Weak Solutions of the Navier-Stokes Equations with Non-zero Boundary Values in an Exterior Domain Satisfying the Strong Energy Inequality

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Abstract. In an exterior domain $\Omega \subset \mathbb{R}^3$ and a time interval $[0, T)$, $0 < T \leq \infty$, consider the instationary Navier-Stokes equations with initial value $u_0 \in L^2_\sigma(\Omega)$ and external force $f = \operatorname{div} F$, $F \in L^2(0, T; L^2(\Omega))$. As is well-known there exists at least one weak solution in the sense of J. Leray and E. Hopf with vanishing boundary values satisfying the strong energy inequality. In this paper, we extend the class of global in time Leray-Hopf weak solutions to the case when $u|_{\partial\Omega} = g$ with non-zero time-dependent boundary values g. Although uniqueness for these solutions cannot be proved, we show the existence of at least one weak solution satisfying the strong energy inequality and a related energy estimate.

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1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with boundary of class $C^{1,1}$, and let $[0, T)$, $0 < T \leq \infty$, be a time interval. In $\Omega \times [0, T)$ we consider the instationary Navier-Stokes system with viscosity $\nu > 0$ and data f, g, u_0 in the form

$$\begin{aligned} u_t - \nu\Delta u + u \cdot \nabla u + \nabla p = f, \quad \operatorname{div} u = 0 \\ u|_{\partial\Omega} = g, \quad u(0) = u_0. \end{aligned} \tag{1.1}$$

For the data we assume the following:

$$
\begin{aligned}
&f = \operatorname{div} F,\ F \in L^2\big(0,T;L^2(\Omega)\big),\ u_0 \in L^2_\sigma(\Omega),\\
&g \in L^4\big(0,T;W^{-\frac{1}{4},4}(\partial\Omega)\big) \cap L^{s_0}\big(0,T;W^{-\frac{1}{q_0},q_0}(\partial\Omega)\big),\\
&\frac{2}{s_0}+\frac{3}{q_0}=1,\ 2<s_0<\infty,\ 3<q_0<\infty;
\end{aligned}
\tag{1.2}
$$

the initial data u_0 has to satisfy further assumptions to be introduced later, see Section 5.

A weak solution u to (1.1) will be constructed in the form $u = v + E$ where E solves an instationary Stokes system with the boundary data g, and v solves a type of Navier-Stokes system with additional perturbation terms related to E but homogeneous Dirichlet data on $\partial\Omega$. Therefore, the problem splits into two almost independent parts, the construction of E as weak (or very weak) solution of a Stokes system and the analysis of a perturbed Navier-Stokes system. It is worth mentioning that the second step needs only very low assumptions on E known from the theory of very weak solutions (E lies in Serrin's class $L^{s_0}(0,T;L^{q_0}(\Omega))$) and from the classical theorem for weak solutions to satisfy the energy identity ($E \in L^4(0,T;L^4(\Omega))$); here the assumptions on g and on u_0 in (1.2) are not explicitly needed.

To be more precise, we have to find first of all a (so-called) very weak solution of the inhomogeneous Stokes system

$$
\begin{aligned}
E_t - \nu\Delta E + \nabla h = f_0, &\quad \operatorname{div} E = 0\\
E\big|_{\partial\Omega} = g, &\quad E(0) = E_0,
\end{aligned}
\tag{1.3}
$$

see [2]–[6] and, for the case of exterior domains, [7], in $\Omega \times [0,T)$ with suitable data $f_0 = \operatorname{div} F_0$ and E_0; here ∇h means the associated pressure. At first sight, it seems to suffice to choose $f_0 = 0$, $F_0 = 0$, but for later application it will be helpful to consider general data f_0, F_0, see Assumption 1.6 to be used in Corollary 1.7 below. Setting

$$
v = u - E,\ \tilde{p} = p - h,\ f_1 = f - f_0,\ v_0 = u_0 - E_0 \tag{1.4}
$$

we write (1.1) as a perturbed Navier-Stokes system with homogeneous boundary data $v\big|_{\partial\Omega} = 0$

$$
\begin{aligned}
v_t - \nu\Delta v + (v+E)\cdot\nabla(v+E) + \nabla\tilde{p} = f_1, &\quad \operatorname{div} v = 0,\\
v\big|_{\partial\Omega} = 0, &\quad v(0) = v_0
\end{aligned}
\tag{1.5}
$$

