NECESSARY AND SUFFICIENT CONDITIONS FOR LOCAL STRONG SOLVABILITY OF THE NAVIER-STOKES EQUATIONS IN EXTERIOR DOMAINS

CHRISTIAN KOMO

Abstract. Consider the instationary Navier-Stokes equations in a smooth exterior domain $\Omega \subseteq \mathbb{R}^3$ with initial value u_0 and external force $f = \operatorname{div} F$. It is an important question to characterize the class of initial values $u_0 \in L^2_\sigma(\Omega)$ that allow a strong solution $u \in L^s(0,T;L^q(\Omega))$ in some interval $[0,T[\,, 0 < T \le \infty$ where s, q with $3 < q < \infty$ and $\frac{2}{s} + \frac{3}{q} = 1$ are so-called Serrin exponents. In (Analysis (Munich) **33** (2013), 101-119) the authors proved that $\int_0^\infty \|e^{-\nu t A} u_0\|_q^s \, dt < \infty$ is necessary and sufficient for the existence of a strong solution $u \in L^s(0,T;L^q(\Omega))\,, 0 < T \le \infty$, if additionally $3 < q \le 8$. In this paper we will show that this result remains true if $q > 8$ and consequently $\int_0^\infty \|e^{-\nu t A} u_0\|_q^s \, dt < \infty$ is the optimal initial value condition to obtain such a strong solution for all possible Serrin exponents s, q.

1. Introduction and main results

In this paper, $\Omega \subseteq \mathbb{R}^3$ is an exterior domain, i.e. an open, connected subset having a nonempty, compact complement in $\mathbb{R}^3$, with smooth boundary $\partial\Omega \in C^{2,1}$, and $[0,T[\,, 0 < T \le \infty$, denotes the time interval. In $[0,T[\times\Omega$ we consider the instationary Navier-Stokes equations

$$\begin{aligned} u_t - \nu\Delta u + u \cdot \nabla u + \nabla p = f &\quad \text{in }]0,T[\times\Omega\,, \\ \operatorname{div} u = 0 &\quad \text{in }]0,T[\times\Omega\,, \\ u = 0 &\quad \text{on }]0,T[\times\partial\Omega\,, \\ u = u_0 &\quad \text{at } t = 0\,, \end{aligned} \tag{1.1}$$

with external force $f = \operatorname{div} F = (\sum_{i=1}^3 \partial_i F_{i,j})_{j=1}^3$, initial value u_0 and constant viscosity $\nu > 0$. First we recall the definition of weak and strong solutions. We introduce the space of test functions by

$$C_0^\infty([0,T[; C_{0,\sigma}^\infty(\Omega)) := \{\, w|_{[0,T[\times\Omega} \;; w \in C_0^\infty(]-1,T[\times\Omega) \;; \operatorname{div} w = 0 \,\}.$$

Date: September 19, 2013.

2000 *Mathematics Subject Classification.* Primary: 76D05; Secondary: 35Q30, 35B65.

Key words and phrases. Instationary Navier-Stokes equations, strong solutions, weak solutions, initial values, Serrin's class, exterior domains.

Definition 1.1. Let $\Omega \subseteq \mathbb{R}^3$ be an arbitrary domain, let $0 < T \leq \infty$, $\nu > 0$, let $f = \operatorname{div} F$ with $F \in L^1_{\text{loc}}([0,T[; L^2(\Omega))$, and $u_0 \in L^2_\sigma(\Omega)$. Then a vector field $u \in LH_T$, where LH_T denotes the *Leray-Hopf class*

$$LH_T := L^\infty_{\text{loc}}([0,T[; L^2_\sigma(\Omega)) \cap L^2_{\text{loc}}([0,T[; W^{1,2}_{0,\sigma}(\Omega)) , \tag{1.2}$$

is called *weak solution* (in the sense of *Leray-Hopf*) of the instationary Navier-Stokes system (1.1) with data f , u_0 if

$$-\langle u, w_t\rangle_{\Omega,T} + \nu\langle\nabla u, \nabla w\rangle_{\Omega,T} + \langle u\cdot\nabla u, w\rangle_{\Omega,T} = \langle u_0, w(0)\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,T}$$

for all test functions $w \in C^\infty_0([0,T[; C^\infty_{0,\sigma}(\Omega))$. In the identity above $\langle\cdot,\cdot\rangle_\Omega$, $\langle\cdot,\cdot\rangle_{\Omega,T}$ denotes the usual L^2-scalar product in Ω and in $]0,T[\times\Omega$, respectively.

If u is such a weak solution, it can be proved (see [18, Chapter V, Theorem 1.7.1]) that there exists a distribution p, called an associated pressure, such that the equality

$$u_t - \nu\Delta u + u\cdot\nabla u + \nabla p = f$$

holds in the sense of distributions in $]0,T[\times\Omega$, see [18, V.1.7]. The existence of a global weak solution satisfying the energy inequality (1.4) below is well known, see [18, Chapter V, Theorem 3.1.1]. For exponents $1 < s , q < \infty$ we introduce the *Serrin number* by

$$\mathcal{S}(s,q) := \frac{2}{s} + \frac{3}{q}.$$

A weak solution of (1.1) is called a *strong solution* if there exist *Serrin exponents* s, q, i.e. exponents $1 < s, q < \infty$ with $\mathcal{S}(s,q) = 1$ such that additionally *Serrin's condition*

$$u \in L^s(0,T; L^q(\Omega)) \tag{1.3}$$

is satisfied. By Serrin's uniqueness Theorem [18, V, Theorem 1.5.1] a weak solution with (1.3) is unique within the class of weak solutions satisfying the energy inequality, i.e. fulfilling

$$\frac{1}{2}\|u(t)\|_2^2 + \nu\int_0^t \|\nabla u\|_2^2\, d\tau \leq \frac{1}{2}\|u_0\|_2^2 - \int_0^t \langle F, \nabla u\rangle_\Omega\, d\tau \tag{1.4}$$

for almost all $t \in [0,T[$. For sufficiently smooth Ω , f , u_0 a strong solution u has the regularity property

$$u \in C^\infty(]0,T[\times\overline{\Omega}) , \quad p \in C^\infty(]0,T[\times\overline{\Omega}) ,$$

see [18, Theorem V.1.8.2].

