Besov space regularity conditions for weak solutions of the Navier-Stokes equations

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Abstract

Consider a bounded domain $\Omega \subseteq \mathbb{R}^3$ with smooth boundary, some initial value $u_0 \in L^2_\sigma(\Omega)$, and a weak solution u of the Navier-Stokes system in $[0,T) \times \Omega$, $0 < T \le \infty$. Our aim is to develop regularity and uniqueness conditions for u which are based on the Besov space

$$B^{q,s}(\Omega) := \left\{ v \in L^2_\sigma(\Omega);\ \|v\|_{B^{q,s}(\Omega)} := \left(\int_0^\infty \left\| e^{-\tau A} v \right\|_q^s \mathrm{d}\tau \right)^{1/s} < \infty \right\}$$

with $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$; here A denotes the Stokes operator. This space, introduced by the authors in [4], [5], is a subspace of the well known Besov space $\mathbb{B}^{-2/s}_{q,s}(\Omega)$, see [1]. Our main results on the regularity of u exploits a variant of the space $B^{q,s}(\Omega)$ in which the integral in time has to be considered only on finite intervals $(0,\delta)$ with $\delta \to 0$. Further we discuss several criteria for uniqueness and local right-hand regularity, in particular, if u satisfies Serrin's limit condition $u \in L^\infty_{\mathrm{loc}}([0,T); L^3_\sigma(\Omega))$. Finally, we obtain a large class of regular weak solutions u defined by a smallness condition $\|u_0\|_{B^{q,s}(\Omega)} \le K$ with some constant $K = K(\Omega, q) > 0$.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$ and $0 < T \leq \infty$. In $[0,T) \times \Omega$ we consider weak and strong solutions of the Navier-Stokes system

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = 0,\ \operatorname{div} u = 0 \qquad (1.1)$$
$$u|_{\partial\Omega} = 0,\ u|_{t=0} = u_0$$

where $u_0 \in L^2_\sigma(\Omega)$ is given and p means the associated pressure. For simplicity, the external force is assumed to vanish.

We set $L^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$, $C^\infty_{0,\sigma}(\Omega) = \{v = (v_1, v_2, v_3) \in C^\infty_0(\Omega);\ \operatorname{div} v = \nabla \cdot v = 0\}$, $1 < q < \infty$.

Definition 1.1 *Let $u_0 \in L^2_\sigma(\Omega)$. Then a vector field*

$$u = (u_1, u_2, u_3) \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; W^{1,2}_0(\Omega)) \qquad (1.2)$$

is called a (Leray-Hopf type) weak solution of the system (1.1), *if the relation*

$$-\langle u, w_t \rangle_{\Omega,T} + \langle \nabla u, \nabla w \rangle_{\Omega,T} - \langle uu, \nabla w \rangle_{\Omega,T} = \langle u_0, w(0) \rangle_\Omega \qquad (1.3)$$

holds for each $w \in C^\infty_0([0,T); C^\infty_{0,\sigma}(\Omega))$, and if the strong energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^t \|\nabla u\|_2^2 \,\mathrm{d}\tau \leq \frac{1}{2}\|u(t_0)\|_2^2 \qquad (1.4)$$

holds for almost all $t_0 \in [0,T)$ including $t_0 = 0$, and all $t \in [t_0, T)$.

Usually, a weak solution u of the system (1.1) is defined with energy inequality (1.4) only for $t_0 = 0$, $u(t_0) = u(0) = u_0$. However, for a bounded domain, classical existence proofs of weak solutions rest on approximation procedures yielding smooth approximate solutions $(u_k)_{k\in\mathbb{N}}$, such that $u_k \to u$, $k \to \infty$, in a certain sense, and that even the strong energy inequality (1.4) holds for almost all $t_0 \in [0,T)$, see [15, V. Theorem 3.6.2].

Moreover, we may assume that, after a redefinition on a null set of $[0,T)$, such a weak solution

$$u : [0,T) \to L^2_\sigma(\Omega) \text{ is weakly continuous.} \qquad (1.5)$$

Thus the initial condition $u|_{t=0} = u(0) = u_0$ is well defined, see [15, V], [17].

The regularity results in this paper are based on the idea to identify u locally at least for almost all $t_0 \in [0,T)$ with a regular Serrin solution in some interval $[t_0, t_0 + \delta) \subseteq [0,T)$, $\delta > 0$. For this purpose we first develop such local Serrin solutions for initial values $u(t_0)$, see Proposition 3.6 below. For this local identification we need that the given weak solution u satisfies the strong energy inequality (1.4).

A weak solution u in Definition 1.1 is called a *strong solution* of (1.1) if Serrin's condition

$$u \in L^s_{\mathrm{loc}}([0,T); L^q(\Omega)),\ 2 < s < \infty,\ 3 < q < \infty,\ \frac{2}{s} + \frac{3}{q} = 1 \tag{1.6}$$

is satisfied. It is well known, see e.g. [15, V. Theorem 1.8.2], that a strong solution u is regular (of class C^∞) in $(0,T)\times\Omega$ and uniquely determined by $u_0 \in L^2_\sigma(\Omega)$.

To present our main results in Section 2 below we will introduce some notation; further notation and preliminaries will be described in Section 3 below. We will use standard notation for Lebesgue and Sobolev spaces as well as for their solenoidal subspaces of vector fields.

We also discuss several regularity and uniqueness conditions for u based on Lorentz spaces. We need the usual Lorentz spaces $L^{r,\gamma}(\Omega)$ with $2 \le r \le \gamma \le \infty$, see [1, (0.16), (0.17)], [18, 1.18.6]. Further we need the solenoidal Lorentz spaces $L^{r,\gamma}_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}}^{\|\cdot\|_{L^{r,\gamma}(\Omega)}}$, see [1, (0.16), (0.17)], [5, Lemma 3.2]. In Proposition 3.3 below further properties of these spaces will be mentioned.

On $L^q_\sigma(\Omega)$, $1 < q < \infty$, let $A = -P\Delta$, $A = A_q$, denote the densely defined Stokes operator which generates an analytic semigroup $\{e^{-tA}; t \ge 0\}$; here $P = P_q$ denotes the Helmholtz projection from $L^q(\Omega)$ onto $L^q_\sigma(\Omega)$.

Let $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$. Then we need the normed space

$$B^{q,s}(\Omega) := \left\{ v \in L^2_\sigma(\Omega);\ \|v\|_{B^{q,s}(\Omega)} := \left(\int_0^\infty \left\| e^{-\tau A} v \right\|_q^s \mathrm{d}\tau \right)^{1/s} < \infty \right\}$$

with norm $\|v\|_{B^{q,s}(\Omega)}$. Equipped with the norm

$$\|v\|_{\widehat{B}^{q,s}(\Omega)} := \|v\|_{B^{q,s}(\Omega)} + \|v\|_2$$

$\widehat{B}^{q,s}(\Omega) = B^{q,s}(\Omega)$ is a well defined Banach space of Besov space type, see Proposition 3.1 below. Moreover, for $\delta > 0$ we consider the normed space

$$B^{q,s}_\delta(\Omega) := \left\{ v \in L^2_\sigma(\Omega);\ \|v\|_{B^{q,s}_\delta(\Omega)} := \left(\int_0^\delta \left\| e^{-\tau A} v \right\|_q^s \mathrm{d}\tau \right)^{1/s} < \infty \right\}$$

as well as the Banach space

$$\widehat{B}^{q,s}_\delta(\Omega) := B^{q,s}_\delta(\Omega) \ \text{ with } \ \|v\|_{\widehat{B}^{q,s}_\delta(\Omega)} := \|v\|_{B^{q,s}_\delta(\Omega)} + \|v\|_2.$$

Finally, we mention the continuous embeddings

$$\begin{aligned} &D(A_2^{1/4}) \subseteq L^3_\sigma(\Omega) = L^{3,3}_\sigma(\Omega) \subseteq L^{3,s}_\sigma(\Omega) \subseteq L^{3,\infty}_\sigma(\Omega), \\ &L^{3,s}_\sigma(\Omega) \subseteq \widehat{B}^{q,s}(\Omega) = B^{q,s}(\Omega) \text{ with } 3 < q \le s < \infty,\ \frac{2}{s} + \frac{3}{q} = 1, \end{aligned} \tag{1.7}$$

see (3.19), (3.20), (3.21).