with the new perturbation terms

$$
(v+E)\cdot\nabla(v+E) = \operatorname{div}\big(v\otimes v + (E\otimes v + v\otimes E) + E\otimes E\big);
$$

here $E \otimes v = (E_i v_j)_{i,j=1,2,3}$ denotes the dyadic product of the vector fields E and v and the divergence is taken columnwise, i.e., $\operatorname{div} E \otimes v = \big(\sum_{i=1}^3 \partial_i(E_i v_j)\big)_{j=1,2,3}$ $(= E \cdot \nabla v$, since $\operatorname{div} E = 0)$.

To deal with Leray-Hopf type weak solutions v of (1.5), see Definition 1.1 below, we need that E has the following properties:

$$\begin{gathered} E \in L^4\big(0,T;L^4(\Omega)\big) \cap L^{s_0}\big(0,T;L^{q_0}(\Omega)\big) \\ \frac{2}{s_0} + \frac{3}{q_0} = 1,\ 2 < s_0 < \infty,\ 3 < q_0 < \infty. \end{gathered} \tag{1.6}$$

Actually, the condition $E \in L^4\big(0,T;L^4(\Omega)\big)$ in (1.6) is needed for estimates of the perturbation term $E \otimes E$ in the space $L^2\big(0,T;L^2(\Omega)\big)$, whereas the second condition $E \in L^{s_0}\big(0,T;L^{q_0}(\Omega)\big)$ of Serrin type will help to estimate the terms $v \otimes E$ and $E \otimes v$. To guarantee (1.6) for the solution E of (1.3) the data f_0, g, E_0 have to satisfy certain assumptions known from the theory of the very weak Stokes system, see Sect. 5. However, looking at (1.5), it suffices to assume (1.6) and $v_0 \in L^2_\sigma(\Omega)$, $f_1 = \operatorname{div} F_1$, $F_1 \in L^2\big(0,T;L^2(\Omega)\big)$, in order to define Leray-Hopf type weak solutions of (1.5); later concrete conditions on g, u_0, E_0 will be described to satisfy these assumptions, see Sect. 4 and 5.

In this respect, this paper mainly deals with the perturbed Navier-Stokes system (1.5) rather than with (1.1).

Definition 1.1 *Let E satisfy* (1.6) *and assume $v_0 \in L^2_\sigma(\Omega)$, $f_1 = \operatorname{div} F_1$, $F_1 \in L^2\big(0,T;L^2(\Omega)\big)$. Then a vector field v on $\Omega \times [0,T)$ is a Leray-Hopf type weak solution of the perturbed Navier-Stokes system* (1.5) *if the following conditions are satisfied:*

(i) $v \in L^\infty_{\text{loc}}\big([0,T);L^2_\sigma(\Omega)\big) \cap L^2_{\text{loc}}\big([0,T);W^{1,2}_0(\Omega)\big)$,

(ii) *for each test function* $w \in C_0^\infty\big([0,T);C^\infty_{0,\sigma}(\Omega)\big)$

$$\begin{aligned} \langle v, w_t\rangle_{\Omega,T} + \nu\,\langle \nabla v, \nabla w\rangle_{\Omega,T} - \langle (v+E)\otimes(v+E), \nabla w\rangle_{\Omega,T} \\ = \langle v_0, w(0)\rangle - \langle F_1, \nabla w\rangle_{\Omega,T}\,, \end{aligned} \tag{1.7}$$

(iii) *the energy inequality*

$$\frac{1}{2}\|v(t)\|_2^2 + \nu \int_0^t \|\nabla v\|_2^2\,d\tau \le \frac{1}{2}\|v_0\|_2^2 - \int_0^t \langle F_1 - (v+E)\otimes E, \nabla v\rangle\,d\tau \tag{1.8}$$

holds for $0 < t < T$.