Up to now, the existence of a strong solution u of (1.1) is only known in a sufficiently small interval $[0,T[$, $0 < T \leq \infty$, and under additional assumptions on Ω, f, and u_0. Since the work [16] there have been found several conditions on u_0 in order to obtain the existence of a strong solution $u \in L^s(0,T; L^q(\Omega))$ of (1.1) in some interval $[0,T[$, $0 < T \leq \infty$, getting weaker step by step, see [1, 9, 10, 13, 15, 17, 18, 19].

Let $A = A_2$ denote the Stokes operator; for more information we refer to the preliminaries. In [7] Farwig et al. considered (1.1) in a smooth bounded domain and proved that (1.10) below is the optimal (weakest possible) condition to obtain a strong solution $u \in L^s(0,T;L^q(\Omega))$ in some interval $[0,T[\,, 0 < T \leq \infty$, for all exponents $1 < s,q < \infty$ with $\mathcal{S}(s,q) = 1$. In [5] the author proved that the condition (1.10) is sufficient for the existence of a strong solution $u \in L^s(0,T;L^q(\Omega))\,, 0 < T \leq \infty$ of (1.1) in a smooth exterior domain where s,q are exponents with $\mathcal{S}(s,q) = 1$ and additionally $3 < q \leq 8$. The corresponding necessity is proven even without restriction on q. The goal of the present paper is to show that this condition (1.10) is sufficient for the existence of such a strong solution in a smooth exterior domain for the 'remaining' exponents s,q, i.e. for $8 < q < \infty\,, \frac{2}{s} + \frac{3}{q} = 1$. Our first main result reads as follows.

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $1 < s < \infty\,, 8 < q < \infty$, be given with $\mathcal{S}(s,q) = 1$. Assume $0 < T \leq \infty\,, \nu > 0$, assume $F \in L^2(0,T;L^2(\Omega))$, and $u_0 \in L^2_\sigma(\Omega)$. Let*

$$E(t) := e^{-\nu t A}u_0 + \int_0^t A^{1/2}e^{-\nu(t-\tau)A}A^{-1/2}P\mathrm{div}F(\tau)\,d\tau\,, \quad a.a.\ t \in [0,T[\,,$$

denote the weak solution of the (linear) Stokes system with initial value u_0 and external force $f = \mathrm{div}F$. Then there exists a constant $\epsilon_* = \epsilon_*(\Omega,q) > 0$ (independent of T,ν,F, and u_0) with the following property: If $E \in L^s(0,T;L^q(\Omega))$ and

$$\|E\|_{2,2;T} + \|E\|_{q,s;T} \leq \epsilon_* \nu^{\frac{1}{2}+\frac{3}{2q}} \tag{1.5}$$

holds, then there exists a strong solution $u \in L^s(0,T;L^q(\Omega))$ of the Navier-Stokes equations (1.1). After a possible redefinition on a set of Lebesgue measure 0, we get that $u : [0,T[\to L^2_\sigma(\Omega)$ is strongly continuous and satisfies the energy equality

$$\frac{1}{2}\|u(t)\|_2^2 + \nu\int_0^t \|\nabla u\|_2^2\,d\tau = \frac{1}{2}\|u_0\|_2^2 - \int_0^t \langle F, \nabla u\rangle_\Omega\,d\tau \tag{1.6}$$

for all $t \in [0,T[$.

For a proof of this theorem we refer to Section 4. The idea is to construct u as a fixed point in $L^s(0,T;L^q_\sigma(\Omega)) \cap L^2(0,T;L^2_\sigma(\Omega))$ of a suitable non-linear problem, see (3.13) below. The estimates needed to solve this fixed point equation with the help of Banach's fixed point theorem are presented in Lemma 3.2. Afterwards we use the Yosida approximation to show that u satisfies (1.2) and is therefore a strong solution of (1.1).

The corollary below presents a smallness conditions on the data u_0, F which imply the existence of a strong solution $u \in L^s(0,T;L^q(\Omega))$ of (1.1). Especially it follows that the condition (1.10) is sufficient for the existence of a strong solution $u \in L^s(0,T';L^q(\Omega))$ in a sufficiently small interval $[0,T'[\,,0<T'\le T$, for exponents s,q with $q>8$ and $\mathcal{S}(s,q)=1$.

Corollary 1.3. *Let $\Omega\subseteq\mathbb{R}^3$ be an exterior domain with $\partial\Omega\in C^{2,1}$, consider $1<s<\infty\,,8<q<\infty$, with $\mathcal{S}(s,q)=1$, and $1<s_*,q_*<\infty$ with $\mathcal{S}(s_*,q_*)=2$ where $\frac{1}{3}+\frac{1}{q}\ge\frac{1}{q_*}\ge\frac{1}{q}$. Further assume $0<T<\infty$, $\nu>0$, assume $F\in L^{s_*}(0,T;L^{q_*}(\Omega))\cap L^2(0,T;L^2(\Omega))$, and $u_0\in L^2_\sigma(\Omega)$. There exists a constant $\epsilon_*=\epsilon_*(\Omega,q,q_*)>0$ (independent of T,ν,F, and u_0) with the following property: If the conditions*

$$\int_0^T \|e^{-\nu\tau A}u_0\|_q^s\,d\tau\le\epsilon_*\nu^{s-1}\,, \tag{1.7}$$

$$\int_0^T \|e^{-\nu\tau A}u_0\|_2^2\,d\tau\le\epsilon_*\nu^{2-\frac{2}{s}}\,, \tag{1.8}$$

$$\nu^{-1-\frac{3}{2q_*}}\|F\|_{q_*,s_*;T}+\nu^{-1-\frac{3}{2q}}T^{\frac{1}{2}}\|F\|_{2,2;T}\le\epsilon_*\,, \tag{1.9}$$

are satisfied, then there exists a strong solution $u\in L^s(0,T;L^q(\Omega))$ of the Navier-Stokes equations (1.1). After a possible redefinition on a set of Lebesgue measure 0, we get that $u:[0,T[\to L^2_\sigma(\Omega)$ is strongly continuous and fulfils (1.6) for all $t\in[0,T[$.