2 Main results on regularity and uniqueness

2.1 Regularity of weak solutions

The following results are sufficient for weak solutions to (1.1) to satisfy Serrin's condition (1.6), i.e. $u \in L^s_{\text{loc}}([0,T); L^q(\Omega))$.

Theorem 2.1 *Let u be a weak solution of the Navier-Stokes system* (1.1) *as in Definition* 1.1, *and let $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$. Then the following properties are sufficient for Serrin's condition* (1.6)*:*

$$u \in L^\infty_{\text{loc}}([0,T); B^{q,s}(\Omega)), \tag{2.1}$$

$$\lim_{t \uparrow t_1} \left(\int_0^{t_1 - t} \|e^{-\tau A} u(t)\|_q^s \, \mathrm{d}\tau \right)^{1/s} = 0 \quad \textit{for each} \;\; t_1 \in (0,T). \tag{2.2}$$

Remarks a) Using Proposition 3.2 below we conclude from (2.1) that $u(t) \in B^{q,s}(\Omega)$ is well defined for each $0 \le t < T$, and that

$$\|u\|_{L^\infty(0,S;B^{q,s}(\Omega))} = \sup_{0 \le t < S} \|u(t)\|_{B^{q,s}(\Omega)} \tag{2.3}$$

is satisfied for each $0 < S < T$. Thus (2.2) is well defined.

b) The condition (2.2) can be written in the form

$$\|u(t)\|_{B^{q,s}_{t_1 - t}(\Omega)} = \left(\int_0^{t_1 - t} \|e^{-\tau A} u(t)\|_q^s \, \mathrm{d}\tau \right)^{1/s} \to 0 \quad \text{as} \;\; t \uparrow t_1. \tag{2.4}$$

The next result rests instead of (2.2) on the slightly stronger condition (2.5).

Corollary 2.2 *Let u, s, q be as in Theorem* 2.1. *Suppose the condition* (2.1)*, and assume that for each $S \in (0,T)$ there exists some $S_0 \in [0,S)$ satisfying*

$$\lim_{\delta \downarrow 0} \|u\|_{L^\infty(S_0,S;B^{q,s}_\delta(\Omega))} = 0. \tag{2.5}$$

Then Serrin's condition (1.6) *is satisfied.*

Remarks a) Assume (2.5) holds for each $S \in (0,T)$ with $S_0 = 0$. Then

$$\lim_{\delta \downarrow 0} \|u\|_{L^\infty(0,S;B^{q,s}_\delta(\Omega))} = 0 \;\text{ holds for each }\; S \in (0,T), \tag{2.6}$$

and it follows from Corollary 2.2 that Serrin's condition (1.6) is satisfied

b) Assume instead of (2.5) that for each $S \in (0,T)$ there exists some $S_0 \in [0,S)$ satisfying

$$\lim_{\delta \downarrow 0} \|u(t)\|_{B^{q,s}_\delta(\Omega)} = 0 \;\text{ uniformly for a.a. }\; t \in (S_0, S).$$

Then Serrin's condition (1.6) is satisfied. This result is obvious from (2.5).

c) Using the estimates in the proof of [4, Theorem 1.3] we obtain the following result:

Let u, s, q be as in Theorem 2.1. Then the conditions (2.1) and (2.2) are sufficient and necessary for the validity of Serrin's condition (1.6). (2.7)

The next result shows the validity of Serrin's condition if $u : [0,T) \to B^{q,s}_{\delta}(\Omega)$ is continuous for some $\delta > 0$ or equivalently for all $\delta > 0$ including $\delta = \infty$; note that this equivalence follows from (3.11) below.

Theorem 2.3 *Let u be a weak solution of the Navier-Stokes system* (1.1), *and let* $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$. *If the condition*

$$u \in C([0,T); B^{q,s}_{\delta}(\Omega)) \;\; \textit{holds with any} \;\; 0 < \delta \leq \infty, \tag{2.8}$$

then Serrin's condition (1.6) *is satisfied.*

The following theorem yields the regularity of u if the norm of $u \in L^{\infty}(0,T; B^{q,s}_{\delta}(\Omega))$, $0 < \delta \leq \infty$ is sufficiently small.

Theorem 2.4 *Let u be a weak solution of the Navier-Stokes system* (1.1) *and let* $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$. *Then there exists a (sufficiently small) constant* $K = K(\Omega, q) > 0$ *such that the condition*

$$\|u\|_{L^{\infty}(0,T;B^{q,s}_{\delta}(\Omega))} \leq K \;\; \textit{with some} \;\; 0 < \delta \leq \infty \tag{2.9}$$

implies Serrin's condition (1.6).

The next corollaries are based on the embedding properties (1.7) and are consequences of Theorems 2.3 and 2.4. Note that $s \geq q$ is needed in this case.

Corollary 2.5 *Let u be a weak solution of the Navier-Stokes system* (1.1) *and let* $3 < q \leq s < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$.

(i) Each of the conditions

$$u \in C([0,T); D(A_2^{1/4})) \;\; \textit{or} \;\; u \in C([0,T); L^3_{\sigma}(\Omega)) \; \textit{or} \, u \in C([0,T); L^{3,s}_{\sigma}(\Omega)) \tag{2.10}$$

is sufficient for Serrin's regularity condition (1.6).

(ii) There exists a constant $K = K(\Omega, q) > 0$ such that each of the conditions

$$\|u\|_{L^{\infty}(0,T;D(A_2^{1/4}))} \leq K \; \textit{or} \; \|u\|_{L^{\infty}(0,T;L^3_{\sigma}(\Omega))} \leq K \; \textit{or} \; \|u\|_{L^{\infty}(0,T;L^{3,s}_{\sigma}(\Omega))} \leq K \tag{2.11}$$

implies Serrin's condition (1.6).

2.2 Uniqueness conditions for weak solution

The regularity results in Subsection 2.1 also show that the given weak solution u is uniquely determined for the initial value $u_0 \in L^2_\sigma(\Omega)$. However, if we are only interested in the uniqueness of u, there are several weaker conditions, see the following results.

Theorem 2.6 *Let u be a weak solution of the Navier-Stokes system* (1.1) *with $u_0 \in L^2_\sigma(\Omega)$, and let $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$. Assume that*

$$u \in L^\infty_{\text{loc}}([0,T); B^{q,s}(\Omega)), \tag{2.12}$$

and that the energy equality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^t \|\nabla u\|_2^2 \,\mathrm{d}\tau = \frac{1}{2}\|u(t_0)\|_2^2 \tag{2.13}$$

holds for each $0 \le t_0 \le t < T$. Then u is uniquely determined by the initial value u_0.

The next lemma yields some sufficient conditions for the energy inequality (1.4) to hold at *every* t_0 and for the energy equality (2.13).

Lemma 2.7 *Let u, u_0 be as in Theorem* 2.6.

(i) If we assume

$$\|u(\cdot)\|_2^2 \in C([0,T)), \tag{2.14}$$

then the energy inequality (1.4) *holds at* every $t_0 \in [0,T)$.

(ii) If either

$$uu \in L^2_{\text{loc}}([0,T); L^2(\Omega)), \tag{2.15}$$

or

$$u \in L^\infty_{\text{loc}}([0,T); L^{3,\infty}_\sigma(\Omega)), \tag{2.16}$$

then the energy equality (2.13) *holds.*

If (2.12) is replaced by (2.17) below, we obtain the following simpler result.

