (\frac{1}{2}, 1]$ the left-sided α-Hölder semi-norm*

$$[E_k(t_1-)]_\alpha = \sup_{\delta > 0} \frac{|E_k(t_1 - \delta) - E_k(t_1)|}{\delta^\alpha}$$

is finite (with the supremum taken only for small $\delta > 0$) or that

$$[E_k(t_1-)]_{1/2} \le \varepsilon_*. \tag{2.14}$$

Then u is regular at t_1.

Proof Obviously the condition $[E_k(t_1-)]_\alpha < \infty$ for $\alpha \in (\frac{1}{2}, 1]$ implies that $[E_k(t_1-)]_{1/2} \le \varepsilon_*$ (with the supremum taken only for small $\delta > 0$). Hence it suffices to assume the second condition. With $r = 2$, $q = 6$ and $s = 4$ we get that

$$\begin{aligned} \int_{t_1-\delta}^{t_1} \|u(\tau)\|_q^r \, \mathrm{d}\tau &\le c \int_{t_1-\delta}^{t_1} \|\nabla u(\tau)\|_2^2 \, \mathrm{d}\tau \\ &\le c\big(E_k(t_1 - \delta) - E_k(t_1)\big) \\ &\le c\varepsilon_* \delta^{1/2} \end{aligned} \tag{2.15}$$

provided we choose only those $\delta > 0$ such that $(EI)_{t_1-\delta}$ holds. We conclude that (2.11) is satisfied and consequently that u is regular at t_1. ■

Note that for the Hölder exponent $\alpha = \frac{1}{2}$ we do need a smallness condition on the local left-sided Hölder seminorm. Actually, if $(t_0, t_1) \subset [0,T)$ is a maximal interval of regularity of a weak solution u, then due to (2.10)

$$\|\nabla u(\tau)\|_2 = \|A^{1/2}u(\tau)\|_2 \ge \varepsilon_*(t_1 - \tau)^{-1/4}, \quad 0 < \tau < t_1.$$

Hence the estimate

$$2c^2 \le \frac{1}{\delta^{1/2}} \int_{t_1-\delta}^{t_1} \|\nabla u(\tau)\|_2^2 \, \mathrm{d}\tau \le \frac{1}{\delta^{1/2}} \big(E_k(t_1-\delta) - E_k(t_1)\big)$$

for a.a. $\delta \in (t_1 - t_0, t_1)$ shows in this case that the condition (2.14) with an arbitrary (not sufficiently small) $\varepsilon_* > 0$ does not imply regularity. For the case including an external force we refer to [13].

For further regularity criteria beyond Serrin's condition we refer to [11].

In the case of a general domain where only the Stokes operator $A = A_2$ on $L^2_\sigma(\Omega)$ is available, we get the following results, cf. [11], [17]:

Theorem 2.11 *Let $\Omega \subset \mathbb{R}^3$ be a general domain and $u \in \mathcal{LH}_T$ be a weak solution satisfying (SEI). Assume that at $t_1 \in (0,T)$ one of the following conditions is satisfied: there exists $0 < \delta < t_1$ such that*

$$⨍_{t_1-\delta}^{t_1} \|A^{1/4} u(\tau)\|_2 \, \mathrm{d}\tau \le \varepsilon_*, \tag{2.16}$$

$$⨍_{t_1-\delta}^{t_1} \|u(\tau)\|_2 \, \|\nabla u(\tau)\|_2 \, \mathrm{d}\tau \le \varepsilon_*, \tag{2.17}$$

$$\Big(\sup_{[t_1-\delta,t_1]} \|u(\cdot)\|_2^2\Big) ⨍_{t_1-\delta}^{t_1} \|\nabla u(\tau)\|_2^2 \, \mathrm{d}\tau \le \varepsilon_*. \tag{2.18}$$

Here $⨍_{t_1-\delta}^{t_1}(\dots)$ denotes the integral mean $\frac{1}{\delta}\int_{t_1-\delta}^{t_1}(\dots)$, and ε_* in (2.16)–(2.18) is an absolute constant independent of the domain. Then u is $L^8(L^4)$-regular at t_1.

The proof of Theorem 2.11 shows that each of the conditions (2.16), (2.17) or (2.18) even yields the global regularity of u on $(t_1 - \varepsilon, T)$. Note that (2.16)–(2.18) are scaling invariant conditions.

2.3 The Energy Equality and Uniqueness

As already mentioned in Remark 1.2 5., a weak solution $u \in \mathcal{LH}_T$ satisfies (EE) if $u \in L^4\big(0,T;L^4(\Omega)\big)$ or if related $L^s(L^q)$-conditions are fulfilled which imply $u \in L^4(L^4)$ by Hölder's inequality and the assumption $u \in \mathcal{LH}_T$. In other words, those $L^s(L^q)$-conditions are closer to Serrin's class as to the class of weak solutions: actually, for those q, s we have $\frac{2}{s} + \frac{3}{q} \le \frac{5}{4} = \frac{2}{4} + \frac{3}{4} \le \frac{3}{2}$. Recently, Cheskidov, Friedlander and Shvydkoy [5] and together with Constantin [6] found conditions which concerning their scaling behavior are of the type $L^3\big(0,T;L^{9/2}(\Omega)\big)$ where

$$\frac{2}{3} + \frac{3}{9/2} = \frac{4}{3} > \frac{5}{4}.$$

Theorem 2.12 *Let* $\Omega \subset \mathbb{R}^3$ *be a* C^2*-domain and* $u \in \mathcal{LH}_T$ *be a weak solution of* (1.1)

(i) Asume that

$$u \in L^3\big(0,T;\mathcal{D}\big(A_2^{5/12}\big)\big).$$

Then u satisfies the energy equality (EE); see [5].

(ii) Let $\Omega = \mathbb{R}^3$ *and let u satisfy*

$$u \in L^3\big(0,T;B^{1/3}_{3,\infty}(\mathbb{R}^3)\big).$$

Then u satisfies (EE); see [6].

We note that $\mathcal{D}\big(A_2^{5/12}\big) \subset L^{9/2}_\sigma(\Omega)$ equals a solenoidal subspace of the Bessel potential space $H^{5/6}(\Omega)$. The second result is available only for the whole space case since the characterization of the Besov space $B^{1/3}_{3,\infty}$ via Littlewood-Paley decomposition is crucially used. Formally, the order $\frac{1}{3}$ of fractional derivatives in $B^{1/3}_{3,\infty}$ is optimal in view of the term $\int_{\mathbb{R}^3}(u\cdot\nabla u)u\,\mathrm{d}x$ which must be shown to vanish.

An intermediate result for domains exploiting derivatives of fractional order $\frac{1}{2}$ was proved in [21]. To meet the same scaling as in Theorem 2.12 the space $\mathcal{D}\big(A_2^{5/12}\big)$ is replaced by $\mathcal{D}\big(A^{1/4}_{18/7}\big) \subset L^{9/2}(\Omega)$.

Theorem 2.13 *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with* $\partial\Omega \in C^{1,1}$ *and let* $u \in \mathcal{LH}_T$ *be a weak solution of* (1.1) *with* $u(0) = u_0 \in L^2_\sigma(\Omega)$ *satisfying*

$$u \in L^3\big(0,T,\mathcal{D}\big(A^{1/4}_{18/7}\big)\big). \tag{2.19}$$

Then u satisfies the energy equality (EE).