A proof of this result can be found in Section 5. It follows from [14, Theorem 1.2 (ii)] that $e^{-\nu\tau A}u_0\in L^q(\Omega)$ for almost all $\tau>0$ and consequently the left hand side of (1.7) is well defined.

The following theorem is a consequence of Corollary 1.3 and [5, Theorem 1.3 and Corollary 1.4]. It states that the condition (1.10) on $u_0\in L^2_\sigma(\Omega)$ defines the largest possible class to get a strong solution $u\in L^s(0,T;L^q(\Omega))$, $0<T\le\infty$ of (1.1) for all Serrin exponents s,q.

Theorem 1.4. *Let $\Omega\subseteq\mathbb{R}^3$ be an exterior domain with $\partial\Omega\in C^{2,1}$, let $1<s,q<\infty$ with $\mathcal{S}(s,q)=1$. Further, let $1<s_*\,,q_*<\infty$ with $\mathcal{S}(s_*,q_*)=2$ where $\frac{1}{3}+\frac{1}{q}\ge\frac{1}{q_*}\ge\frac{1}{q}$, let $0<T\le\infty\,,\nu>0$, assume $F\in L^{s_*}(0,T;L^{q_*}(\Omega))\cap L^2(0,T;L^2(\Omega))$, and $u_0\in L^2_\sigma(\Omega)$. Then*

$$\int_0^\infty \|e^{-\nu\tau A}u_0\|_q^s\,d\tau<\infty \tag{1.10}$$

is a necessary and sufficient condition for the existence of a strong solution $u\in L^s(0,T';L^q(\Omega))$ of (1.1) in some interval $[0,T'[\,,0<T'\le T$.

After some preliminaries in Section 2 we deal with the construction of a fixed point needed for the proof of Theorem 1.2. In Section 4 we will prove this theorem and the goal of the last section is to show that Corollary 1.3 holds.

2. Preliminaries

Given an open set $\Omega\subseteq\mathbb{R}^n\,,n\in\mathbb{N}$, and $1\le q\le\infty\,,k\in\mathbb{N}$, we need the usual Lebesgue and Sobolev spaces, $L^q(\Omega)$ and $W^{k,q}(\Omega)$ with norm $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and $\|\cdot\|_{W^{k,q}(\Omega)} = \|\cdot\|_{k,q}$, respectively. For two measurable functions f, g with the property $f\cdot g \in L^1(\Omega)$, where $f\cdot g$ means the usual scalar product of vector or matrix fields, we set $\langle f,g\rangle_\Omega := \int_\Omega f(x)\cdot g(x)\,dx$. Note that the same symbol $L^q(\Omega)$ etc. will be used for spaces of scalar-, vector- or matrix-valued functions. Let $C^m(\Omega)\,, m = 0,1,\dots,\infty$, denote the usual space of functions for which all partial derivatives of order $|\alpha| \le m$ ($|\alpha| < \infty$ when $m = \infty$) exist and are continuous. As usual, $C_0^m(\Omega)$ is the set of all functions from $C^m(\Omega)$ with compact support in Ω. Further we introduce $C^\infty_{0,\sigma}(\Omega) := \{\, v \in C_0^\infty(\Omega);\ \operatorname{div} v = 0\,\}$ as the space of smooth solenoidal vector fields. For $1 < q < \infty$ we define the spaces $L^q_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$ and $W^{1,2}_{0,\sigma}(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}}}$. For $1 \le q \le \infty$ let q' be the dual exponent such that $\frac{1}{q} + \frac{1}{q'} = 1$. It is well known that $L^q_\sigma(\Omega)' \cong L^{q'}_\sigma(\Omega)\,, 1 < q < \infty$, using the standard pairing $\langle\cdot,\cdot\rangle_\Omega$.

Given a Banach space X, $1 \le p \le \infty$, and an interval $]0,T[$ we denote by $L^p(0,T;X)$ the space of (equivalence classes of) strongly measurable functions $f :]0,T[\to X$ such that $\|f\|_p := \left(\int_0^T \|f(t)\|_X^p\,dt\right)^{\frac{1}{p}} < \infty$ if $1 \le p < \infty$ and $\|f\|_\infty := \operatorname{ess\,sup}_{t\in]0,T[} \|f(t)\|_X$ if $p = \infty$. Moreover, we define the set of *locally integrable* functions

$$\begin{aligned} L^p_{\mathrm{loc}}([0,T[;X) := \{\, u : [0,T[\to X \text{ strongly measurable}, \\ u \in L^p(0,T';X) \text{ for all } 0 < T' < T\}. \end{aligned}$$

If $X = L^q(\Omega)$, $1 \le q \le \infty$, the norm in $L^p(0,T;L^q(\Omega))$ will be denoted by $\|f\|_{q,p;T}$.