Our first main result reads as follows:

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{1,1}$, let $f_1 = \operatorname{div} F_1$, $F_1 \in L^2\big(0,T;L^2(\Omega)\big)$, $v_0 \in L^2_\sigma(\Omega)$, and assume that E satisfies* (1.6). *Then the perturbed Navier-Stokes system*

$$\begin{aligned} v_t - \nu\Delta v + (v+E)\cdot\nabla(v+E) + \nabla\tilde{p} = f_1, \quad \operatorname{div} v = 0, \\ v\big|_{\partial\Omega} = 0, \quad v(0) = v_0 \end{aligned} \tag{1.9}$$

has at least one Leray-Hopf type weak solution v in the sense of Definition 1.1. Moreover, v satisfies the energy estimate

$$\begin{aligned}\|v(t)\|_2^2 + \nu \int_0^t \|\nabla v\|_2^2\, d\tau \le c\Big(\|v_0\|_2^2 + \\ + \frac{4}{\nu}\int_0^t (\|F_1\|_2^2 + \|E\|_4^4)\, d\tau\Big) \exp\Big(\frac{c}{\nu^{s_0-1}}\int_0^t \|E\|_{q_0}^{s_0}\, d\tau\Big)\end{aligned} \tag{1.10}$$

for all $0 < t < T$; here $c = c(\Omega, q_0) > 0$ means a constant.

Remark 1.3 *(i) In view of* (1.1), (1.3)–(1.5) *and Definition 1.1 $u = v + E$ is called a Leray-Hopf type weak solution of* (1.1) *with boundary data $u|_{\partial\Omega} = g = E|_{\partial\Omega}$ and initial value $u(0) = v_0 + E_0$. In the most general setting of very weak solutions, cf. [23], [24], these terms are not well-defined separately from each other and from f, but have to be interpreted in the generalized sense that $v = u - E$ satisfies $v|_{\partial\Omega} = 0$ and $v(0) = v_0$. For a more concrete situation and assumptions on g, u_0 we refer to Sect. 5.*

(ii) The weak solution v in Theorem 1.2 may be modified on a null set of $(0,T)$ such that $v : [0,T) \to L^2_\sigma(\Omega)$ is weakly continuous. Hence $v(0) = v_0$ is well-defined, $v|_{\partial\Omega} = 0$ is well-defined for a.a. $t \in [0,T)$ in the sense of traces, and there exists a distribution $\tilde p$ on $\Omega \times (0,T)$ such that

$$v_t - \nu\Delta v + (v+E)\cdot\nabla(v+E) + \nabla\tilde p = f_1.$$

In Theorem 1.2, cf. [5], there is still missing the so-called *strong energy inequality* where the initial time $t_0 = 0$ in (1.8) is replaced by an arbitrary initial time t_0. In fact, as is well known, it is an open problem whether the strong energy inequality holds for all weak solutions and *all* t_0. Here we will prove the usual strong energy inequality, i.e. for a.a. t_0, following ideas of the seminal paper of T. Miyakawa and H. Sohr [21]. Since in that proof we have to (construct and) control the pressure, we need stronger assumptions on the field E.

Theorem 1.4 *In the situation of Theorem 1.2 additionally assume that $f_1 \in L^2\big(0,T; L^2(\Omega)\big)$ and with exponents $1 < s_1 < \infty$, $2 < q_1 < \infty$*

$$\nabla E \in L^{s_1}\big(0,T; L^{q_1}(\Omega)\big) \quad \text{where} \quad \frac12 < \frac{1}{q_0} + \frac{1}{q_1} < \frac56,\ \frac{1}{s_0} + \frac{1}{s_1} < 1. \tag{1.11}$$

Then the perturbed Navier-Stokes system (1.9) *has a weak solution satisfying the* strong energy inequality

$$\frac12\|v(t)\|_2^2 + \nu\int_{t_0}^t \|\nabla v\|_2^2\, d\tau \le \frac12\|v(t_0)\|_2^2 - \int_{t_0}^t \langle F_1 - (v+E)\otimes E, \nabla v\rangle\, d\tau \tag{1.12}$$

and the strong energy estimate

$$
\begin{aligned}
\|v(t)\|_2^2 + \nu \int_{t_0}^{t} \|\nabla v\|_2^2 \, d\tau \le c\Big(\|v(t_0)\|_2^2 + \\
+ \frac{4}{\nu} \int_{t_0}^{t} (\|F_1\|_2^2 + \|E\|_4^4) \, d\tau \Big) \exp\Big(\frac{c}{\nu^{s_0 - 1}} \int_{t_0}^{t} \|E\|_{q_0}^{s_0} \, d\tau \Big)
\end{aligned}
\tag{1.13}
$$

for almost all $t_0 \in (0,T)$ *and all* $t \in (t_0, T)$.