This theorem will be proved in §3.3. It can be extended to general unbounded domains $\Omega \subset \mathbb{R}^3$ of uniform $C^{1,1}$-type where $\mathcal{D}\big(A^{1/2}_{18/7}\big)$ has to be replaced by $\mathcal{D}\big((I+\tilde{A}_{18/7})^{1/2}\big)$ and $\tilde{}$ indicates that the space $L^q(\Omega)$, $q > 2$, is replaced by $\tilde{L}^q(\Omega) = L^q(\Omega) \cap L^2(\Omega)$. For further details we refer to [21] where also a generalization to domains of fractional powers of the Stokes operator on Lorentz spaces can be found.

The final problem to be addressed is the question of uniqueness of weak solutions. The regularity results of §§2.1 and 2.2 will yield uniqueness, but there are several weaker conditions for this result.

Theorem 2.14 *Let* $u \in \mathcal{LH}_T$ *be a weak solution of the Navier-Stokes system* (1.1) *in a bounded smooth domain* $\Omega \subset \mathbb{R}^3$ *with* $u_0 \in L^2_\sigma(\Omega)$.

(i) Assume that for some $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$ *there holds* $u(t_0) \in \mathcal{B}^{q,s}(\Omega)$ *for all* $t_0 \in [0,T)$, *and that u satisfies the energy equality* (EE). *Then u is uniquely determined by the initial value* u_0 *within the class of all weak solutions satisfying* (SEI).

(ii) If $u \in L^4_{\text{loc}}\big([0,T); L^4(\Omega)\big)$ *or*

$$u \in L^\infty_{\text{loc}}\big([0,T); L^{3,\infty}_\sigma(\Omega)\big), \tag{2.20}$$

then u *even satisfies (EE).*

Remark 2.15 Let u be a weak solution of (1.1) as in Theorem 2.14. It is interesting to discuss uniqueness and regularity properties of u with the Serrin condition $u \in L^s_{\text{loc}}([0,T); L^q(\Omega))$, $\frac{2}{s} + \frac{3}{q} = 1$ in the limit case $s = \infty, q = 3$. In this case, the arguments in the proof of Lemma 2.4 (i) show that $u(t) \in L^3(\Omega)$ for each $t \in [0,T)$. Since $L^3(\Omega) \subset L^{3,\infty}(\Omega) \cap \mathcal{B}^{q,s}(\Omega)$, Theorem 2.14 yields the uniqueness property for u. On the other hand, from Corollary 2.8 (iii) we see that the stronger assumption $u \in C\big([0,T); L^3_\sigma(\Omega)\big)$ is sufficient to get the regularity $u \in L^s_{\text{loc}}\big([0,T); L^q(\Omega)\big)$ with certain $2 < s < \infty$, $3 < q < \infty$ such that $\frac{2}{s} + \frac{3}{q} = 1$.

Furthermore, for $u \in L^\infty_{\text{loc}}\big([0,T); L^3_\sigma(\Omega)\big)$ we get the local right-side regularity property for each $t \in [0,T)$, see Theorem 2.5 (ii). Hence Theorem 2.5 is a slightly weaker result than that in a series of papers on the celebrated $L^\infty\big(0,T; L^3(\Omega)\big)$-regularity result of Seregin et al. We refer to [42] for domains with a flat boundary, and to [39] where in domains with curved boundaries some additional condition on the pressure had to be assumed.

3 Proof of the Main Results

3.1 Preliminaries

Before coming to the proof of the main result, Theorem 2.1, we start with some preliminaries.

The Stokes operator A_q, $1 < q < \infty$, being sectorial and generating a bounded analytic semigroup, admits fractional powers A^α_q, $-1 \le \alpha \le 1$. First we consider the case of a bounded domain $\Omega \subset \mathbb{R}^3$ of class $C^{1,1}$. Then A^α_q, $0 \le \alpha \le 1$, is an injective, closed and densely defined operator with domain $\mathcal{D}(A^\alpha_q) \subset L^q_\sigma(\Omega)$ and range $\mathcal{R}(A^\alpha_q) = L^q_\sigma(\Omega)$ such that $\mathcal{D}(A_q) \subset \mathcal{D}(A^\alpha_q) \subset \mathcal{D}(A^\beta_q) \subset L^q_\sigma(\Omega)$ for $0 \le \beta \le \alpha \le 1$. For $-1 \le \alpha < 0$ we define the bounded operators $A^\alpha_q = (A^{-\alpha}_q)^{-1} : L^q_\sigma(\Omega) \to \mathcal{R}(A^\alpha_q) = \mathcal{D}(A^{-\alpha}_q)$. Important properties are the embeddings

$$\|v\|_q \le c\|A^\alpha_r v\|_r, \quad v \in \mathcal{D}(A^\alpha_r), \quad 1 < r \le q < \infty, \quad 2\alpha + \frac{3}{q} = \frac{3}{r} \tag{3.1}$$

$$\|\nabla v\|_q \sim \|A^{1/2}_q v\|_q, \quad v \in \mathcal{D}(A^{1/2}_q) = W^{1,q}_{0,\sigma}(\Omega), \quad 1 < q < \infty; \tag{3.2}$$

moreover, $\|\nabla v\|_2 = \|A^{1/2}_2 v\|_2$ for $v \in W^{1,2}_{0,\sigma}(\Omega)$. From semigroup theory and the fact that $\sigma(A_q) \subset (0,\infty)$ we know that there exist $c = c(q,\Omega) > 0$ and $\delta = \delta(q,\Omega) > 0$ such that

$$\|A^\alpha_q e^{-tA_q} v\|_q \le c t^{-\alpha} e^{-\delta t}\|v\|_q, \quad v \in L^q_\sigma(\Omega), \quad 0 \le \alpha \le 1. \tag{3.3}$$

When $\Omega \subset \mathbb{R}^3$ is an exterior domain with $\partial\Omega \in C^{1,1}$, then

$$
\begin{aligned}
\|v\|_q &\le c\|A_r^\alpha v\|_r, \quad 0 \le \alpha \le \frac{1}{2}, \quad 1 < r < 3\\
\|v\|_q &\le c\|A_r^\alpha v\|_r, \quad 0 \le \alpha \le 1, \quad 1 < r < \frac{3}{2}
\end{aligned}
\tag{3.4}
$$

where $2\alpha + \frac{3}{q} = \frac{3}{r}$. Furthermore, the equivalence of norms in (3.2) is to be replaced by

$$
\|A_q^{1/2}v\|_q \le c\|\nabla v\|_q, \ 1 < q < \infty; \quad \|\nabla v\|_q \le c\|A_q^{1/2}v\|_q, \ 1 < q < 3. \tag{3.5}
$$

Since $0 \in \sigma(A) = [0,\infty)$, inequality (3.3) does hold only with $\delta = 0$, cf. [4], [28].