Fix an exterior domain $\Omega \subseteq \mathbb{R}^3$ with $\partial\Omega \in C^{2,1}$. Let $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$, $1 < q < \infty$, be the *Helmholtz projection* with range $\mathcal{R}(P_q) = L^q_\sigma(\Omega)$ and null space $\mathcal{N}(P_q) = \{\nabla p \in L^q(\Omega)\,; p \in L^q_{\mathrm{loc}}(\overline{\Omega})\}$. This operator is *consistent* in the sense that $P_q f = P_r f$ for $f \in L^q(\Omega) \cap L^r(\Omega)$. Furthermore, we get for the adjoint operator $P_q' \cong P_{q'}$ which means that $\langle P_q f, g\rangle_\Omega = \langle f, P_{q'} g\rangle_\Omega$ for all $f \in L^q(\Omega)$, $g \in L^{q'}(\Omega)$. For $1 < q < \infty$ we define the *Stokes operator* by

$$\mathcal{D}(A_q) = L^q_\sigma(\Omega) \cap W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega), \quad A_q u := -P_q \Delta u\ , u \in \mathcal{D}(A_q).$$

The Stokes operator is consistent in the sense that for $1 < q, r < \infty$

$$A_q u = A_r u \quad \forall u \in \mathcal{D}(A_q) \cap \mathcal{D}(A_r). \tag{2.1}$$

Throughout this paper we will write $A = A_2$. It is well known that $-A_q$ generates a uniformly bounded analytic semigroup $\{\, e^{-tA_q} : t \ge 0\,\}$ on $L^q_\sigma(\Omega)$ satisfying the decay estimate

$$\|A_q^\alpha e^{-tA_q}\|_q \le c\, t^{-\alpha}, \quad t > 0\,, \tag{2.2}$$

where $\alpha \ge 0\,, q > 1$, and $c = c(\Omega, q, \alpha) > 0$.

For $\alpha \in [-1,1]$ the fractional power $A_q^\alpha : \mathcal{D}(A_q^\alpha) \to L^q_\sigma(\Omega)$ with dense domain $\mathcal{D}(A_q^\alpha) \subseteq L^q_\sigma(\Omega)$ and dense range $\mathcal{R}(A_q^\alpha) \subseteq L^q_\sigma(\Omega)$ is a well defined, injective, closed operator such that

$$(A_q^\alpha)^{-1} = A_q^{-\alpha}, \quad \mathcal{R}(A_q^\alpha) = \mathcal{D}(A_q^{-\alpha}), \text{ and } \quad (A_q^\alpha)' = A_{q'}^\alpha.$$

In general, $\mathcal{D}(A_q^\alpha)$ will be equipped with the graph norm $\|u\|_{\mathcal{D}(A_q^\alpha)} := \|u\|_q + \|A_q^\alpha u\|_q$ for $u \in \mathcal{D}(A_q^\alpha)$ which makes $\mathcal{D}(A_q^\alpha)$ to a Banach space since A_q^α is closed. There holds $\mathcal{D}(A_q) \subseteq \mathcal{D}(A_q^\alpha) \subseteq \mathcal{D}(A_q^\beta) \subseteq L^q_\sigma(\Omega)$ for $0 \le \beta \le \alpha \le 1$. Furthermore we have $\mathcal{D}(A^{1/2}) = W^{1,2}_{0,\sigma}(\Omega)$ and $\|\nabla u\|_2 = \|A^{1/2}u\|_2$ for all $u \in \mathcal{D}(A^{1/2})$. There holds

$$\|u\|_\gamma \le c\|A_q^\alpha u\|_q \quad \text{where } 0 \le \alpha \le \frac{1}{2}, 1 < q < 3, 2\alpha + \frac{3}{\gamma} = \frac{3}{q}, \tag{2.3}$$

for all $u \in \mathcal{D}(A_q^\alpha)$ with a constant $c = c(\Omega, q, \alpha) > 0$. Concerning further information on the Helmholtz projection and the Stokes operator in exterior domains we refer to [2, 3, 11, 12, 14].

3. Construction of a suitable fixed point

The proof of Theorem 1.2 is essentially based on the construction of a fixed point of (3.13) below. Throughout this paper we will essentially make use of the following lemma.

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $p > \frac{3}{2}$, $F \in L^p(\Omega)$. Choose $r, \sigma \ge 0$ with*

$$2\sigma + \frac{3}{r} = \frac{3}{p}, \quad 0 \le \sigma \le \frac{1}{2}. \tag{3.1}$$

There exists a unique element in $L^r_\sigma(\Omega)$ denoted by $A_r^{-1/2-\sigma}P_r\mathrm{div}F \in L^r_\sigma(\Omega)$ with

$$\langle A_r^{-1/2-\sigma}P_r\mathrm{div}F, A_{r'}^{1/2+\sigma}w\rangle_\Omega = -\langle F, \nabla w\rangle_\Omega \tag{3.2}$$

for all $w \in \mathcal{D}(A_{r'}^{1/2+\sigma})$. There holds

$$\|A_r^{-1/2-\sigma}P_r\mathrm{div}F\|_r \le c\|F\|_p \tag{3.3}$$

with a constant $c = c(\Omega, p, r) > 0$.

Proof. This is [5, Lemma 3.1]. □

The next lemma presents estimates which will be frequently used in Theorem 3.3.

Lemma 3.2. *Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let exponents $2 < s < \infty, 6 < q < \infty$, be given such that $\mathcal{S}(s,q) = 1$. Further assume $0 < T \le \infty$ and $\nu > 0$. Define $\alpha := \frac{1}{2} + \frac{3}{2q}$ and the Banach space*

$$X := L^s(0,T; L^q_\sigma(\Omega)) \cap L^2(0,T; L^2_\sigma(\Omega)), \quad \|u\|_X := \|u\|_{q,s;T} + \|u\|_{2,2;T}.$$

For $u, v \in X$ the expression

$$\mathcal{F}(u,v)(t) := -\int_0^t A_q^{\alpha} e^{-\nu(t-\tau)A_q} A_q^{-\alpha} P_q \mathrm{div}(u \otimes v)(\tau)\, d\tau \tag{3.4}$$

is well defined as element of $L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega)$ for almost all $t \in [0,T[$. There holds that $\mathcal{F} : X \times X \to X$ is a continuous bilinear form satisfying the estimate

$$\|\mathcal{F}(u,v)\|_X \le c\nu^{-\alpha} \|u\|_X \|v\|_X \tag{3.5}$$

for all $u, v \in X$ with a constant $c = c(\Omega, q) > 0$.