Corollary 2.8 *Let u be a weak solution of the Navier-Stokes system* (1.1) *with initial value $u_0 \in L^2_\sigma(\Omega)$, and let $3 < q \le s < \infty$ be given such that $\frac{2}{s} + \frac{3}{q} = 1$. Assume that one of the following conditions is satisfied:*

$$\begin{aligned} &u \in L^\infty_{\text{loc}}([0,T); D(A_2^{1/4})) \ \ \textit{or} \ \ u \in L^\infty_{\text{loc}}([0,T); L^3_\sigma(\Omega)) \\ &\textit{or} \ \ u \in L^\infty_{\text{loc}}([0,T); L^{3,s}_\sigma(\Omega)). \end{aligned} \tag{2.17}$$

Then u is uniquely determined by the initial value u_0.

Remark Let u be a weak solution of (1.1) as in Corollary 2.8. It is interesting to discuss uniqueness and regularity properties of u if the Serrin condition $u \in L^s_{\text{loc}}([0,T); L^q(\Omega))$, $\frac{2}{s} + \frac{3}{q} = 1$ holds in the limit case $s = \infty, q = 3$. In this case, Corollary 2.8 yields the uniqueness property for u, and from Corollary 2.5 we see that the stronger assumption $u \in C([0,T); L^3_\sigma(\Omega))$ or the smallness condition $\|u\|_{L^\infty(0,T;L^3_\sigma(\Omega))} \leq K$ are sufficient for Serrin's regularity class $u \in L^s_{\text{loc}}([0,T); L^q(\Omega))$ with certain $2 < s < \infty,\ 3 < q < \infty,\ \frac{2}{s} + \frac{3}{q} = 1$. Further, in the limit case $u \in L^\infty_{\text{loc}}([0,T); L^3_\sigma(\Omega))$ we can prove, instead of the complete regularity, the local right-side regularity property of u for each $t \in [0,T)$, see Theorem 2.10 (ii) below. Hence Theorem 2.10 is a slightly weaker result than that in [12] where, on the other hand, in domains with curved boundaries some additional condition on the pressure had to be assumed; we also refer to [13] for domains with a flat boundary.

2.3 Local regularity results

Definition 2.9 *Let u be a weak solution of the Navier-Stokes system* (1.1) *and let* $2 < s < \infty,\ 3 < q < \infty,\ \frac{2}{s} + \frac{3}{q} = 1$.

(i) A point $t \in (0,T)$ is called an $L^s(L^q)$-**regular point** *of u if there exists some interval $(t-\varepsilon, t+\varepsilon) \subseteq (0,T)$ with $\varepsilon > 0$ such that $u \in L^s(t-\varepsilon, t+\varepsilon; L^q(\Omega))$. Moreover, $t = 0$ is called* $L^s(L^q)$-**regular** *for u if there exists some $\varepsilon > 0$ with $(0,\varepsilon) \subseteq (0,T)$ and $u \in L^s(0,\varepsilon; L^q(\Omega))$.*

(ii) A point $t \in [0,T)$ is called **local right-side** $L^s(L^q)$-**regular** *for u if there is some $\varepsilon > 0$, $t < t+\varepsilon < T$, such that $u \in L^s(t, t+\varepsilon; L^q(\Omega))$. By analogy, a point $t \in (0,T)$ is called* **local left-side** $L^s(L^q)$-**regular** *for u if there is some $\varepsilon > 0$, $t - \varepsilon > 0$, such that $u \in L^s(t-\varepsilon, t; L^q(\Omega))$.*

(iii) A point $t \in [0,T)$ is called a **singular point** *for u if it is not regular.*

Note that in Definition 2.9 $\varepsilon = \varepsilon(t) > 0$ may depend on $t \in [0,T)$. We know from [5, Proposition 1.2] that $t = 0$ is regular if and only if $u_0 \in B^{q,s}(\Omega)$.

The following local $L^s(L^q)$-result extends the result in [5, Theorem 4.4], where the energy equality is supposed. We will see that the conditions (2.12), (2.13) in Theorem 2.6 are sufficient to obtain the local right-side $L^s(L^q)$-regularity condition for each $t \in [0,T)$. Since $\varepsilon = \varepsilon(t) > 0$ may depend on $t \in [0,T)$ we do not know whether $u \in L^s_{\text{loc}}([0,T); L^q(\Omega))$ is satisfied. Thus we only obtain a partial regularity result.

Theorem 2.10 *Let u be a weak solution of the Navier-Stokes system* (1.1), *and let* $2 < s < \infty,\ 3 < q < \infty,\ \frac{2}{s} + \frac{3}{q} = 1$.

(i) Assume that $u \in L^\infty_{\text{loc}}([0,T); B^{q,s}(\Omega))$ and that u satisfies the energy inequality (2.13) *(i.e., for each $t_0 \in [0,T)$ and each $t \in [t_0,T)$). Then u satisfies the local right-side $L^s(L^q)$-regularity condition $u \in L^s(t, t+\varepsilon; L^q(\Omega))$ for each $t \in [0,T)$ with $\varepsilon = \varepsilon(t) > 0$, $t + \varepsilon < T$.*

(ii) Assume that either $u \in L^\infty_{\text{loc}}([0,T); D(A_2^{1/4}))$ *or* $u \in L^\infty_{\text{loc}}([0,T); L^3_\sigma(\Omega))$ *or* $u \in L^\infty_{\text{loc}}([0,T); L^{3,s}_\sigma(\Omega))$ *where additionally* $s \geq q$*. Then* u *satisfies the local right-side* $L^s(L^q)$ *regularity condition as in (i).*

The next result shows that the local left-side $L^s(L^q)$-regularity condition for $u \in L^\infty_{\text{loc}}([0,T); B^{q,s}(\Omega))$ even yields the complete regularity of u.

Theorem 2.11 *Let* u *be a weak solution of the Navier-Stokes system* (1.1) *as in Definition* 1.1*, and let* $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$*. Assume that*

$$u \in L^\infty_{\text{loc}}([0,T); B^{q,s}(\Omega)) \tag{2.18}$$

or that

$$u(t) \in B^{q,s}(\Omega) \text{ for each } t \in [0,T). \tag{2.19}$$

Further assume that each $t \in (0,T)$ *is local left-side* $L^s(L^q)$*-regular for* u*. Then* u *satisfies Serrin's condition* (1.6)*.*

It is interesting to note that the corresponding result for local right-side points cannot be proved. We know that (2.19) is a corollary of (2.18), see Proposition 3.2.

Corollary 2.12 *Let* u *be a weak solution of the Navier-Stokes system* (1.1) *with* $u_0 \in L^2_\sigma(\Omega)$*. Then the set* $\mathcal{R} \subseteq [0,T)$ *of regular points of* u *is dense in* $[0,T)$*, and the set* $[0,T) \setminus \mathcal{R}$ *of singular points of* u *is a null set.*

2.4 A large class of regular weak solutions

The following result yields a regular weak solution u of (1.1) if the initial value norm $\|u_0\|_{B^{q,s}(\Omega)}$ is sufficiently small.

Theorem 2.13 *Let* $u_0 \in B^{q,s}(\Omega)$ *with* $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$*. Assume that*

$$\|u_0\|_{B^{q,s}(\Omega)} \leq \varepsilon_* \tag{2.20}$$

with $\varepsilon_* = \varepsilon_*(\Omega, q) > 0$ *as in Proposition* 3.6 *below. Let* u *be a weak solution of the Navier-Stokes system* (1.1) *with initial value* $u_0 \in B^{q,s}(\Omega)$*. Then* u *satisfies Serrin's condition* (1.6) *on* $[0,T)$*.*

3 Preliminaries

In this section $\Omega \subseteq \mathbb{R}^3$ is always a bounded domain with $\partial\Omega \in C^{2,1}$. Moreover, $[0,T)$ with $0 < T \le \infty$ will be a time interval. By $\langle\cdot,\cdot\rangle_\Omega$ we denote the pairing of vector fields in Ω, and $\langle\cdot,\cdot\rangle_{\Omega,T}$ means the corresponding pairing in $[0,T)\times\Omega$. Let $C^\infty_{0,\sigma}(\Omega) = \{v \in C_0^\infty(\Omega);\ \operatorname{div} v = 0\}$ and $L^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$, $1 < q < \infty$, where $\|\cdot\|_q$ means the norm in the Lebesgue space $L^q(\Omega)$. Further, $W^{k,q}(\Omega)$, $k \in \mathbb{N}$, and $W^{k,q}_0(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{k,q}(\Omega)}}$ are the usual Sobolev spaces.