A weak solution to the (Navier-)Stokes system can be considered as a mild solution of the related (non-)linear integral equation

$$
u(t) = e^{-tA}u_0 + A^{1/2}\int_0^t e^{-(t-\tau)A}\big(A^{-1/2}P\,\mathrm{div}\,\big)(F - u\otimes u)(\tau)\,\mathrm{d}\tau, \tag{3.6}
$$

cf. (1.8). To understand this formula we have to explain the formal operator $A^{-1/2}P\,\mathrm{div}$ on matrix-valued $L^q(\Omega)$-functions F. Actually, we define for $\varphi \in \mathcal{D}(A_{q'}^{-1/2}) \subseteq L^{q'}_\sigma(\Omega)$

$$
\langle A^{-1/2}P\,\mathrm{div}\,F, \varphi\rangle := -\langle F, \nabla A_{q'}^{-1/2}\varphi\rangle_\Omega \tag{3.7}
$$

admitting the estimate $\big|\langle F, \nabla A_{q'}^{-1/2}\varphi\rangle\big| \le \|F\|_q \|\nabla A_{q'}^{-1/2}\varphi\|_{q'} \le c_{q'}\|\varphi\|_{q'}$. Hence

$$
A^{-1/2}P\,\mathrm{div} \in \mathcal{L}\big(L^q(\Omega)\big), \quad \|A^{-1/2}P\,\mathrm{div}\,\|_{\mathcal{L}(L^q)} \le c_{q'}(\Omega). \tag{3.8}
$$

Note that (3.8) holds for all $1 < q < \infty$ when Ω is bounded, but for $q > \frac{3}{2}$ $(1 < q' < 3)$ only when Ω is an exterior domain.

For a further analysis of (3.6) we need the notion of maximal regularity. Consider the abstract inhomogeneous linear Cauchy problem

$$
u_t + Au = f \quad \text{ in } (0,T), \quad u(0) = u_0 \tag{3.9}
$$

where $-A$ is the generator of an analytic semigroup e^{-tA} on a Banach space X. Then A is said to admit *maximal L^p-regularity* on $[0,T)$, $1 < p < \infty$, $0 < T \le \infty$, if for suitable u_0 and all $f \in L^p(0,T;X)$ the solution of (3.9) given by the variation of constants formula

$$
u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}f(\tau)\,\mathrm{d}\tau, \quad t \in [0,T), \tag{3.10}
$$

is differentiable a.e., $u(t) \in \mathcal{D}(A)$ a.e. and u_t, $Au \in L^p(0,T;X)$. In this case the closed graph theorem yields a constant $C = C(p,T) > 0$ independent of f such that

$$
\|u_t; Au\|_{L^p(0,T;X)} \le C\bigg(\int_0^T \|Ae^{-tA}u_0\|_X^p\,\mathrm{d}t\bigg)^{1/p} + C\|f\|_{L^p(0,T;X)}. \tag{3.11}
$$

It is well-known that maximal L^p-regularity for one $p \in (1,\infty)$ implies maximal L^p-regularity for all $p \in (1,\infty)$.

The Stokes operator $A = A_q$ on $X = L^q_\sigma(\Omega)$, $1 < q < \infty$, has maximal L^p-regularity on $[0,\infty)$ for bounded and exterior domains $\Omega \subset \mathbb{R}^3$ of class $C^{1,1}$; see e.g. [28], [37], [49]. In particular, for any $f \in L^s\big(0,\infty; L^q(\Omega)\big)$ the Cauchy problem (3.9) with $A_q = -P_q\Delta$, $u_0 = 0$ and f replaced by $P_q f$ has a unique (strong) solution $u \in C^0_{\rm loc}[0,\infty); L^q_\sigma(\Omega))$ such that $u_t,\ Au \in L^s\big(0,\infty; L^q(\Omega)\big)$, $1 < s < \infty$, and

$$\|u_t; Au\|_{q,s;T} \le c\|f\|_{q,s;T}. \tag{3.12}$$

Moreover, if Ω is bounded and $1 < q < \infty$ or Ω is an exterior domain and $1 < q < \frac{3}{2}$, there exists an associated pressure p such that

$$\|u_t; \nabla^2 u; \nabla p\|_{q,s;T} \le c\|f\|_{q,s;T}. \tag{3.13}$$

For $u_0 \in \big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1-1/s,s} \subset \mathcal{D}(A_q)$, a real interpolation space with norm

$$|||u_0|||_{1-1/s,s,q} := \|u_0\|_q + \Big(\int_0^\infty \|Ae^{-tA}u_0\|_q^s \,\mathrm{d}\tau\Big)^{1/s},$$

the Cauchy problem (3.9) has a unique solution as above; however, in (3.12), (3.13) the additional term $|||u_0|||_{1-1/s,s,q}$ is needed on the right-hand side. For details on the above real interpolation we refer to [50, Chapters 1.13, 1.14]. The integration over $(0,\infty)$ in the norm $\|\cdot\|_{1-1/s,s,q}$ may be replaced by an integration over $(0,\delta)$ for any $\delta > 0$, and, if Ω is bounded, the term $\|\cdot\|_q$ may be omitted [50, Theorem 1.14.5]; cf. the discussion of $\|u_0\|_{\mathbb{B}^{-2/s}_{q,s}}$ in § 2.1.

The relation between a weak solution u of the instationary Stokes system (omitting the term $u\cdot\nabla u$ in (1.1)) with force term $f = \operatorname{div} F$ and initial value u_0 and the notion of a mild solution, cf. (1.8), is given by the representation

$$u(t) = e^{-tA}u_0 + A^{1/2}\int_0^t e^{-(t-\tau)A}(A^{-1/2}P\operatorname{div})F(\tau)\,\mathrm{d}\tau. \tag{3.14}$$

Due to [47, Ch. IV, Theorem 2.4.1, Lemma 2.4.2] for any $u_0 \in L^2_\sigma(\Omega)$ and $F \in L^2\big(0,T; L^2(\Omega)\big)$, the vector field u in (3.14) is the unique weak solution $u \in \mathcal{LH}_T$ of the instationary Stokes system. Moreover, u satisfies the energy equality (EE), cf. (1.12), leading to the estimate

$$\frac{1}{2}\|u\|^2_{2,\infty;T} + \|\nabla u\|^2_{2,2;T} \le 2\|u_0\|_2^2 + 4\|F\|^2_{2,2;T}. \tag{3.15}$$

Obviously, the closed operator $A^{1/2}$ may be commuted with the integral in (3.14). Working for a moment with $v(t) = A^{-1/2}u(t)$ we see that $v_t + Av = A^{-1/2}P\operatorname{div} F$. Hence (3.14) is equivalent to the identities

$$(A^{-1/2}u)_t + A^{1/2}u = A^{-1/2}P\operatorname{div}F \quad \text{on } (0,T), \quad u(0) = u_0 \tag{3.16}$$

and

$$u(t) = e^{-tA}u_0 + \int_0^t A^{1/2}e^{-(t-\tau)A}\big((A^{-1/2}u)_t + A^{1/2}u\big)(\tau)\,\mathrm{d}\tau. \tag{3.17}$$

3.2 Proof of Theorem 2.1

Proof of Theorem 2.1 (cf. [17]) The starting point to construct a strong solution $u \in L^s\big(0,T;L^q(\Omega)\big)$ is the representation (3.14) where by (3.8)

$$\tilde F := A^{-1/2}P\,\mathrm{div}\,F \in L^{s/2}(0,T;L^{q/2}(\Omega))$$

for $F \in L^{s/2}\big(0,T;L^{q/2}(\Omega)\big)$. Due to (3.1) with $r=\frac{q}{2}$ and $\alpha=\frac{3}{2q}<\frac12$ and by (3.3) we get that

$$\|u(t)\|_q \le \|e^{-tA}u_0\|_q + c\int_0^t (t-\tau)^{-\alpha-1/2}\|\tilde F(\tau)\|_{q/2}\,\mathrm{d}\tau \tag{3.18}$$

where $\alpha+\frac12=\frac{3}{2q}+\frac12=1-\frac1s$. Then the Hardy-Littlewood inequality (see e.g. [47, Ch. II, Lemma 3.3.2]) implies that

$$\|u\|_{q,s;T} \le \|e^{-tA}u_0\|_{q,s;T} + C\|\tilde F\|_{q/2,s/2;T}. \tag{3.19}$$