***Proof*. Step 1.** We apply [5, Lemma 3.2] with with $r = q, \sigma = 3/2q$, $\beta = \alpha$ and $F := u \otimes v \in L^{\frac{s}{2}}(0,T; L^{\frac{q}{2}}(\Omega))$ to see that $\mathcal{F}(u,v)(t) \in L^q_\sigma(\Omega)$ is well defined for a.a. $t \in [0,T[$. Further, since $\mathcal{S}(s,q) = \mathcal{S}(\frac{s}{2}, \frac{q}{2}) - 1$ we get from [5, (3.12)]

$$\|\mathcal{F}(u,v)\|_{q,s;T} \le c\nu^{-\alpha} \|u \otimes v\|_{\frac{q}{2},\frac{s}{2};T} \le c\nu^{-\alpha} \|u\|_{q,s;T} \|v\|_{q,s;T}. \tag{3.6}$$

Since all exponents in Lemma 3.2 are uniquely determined by q it follows that the constant c in (3.6) depends only on Ω, q.

Step 2. Define $1 < q_2\, , s_2 < \infty$ such that

$$\frac{1}{q_2} = \frac{1}{q} + \frac{1}{2}\,, \quad \frac{1}{s_2} = \frac{1}{s} + \frac{1}{2}. \tag{3.7}$$

From $u, v \in X$ and Hölder's inequality we get $u \otimes v \in L^{s_2}(0,T; L^{q_2}(\Omega))$. Further

$$2 \cdot \frac{3}{2q} + \frac{3}{2} = \frac{3}{q_2}. \tag{3.8}$$

From $q > 6$ it follows $q_2 > \frac{3}{2}$. Therefore, we obtain from (3.3), (3.8) that

$$\|A^{-\alpha} P \mathrm{div}\big(u \otimes v\big)(t)\|_2 \le c(\Omega, q) \|(u \otimes v)(t)\|_{q_2} \tag{3.9}$$

for a.a. $t \in [0,T[$ and consequently $A^{-\alpha} P \mathrm{div}(u \otimes v) \in L^{s_2}(0,T; L^2(\Omega))$. Therefore, the consistence of the Stokes operator (see (2.1)) applied to (3.4) yields

$$\mathcal{F}(u,v)(t) = -\int_0^t A^{\alpha} e^{-\nu(t-\tau)A} A^{-\alpha} P \mathrm{div}\big(u \otimes v\big)(\tau)\, d\tau \tag{3.10}$$

for almost all $t \in [0,T[$. We get from [5, (3.12)] in combination with (3.8), (3.10) that

$$\|\mathcal{F}(u,v)\|_{2,2;T} \le c\nu^{-\alpha} \|u \otimes v\|_{q_2,s_2;T} \le c(\Omega,q)\nu^{-\alpha} \|u\|_{2,2;T} \|v\|_{q,s;T} \tag{3.11}$$

Combining (3.6), (3.11) yields

$$\|\mathcal{F}(u,v)\|_X \le c\nu^{-\alpha} \|u\|_X \|v\|_X$$

for all $u, v \in X$ with a constant $c = c(\Omega, q) > 0$. □

The following fixed point result is needed for the construction of the strong solution u in Theorem 1.2.

Theorem 3.3. *Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let exponents $2 < s < \infty$, $6 < q < \infty$ be given with $\mathcal{S}(s,q) = 1$. Further assume $0 < T \leq \infty$, $\nu > 0$ and $E \in L^s(0,T; L^q_\sigma(\Omega)) \cap L^2(0,T; L^2_\sigma(\Omega))$. Define $\alpha := \frac{1}{2} + \frac{3}{2q}$. Then there exists a constant $\epsilon_* = \epsilon_*(\Omega, q) > 0$ with the following property: If*

$$\|E\|_{q,s;T} + \|E\|_{2,2;T} \leq \epsilon_* \nu^\alpha \tag{3.12}$$

then there exists $u \in L^s(0,T; L^q_\sigma(\Omega)) \cap L^2(0,T; L^2_\sigma(\Omega))$ satisfying

$$u(t) = E(t) - \int_0^t A_q^\alpha e^{-\nu(t-\tau)A_q} A_q^{-\alpha} P_q \mathrm{div}(u \otimes u)(\tau)\, d\tau \tag{3.13}$$

for almost all $t \in [0, T[$ and

$$\|u\|_{q,s;T} + \|u\|_{2,2;T} \leq 2(\|E\|_{q,s;T} + \|E\|_{2,2;T}). \tag{3.14}$$

Proof. With no loss of generality assume $E \neq 0$. Let X, let $\mathcal{F} : X \times X \to X$ be defined as in Lemma 3.2, and let $c = c(\Omega, q) > 0$ be the constant obtained in (3.5). Let R be the smallest positive root of the polynomial $c\nu^{-\alpha}x^2 - x + \|E\|_X$, i.e.

$$R = \frac{1 - \sqrt{1 - 4c\nu^{-\alpha}\|E\|_X}}{2c\nu^{-\alpha}} = \frac{2\|E\|_X}{1 + \sqrt{1 - 4c\nu^{-\alpha}\|E\|_X}}.$$