The Bochner space $L^s(0,T;L^q(\Omega))$, $1 < s,q < \infty$, has the norm

$$\|v\|_{L^s(0,T;L^q(\Omega))} = \|v\|_{q,s,T} = \|v\|_{q,s,(0,T)} = \left(\int_0^T \|v\|_q^s \,\mathrm{d}\tau\right)^{1/s},\quad v \in L^s(0,T;L^q(\Omega)),$$

and $L^\infty(0,T;L^q(\Omega))$ has the norm

$$\|v\|_{L^\infty(0,T;L^q(\Omega))} = \inf_{\mathcal{F}\subseteq[0,T),|\mathcal{F}|=0}\big(\sup_{t\in[0,T)\setminus\mathcal{F}} \|v(t)\|_{L^q(\Omega)}\big),$$

where $\mathcal{F} \subseteq [0,T)$ runs through the set of all Lebesgue null subsets of $[0,T)$, i.e. with $|\mathcal{F}| = 0$. A calculation shows that for each $v \in L^\infty(0,T;L^q(\Omega))$ there is a null set $\mathcal{F} \subseteq [0,T)$, $|\mathcal{F}| = 0$, depending on v, such that

$$\|v\|_{L^\infty(0,T;L^q(\Omega))} = \sup_{t\in[0,T)\setminus\mathcal{F}} \|v(t)\|_{L^q(\Omega)}. \tag{3.1}$$

Let $P = P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$, $1 < q < \infty$, denote the Helmholtz projection, and let $A = A_q = -P_q\Delta : D(A_q) \to L^q_\sigma(\Omega)$ be the Stokes operator with domain $D(A_q) = W^{2,q}(\Omega)\cap W^{1,q}_0(\Omega)\cap L^q_\sigma(\Omega)$ and range $R(A_q) = L^q_\sigma(\Omega)$. Then $A_q^\alpha : D(A_q^\alpha) \to L^q_\sigma(\Omega)$, $-1 \le \alpha \le 1$, denotes the fractional power of A_q. It holds $D(A_q) \subseteq D(A_q^\alpha) \subseteq L^q_\sigma(\Omega)$, $R(A_q^\alpha) = L^q_\sigma(\Omega)$ if $0 \le \alpha \le 1$, see [6], [9]. In particular, $D(A_q^{1/2}) = W^{1,q}_0(\Omega)\cap L^q_\sigma(\Omega)$ with equivalent norms $\|A_q^{1/2}v\|_q \approx \|\nabla v\|_q$, $v \in D(A_q^{1/2})$. Moreover, with $C = C(\Omega,q) > 0$ we need the following important embedding estimates:

$$\|v\|_q \le C\|A_\gamma^\alpha v\|_\gamma,\ v \in D(A_\gamma^\alpha),\ 1 < \gamma \le q,\ 2\alpha + \frac{3}{q} = \frac{3}{\gamma},\ 0 \le \alpha \le 1, \tag{3.2}$$

$$\|A^{-\alpha}v\|_q \le C\|v\|_2,\ v \in L^2_\sigma(\Omega),\ 0 < \alpha < \frac{3}{4},\ 2\alpha + \frac{3}{q} = \frac{3}{2}. \tag{3.3}$$

Let $e^{-tA_q} : L^q_\sigma(\Omega) \to L^q_\sigma(\Omega)$, $0 \le t < \infty$, denote the analytic semigroup generated by the Stokes operator A_q. Since Ω is bounded, we obtain with some $\kappa = \kappa(\Omega) > 0$ and $C = C(\Omega,\alpha) > 0$ the estimates

$$\|A^\alpha e^{-tA}v\|_q \le Ct^{-\alpha}e^{-\kappa t}\|v\|_q,\ v \in L^q_\sigma(\Omega),\ 0 \le \alpha \le 1,\ t > 0, \tag{3.4}$$

$$\|e^{-tA}v\|_q \le Ct^{-\alpha}e^{-\kappa t}\|A^{-\alpha}v\|_q \le Ct^{-\alpha}e^{-\kappa t}\|v\|_2,\ v \in L^2_\sigma(\Omega), \tag{3.5}$$

where in (3.5) the parameters α, q are assumed to satisfy the conditions in (3.3), see [4], [5].

For $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$ we define the normed space

$$B^{q,s}(\Omega) := \left\{ v \in L^2_\sigma(\Omega);\ \|v\|_{B^{q,s}(\Omega)} := \left(\int_0^\infty \left\| e^{-\tau A} v \right\|_q^s \mathrm{d}\tau \right)^{1/s} < \infty \right\}. \tag{3.6}$$

Since $q > 2$ and $v \in L^2_\sigma(\Omega)$, we conclude from (3.5) that the function $\tau \mapsto \|e^{-\tau A} v\|_q^s$ is well defined for $\tau \in (0, \infty)$. Thus the condition $v \in B^{q,s}(\Omega)$ is well defined and means that $\tau \mapsto \|e^{-\tau A} v\|_q^s$ is integrable on $[0, \infty)$. The space $B^{q,s}(\Omega)$ with norm $\|v\|_{B^{q,s}(\Omega)}$ is a normed space. Additionally,

$$\widehat{B}^{q,s}(\Omega) := B^{q,s}(\Omega) \ \text{ with } \ \|v\|_{\widehat{B}^{q,s}(\Omega)} := \|v\|_{B^{q,s}(\Omega)} + \|v\|_2, \tag{3.7}$$

is a well defined Banach space, see Proposition 3.1 below.

Let $\mathbb{B}^{2/s}_{q',s'}(\Omega)$, $s' = \frac{s}{s-1}$, $q' = \frac{q}{q-1}$, denote a well known closed solenoidal subspace of the usual Besov space $B^{2/s}_{q',s'}(\Omega)$ introduced in [1], see also [4, (3.1)], [5, Lemma 3.1], and let $\mathbb{B}^{-2/s}_{q,s}(\Omega)$ be its dual space. Using [4], [5], we obtain for $v \in L^2_\sigma(\Omega)$ that $\|v\|_{B^{q,s}(\Omega)} + \|v\|_2$ and $\|v\|_{\mathbb{B}^{-2/s}_{q,s}(\Omega)} + \|v\|_2$ as well as $\|v\|_{\widehat{B}^{q,s}(\Omega)}$ are equivalent. Therefore we obtain that

$$\widehat{B}^{q,s}(\Omega) = B^{q,s}(\Omega) = \mathbb{B}^{-2/s}_{q,s}(\Omega) \cap L^2_\sigma(\Omega) \subseteq \mathbb{B}^{-2/s}_{q,s}(\Omega), \tag{3.8}$$

and we call $\widehat{B}^{q,s}(\Omega)$ a Besov space.