Let $u_0=0$. Then a direct application of the maximal regularity estimate (3.12) to $v(t)=A^{-1/2}u(t)$, cf. (3.16), (3.17), yields the estimate

$$\|(A^{-1/2}u)_t;A^{1/2}u\|_{q/2,s/2;T} \le C\|\tilde F\|_{q/2,s/2;T} \tag{3.20}$$

with $\tilde F=(A^{-1/2}u)_t+A^{1/2}u$. In view of (3.20) we define the Banach space

$$X_T=\Big\{v:[0,T)\to L^{q/2}_\sigma(\Omega):\\ \big(A^{-1/2}_{q/2}v\big)_t,\ A^{1/2}_{q/2}v\in L^{s/2}\big(0,T;L^{q/2}_\sigma(\Omega)\big),\ A^{-1/2}_{q/2}v(0)=0\Big\}$$

equipped with the norm

$$\|v\|_{X_T}=\big\|(A^{-1/2}_{q/2}v)_t;A^{1/2}_{q/2}v\big\|_{q/2,s/2;T}$$

which by (3.20) is equivalent to the norm $\big\|(A^{-1/2}_{q/2}v)_t+A^{1/2}_{q/2}v\big\|_{q/2,s/2;T}$.

In our application to the Navier-Stokes system (1.1) with $u_0\in L^2_\sigma(\Omega)\cap\mathcal{B}^{q,s}$ let $E_0(t):=e^{-tA}u_0$, which by assumption satisfies $E_0\in L^s(L^q)$, and let

$$E_1(t)=\int_0^t A^{1/2}e^{-(t-\tau)A}A^{-1/2}P\,\mathrm{div}\,F(\tau)\,\mathrm{d}\tau \tag{3.21}$$

satisfying $E_1\in L^s\big(0,T;L^q_\sigma(\Omega)\big)$ by (3.19). Then $\tilde u=u-E$, with $E:=E_0+E_1\in L^s(L^q)$, is a solution of the fixed point problem $\tilde u=\mathcal{F}\tilde u$ where

$$\mathcal{F}\tilde u(t)=-\int_0^t A^{1/2}e^{-(t-\tau)A}\big(A^{-1/2}P\,\mathrm{div}\big)(\tilde u+E)\otimes(\tilde u+E)(\tau)\,\mathrm{d}\tau. \tag{3.22}$$

By (3.19) we conclude that $\mathcal{F}\tilde u\in L^s(L^q)$ for $\tilde u\in L^s(L^q)$ and

$$\|\mathcal{F}\tilde u\|_{q,s;T}\le c\big(\|\tilde u\|^2_{q,s;T}+\|E\|^2_{q,s;T}\big)$$

where $\|E\|_{q,s;T}$ can be estimated by F in $L^{s/2}(L^{q/2})$ and by $\|E_0\|_{q,s;\infty}$. Due to the quadratic term $\tilde{u}+E$ in $\mathcal{F}$ it is easy to show that $\mathcal{F}$ defines a (strictly) contractive map from a closed ball $\overline{B} \subset X_T$ into itself provided that $\|E\|_{q,s;T}$ is sufficiently small. This smallness condition is guaranteed by the assumption (2.5). Now Banach's fixed point theorem yields the existence of a unique fixed point $\tilde{u}$ of $\mathcal{F}$ in $\overline{B}$. Then $u = \tilde{u} + E \in L^s(L^q)$ is a solution of (1.1) in the sense of the integral equation (3.6).

In the second step of the proof we have to show that $u = \tilde{u} + E \in L^s(L^q)$ is also a weak solution, i.e. $u \in \mathcal{LH}_T$. To prove that $\tilde{u} \in L^2(W_0^{1,2})$ we use the Yosida approximation operators $J_n = \big(I + \frac{1}{n}A^{1/2}\big)^{-1} : L^q_\sigma(\Omega) \to W^{1,q}_{0,\sigma}(\Omega)$. Note that the family of operators $(J_n)_{n\in\mathbb{N}}$ and $\big(\frac{1}{n}A_n^{1/2}J_n\big)_{n\in\mathbb{N}}$ are uniformly bounded in $\mathcal{L}\big(L^q_\sigma(\Omega)\big)$ and that $J_n v \to v$ as $n \to \infty$ for every $v \in L^q_\sigma(\Omega)$, $1 < q < \infty$.

Writing (3.22) with $\tilde{u} = u - E = \mathcal{F}(\tilde{u})$ in the form

$$\tilde{u}(t) = -\int_0^t A^{1/2} e^{-(t-\tau)A} \big(A^{-1/2} P \operatorname{div}\big)(u \otimes \tilde{u} + u \otimes E)(\tau)\, \mathrm{d}\tau, \qquad (3.23)$$

we apply J_n which commutes with $A^{\pm 1/2}$ and $e^{-(t-\tau)A}$. Thus, with $\tilde{u}_n := J_n \tilde{u}$, we are led to the term

$$J_n P \operatorname{div}(u \otimes \tilde{u}) = J_n P(u \cdot \nabla \tilde{u}_n) + \Big(\frac{1}{n}A^{1/2}J_n\Big)\big(A^{-1/2}P \operatorname{div}\big)\big(u \otimes A^{1/2}\tilde{u}_n\big)$$

which can be estimated in $L^r(\Omega)$, $\frac{1}{r} = \frac{1}{2} + \frac{1}{q}$, as follows:

$$\|J_n P \operatorname{div}(u\, \tilde{u})\|_r \le c\|u\|_q\, \|A^{1/2}\tilde{u}_n\|_2;$$

here we used (3.2), the uniform boundedness of the operators J_n, P, $\frac{1}{n}A^{1/2}J_n$ and $A^{-1/2}P \operatorname{div}$, and Hölder's inequality. Moreover,

$$\|J_n P \operatorname{div}(u \otimes E)\|_r \le c\|u\|_q\, \|A^{1/2}E\|_2.$$

At this moment, we already mention that $A^{1/2}E = A^{1/2}(e^{-tA}u_0) + A^{1/2}E_1$ lies in $L^2(L^2)$ and satisfies due to (3.15) the estimate

$$\|A^{1/2}E\|_{2,2;T} \le c\big(\|u_0\|_2 + \|F\|_{2,2;T}\big).$$

We conclude that the operators $A^{1/2}$ and $A^{-1/2}$ in (3.23) cancel each other and that $A^{1/2}\tilde{u}_n = A^{1/2}J_n\tilde{u}$ has the representation

$$A^{1/2}\tilde{u}_n(t) = -\int_0^t A^{1/2}e^{-(t-\tau)A} J_n P \operatorname{div}(u \otimes \tilde{u} + u \otimes E)(\tau)\, \mathrm{d}\tau.$$