It is easy to see $\|E\|_X < R < 2\|E\|_X$. We introduce the closed ball $\mathcal{B} := \{u \in X; \|u\|_X \leq R\}$ and define

$$T : X \to X, \quad Tu := E + \mathcal{F}(u,u).$$

Define $\epsilon_* := \frac{1}{8c}$. To finish the proof we show $T(\mathcal{B}) \subseteq \mathcal{B}$ and that T is a strict contraction on $\mathcal{B}$. To begin with, we obtain

$$\|Tu\|_X \leq \|E\|_X + c\nu^{-\alpha}R^2 = R$$

for all $u \in \mathcal{B}$. Thus $T(\mathcal{B}) \subseteq \mathcal{B}$. Further

$$\begin{aligned} \|Tu - Tv\|_X &= \|\mathcal{F}(u, u-v) + \mathcal{F}(u-v, v)\|_X \\ &\leq c\nu^{-\alpha}\big(\|u\|_X + \|v\|_X\big)\|u - v\|_X \\ &\leq 2c\nu^{-\alpha}R\|u - v\|_X \end{aligned}$$

for all $u, v \in \mathcal{B}$. By construction $2c\nu^{-\alpha}R \leq 4c\nu^{-\alpha}\|E\|_X < 1$ and consequently $T : \mathcal{B} \to \mathcal{B}$ is a strict contraction. By Banach's fixed point theorem there exists $u \in \mathcal{B}$ with $T(u) = u$. Especially

$$\|u\|_X \leq R < 2\|E\|_X.$$

□

We remark that u is constructed as a very weak solution of (1.1) with the additional property $u \in L^2(0,T; L^2(\Omega))$. We refer to [1, 4, 6] for the notion of a very weak solution and their properties. This L^2-integrability is needed for the application of the Yosida approximation in the proof of Theorem 1.2. In the case $3 < q < \frac{24}{7}$ it is proved in [5] that a solution in $u \in L^s(0,T; L^q_\sigma(\Omega))$ of (3.13) automatically satisfies

$u \in L^2(0,T;L^2(\Omega))$. The proof is based on imbedding properties which cannot be used in the case $q > 8$.

4. Proof of Theorem 1.2

Step 1. We use [18, IV, Theorems 2.3.1 and 2.4.1] to get $E \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$ and that E is a weak solution to the (linear) Stokes system with external force $f = \mathrm{div} F$ and initial value u_0. Define $\alpha := \frac{1}{2} + \frac{3}{2q}$. Let $\epsilon_* = \epsilon_*(\Omega, q) > 0$ be the constant from Theorem 3.3. It follows that if

$$\|E\|_{q,s;T} + \|E\|_{2,2;T} \le \epsilon_* \nu^\alpha$$

holds then there exists $u \in L^s(0,T;L^q_\sigma(\Omega)) \cap L^2(0,T;L^2_\sigma(\Omega))$ satisfying

$$u(t) = E(t) - \int_0^t A_q^\alpha e^{-\nu(t-\tau)A_q} A_q^{-\alpha} P_q \mathrm{div}(u \otimes u)(\tau)\, d\tau \tag{4.1}$$

for a.a. $t \in [0,T[$ and

$$\|u\|_{q,s;T} \le 2\|E\|_X. \tag{4.2}$$

In the following assume that $u \in L^s(0,T;L^q_\sigma(\Omega)) \cap L^2(0,T;L^2_\sigma(\Omega))$ fulfils (4.1), (4.2). We have to prove, after a possible reduction of ϵ_* (see the discussion following (4.7)), that u satisfies (1.2).

Step 2. By construction $u \otimes u \in L^{s_2}(0,T;L^{q_2}(\Omega))$ where $1 < s_2, q_2 < \infty$ are defined by

$$\frac{1}{s_2} = \frac{1}{2} + \frac{1}{s}\,, \quad \frac{1}{q_2} = \frac{1}{2} + \frac{1}{q}.$$

Define $\tilde{u} := u - E$. Thus

$$\tilde{u}(t) = -\int_0^t A_q^\alpha e^{-\nu(t-\tau)A_q} A_q^{-\alpha} P_q \mathrm{div}((\tilde{u} + E) \otimes (\tilde{u} + E))(\tau)\, d\tau \tag{4.3}$$

for almost all $t \in [0,T[$. We apply [5, (3.11)] with $r_1 = q/2\,, r_2 = q$ and $F := u \otimes u \in L^{\frac{s}{2}}(0,T;L^{\frac{q}{2}}(\Omega))$ to (4.3) and afterwards, we use the consistence of the Stokes operator in combination with $F \in L^{s_2}(0,T;L^{q_2}(\Omega))$ and obtain

$$\begin{aligned}\tilde{u}(t) &= -\int_0^t A_{q/2}^{1/2} e^{-\nu(t-\tau)A_{q/2}} A_{q/2}^{-1/2} P_{q/2} \mathrm{div}\big((\tilde{u} + E) \otimes (\tilde{u} + E)\big)(\tau)\, d\tau \\ &= -\int_0^t A_{q_2}^{1/2} e^{-\nu(t-\tau)A_{q_2}} A_{q_2}^{-1/2} P_{q_2} \mathrm{div}\big((\tilde{u} + E) \otimes (\tilde{u} + E)\big)(\tau)\, d\tau\end{aligned} \tag{4.4}$$

for a.a. $t \in [0,T[$.

To prove $\nabla u \in L^2(0,T;L^2(\Omega))$ we will slightly modify the arguments presented in [5] respectively [7]. Let $J_n := (I + \frac{1}{n} A_{q_2}^{1/2})^{-1}\,, n \in \mathbb{N}$, be the

Yosida approximation of I in $L^{q_2}_\sigma(\Omega)$, so that $\tilde{u} = J_n\tilde{u} + \frac{1}{n}A^{1/2}_{q_2}J_n\tilde{u}$. For further properties of J_n we refer to [18, II, Section 3.4]. There holds

$$\begin{aligned}&A^{1/2}_{q_2}J_nA^{-1/2}_{q_2}P_{q_2}\mathrm{div}\big(u\otimes(\tilde{u}+E)\big)\\&= J_nP_{q_2}\big(u\cdot\nabla J_n\tilde{u} + u\cdot\nabla E\big) + \frac{1}{n}A^{1/2}_{q_2}J_nA^{-1/2}_{q_2}P_{q_2}\mathrm{div}\big(u\otimes A^{1/2}_{q_2}J_n\tilde{u}\big).\end{aligned}$$