Further we need, for $\delta > 0$, the normed space

$$B^{q,s}_\delta(\Omega) := \left\{ v \in L^2_\sigma(\Omega);\ \|v\|_{B^{q,s}_\delta(\Omega)} := \left(\int_0^\delta \|e^{-\tau A} v\|_q^s \, \mathrm{d}\tau \right)^{1/s} < \infty \right\}. \tag{3.9}$$

Correspondingly, we obtain that

$$\widehat{B}^{q,s}_\delta(\Omega) := B^{q,s}_\delta(\Omega) \ \text{ with } \ \|v\|_{\widehat{B}^{q,s}_\delta(\Omega)} := \|v\|_{B^{q,s}_\delta(\Omega)} + \|v\|_2 \tag{3.10}$$

is a Banach space. Using the estimates (3.2)–(3.5) we easily get for the normed spaces $B^{q,s}(\Omega)$, $B^{q,s}_\delta(\Omega)$ and the Banach spaces $\widehat{B}^{q,s}(\Omega)$, $\widehat{B}^{q,s}_\delta(\Omega)$ the following results:

$$\begin{aligned} \widehat{B}^{q,s}(\Omega) &= \widehat{B}^{q,s}_\delta(\Omega) \text{ with equivalent norms } \|\cdot\|_{\widehat{B}^{q,s}_\delta(\Omega)} \le \|\cdot\|_{\widehat{B}^{q,s}(\Omega)}, \\ B^{q,s}(\Omega) &= B^{q,s}_\delta(\Omega) \text{ with equivalent norms } \|\cdot\|_{B^{q,s}_\delta(\Omega)} \le \|\cdot\|_{B^{q,s}(\Omega)}. \end{aligned} \tag{3.11}$$

Next we will analyze the structure of the spaces $B^{q,s}(\Omega)$, $B^{q,s}_\delta(\Omega)$, etc. Any $v \in B^{q,s}(\Omega)$ can be identified with an element of $L^2_\sigma(\Omega)$ and with the function $\tau \mapsto e^{-\tau A} v$, such that $B^{q,s}(\Omega)$ can be considered as a subspace of $L^s(0, \infty; L^q(\Omega))$.

Due to this point of view we will identify each (continuous linear) functional on $\widehat{B}^{q,s}(\Omega)$ with a pair (F,f) where $F \in L^{s'}(0,\infty; L^{q'}(\Omega))$ and $f \in L^2_\sigma(\Omega)$ in the sense

$$\langle\langle v,(F,f)\rangle\rangle = \int_0^\infty \langle e^{-\tau A}v, F(\tau)\rangle_\Omega \,\mathrm{d}\tau + \langle v,f\rangle_\Omega, \quad v \in \widehat{B}^{q,s}(\Omega). \tag{3.12}$$

Proposition 3.1 *$\widehat{B}^{q,s}(\Omega)$ and $\widehat{B}^{q,s}_\delta(\Omega)$ are reflexive Banach spaces.*

Proof Since $\mathbb{B}^{-2/s}_{q,s}(\Omega)$ and $L^2_\sigma(\Omega)$ are reflexive Banach spaces, the same holds for $\widehat{B}^{q,s}(\Omega) = \mathbb{B}^{-2/s}_{q,s}(\Omega) \cap L^2_\sigma(\Omega)$ as well as for $\widehat{B}^{q,s}_\delta(\Omega)$. ∎

Finally we investigate properties of weak solutions u with values $u(t)$ in $B^{q,s}(\Omega))$, $0<t<T$.

Proposition 3.2 *Let u be a weak solution of the Navier-Stokes system* (1.1)*, and assume that $u \in L^\infty_{\mathrm{loc}}([0,T); B^{q,s}(\Omega))$. Then $u(t) \in B^{q,s}(\Omega)$ is well defined for each $t \in [0,T)$ and it holds*

$$\|u\|_{L^\infty(0,S;B^{q,s}(\Omega))} = \sup_{t\in[0,S)} \|u(t)\|_{B^{q,s}(\Omega)}, \quad 0<S<T. \tag{3.13}$$

The corresponding result holds with $B^{q,s}(\Omega)$ replaced by $B^{q,s}_\delta(\Omega)$, $\delta>0$.

Proof Let $\mathcal{F} \subseteq [0,S)$ be a Lebesgue null set such that $\|u\|_{L^\infty(0,S;B^{q,s}(\Omega)} = \sup_{t\in[0,S)\setminus\mathcal{F}} \|u(t)\|_{B^{q,s}(\Omega))}$, cf. (3.1). Let $t_0 \in [0,S)$ be given. Then we choose a decreasing sequence $(t_j)_{j\in\mathbb{N}} \subset (0,S)\setminus\mathcal{F}$ such that $t_0 = \lim_{j\to\infty} t_j$ satisfying $u(t_j) \in B^{q,s}(\Omega)$ for each $j\in\mathbb{N}$ and $\sup_j \|u(t_j)\|_{B^{q,s}(\Omega)} < \infty$. Due to (1.2) also $(\|u(t_j)\|_2)_{j\in\mathbb{N}}$ is bounded so that $(u(t_j))_{j\in\mathbb{N}}$ is bounded even in $\widehat{B}^{q,s}(\Omega)$. Since $\widehat{B}^{q,s}(\Omega)$ is reflexive, see Proposition 3.1, we obtain some $u_0 \in \widehat{B}^{q,s}(\Omega)$ such that (omitting the notion of subsequences) $u(t_j) \rightharpoonup u_0$ converges weakly to u_0 in $\widehat{B}^{q,s}(\Omega)$. Moreover, in view of the calculation (3.12) it holds

$$\|u_0\|_{B^{q,s}(\Omega)} \le \liminf_{j\to\infty} \|u_j\|_{B^{q,s}(\Omega)} \le \sup_j \|u_j\|_{B^{q,s}(\Omega)} < \infty \tag{3.14}$$

and $u(t_j) \rightharpoonup u_0$ in $L^2_\sigma(\Omega$. Finally, due to (1.5), $u(t_j) \rightharpoonup u(t_0)$ in $L^2_\sigma(\Omega$. Hence $u_0 = u(t_0) \in B^{q,s}(\Omega)$.

Thus we obtain from (3.14) that

$$\|u(t_0)\|_{B^{q,s}(\Omega)} \le \sup_j \|u(t_j)\|_{B^{q,s}(\Omega)} \le \|u\|_{L^\infty(0,S;B^{q,s}(\Omega))}.$$

This shows that $u(t) \in B^{q,s}(\Omega)$ is well defined for each $t\in[0,S)$, and that (3.13) is satisfied. By analogy, we get the proof with $B^{q,s}(\Omega)$ replaced by $B^{q,s}_\delta(\Omega)$. ∎

The next proposition concerns the Lorentz spaces $L^{3,s}_\sigma(\Omega) \subseteq L^{3,\infty}_\sigma(\Omega)$. Note that $\|\cdot\|_{\mathbb{B}^{-2/s}_{q,s}(\Omega)} + \|\cdot\|_2$ means the norm of $\mathbb{B}^{-2/s}_{q,s}(\Omega) \cap L^2_\sigma(\Omega)$.

Proposition 3.3 *Let $2 < s < \infty$, $3 < q < \infty$, $s \geq q$, $\frac{2}{s} + \frac{3}{q} = 1$. Then there hold the following continuous embeddings with constants $C = C(\Omega, s) > 0$.*

$$\begin{aligned}
&a) \quad D(A_2^{1/4}) \subseteq L^3_\sigma(\Omega) \subseteq L^{3,s}_\sigma(\Omega) \subseteq \mathbb{B}^{-2/s}_{q,s}(\Omega), && (3.15)\\
&b) \quad L^3_\sigma(\Omega) = L^{3,3}_\sigma(\Omega),\ L^2_\sigma(\Omega) = L^{2,2}_\sigma(\Omega), && (3.16)\\
&c) \quad L^{3,s}_\sigma(\Omega) \subseteq L^2_\sigma(\Omega), && (3.17)\\
&d) \quad L^{3,s}_\sigma(\Omega) \subseteq \mathbb{B}^{-2/s}_{q,s}(\Omega) \cap L^2_\sigma(\Omega) = \widehat{B}^{q,s}(\Omega), && (3.18)\\
&e) \quad L^{3,s}_\sigma(\Omega) \subseteq L^{3,\infty}_\sigma(\Omega). && (3.19)
\end{aligned}$$

Proof a) The first result in (3.15) is proved by (3.2). The other embeddings are contained in [1, (0.16)] and [5, Lemma 3.2].

b) See [18, 1.18.6, (5a)].