As for the proof of (3.18), (3.19) we use (3.1) (with $\frac{3}{r} = \frac{3}{2} + 2\alpha$, $\alpha = \frac{3}{2q} < \frac{1}{2}$) and the Hardy-Littlewood inequality (with $\frac{1}{s} + \frac{1}{2} + (\alpha + \frac{1}{2}) = \frac{1}{2} + 1$) to get the estimate

$$\|A^{1/2}\tilde{u}_n(t)\|_2 \le c\int_0^t (t-\tau)^{-\alpha-1/2}\|u\|_q\big(\|A^{1/2}\tilde{u}_n\|_2 + \|A^{1/2}E\|_2\big)\, \mathrm{d}\tau,$$

$$\|A^{1/2}\tilde{u}_n\|_{2,2;T} \le c_0\|u\|_{q,s;T}\big(\|A^{1/2}\tilde{u}_n\|_{2,2;T} + \|A^{1/2}E\|_{2,2;T}\big),$$

with an absolute constant $c = c(q, \Omega) > 0$ independent of T. Choosing $T_1 \in (0, T]$ sufficiently small such that $c_0\|u\|_{q,s;T_1} \le \frac{1}{2}$ we obtain the uniform bound

$$\|A^{1/2}\tilde{u}_n\|_{2,2;T_1} \le 2c_0\|u\|_{q,s;T_1}\, \|A^{1/2}E\|_{2,2;T_1}, \quad n \in \mathbb{N}.$$

Since $(\tilde{u}_n) \subset L^s\big(0, T_1; L^q(\Omega)\big)$ is bounded, we get that $\nabla\tilde{u} \in L^2\big(0, T_1; L^2(\Omega)\big)$; for this argument we refer to the implication (3.1.8) $\Rightarrow$ (3.1.9) in [47, Ch. II, 3.1]. Repeating this step finitely many times we show that $\tilde{u}$, $u = \tilde{u} + E \in L^2\big(0, T; W_0^{1,2}(\Omega)\big)$ for any finite T.

Finally we have to show that $\tilde{u}, u \in L^\infty(L^2)$. Arguing formally and omitting details, cf. [17, pp. 102f], we consider $\tilde{u}$ as a solution of the instationary Stokes system with right-hand side $-P(u \cdot \nabla u) \in L^{s_1}(L^{q_1})$ where $\frac{1}{s_1} = \frac{1}{2} + \frac{1}{s}$, $\frac{1}{q_1} = \frac{1}{2} + \frac{1}{q}$ since $u \in L^s(L^q)$ and $\nabla u \in L^2(L^2)$. The corresponding "Serrin number" $\frac{2}{s_1} + \frac{3}{q_1} = \left(\frac{2}{2} + \frac{3}{2}\right) + \left(\frac{2}{s} + \frac{3}{q}\right)$ equals $\frac{7}{2}$ so that maximal regularity estimates yield the result $\tilde{u}_t$, $A\tilde{u} \in L^{s_2}(L^{q_2})$ with Serrin number $\frac{2}{s_2} + \frac{3}{q_2} = \frac{7}{2}$. Then Sobolev's embedding theorem formally implies that $\tilde{u} \in L^{s_3}(L^{q_3})$ with $\frac{2}{s_3} + \frac{3}{q_3} = \frac{7}{2} - 2 = \frac{3}{2}$. A rigorous analysis proves that $\frac{1}{s_3} = \frac{1}{2} - \frac{1}{s}$, $\frac{1}{q_3} = \frac{1}{2} - \frac{1}{q}$ are admissible. Since also $E \in L^{s_3}(L^{q_3})$ we have $u \in L^{s_3}(L^{q_3}) \cap L^s(L^q)$, and by Hölder's inequality $u \in L^4(L^4)$.

We conclude that u solves (3.6) and can be considered as a solution of the Stokes system with right-hand side $\operatorname{div}(F - u \otimes u)$ where $F - u \otimes u \in L^2(L^2)$. Then (3.6) implies that u is also a solution of the variational problem (1.3). Testing (1.3) with $w = u$ we get that $\langle u \otimes u, \nabla u\rangle_{\Omega,T} = 0$ and that u satisfies (EE). In particular, $u \in L^\infty(L^2)$.

Now the proof of the sufficiency of the condition (2.1) to guarantee the existence of a local strong solution is complete.

To prove necessity assume that $F \in L^{s/2}(L^{q/2})$ and that $u \in L^s(L^q)$. Then $\tilde{u} = u - E$ has the representation

$$\tilde{u}(t) = -\int_0^t A^{1/2}e^{-(t-\tau)A}A^{-1/2}P\,\operatorname{div}(u \otimes u)(\tau)\,\mathrm{d}\tau$$

where $u \otimes u \in L^{s/2}(L^{q/2})$. As in (3.19) we conclude that $\tilde{u} \in L^s(L^q)$ so that also $E = u - \tilde{u} \in L^s(L^q)$. The term E_1, see (3.21) satisfies $E_1 \in L^s(L^q)$ due to (3.19) since $F \in L^{s/2}(L^{q/2})$. Hence also $E_0 = e^{-tA}u_0 = E - E_1 \in L^s(L^q)$. ■

3.3 Proof of Further Results

Proof of Lemma 2.4 (i) For the proof it is more convenient to consider u as a function with values in the intersection space $\widehat{\mathcal{B}}^{q,s}_\delta(\Omega) = L^2_\sigma(\Omega) \cap \mathcal{B}^{q,s}_\delta(\Omega)$. Obviously, $\widehat{\mathcal{B}}^{q,s}_\delta(\Omega)$ is a reflexive, separable Banach space when equipped with the norm $\|\cdot\|_2 + \|\cdot\|_{\mathcal{B}^{q,s}_\delta}$. It is easy to see that there exists a Lebesgue null set $N \subset [0, T]$ such that $\|u\|_{L^\infty(0,T;\widehat{\mathcal{B}}^{q,s}_\delta)} = \sup_{t \in [0,T] \setminus N} \|u(t)\|_{\widehat{\mathcal{B}}^{q,s}_\delta}$. A similar result holds when the norm in $\widehat{\mathcal{B}}^{q,s}_\delta$ is replaced by the norm in $\mathcal{B}^{q,s}_\delta$.

Consider any $t_0 \in N$ and choose a sequence $(t_j) \subset (0, T) \setminus N$ such that $t_j \to t_0$ as $j \to \infty$. Thus $\big(u(t_j)\big)$ is a bounded sequence in $\widehat{\mathcal{B}}^{q,s}_\delta(\Omega) \subset L^2_\sigma(\Omega)$.

Due to (1.6) we get that $u(t_j) \rightharpoonup u(t_0) \in L^2_\sigma(\Omega)$. Moreover, by the reflexivity of $\widehat{\mathcal{B}}^{q,s}_\delta(\Omega)$ we may conclude by standard arguments that $u(t_j) \rightharpoonup u(t_0)$ in $\mathcal{B}^{q,s}_\delta(\Omega)$. Hence $u(t_0) \in \widehat{\mathcal{B}}^{q,s}_\delta(\Omega)$ and $\|u(t_0)\|_{\mathcal{B}^{q,s}_\delta} \le \liminf_{j\to\infty} \|u(t_j)\|_{\mathcal{B}^{q,s}_\delta}$.

The same ideas prove that $u \in C^0_w\big([0,T]; \mathcal{B}^{q,s}_\delta(\Omega)\big)$. In particular, $\|u(\cdot)\|_{\mathcal{B}^{q,s}_\delta}$ is lower semi-continuous.