Consequently, we apply $A_{q_2}J_n$ to (4.4) and obtain

$$\begin{aligned}&A^{1/2}_{q_2}J_n\tilde{u}(t)\\&\quad= -\int_0^t A^{1/2}_{q_2}e^{-\nu(t-\tau)A_{q_2}}\Big(J_nP_{q_2}\big(u\cdot\nabla J_n\tilde{u}+u\cdot\nabla E\big)\\&\qquad+\frac{1}{n}A^{1/2}_{q_2}J_nA^{-1/2}_{q_2}P_{q_2}\mathrm{div}\big(u\otimes A^{1/2}_{q_2}J_n\tilde{u}\big)\Big)\,d\tau\end{aligned}\tag{4.5}$$

for almost all $t\in[0,T[$. We use $2\cdot\frac{3}{2q}+\frac{3}{2}=\frac{3}{q_2}, 1<q_2<3$, in combination with (2.2), (2.3) and the boundedness of the sequences $(\frac{1}{n}A^{1/2}_{q_2}J_n)_{n\in\mathbb{N}}$, $(J_nP_{q_2})_{n\in\mathbb{N}}$ to obtain the estimate

$$\begin{aligned}\|A^{1/2}_{q_2}J_n\tilde{u}(t)\|_2 &\le c\|A^{3/2q}_{q_2}A^{1/2}_{q_2}J_n\tilde{u}\|_{q_2}\\&\le c\nu^{-\alpha}\int_0^t|t-\tau|^{-\alpha}\Big(\|J_nP_{q_2}\big(u\cdot\nabla J_n\tilde{u}+u\cdot\nabla E\big)\|_{q_2}\\&\quad+\big\|\frac{1}{n}A^{1/2}_{q_2}J_nA^{-1/2}_{q_2}P_{q_2}\mathrm{div}\big(u\otimes A^{1/2}_{q_2}J_n\tilde{u}\big)\big\|_{q_2}\Big)d\tau\\&\le c\nu^{-\alpha}\int_0^T|t-\tau|^{-\alpha}\Big(\|u\cdot\nabla J_n\tilde{u}\|_{q_2}+\|u\cdot\nabla E\|_{q_2}\\&\quad+\|u\otimes A^{1/2}_{q_2}J_n\tilde{u}\|_{q_2}\Big)\,d\tau\end{aligned}\tag{4.6}$$

with a constant $c=c(\Omega,q)>0$. The Hardy-Littlewood inequality (see [20, Ch. V, 1.2]) with $(1-\alpha)+\frac{1}{2}=\frac{1}{s_2}$ applied to (4.6) combined with Hölder's inequality and $\|\nabla J_n\tilde{u}\|_{2,2;T}=\|A^{1/2}_{q_2}J_n\tilde{u}\|_{2,2;T}$ yields

$$\begin{aligned}&\|A^{1/2}_{q_2}J_n\tilde{u}\|_{2,2;T}\\&\le c\nu^{-\alpha}\Big(\|u\cdot\nabla J_n\tilde{u}\|_{q_2,s_2;T}+\|u\cdot\nabla E\|_{q_2,s_2;T}+\|u\otimes A^{1/2}_{q_2}J_n\tilde{u}\|_{q_2,s_2;T}\Big)\\&\le c\nu^{-\alpha}\|u\|_{q,s;T}\big(\|\nabla J_n\tilde{u}\|_{2,2;T}+\|\nabla E\|_{2,2;T}+\|A^{1/2}_{q_2}J_n\tilde{u}\|_{2,2;T}\big)\\&\le c_*\nu^{-\alpha}\|u\|_{q,s;T}\big(\|A^{1/2}_{q_2}J_n\tilde{u}\|_{2,2;T}+\|\nabla E\|_{2,2;T}\big)\end{aligned}\tag{4.7}$$

with a fixed constant $c_*=c_*(\Omega,q)>0$. Replacing ϵ_* by $\min\{\epsilon_*,\frac{1}{4c_*}\}$ and using (1.5) yields

$$c_*\nu^{-\alpha}\|u\|_{q,s;T}\le 2c_*\nu^{-\alpha}\|E\|_X\le\frac{1}{2}.\tag{4.8}$$

We can apply the absorption principle to (4.7) and get

$$\|A_{q_2}^{1/2} J_n \tilde{u}\|_{2,2;T} \le c\nu^{-\alpha} \|u\|_{q,s;T} \|\nabla E\|_{2,2;T} \tag{4.9}$$

with a constant $c = c(\Omega, q) > 0$ independent of $n \in \mathbb{N}$. By a functional analytic argument (see [18, II. (3.1.8), (3.1.9)]) in combination with the consistence of the Stokes operator it follows $\tilde{u}(t) \in \mathcal{D}(A^{1/2})$ for a.a. $t \in [0, T[$ and $A^{1/2}\tilde{u} \in L^2(0, T; L^2(\Omega))$. Therefore $\nabla\tilde{u} \in L^2(0, T; L^2(\Omega))$.

Step 3. Since $\nabla u \in L^2(0, T; L^2(\Omega))$ we can write

$$\tilde{u}(t) = -\int_0^t e^{-\nu(t-\tau)A_{q_2}} P_{q_2}\big(u \cdot \nabla u\big)\, d\tau \tag{4.10}$$

for a.a. $t \in [0, T[$. The same argumentation as in [7, page 640] shows that (4.10) implies $u \otimes u \in L^2(0, T; L^2(\Omega))$. A careful inspection shows that this proof remains true although we consider an exterior domain instead of a bounded domain. Therefore

$$\tilde{u}(t) = -\int_0^t A^{1/2} e^{-\nu(t-\tau)A} A^{-1/2} P\mathrm{div}(u \otimes u)(\tau)\, d\tau$$

for a.a. $t \in [0, T[$. Consequently, $\tilde{u}$ can be considered as a weak solution of the instationary Stokes system with initial value 0 and external force $f = -\mathrm{div}(u \otimes u)$ where $u \otimes u \in L^2(0, T; L^2(\Omega))$. Then linear theory (see [18, IV, Theorems 2.3.1 and 2.4.1]) implies that $\tilde{u}$ satisfies (1.2). Thus $u = \tilde{u} + E$ satisfies (1.2). Altogether u is a strong solution of (1.1).
□