Remark 1.5 *Assume* $T = \infty$ *and*

$$
E \in L^4\big(0,\infty; L^4(\Omega)\big) \cap L^{s_0}_{\mathrm{loc}}\big([0,\infty); L^{q_0}(\Omega)\big), \; 2 < s_0 < \infty,
$$

such that $\int_0^t \|E\|_{q_0}^{s_0} \, d\tau$ *is increasing at least linearly as* $t \to \infty$*; we note that in this case the proof in Sect. 4 will easily show the existence of a weak solution* v *in* $(0,\infty)$*. Then the energy estimate* (1.10) *yields for the kinetic energy* $\frac{1}{2}\|v(t)\|_2^2$ *only an exponentially increasing bound as* $t \to \infty$*. This worst case estimate reflects the fact that nonzero boundary values could imply a permanent flux of energy through the boundary into the domain.*

To avoid the situation described in Remark 1.5 above we consider an assumption on E known as Leray's inequality in the context of stationary Navier-Stokes equations in multiply connected domains:

Assumption 1.6 *Let* $E \in L^\infty\big(0,\infty; L^4(\Omega)\big)$ *satisfy for a.a.* $t \in (0,\infty)$ *the condition*

$$
\Big| \int_\Omega w_1 \otimes E(t) \cdot \nabla w_2 \, dx \Big| \le \frac{\nu}{4} \|\nabla w_1\|_2 \, \|\nabla w_2\|_2 \,, \quad w_1, w_2 \in W_0^{1,2}(\Omega) \cap L^2_\sigma(\Omega) \,. \tag{1.14}
$$

We recall that in a (multiply connected) exterior domain $\Omega \subset \mathbb{R}^3$ with boundary components $\Gamma_1, \dots, \Gamma_L$ ($L \in \mathbb{N}$), i.e., $\partial\Omega = \bigcup_{j=1}^L \Gamma_j \in C^{1,1}$, to any boundary data $g \in W^{1/2,2}(\partial\Omega)$ satisfying the *restricted flux condition*

$$
\int_{\Gamma_j} g \cdot N \, do = 0, \quad j = 1, \dots, L,
$$

and any $\varepsilon > 0$ there exists a solenoidal extension $E_\varepsilon \in W^{1,2}(\Omega)$ of g with compact support in $\overline{\Omega}$ such that

$$
\Big| \int_\Omega w E_\varepsilon \cdot \nabla w \, dx \Big| \le \varepsilon \, \|\nabla w\|_2^2 \quad \text{for all } w \in W_0^{1,2}(\Omega) \cap L^2_\sigma(\Omega). \tag{1.15}
$$

Corollary 1.7 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{1,1}$, let $v_0 \in L^2_\sigma(\Omega)$ and let*

$$f_1 = \operatorname{div} F_1, \; F_1 \in L^2\big(0,\infty; L^2(\Omega)\big). \tag{1.16}$$

Furthermore, let E satisfy Assumption 1.6. Then the perturbed Navier-Stokes system (1.9) *has a global in time Leray-Hopf type weak solution v satisfying the energy estimate*

$$\|v(t)\|_2^2 + \nu \int_0^t \|\nabla v\|_2^2 \, d\tau \le \|v_0\|_2^2 + \frac{2}{\nu} \int_0^t \big(\|F_1\|_2^2 + \|E\|_4^4\big)\, d\tau \, . \tag{1.17}$$

There are many applications of the two-dimensional Navier-Stokes system with nonhomogeneous boundary values in optimal control theory since the $2D$-system admits global smooth solutions and uniqueness. For the three-dimensional case there are only few results on the existence of global or weak solutions. We mention the existence of local in time strong solutions by A.V. Fursikov, M.D. Gunzburger and L.S. Hou [12], and results in a scale of Besov spaces by G. Grubb [17]. The existence of global in time weak solutions is proved by J.-P. Raymond [22] for boundary data in a fractional Sobolev space on $\partial\Omega \times (0,T)$ with derivatives in space and time of fractional order $3/4$ for domains with boundary $\partial\Omega \in C^3$. For first results on global weak solutions of Leray-Hopf type in bounded domains see R. Farwig, H. Kozono and H. Sohr [8] for time-independent boundary data and [9], [10] for time-dependent boundary (and even a prescribed non-zero divergence [9]). The existence of weak, mild and strong time-periodic solutions in a bounded domain with non-zero boundary values constant in time was recently shown by R. Farwig and T. Okabe [11]. The first result on weak solutions with nonhomogeneous boundary data in an exterior domain as in Theorem 1.2 was published as part of the proceedings of a conference in Kobe (2009), see [5]; however, the question of weak solutions satisfying the strong energy inequality was left open.