(ii) We consider $t \in [0,T]$ such that $u(t) \notin \mathcal{B}^{q,s}_\delta(\Omega)$ and choose any sequence $(t_j) \subset [0,T]$ with $t_j \to t$ as $j \to \infty$. Since $\big(u(t_j)\big) \subset L^2_\sigma(\Omega)$ is bounded, we get from (3.1), (3.3) with $2\alpha + \frac{3}{q} = \frac{3}{2}$ that $\|e^{-\tau A}u(t_j)\|_q \le c\tau^{-\alpha}\|u(t_j)\|_2 \le C$ with a constant $C = C(\tau) > 0$. Moreover, the weak continuity of $u(\cdot)$ in $L^2_\sigma(\Omega)$ implies the convergence

$$\langle e^{-\tau A}u(t_j), \varphi\rangle = \langle u(t_j), e^{-\tau A}\varphi\rangle \to \langle u(t), e^{-\tau A}\varphi\rangle = \langle e^{-\tau A}u(t), \varphi\rangle$$

for all φ in the dense subset $L^2_\sigma(\Omega) \cap L^{q'}_\sigma(\Omega)$ of $L^{q'}_\sigma(\Omega)$. Summarizing the last two arguments we conclude that $e^{-\tau A}u(t_j) \rightharpoonup e^{-\tau A}u(t)$ in $L^q_\sigma(\Omega)$; in particular, $\|e^{-\tau A}u(t)\|_q \le \liminf_j \|e^{-\tau A}u(t_j)\|_q$. Hence by Fatou's lemma

$$\int_0^\delta \|e^{-\tau A}u(t)\|_q^s \,\mathrm{d}\tau \le \int_0^\delta \liminf_j \|e^{-\tau A}u(t_j)\|_q^s \,\mathrm{d}\tau \le \liminf_j \int_0^\delta \|e^{-\tau A}u(t_j)\|_q^s \,\mathrm{d}\tau,$$

i.e., $\infty = \|u(t)\|_{\mathcal{B}^{q,s}_\delta} \le \liminf_j \|u(t_j)\|_{\mathcal{B}^{q,s}_\delta}$. Thus $\|u(t')\|_{\mathcal{B}^{q,s}_\delta} \to \|u(t)\|_{\mathcal{B}^{q,s}_\delta}$ as $t' \to t$. Similar arguments apply when $u(t) \in \mathcal{B}^{q,s}_\delta(\Omega)$.

(iii) Assume that $\delta(t) \in [0,\infty)$. Then for $\beta > \delta(t)$ and any sequence $(t_j) \subset [0,T]$ with $t_j \to t$ as $j \to \infty$

$$\varepsilon_* < \int_0^\beta \|e^{-\tau A}u(t)\|_q^s \,\mathrm{d}\tau \le \liminf_j \int_0^\beta \|e^{-\tau A}u(t_j)\|_q^s \,\mathrm{d}\tau$$

which implies that $\int_0^\beta \|e^{-\tau A}u(t_j)\|_q^s \,\mathrm{d}\tau > \varepsilon_*$ for large j. Hence $\delta(t_j) < \beta$, i.e., $\delta(\cdot)$ is upper semi-continuous at t. The case $\delta(t) = \infty$ is trivial. ∎

Proof of Theorem 2.5 (i) Let $t_1 \in (0,T)$. To show that t_1 is a left-sided regular point of u we find due to the assumption (2.8) and (SEI) $t < t_1$ such that $\|u(t)\|_{\mathcal{B}^{q,s}_{t_1-t}} \le \|u(t)\|_{\mathcal{B}^{q,s}_{\delta(t)}} \le \varepsilon_*$ and that $(EI)_t$ holds. Here $\varepsilon_* > 0$ is the constant from Theorem 2.1, see (2.5). By Serrin's uniqueness theorem and Theorem 2.1 we conclude that $u \in L^s\big(t_0, t_1; L^q(\Omega)\big)$. Hence u is left-sided regular in t_1.

(ii) Since $u(t_1) \in \mathcal{B}^{q,s}_\infty(\Omega)$, there exists a strong solution $v \in L^s\big(t_1, t_1 + \varepsilon; L^q(\Omega)\big)$, $\varepsilon > 0$, of (1.1) with initial value $v(t_1) = u(t_1)$. Moreover, by assumption the energy inequality for u holds with initial time t_1. Hence Serrin's uniqueness theorem implies that $v = u$ in $[t_1, t_1 + \varepsilon)$, and u is right-sided regular in t_1.

(iii) To combine the results from (i) and (ii), in particular to apply (ii), it suffices to prove that u satisfies $(EI)_{t_1}$ at any t_1. Let $t_1 \in (0,T)$ be an instant where the validity of the energy inequality is not guaranteed by (SEI). By (i) t_1 is a left-sided regular point for u and, consequently, $u \in L^s(t_0, t_1; L^q(\Omega))$ for some $0 < t_0 < t_1$. Therefore, u satisfies (EE) for all initial times $t_0' \in (t_0, t_1)$, in particular

$$\frac{1}{2}\|u(t_1)\|_2^2 + \int_{t_0'}^{t_1} \|\nabla u\|_2^2 \, \mathrm{d}\tau = \frac{1}{2}\|u(t_0')\|_2^2.$$

Thus $\lim_{t_0' \nearrow t_1} \|u(t_0')\|_2^2 = \|u(t_1)\|_2^2$ for $t_0' \in (t_0, t_1)$. Moreover, by (SEI), there is a sequence $t_j \nearrow t_1$ such that

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_j}^{t} \|\nabla u\|_2^2 \, \mathrm{d}\tau \leq \frac{1}{2}\|u(t_j)\|_2^2, \quad t_j \leq t < T.$$

Passing to the limit $t_j \nearrow t_1$ we get that u satisfies $(EI)_{t_1}$.

Finally, since $u(0) = u_0 \in \mathcal{B}_\infty^{q,s}(\Omega)$, we know that $u \in L^s\big(0, \varepsilon_0; L^q(\Omega)\big)$ for some $\varepsilon_0 > 0$. Now an elementary compactness argument proves that $u \in L^s_{\mathrm{loc}}\big([0,T); L^q(\Omega)\big)$.

(iv) Under the assumption (2.9) for t_1 we get for a.a. t in a left-sided neighborhood of t_1 that $\|u(t)|_{\mathcal{B}^{q,s}_{t_1 - t}} < \varepsilon_*$. Hence $\delta(t) > t_1 - t$ for these t; moreover, we may assume $(EI)_t$. By the above arguments we conclude that u lies in Serrin's class $L^s(L^q)$ on the interval $(t, t+\delta(t)) \supset (t, t_1)$, i.e., u is $L^s(L^q)$-regular at t_1. ∎

Proof of Corollary 2.7 (i) Fix $\alpha \in (\frac{1}{4}, \frac{1}{2}]$ and let $q \geq q_0 \geq 2$ satisfy $2\alpha + \frac{3}{q} = \frac{3}{q_0}$. By Theorem 2.5 (i) we get for all $t \in (0, t_1)$ where $(EI)_t$ is satisfied that

$$\begin{aligned}
\epsilon_* < \int_0^{t_1 - t} \|e^{-\tau A} u(t)\|_q^s \, \mathrm{d}\tau &\leq c \int_0^{t_1 - t} \|A^\alpha e^{-\tau A} u(t)\|_{q_0}^s \, \mathrm{d}\tau \\
&\leq c \int_0^{t_1 - t} \tau^{-\frac{s}{2}(\frac{3}{2} - \frac{3}{q_0})} \|A^\alpha u(t)\|_2^s \, \mathrm{d}\tau \\
&= c (t_1 - t)^{-\frac{s}{4} + \alpha s} \|A^\alpha u(t)\|_2^s
\end{aligned}$$

since $-\frac{s}{2}(\frac{3}{2} - \frac{3}{q_0}) = -\frac{s}{4} + \alpha s - 1 > -1$. Hence $\|A^\alpha u(t)\|_2 > \epsilon_* (t_1 - t)^{\frac{1}{4} - \alpha}$ for a.a. $t \in (0, t_1)$.