c) The result (3.17) rests on [18, 1.3.3, (e), (f), and 1.18.6, Theorem 2]: Since $L^6(\Omega) \subset L^2(\Omega)$ for the bounded domain Ω, we get that $(L^2(\Omega, L^6(\Omega)_{1/2,s} = L^{3,s}(\Omega) \subset (L^2(\Omega), L^2(\Omega)_{1/2,s} = L^2(\Omega)$ with a continuous embedding. Hence also $L^{3,s}_\sigma(\Omega) \subset L^2_\sigma(\Omega)$ by the density of the space $C^\infty_{0,\sigma}(\Omega)$ in both spaces $L^{3,s}_\sigma(\Omega)$ and $L^2_\sigma(\Omega)$, see Section 1.

d) This result is obtained combining (3.15) with (3.17).

e) [18, 1.3.3 (3)] yields (3.19). ∎

By Proposition 3.3 we get for $3 < q \leq s < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$, the continuous embeddings

$$D(A_2^{1/4}) \subseteq L^3_\sigma(\Omega) \subseteq L^{3,s}_\sigma(\Omega) \subseteq \widehat{B}^{q,s}(\Omega) = B^{q,s}(\Omega) \tag{3.20}$$

with the estimate

$$\|v\|_{B^{q,s}(\Omega)} \leq \|v\|_{\widehat{B}^{q,s}(\Omega)} \leq C\|v\|_{L^{3,s}_\sigma(\Omega)} \leq C\|v\|_{L^3_\sigma(\Omega)} \leq C\|v\|_{D(A_2^{1/4})} \tag{3.21}$$

for all $v \in D(A_2^{1/4})$, where $C = C(\Omega, q) > 0$, see [5, Lemma 3.2], [1, (0.16), (0.17)]. Hence, by (3.20), (3.21), for $0 < T < \infty$, we also obtain the continuous embeddings

$$\begin{aligned} C([0,T]; D(A_2^{1/4})) \subset C([0,T]; L^3_\sigma(\Omega)) \subset C([0,T]; L^{3,s}_\sigma(\Omega)) \\ \subset C([0,T]; B^{q,s}(\Omega)). \end{aligned} \tag{3.22}$$

Next we recall the classical uniqueness result by Serrin and Masuda, see [15, V. Theorem 1.5.1].

Proposition 3.4 *Let $u, \tilde{u}$ be two weak solutions of the Navier-Stokes system* (1.1) *in the sense of Definition 1.1 with the same initial value $u_0 \in L^2_\sigma(\Omega)$, and assume that $u \in L^s_{\text{loc}}([0,T); L^q(\Omega))$ is satisfied with $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$. Then it holds $u = \tilde{u}$ in $[0,T)$.*

Lemma 3.5 *Let u be a weak solution of the Navier-Stokes system* (1.1) *and choose $t_0 \in (0,T)$ such that the energy inequality* (1.4) *is satisfied for $t_0 \le t < T$. Then $\tilde{u}$ defined by*

$$\tilde{u}(t) := u(t+t_0) \quad \textit{for each} \quad t \in [0, T-t_0)$$

is a well defined weak solution of the system (1.1) *with initial value $\tilde{u}_0 = u(t_0)$ and with T replaced by $T - t_0$.*

Proof Let $\widetilde{w} \in C_0^\infty([t_0,T); C_{0,\sigma}^\infty(\Omega))$. Then we use in (1.3) test functions w_j, $j \in \mathbb{N}$, defined as follows: Choose $t_j = t_0\big(1 - \frac{1}{j}\big)$, and set $w_j(t) = 0$ for $0 \le t \le t_j$, $w_j(t) = \frac{t-t_j}{t_0-t_j}\widetilde{w}(t_0)$ for $t_j \le t \le t_0$ and $w_j(t) = \tilde{w}(t)$ for $t_0 \le t < T$. Omitting a smoothing procedure we can use w_j as test functions in (1.3) and let $j \to \infty$. Then a calculation and (1.5) show that $\tilde{u}$ with $\tilde{u}_0 = u(t_0)$ is a well defined weak solution as in Definition 1.1 with u_0 replaced by $\tilde{u}_0$ and with T replaced by $T - t_0$. ∎

The next proposition is a local regularity result from [5, Proposition 1.2, Corollary 1.3], see also [4, Theorems 1.1 and 1.2].

Proposition 3.6 *Let u be a weak solution of the Navier-Stokes system* (1.1) *on $[0,T)$, $0 < T \le \infty$, with $u_0 \in L^2_\sigma(\Omega)$, and let $t_0 \in [0,T)$ be given such that the energy inequality* (1.4) *is valid for t_0 and $t \in [t_0, T)$. Then there is a constant $\varepsilon_* = \varepsilon_*(\Omega, q) > 0$ with the following property: If for $0 < \delta < \infty$ with $t_0 + \delta < T$, and with $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$,*

$$\left(\int_0^\delta \| e^{-\tau A} u(t_0) \|_q^s \, \mathrm{d}\tau \right)^{1/s} \le \varepsilon_*, \tag{3.23}$$

then u satisfies locally the Serrin condition

$$u \in L^s(t_0, t_0 + \delta; L^q(\Omega)). \tag{3.24}$$

Proof In [4, Theorem 1.2], Proposition 3.6 has been shown only for $t_0 = 0$ and is based on Banach's fixed point principle. The same arguments can be used replacing $[0,T)$ by $[t_0,T)$ and $u_0 = u(0)$ by $\tilde{u}_0 = u(t_0)$. Here we need Lemma 3.5 to treat u as a weak solution of (1.1) in the interval $[t_0, T)$. To identify the weak solution on $[t_0, t_0 + \delta)$ with initial value $u(t_0)$ by Proposition 3.4 with the given weak solution u we need that u satisfies the strong energy inequality (1.4) for t_0. The proof of this proposition also follows from [5, Corollary 1.3]. ∎

Now we discuss some sufficient conditions on a weak solution u to satisfy the strong energy (in-)equality for *each* $t_0 \in [0,T)$.

Proposition 3.7 *Let u be a weak solution of the Navier-Stokes system* (1.1) *as in Definition* 1.1, *and assume additionally that*

$$t \longmapsto \|u(t)\|_2^2, \; t \in [0,T), \; \textit{is a continuous function.}$$

Then the strong energy inequality holds at each *$t_0 \in [0,T)$ and each $t \in [t_0, T)$.*

Proof Starting from (1.4) with $t_0 = 0$ and for almost all $t_0 \in [0,T)$, let us consider any $t_0 \in [0,T)$ and choose a strictly decreasing sequence $(t_j)_{j\in\mathbb{N}} \subset (t_0, t)$ with $t_0 = \lim_{j\to\infty} t_j$ such that for all $T > t > t_j$

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_j}^t \|\nabla u\|_2^2 \,\mathrm{d}\tau \le \frac{1}{2}\|u(t_j)\|_2^2.$$

Then we take the limit $j \to \infty$ on both sides and obtain the result. ∎

Proposition 3.8 *Let u be a weak solution of the Navier-Stokes system* (1.1) *as in Definition 1.1.*

(i) Assume that

$$uu \in L^2_{\mathrm{loc}}([0,T); L^2(\Omega)). \tag{3.25}$$

Then, after redefinition on a null set of $[0,T)$, *we obtain the* energy equality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^t \|\nabla u\|_2^2 \,\mathrm{d}\tau = \frac{1}{2}\|u(t_0)\|_2^2 \tag{3.26}$$

for each $t_0 \in [0,T)$ *and each* $t \in [t_0, T)$.