After introducing some notation and preliminaries in Sect. 2 we construct approximate solutions (v_m), $m \in \mathbb{N}$, of the perturbed Navier-Stokes system using Yosida operators in Sect. 3. The passage to the limit will be performed in Sect. 4 where we prove the existence of weak solutions to the perturbed Navier-Stokes system and also the strong energy inequality. Finally, in Sect. 5 we discuss the construction of suitable vector fields E.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{1,1}$, let $0 < T \le \infty$ and $1 \le q,\ s \le \infty$ with conjugate exponents $1 \le q',\ s' \le \infty$. We will use standard notation for Lebesgue spaces $(L^q(\Omega), \|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q)$ and for Bochner spaces

$\big(L^s\big(0,T;L^q(\Omega)\big), \|\cdot\|_{L^s(0,T;L^q(\Omega))} = \|\cdot\|_{q,s;T}\big)$. Here $v \in L^s_{\rm loc}\big([0,T);L^q(\Omega)\big)$ means that $v \in L^s\big(0,T';L^q(\Omega)\big)$ for each finite $0 < T' < T$. The pairing of functions (or vector fields) in Ω and $\Omega \times (0,T)$ is denoted by $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\Omega,T}$, respectively. Sobolev spaces are denoted by $(W^{m,q}(\Omega), \|\cdot\|_{W^{m,q}}), m \in \mathbb{N}$, the corresponding trace space by $(W^{m-1/q,q}(\partial\Omega), \|\cdot\|_{W^{m-1/q,q}})$ when $1 < q < \infty$. The dual space to $W^{1-1/q',q'}(\partial\Omega)$ is denoted by $W^{-1/q,q}(\partial\Omega)$, the corresponding pairing is $\langle\cdot,\cdot\rangle_{\partial\Omega}$. Concerning smooth functions we need the spaces $C_0^\infty(\Omega), C_{0,\sigma}^\infty(\Omega) = \{v \in C_0^\infty(\Omega) : \operatorname{div} v = 0\}$, and in the context of very weak solutions

$$C^2_{0,\sigma}(\overline{\Omega}) = \{w \in C^2(\overline{\Omega}) : \operatorname{div} w = 0,\ \operatorname{supp} w \text{ compact in } \overline{\Omega},\ w = 0 \text{ on } \partial\Omega\},$$

see Sect. 5. Note that $W_0^{1,q}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,q}}}$, with dual space denoted by $W^{-1,q'}(\Omega)$, and that $L^q_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$. By $w \in C_0^\infty([0,T);C^\infty_{0,\sigma}(\Omega))$, the space of test functions in Definition 1.1, we mean that $w \in C_0^\infty([0,T) \times \Omega)$ satisfies $\operatorname{div}_x w = 0$ for all $t \in [0,T)$ (taking the divergence with respect to $x \in \Omega$).

For $1 < q < \infty$ let $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$ be the Helmholtz projection and let

$$A_q = -P_q\Delta : \mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega) \subset L^q_\sigma(\Omega) \to L^q_\sigma(\Omega)$$

denote the Stokes operator. For $-1 \le \alpha \le 1$ its fractional powers $A_q^\alpha : \mathcal{D}(A_q^\alpha) \to L^q_\sigma(\Omega)$ are well-defined injective, densely defined closed operators; for $0 \le \alpha \le 1$ we know that $\mathcal{D}(A_q) \subseteq \mathcal{D}(A_q^\alpha) \subseteq L^q_\sigma(\Omega)$ and the range $\mathcal{R}(A_q^\alpha)$ is dense in $L^q_\sigma(\Omega)$; finally, $(A_q^\alpha)^{-1} = A_q^{-\alpha}$ for $-1 \le \alpha \le 1$. In particular, one has