(ii) Since $u(t_1) \notin \mathcal{B}^{q,s}(\Omega)$, we have $\|u(t_1)\|_{\mathcal{B}^{q,s}_\delta} = \infty$ for each $\delta > 0$. The lower semi-continuity of the map $t \mapsto \|u(t)\|_{\mathcal{B}^{q,s}_\delta}$ implies that $\|u(t)\|_{\mathcal{B}^{q,s}_\delta} \to \infty$ as $t \searrow t_1$. Moreover, due to the embeddings (2.6) we get that $\|u(t)\|_{\mathcal{B}^{q,s}_\delta} \leq c\|u(t)\|_3 \leq c\|u(t)\|_{\mathcal{D}(A^{1/4})} \leq c\|u(t)\|_{\mathcal{D}(A^{1/2})}$. ∎

Proof of Theorem 2.9 (i) Assuming (i) we find $\delta > 0$ such that with $t_0 = t_1 - \delta$ and $T' = t_1 + \delta$

$$\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} (T' - \tau)^{r/s} \|u(\tau)\|_q^r \, \mathrm{d}\tau \leq \frac{2^{r/s}}{\delta^{1 - r/s}} \int_{t_0}^{t_1} \|u(\tau)\|_q^r \, \mathrm{d}\tau \leq \tilde{\varepsilon}_*,$$

i.e., (2.12) is satisfied. Thus it suffices to prove (ii).

(ii) Let us check condition (2.8) in Theorem 2.5 (i), i.e., find $t \in (0, t_1)$ such that $\|u(t)\|_{\mathcal{B}^{q,s}_{t_1-t}} \le \varepsilon_*$ and that $(EI)_t$ is satisfied. Now from (2.12) there exists $t \in (t_0, t_1)$ such that

$$(T' - t)^{r/s} \|u(t)\|_q^r \le \tilde{\varepsilon}_*$$

or, equivalently, $(T' - t)\|u(t)\|_q^s \le \varepsilon_*^{s/r}$ and that $(EI)_t$ holds. Hence, employing the boundedness of the semigroup $e^{-\tau A}$ on $L^q_\sigma(\Omega)$,

$$\int_0^{T'-t} \|e^{-\tau A} u(t)\|_q^s \, \mathrm{d}\tau \le C(T' - t)\|u(t)\|_q^s \le \varepsilon_*$$

with an appropriately chosen $\tilde{\varepsilon}_*$ in (2.12). ■

Proof of Theorem 2.11 By the moment inequality of interpolation theory and the identity $\|\nabla u\|_2 = \|A^{1/2} u\|_2$ (cf. (3.2) with $q = 2$) which holds for any domain we get that

$$\begin{aligned}
\left(\fint_{t_1-\delta}^{t_1} \|A^{1/4} u\|_2 \, \mathrm{d}\tau \right)^2 &\le \left(\fint_{t_1-\delta}^{t_1} \|u\|_2^{1/2} \, \|A^{1/2} u\|_2^{1/2} \, \mathrm{d}\tau \right)^2 \\
&\le \fint_{t_1-\delta}^{t_1} \|u\|_2 \, \|\nabla u\|_2 \, \mathrm{d}\tau \\
&\le \sup_{[t_1-\delta, t_1]} \|u(\cdot)\|_2 \left(\fint_{t_1-\delta}^{t_1} \|\nabla u\|_2^2 \, \mathrm{d}\tau \right)^{1/2}.
\end{aligned}$$

Hence it suffices to consider (2.16) only. From (2.16) we find $t \in (t_1 - \delta, t_1)$ where u satisfies $\|A^{1/4} u(t)\|_2 \le \varepsilon_*$ and $(EI)_t$. Next we need the estimate

$$\left(\int_0^\infty \|e^{-\tau A} u(t)\|_4^8 \, \mathrm{d}\tau \right)^{1/8} \le c \|A^{1/4} u(t)\|_2$$

which is based on L^2-arguments only, cf. [17, Proof of Theorem 4.1], and holds with a constant $c > 0$ independent of the domain. As in the case of smooth bounded domains the condition $\|e^{-\tau A} u(t)\|_{4,8;\infty} \le \varepsilon_*$ suffices to prove that the weak solution u is $L^8(L^4)$-regular, cf. [17, Theorem 4.1]. ■

Proof of Theorem 2.13 The main idea to analyze the term $\int_\Omega (u \cdot \nabla u) u \, \mathrm{d}x$ is the splitting of the third factor u into a low frequency and a high frequency part, u_ℓ and u_h, respectively. For u_ℓ we take

$$u_\ell(t) = e^{-\delta A} u(t), \quad 0 < \delta < 1,$$

and use it as (admissible) test function in (1.3). By Lemma (3.2) below we get that for $0 < t < T$

$$\frac{1}{2}\langle u(t), e^{-\delta A}u(t)\rangle + \int_0^t \langle A^{1/2}u, e^{-\delta A}A^{1/2}u\rangle \,\mathrm{d}\tau = \frac{1}{2}\langle u_0, e^{-\delta A}u_0\rangle + \int_0^t \langle u\cdot\nabla u, e^{-\delta A}u\rangle \,\mathrm{d}\tau.$$

Since $A^{1/2}u \in L^2\big(0,T; L^2_\sigma(\Omega)\big)$ we pass to the limit $\delta \to 0$ to get that

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2 \,\mathrm{d}\tau = \frac{1}{2}\|u_0\|_2^2 + \lim_{\delta\to 0}\int_0^t \langle u\cdot\nabla u, e^{-\delta A}u\rangle \,\mathrm{d}\tau.$$

To discuss the last term we write $\nabla u = \nabla u_\ell + \nabla u_h$, use that $\langle u\cdot\nabla u_\ell, u_\ell\rangle = 0$ and obtain, after an integration by parts, the estimate

$$\begin{aligned}
|\langle u\cdot\nabla u, u_\ell\rangle| &= |\langle u\cdot\nabla u_\ell, u_h\rangle| \le \|u\|_{9/2}\,\|\nabla u_\ell\|_{18/7}\,\|u_h\|_{18/7} \\
&\le c\|u\|_{9/2}\,\|A^{1/2}u_\ell\|_{18/7}\,\|u_h\|_{18/7} \\
&\le c\|A^{1/4}u\|_{18/7}\big(\delta^{-1/4}\|A^{1/4}u\|_{18/7}\big)\big(\delta^{1/4}\|A^{1/4}u\|_{18/7}\big) \\
&= c\|A^{1/4}u\|_{18/7}^3;
\end{aligned}$$

in the second last step we used the embedding (3.1) with $\alpha = \frac{1}{4}$, the estimate $\|A^{1/2}u_\ell\|_{18/7} \le c\delta^{-1/4}\|A^{1/4}u\|_{18/7}$, see (3.3) with $\alpha = \frac{1}{4}$, and the identity $u_h = u - e^{-\delta A}u = \int_0^\delta Ae^{-\tau A}u\,\mathrm{d}\tau$ for $u \in \mathcal{D}\big(A^{1/4}_{18/7}\big)$ together with (3.3). Moreover, since $\|u_h\|_3 \le c\delta^{1/4}\|A^{1/4}u\|_3 \le c\delta^{1/4}\|A^{1/2}u\|_2$ by (3.3) and (3.1), we get the pointwise convergence

$$|\langle u\cdot\nabla u, u_\ell\rangle - \langle u\cdot\nabla u, u\rangle| \le \|u\|_6\,\|\nabla u\|_2\; c\delta^{1/4}\|\nabla u\|_2 \to 0$$

as $\delta \to 0$. Hence the assumption (2.19), i.e., $\|A^{1/4}u\|_{18/7}^3 \in L^1(0,T)$, and Lebesgue's convergence theorem yield the convergence

$$\int_0^t \langle u\cdot\nabla u, e^{-\delta A}u\rangle \,\mathrm{d}\tau \to \int_0^t \langle u\cdot\nabla u, u\rangle \,\mathrm{d}\tau = 0.$$