(ii) Assume that one of the following conditions is satisfied: either

$$u \in L^\infty_{\mathrm{loc}}([0,T); L^3_\sigma(\Omega)) \tag{3.27}$$

or

$$u \in L^\infty_{\mathrm{loc}}([0,T); L^{3,\infty}_\sigma(\Omega)) \tag{3.28}$$

or

$$u \in L^\infty_{\mathrm{loc}}([0,T); D(A_2^{1/4})) \tag{3.29}$$

is valid. Then (3.25) *is satisfied, and consequently the energy equality* (3.26) *holds.*

Proof (i) Since $u \cdot \nabla u = \operatorname{div} F$ with $F = uu$, we can write (1.1) as a Stokes system with external force $f = -\operatorname{div} F$. Using $\langle \operatorname{div} u(t)u(t), u(t)\rangle_\Omega = -\frac{1}{2}\langle u(t), \nabla |u(t)|^2\rangle_\Omega = 0$, and (3.25), we see that (3.26) is the well defined energy equality for this Stokes system.

(ii) By Hölder's inequality and (1.2) we obtain for $0 < T' < T$ that $\|uu\|_{2,2,T'} \le C\|u\|_{3,\infty,T'}\|u\|_{6,2,T'} < \infty$. Hence (3.27) yields (3.25).

In the case (3.28) we use Hölder's inequality in Lorentz spaces ([10, Lemma 2.2]), namely

$$\|vw\|_{L^2(\Omega)} \le c\|vw\|_{L^{2,2}(\Omega)} \le c\|v\|_{L^{3,\infty}(\Omega)}\|w\|_{L^{6,2}(\Omega)}, \; c = c(\Omega) > 0,$$

and the Sobolev embedding $W^{1,2}_0(\Omega) \subset L^{6,2}(\Omega)$, see [10, Lemma 2.1 (1)], to get that $\|uu\|_{L^2(\Omega)} \le c\|u\|_{L^{3,\infty}(\Omega)}\|u\|_{W^{1,2}_0(\Omega)}$. Now we may proceed as in the previous case.

Assuming (3.29) we know from (3.21) that $L^\infty(0,T'; D(A_2^{1/4})) \subseteq L^\infty(0,T'; L^3_\sigma(\Omega))$ for $0 < T' < T$. Hence we proceed as for (3.27). ∎

4 Proof of the main results

Proof of Theorem 2.1 First we show that each point $t_1 \in (0,T)$ is local left-side $L^s(L^q)$-regular for u. Let $t_1 \in (0,T)$ and let $\varepsilon^* = \varepsilon^*(\Omega, q) > 0$ be given as in Proposition 3.6. Then, using (2.2), there exists some $0 < \delta_0 < t_1$ such that

$$\left(\int_0^{t_1-t} \|e^{-\tau A}u(t)\|_q^s \,\mathrm{d}\tau\right)^{1/s} \leq \varepsilon^* \tag{4.1}$$

is satisfied for each $t < t_1$ with $t_1 - t \leq \delta_0$. In particular, choose $t = t_0 \in [t_1 - \delta_0, t_1)$ such that also the energy inequality (1.4) is satisfied for t_0. Using Proposition 3.6 we conclude that

$$u \in L^s(t_0, t_1; L^q(\Omega)). \tag{4.2}$$

Thus each $t_1 \in (0,T)$ is a local left-side $L^s(L^q)$-regular point.

In a second step we show that each $t_1 \in [0,T)$ is local right-side $L^s(L^q)$-regular. Let $t_1 \in [0,T)$ be given. In order to apply Proposition 3.6 with $u(t_0)$ replaced by $u(t_1)$ we have to show that the energy inequality (1.4) holds, here in the form

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_1}^t \|\nabla u\|_2^2 \,\mathrm{d}\tau \leq \frac{1}{2}\|u(t_1)\|_2^2, \; t_1 \leq t < T. \tag{4.3}$$

To prove this we first conclude from (4.2) with $t_1 = t_0 + \delta_0$ that $uu \in L^2(t_0, t_1; L^2(\Omega))$. This follows from Hölder's inequality

$$\|uu\|_{2,2,(t_0,t_1)} \leq C\|u\|_{q,s,(t_0,t_1)}\|u\|_{q_1,s_1,(t_0,t_1)}, \; C > 0, \tag{4.4}$$

with $q_1, s_1 > 2$, $\frac{1}{2} = \frac{1}{q} + \frac{1}{q_1}$, $\frac{1}{2} = \frac{1}{s} + \frac{1}{s_1}$, $\frac{2}{s_1} + \frac{3}{q_1} = \frac{3}{2}$, see [15, V. (1.8.11)].

Then we obtain from Proposition 3.8 that the energy equality

$$\frac{1}{2}\|u(t_1)\|_2^2 + \int_{t_0'}^{t_1} \|\nabla u\|_2^2 \,\mathrm{d}\tau = \frac{1}{2}\|u(t_0')\|_2^2 \tag{4.5}$$

is satisfied for each $t_0' \in [t_0, t_1)$. Thus it holds

$$\lim_{t_0' \uparrow t_1} \|u(t_0')\|_2^2 = \|u(t_1)\|_2^2. \tag{4.6}$$

Next we use (1.4) with a sequence $t_j' \in (t_0, t_1)$, $t_j' < t_{j+1}'$, $j \in \mathbb{N}$, $\lim_{j\to\infty} t_j' = t_1$, such that

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_j'}^t \|\nabla u\|_2^2 \,\mathrm{d}\tau \leq \frac{1}{2}\|u(t_j')\|_2^2, \; t_1 \leq t < T \tag{4.7}$$

is satisfied for $j \in \mathbb{N}$. Using (4.6) and the same argument as in the proof of Proposition 3.7, we pass to the limit $j \to \infty$ on both sides of (4.7) and obtain that

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_1}^t \|\nabla u\|_2^2 \,\mathrm{d}\tau \leq \frac{1}{2}\|u(t_1)\|_2^2, \; t_1 \leq t < T \tag{4.8}$$

is satisfied.

Further we observe, using Proposition 3.2, that $u(t) \in B^{q,s}(\Omega)$ is well defined for each $t \in [0, T)$. Thus we can apply Proposition 3.6 for each initial value $u(t_1)$, $t_1 \in [0, T)$. This shows that each $t_1 \in [0, T)$ is local right-side $L^s(L^q)$-regular.

Together with the first step of the proof we conclude that each $t_1 \in [0, T)$ is $L^s(L^q)$-regular. Thus $u \in L^s(0, S; L^q(\Omega))$ for each $S \in (0, T)$ and Serrin's condition (1.6) is satisfied. This completes the proof of Theorem 2.1. ∎

Proof of Corollary 2.2 Assume it holds (2.1) and (2.5). Then we have to show the validity of (2.2). Let $t_1 \in (0, T)$, $0 < t < t_1$, be given. Then we set $S = t_1$ and we choose $S_0 \in [0, S)$ satisfying (2.5). Let $\varepsilon > 0$ be given and choose $0 < \delta_0 < t_1$ such that $\|u\|_{L^\infty(S_0,S;B^{q,s}_\delta(\Omega))} \le \varepsilon$ holds for each $0 < \delta \le \delta_0$. Then for each $0 < t < t_1$ with $t_1 - t \le \delta_0$ we obtain, using the argument as in (3.13), that

$$\|u(t)\|_{B^{q,s}_{t_1-t}(\Omega)} \le \|u(t)\|_{B^{q,s}_{\delta_0}(\Omega)} \le \|u\|_{L^\infty(S_0,S;B^{q,s}_{\delta_0}(\Omega))} \le \varepsilon$$

is satisfied. ∎

Proof of Theorem 2.3 Let (2.8) with $0 < \delta \le \infty$ be given. We show that the conditions (2.1), (2.2) are satisfied. Let $0 < S < T$. Then, using (3.13) and (3.11) we obtain that

$$\|u\|_{L^\infty(0,S;B^{q,s}(\Omega))} \le C\|u\|_{L^\infty(0,S;B^{q,s}_\delta(\Omega))} < \infty$$

with $C > 0$. Thus (2.1) is satisfied.