$$\|A^{1/2}u\|_q \le c\|\nabla u\|_q, \qquad 1 < q < \infty, \tag{2.1}$$

$$\|\nabla u\|_q \le c\|A^{1/2}u\|_q, \quad 1 < q < 3, \tag{2.2}$$

with constants $c = c\,(q,\Omega) > 0$; for $A = A_2$, one has $\|A^{1/2}v\|_2 = \|\nabla v\|_2$ for $v \in \mathcal{D}(A^{1/2})$. Moreover, we note the following embedding estimates (with constants $c = c(q,\Omega) > 0$):

$$\|v\|_\gamma \le c\|A^\alpha v\|_q, \quad 0 \le \alpha \le \frac{1}{2},\ 1 \le q < 3,\ 2\alpha + \frac{3}{\gamma} = \frac{3}{q},\ v \in \mathcal{D}(A^\alpha) \tag{2.3}$$

$$\|A^\alpha v\|_2 \le \|Av\|_2^\alpha\,\|v\|_2^{1-\alpha}, \quad 0 \le \alpha \le 1,\ v \in \mathcal{D}(A) \tag{2.4}$$

$$\|v\|_q \le \|v\|_6^\beta\|v\|_2^{1-\beta} \le c\|\nabla v\|_2^\beta\|v\|_2^{1-\beta},\ 2 \le q \le 6,\ \beta = \frac{3}{2} - \frac{3}{q},\ v \in W_0^{1,2}(\Omega) \tag{2.5}$$

Recall that $-A_q$ generates a bounded analytic semigroup $\{e^{-tA_q} : t \ge 0\}$ on $L^q_\sigma(\Omega)$ satisfying the estimate

$$\|A_q^\alpha e^{-tA_q}v\|_q \le c\,t^{-\alpha}\|v\|_q, \quad 0 \le \alpha \le 1,\ t > 0,\ v \in L^q_\sigma(\Omega) \tag{2.6}$$

with a constant $c = c\,(q,\Omega) > 0$ and $c = 1$ if $q = 2$.

The Stokes operator A_q has the property of *maximal regularity*: For all $1 < s,\, q < \infty$ and $f \in L^s\big(0,T;L^q(\Omega)\big)$ the instationary Stokes system

$$v_t + \nu A_q v = P_q f, \quad v(0) = 0$$

has a unique solution $v \in C^0\big([0,T);L^q(\Omega)\big)$ such that $v_t, A_q v \in L^s\big(0,T;L^q_\sigma(\Omega)\big)$ and v satisfies the *a priori* estimate

$$\|v_t\|_{q,s;T} + \|\nu A_q v\|_{q,s;T} \le c\|f\|_{q,s;T}. \tag{2.7}$$

This solution has the representation

$$v(t) = \int_0^t e^{-\nu(t-\tau)A_q} P_q f(\tau)\, d\tau. \tag{2.8}$$

Moreover, there exists a pressure $p \in L^1_{\text{loc}}(\Omega \times (0,T))$ such that

$$v_t - \nu\Delta v + \nabla p = f$$

and $\|\nabla p\|_{q,s;T} \le c\|f\|_{q,s;T}$. When $1 < q < 3$, then there exists a unique pressure function $p \in L^s\big(0,T;L^{q_*}(\Omega)\big)$ where $\frac{1}{q_*} = \frac{1}{q} - \frac{1}{3}$ and

$$\|p\|_{q_*,s;T} \le c\|\nabla p\|_{q,s;T} \le \|f\|_{q,s;T}. \tag{2.9}$$

In (2.7), (2.9) the constant $c = c(\Omega, q, s) > 0$ is independent of f, T and ν.

To find approximate solutions of the Navier-Stokes system in Sect. 3 we need Yosida's approximation operators

$$J_m = \Big(I + \frac{1}{m}A^{1/2}\Big)^{-1},\ m \in \mathbb{N},$$

where I denotes the identity on $L^2_\sigma(\Omega)$. The following properties are well-known:

$$\begin{aligned}
&\|J_m v\|_2 \le \|v\|_2,\ \Big\|\frac{1}{m}A^{1/2}J_m v\Big\|_2 \le \|v\|_2,\\
&\lim_{m\to\infty} J_m v = v \text{ for all } v \in L^2_\sigma(\Omega),\\
&\|\nabla J_m v\|_2 \le \|\nabla v\|_2 \text{ for all } v \in W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega) = \mathcal{D}(A^{1/2});
\end{aligned} \tag{2.10}$$

for the proof of the last inequality we use that $\|A^{1/2}v\|_2 = \|\nabla v\|_2$ and the commutativity of J_m with $A^{1/2}$. Moreover, since $A_q u = A_2 u$ for $u \in \mathcal{D}(A_q) \cap \mathcal{D}(A_2)$, the family of Yosida operators is also bounded uniformly in $m \in \mathbb{N}$ on $L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega)$ with respect to $\|\cdot\|_q$, $1 < q < \infty$.