5. Proof of Corollary 1.3

Let $\epsilon_* = \epsilon_*(\Omega, q) > 0$ be the constant obtained in Theorem 1.2. Define

$$E_1(t) + E_2(t) := e^{-\nu t A} u_0 + \int_0^t A^{1/2} e^{-\nu(t-\tau)A} A^{-1/2} P\mathrm{div}F(\tau)\, d\tau$$

for a.a. $t \in [0, T[$. Assumption (1.7) yields $E_1 \in L^s(0, T; L^q(\Omega))$. From [18, IV, Lemma 2.4.2 d)] we get

$$\|E_2\|_{2,2;T} \le T^{1/2} \|E_2\|_{2,\infty;T} \le T^{1/2} \sqrt{\frac{8}{\nu}} \|F\|_{2,2;T}. \tag{5.1}$$

We use (2.1) to obtain

$$E_2(t) = \int_0^t A_{q_*}^{1/2} e^{-\nu(t-\tau)A_{q_*}} A_{q_*}^{-1/2} P_{q_*} \mathrm{div}F(\tau)\, d\tau$$

for almost all $t \in [0, T[$. Choose $0 \le \sigma \le \frac{1}{2}$ with $2\sigma + \frac{3}{q} = \frac{3}{q_*}$. From [5, (3.12)] it follows $E_2 \in L^s(0, T; L^q(\Omega))$ and

$$\|E_2\|_{q,s;T} \le c\nu^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{q_*} - \frac{1}{q})} \|F\|_{q_*,s_*;T}$$

with a constant $c = c(\Omega, q, q_*) > 0$. Altogether

$$\|E\|_{2,2;T} + \|E\|_{q,s;T} \le \|e^{-\nu t A}u_0\|_{2,2;T} + \|e^{-\nu t A}u_0\|_{q,s;T} + c\nu^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q_*}-\frac{1}{q})}\|F\|_{q_*,s_*;T} + cT^{1/2}\nu^{-1/2}\|F\|_{2,2;T} \tag{5.2}$$

with $c = c(\Omega, q, q_*) > 0$. Looking at (5.2) it follows that there exists a constant $K_* = K_*(\Omega, q, q_*) > 0$ such that if the conditions (1.7), (1.8), (1.9) are fulfilled where ϵ_* is replaced by K_*, then (1.5) holds. Consequently, the proof of the corollary is completed. □

Acknowledgment. The author thanks Reinhard Farwig for his kind support.

References

[1] H. Amann: *On the strong solvability of the Navier-Stokes equations.* J. Math. Fluid. Mech. **2** (2000), 16-98

[2] W. Borchers and T. Miyakawa: *Algebraic L^2-decay for Navier-Stokes flow in exterior domains.* Acta Math. **165** (1990), 189-227

[3] W. Borchers and H. Sohr: *On the semigroup of the Stokes operator for exterior domains.* Math. Z. **196** (1987), 415-425

[4] R. Farwig and C. Komo: *Regularity of weak solutions to the Navier-Stokes equations in exterior domains.* NoDEA Nonlinear Differential Equations Appl. **17** (2010), 303-321

[5] R. Farwig and C. Komo: *Optimal initial value conditions for local strong solutions of the Navier-Stokes equations in exterior domains.* Analysis (Munich) **33** (2013), 101-119

[6] R. Farwig, H. Kozono and H. Sohr: *Very weak solutions of the Navier-Stokes equations in exterior domains with nonhomogeneous data.* J. Math. Soc. Japan **59** (2007), 127-150

[7] R. Farwig, H. Sohr and W. Varnhorn: *On optimal initial value conditions for local strong solutions of the Navier-Stokes equations.* Ann. Univ. Ferrara **55** (2009), 89-110

[8] R. Farwig, H. Sohr and W. Varnhorn. *Necessary and sufficient conditions on local strong solvability of the Navier-Stokes system.* Appl. Anal. **90** (2011), 47-58

[9] H. Fujita and T. Kato: *On the Navier-Stokes initial value problem.* Arch. Rational Mech. Anal. **16** (1964), 269-315

[10] Y. Giga: *Solution for semilinear parabolic equations in L^p and regularity of weak solutions for the Navier-Stokes system.* J. Differential Equations **61** (1986), 186-212

[11] Y. Giga and H. Sohr: *On the Stokes operator in exterior domains.* J. Fac. Sci. Univ. Tokyo, Sec. IA **36** (1989), 103-130

[12] Y. Giga and H. Sohr: *Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains.* J. Funct. Anal. **102** (1991), 72-94

[13] J. Heywood: *The Navier-Stokes equations: on the existence, regularity and decay of solutions.* Indiana Univ. Math. J. **29** (1980), 639-681

[14] H. Iwashita: *$L_q - L_r$ estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L_q spaces.* Math. Ann. **285** (1989), 265-288

[15] T. Kato: *Strong L^p solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions*. Math. Z. **187** (1984), 471-480

[16] A. Kiselev and O. Ladyzenskaya. *On the existence and uniqueness of solutions of the non-stationary problems for flows of non-compressible fluids* Amer. Math. Soc. Transl. Ser. 2, **24** (1963), 79-106

[17] T. Miyakawa: *On the initial value problem for the Navier-Stokes equations in L^p-spaces*. Hiroshima Math. J. **11** (1981), 9-20

[18] H. Sohr: *The Navier-Stokes Equations: An elementary functional analytic approach*. Birkhäuser Verlag, Basel, 2001

[19] V. Solonnikov: *Estimates for solutions of nonstationary Navier-Stokes equations*. J. Soviet Math. **8** (1977), 467-529

[20] E. Stein: *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, 1970

Christian Komo, Darmstadt University of Technology, Department of Mathematics, 64283 Darmstadt, Germany

E-mail address: komo@mathematik.tu-darmstadt.de