To prove (2.2) let $t_1 \in (0, T)$ and let $\varepsilon > 0$. Then, using (2.8), we choose $0 < \delta_0 < t_1$, $\delta_0 \le \delta$ such that

$$\|u(t) - u(t_1)\|_{B^{q,s}_{t_1-t}(\Omega)} \le \|u(t) - u(t_1)\|_{B^{q,s}_\delta(\Omega)} \le \frac{\varepsilon}{2},$$
$$\|u(t_1)\|_{B^{q,s}_{t_1-t}(\Omega)} \le \|u(t_1)\|_{B^{q,s}_{\delta_0}(\Omega)} \le \frac{\varepsilon}{2}$$

hold for each $t \in (0, t_1)$ with $t_1 - t \le \delta_0$. Consequently,

$$\begin{aligned}\|u(t)\|_{B^{q,s}_{t_1-t}(\Omega)} &\le \|u(t) - u(t_1)\|_{B^{q,s}_{t_1-t}(\Omega)} + \|u(t_1)\|_{B^{q,s}_{t_1-t}(\Omega)} \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

for each $t \in (0, t_1)$ with $t_1 - t \le \delta_0$. Thus (2.2) is satisfied. ∎

Proof of Theorem 2.4. Choose $K := \varepsilon_* = \varepsilon_*(\Omega, q) > 0$ with ε_* as in Proposition 3.6. Using Proposition 3.2 with $B^{q,s}(\Omega)$ replaced by $B^{q,s}_\delta(\Omega)$, $\delta > 0$, we see that $u(t) \in B^{q,s}_\delta(\Omega)$ is well defined for each $t \in [0, T)$, and that

$$\sup_{t\in[0,T)} \|u(t)\|_{B^{q,s}_\delta(\Omega)} = \|u\|_{L^\infty(0,T;B^{q,s}_\delta(\Omega))} \le \varepsilon_* = K.$$

Hence for each $\delta > 0$ and almost all $t_0 \in [0,T)$ it holds the energy inequality (1.4) and we have $\|u(t_0)\|_{B^{q,s}_\delta(\Omega)} \le \varepsilon_*$. Now Proposition 3.6 implies that $u \in L^s_{\text{loc}}([0,T); L^q(\Omega))$. ∎

Proof of Corollary 2.5 This result follows from Theorem 2.3 and Theorem 2.4, using the embedding properties (1.7). ∎

Proof of Theorem 2.6 Assume that in addition to the weak solution u satisfying (2.12), (2.13) there is another weak solution $\tilde{u}$ for the same initial value u_0. By Proposition 3.6 for u with $t_0 = 0$ we obtain that $u \in L^s(0,\delta; L^q(\Omega))$ with some $0 < \delta < T$. Then Proposition 3.4 implies that $u(t) = \tilde{u}(t)$ holds for $0 \le t < \delta$.

Let $[0,T_1)$, $0 < T_1 \le T$, be the largest half open interval such that $u(t) = \tilde{u}(t)$ is satisfied for each $t \in [0,T_1)$. Assume that $T_1 < T$. Due to Proposition 3.2, $u(T_1) \in B^{q,s}(\Omega)$ is well defined. Choose an increasing sequence $(t_j)_{j\in\mathbb{N}} \in (0,T_1)$ with $\lim\limits_{j\to\infty} t_j = T_1$, and use the weak continuity (1.5) for both u and $\tilde{u}$ at each t_j. Then $u(t_j) = \tilde{u}(t_j)$ converges weakly in $L^2_\sigma(\Omega)$ for $j \to \infty$ to $u(T_1) = \tilde{u}(T_1)$.

Now let $t_0 := T_1$. Obviously, by (2.13), u satisfies the energy inequality at t_0. Using Proposition 3.6 for the initial value $u(t_0) \in B^{q,s}(\Omega) = B^{q,s}_\delta(\Omega)$ with some $\delta > 0$, $t_0 + \delta < T$, such that $\|u(t_0)\|_{B^{q,s}_\delta(\Omega)} \le \varepsilon_*$ is satisfied, we conclude that $u \in L^s(t_0, t_0+\delta; L^q(\Omega))$. Moreover, since $u = \tilde{u}$ on $[0,t_0]$ and since $\tilde{u}$ satisfies the strong energy inequality, the energy equality (2.13) for u on $[0,t_0]$ implies that $\tilde{u}$ satisfies the energy inequality with initial time t_0. Hence Proposition 3.4 implies that $u(t) = \tilde{u}(t)$ in $[t_0, t_0+\delta)$. Thus $t_0 + \delta = T_1 + \delta > T_1$ and $u = \tilde{u}$ in $[0, T_1+\delta)$. This is a contradiction to the construction of T_1. This proves Theorem 2.6. ∎

Proof of Lemma 2.7 (i) If (2.14) holds, then Proposition 3.7 yields the result.

(ii) In the case of (2.15) and (2.16), we use Proposition 3.8 (i) and (ii), respectively, to get the energy equality. ∎

Proof of Corollary 2.8 By the embeddings (3.20) and Lemma 2.7 (ii) we see that the conditions (2.12) and (2.13) are satisfied. Then Theorem 2.6 yields the result. ∎

Proof of Theorem 2.10 (i) Due to Proposition 3.2 we know that $u(t_0) \in B^{q,s}(\Omega) = B^{q,s}_\delta(\Omega)$ for each $t_0 \in [0,T)$ and $\delta > 0$. By (2.13) we obtain from Proposition 3.6 that $u \in L^s(t_0, t_0+\delta; L^q(\Omega))$ for each $t_0 \in [0,T)$ with some $\delta = \delta(t_0) > 0$, $t_0 + \delta < T$. This proves part (i).

(ii) Using Propositions 3.8 and 3.3, (3.19) we conclude from the assumptions that the energy equality (3.20) is satisfied for each $t_0 \in [0,T)$, $t \in [t_0, T)$. Further we obtain from the embedding properties (3.20) that $u \in L^\infty_{\text{loc}}([0,T); B^{q,s}(\Omega))$ holds as in (i). Now the result follows as in (i). ∎

Proof of Theorem 2.11 Recall that (2.18) implies (2.19); thus we only assume (2.19). Assuming the local left-side $L^s(L^q)$-regularity for each $t \in (0,T)$ we can argue as in the proof of Theorem 2.1, part b). This shows that each $t_1 \in [0,T)$ is local right-side $L^s(L^q)$-regular. Thus each $t \in [0,T)$ is regular and we obtain the Serrin condition (1.6). ■

Proof of Corollary 2.12 Let u be the given weak solution. Then $u(t_0) \in L^2_\sigma(\Omega)$ is well defined for each $t_0 \in [0,T)$ and $u(t_0) \in W_0^{1,2}(\Omega) \subset L^3(\Omega)$ holds for almost all $t_0 \in [0,T)$, cf. (1.5) and (1.2). Thus using (3.20) we see that $u(t_0) \in L^3_\sigma(\Omega) \subseteq B^{q,s}(\Omega) = B^{q,s}_\delta(\Omega)$ for a.a. $t_0 \in [0,T)$. Moreover, we may assume that the energy inequality (1.4) is satisfied for t_0 and all $t \in (t_0,T)$. Proposition 3.6 implies for each such t_0 that for the initial value $u(t_0) \in B^{q,s}(\Omega)$ there exists some $\delta(t_0) > 0$ such that the weak solution u is a strong solution in $L^s(t_0, t_0 + \delta(t_0); L^q(\Omega))$. Hence each such interval $(t_0, t_0 + \delta(t_0)) \subseteq [0,T)$ consists of regular points. Since this holds for a.a. $t_0 \in [0,T)$, the union of these intervals is an open dense set of regular points. ■

Proof of Theorem 2.13 Use Proposition 3.6 with $T \leq \infty$, $t_0 = 0$, $u(t_0) = u(0) = u_0 \in B^{q,s}(\Omega)$. Then we obtain from (2.20) that (3.23) and (3.24) hold for each $0 < \delta < T$. Thus Serrin's condition (1.6) is satisfied for $T \leq \infty$. This proves Theorem 2.13. ■

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