Now Theorem 2.13 is proved. ■

Lemma 3.1 *Let $u \in \mathcal{LH}_T$ be a weak solution of* (1.1) *with $u_0 \in L^2_\sigma(\Omega)$ and let $S \in \mathcal{L}\big(L^2_\sigma(\Omega)\big)$ be a self-adjoint operator satisfying the estimate $\|Sv\|_{\mathcal{D}(A_2)} \le c\|v\|_2$ for all $v \in L^2_\sigma(\Omega)$ and commuting with $A^{1/2}$ on $\mathcal{D}(A_2)$. Then for all $0 \le t < T$*

$$\frac{1}{2}\langle u(t), S\,u(t)\rangle + \int_0^t \langle S\,A^{1/2}u, A^{1/2}u\rangle \,\mathrm{d}\tau = \frac{1}{2}\langle u_0, S\,u_0\rangle - \int_0^t \langle u\cdot\nabla u, S\,u\rangle \,\mathrm{d}\tau.$$

Proof Following an approximation argument of Serrin [43] let $0 \le \varrho \in C_0^\infty(\mathbb{R})$ be an even cut-off function with $\int \varrho \, \mathrm{d}\tau = 1$ and let $\varrho_\varepsilon(\tau) = \frac{1}{\varepsilon}\varrho(\frac{\tau}{\varepsilon})$, $\varepsilon > 0$. Then fix $0 < t < T$, define the convolution

$$u_\varepsilon(\tau) := \int_0^t \varrho_\varepsilon(\tau - s) u(s) \, \mathrm{d}s$$

and note that $u_\varepsilon \to u$ and $A^{1/2}u_\varepsilon \to A^{1/2}u$ in $L^2\big(0,t;L^2(\Omega)\big)$ as $\varepsilon \to 0$. Since S is self-adjoint, an elementary calculation using symmetry arguments and a change of variables imply that

$$\int_0^t \langle u, \partial_\tau S\, u_\varepsilon \rangle \, \mathrm{d}\tau = \int_0^t \int_0^t \langle u(\tau), \partial_\tau \varrho_\varepsilon(\tau - s) S\, u(s) \rangle \, \mathrm{d}\tau \, \mathrm{d}s = 0.$$

Moreover, the weak L^2-continuity of u with respect to time yield the convergences

$$\langle u(t), S\, u_\varepsilon(t) \rangle \to \frac{1}{2} \langle u(t), S\, u(t) \rangle, \quad \langle u_0, S(u_0)_\varepsilon \rangle \to \frac{1}{2} \langle u_0, S\, u_0 \rangle$$

as $\varepsilon \to 0$. We also note that

$$\begin{aligned} \int_0^t \langle \nabla u, \nabla S\, u_\varepsilon \rangle \, \mathrm{d}\tau &= \int_0^t \langle A^{1/2} u, A^{1/2} S\, u_\varepsilon \rangle \, \mathrm{d}\tau \\ &= \int_0^t \langle S\, A^{1/2} u, A^{1/2} u_\varepsilon \rangle \, \mathrm{d}\tau \to \int_0^t \langle S\, A^{1/2} u, A^{1/2} u \rangle \, \mathrm{d}\tau \end{aligned}$$

and that

$$\begin{aligned} \Big| \int_0^t \langle u \cdot \nabla u, S(u_\varepsilon - u) \rangle \, \mathrm{d}\tau \Big| &\le c \|u\|_{2,\infty;T} \|\nabla u\|_{2,2;T} \left(\int_0^t \|S(u_\varepsilon - u)\|^2_{\mathcal{D}(A_2)} \, \mathrm{d}\tau \right)^{1/2} \\ &\le c \|u\|_{2,\infty;T} \|\nabla u\|_{2,2;T} \; \|u_\varepsilon - u\|_{2,2;t} \end{aligned}$$

converges to 0 as $\varepsilon \to 0$. Now the assertion follows from (1.3) with the admissible test function $w = S\, u_\varepsilon$. ∎

Proof of Theorem 2.14 (i) Assume that u satisfies (EE) and $u(t_0) \in \mathcal{B}^{q,s}(\Omega)$ for all $t_0 \in [0,T)$. Moreover, let $\tilde{u}$ be another weak solution satisfying (SEI) for the same initial value $u_0 \in L^2_\sigma(\Omega)$. We obtain that $u \in L^s\big(0,\delta;L^q(\Omega)\big)$ with some $0 < \delta < T$. Then Serrin's uniquenes theorem implies that $u(t) = \tilde{u}(t)$ for $0 \le t < \delta$.

Let $[0,t_0)$, $0 < t_0 \le T$, be the largest half open interval such that $u(t) = \tilde{u}(t)$ is satisfied for each $t \in [0,t_0)$. If $t_0 < T$, then the weak L^2-continuity in time (1.6) implies that $u(t_0) = \tilde{u}(t_0)$.

Since u satisfies $u(t_0) \in \mathcal{B}^{q,s}(\Omega)$ and $(EI)_{t_0}$ we conclude that $u \in L^s\big(t_0, t_0 + \delta; L^q(\Omega)\big)$. Moreover, since $\tilde{u}$ satisfies (SEI), hence $(EI)_{t_j}$ for a sequence (t_j) with $t_j \nearrow t_0$, and since $u = \tilde{u}$ on $[0, t_0]$ satisfies (EE), we conclude that $\|\tilde{u}(t_j)\|_2 \to \|\tilde{u}(t_0)\|_2$. These arguments imply that $\tilde{u}$ satisfies $(EI)_{t_0}$. Consequently, by Serrin's uniquenes theorem, $u = \tilde{u}$ in $[0, T_1 + \delta)$. This is a contradiction to the construction of t_0.

(ii) For $u \in L^4(L^4)$ Remark 1.2 5. yields (EE). Now assume that u satisfies (2.20). By Hölder's inequality in Lorentz spaces ([34, Lemma 2.1]) and Sobolev's embedding $W_0^{1,2}(\Omega) \subset L^{6,2}(\Omega)$ ([34, Lemma 2.2]) we get that

$$\begin{aligned} \|uu\|_{L^2} \le c\|uu\|_{L^{2,2}} &\le c\|u\|_{L^{3,\infty}}\|u\|_{L^{6,2}} \\ &\le c\|u\|_{L^{3,\infty}}\|u\|_{W^{1,2}}, \end{aligned}$$

where $c = c(\Omega) > 0$. Hence $u \in L^4(L^4)$. Now we proceed as above. ■

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