For these and further properties of the Stokes operator and Yosida's approximation we refer e.g. to [14], [15], [20] and [25].

3 The approximate system

As in the classical case of vanishing Dirichlet data our proof rests on three steps: (1) an approximation procedure yielding a sequence of solutions, (v_m), (2) an energy estimate for v_m with bounds independent of $m \in \mathbb{N}$ to show that each v_m exists on the maximal interval of existence, $[0,T)$, and (3) weak and strong convergence properties of a suitable subsequence of (v_m) to construct a weak solution of the Navier-Stokes equation. Here we have to take into account that for an exterior domain, compact Sobolev embeddings do hold only for bounded subdomains $\Omega' \subset \Omega$.

For step (1) we use the Yosida approximation procedure in (1.9) yielding the *approximate perturbed Navier-Stokes system*

$$\begin{aligned} v_t - \nu\Delta v + (J_m v + E)\cdot\nabla(v+E) + \nabla\tilde{p} = f_1, \quad \operatorname{div} v = 0 \\ v|_{\partial\Omega} = 0, \quad v(0) = v_0 \end{aligned} \tag{3.1}$$

where $v_0 \in L^2_\sigma(\Omega)$, $f_1 = \operatorname{div} F_1$, $F_1 \in L^2\big(0,T;L^2(\Omega)\big)$, and E satisfies (1.6).

Definition 3.1 *A vector field*

$$v = v_m \in L^\infty_{\text{loc}}\big([0,T);L^2_\sigma(\Omega)\big) \cap L^2_{\text{loc}}\big([0,T);W^{1,2}_0(\Omega)\big) \tag{3.2}$$

is called a (Leray-Hopf type) weak solution of (3.1) *if the relation*

$$\begin{aligned} -\langle v, w_t\rangle_{\Omega,T} + \nu\langle\nabla v,\nabla w\rangle_{\Omega,T} - \langle (J_m v + E)\otimes(v+E),\nabla w\rangle_{\Omega,T} \\ = \langle v_0, w(0)\rangle - \langle F_1,\nabla w\rangle_{\Omega,T} \end{aligned} \tag{3.3}$$

is satisfied for every $w \in C_0^\infty\big([0,T;C^\infty_{0,\sigma}(\Omega)\big)$ and the energy inequality

$$\frac{1}{2}\|v(t)\|_2^2 + \nu\int_0^t \|\nabla v\|_2^2\,d\tau \le \frac{1}{2}\|v_0\|_2^2 - \int_0^t \langle F_1 - (J_m v + E)\otimes E, \nabla v\rangle\,d\tau, \tag{3.4}$$

$0 \le t < T$, holds.

Lemma 3.2 *Let $v_0 \in L^2_\sigma(\Omega)$, $f_1 = \operatorname{div} F_1$, $F_1 \in L^2\big(0,T;L^2(\Omega)\big)$, and E satisfying* (1.6) *be given. Then there exists some $T' = T'(\nu, v_0, F_1, E, m) \in (0, \min(1,T)]$ such that* (3.1) *has a unique weak solution $v = v_m$ on $\Omega\times(0,T')$.*

Proof In the proof of this *local* existence result we assume without loss of generality that $\nu = 1$.

Let $v = v_m$ be a solution of (3.1) on $\Omega\times(0,T')$, $0 < T' \le 1$. Hence v is contained in the space

$$X_{T'} := L^\infty\big(0,T';L^2_\sigma(\Omega)\big) \cap L^2\big(0,T';W^{1,2}_0(\Omega)\big)$$

with

$$\|v\|_{X_{T'}} := \|v\|_{2,\infty;T'} + \|A^{\frac{1}{2}} v\|_{2,2;T'} < \infty.$$

Note that an element $v \in X_{T'}$ satisfies $v \in L^s\big(0,T';L^q(\Omega)\big)$ for all $\frac{2}{s}+\frac{3}{q}=\frac{3}{2}$ and the estimate $\|v\|_{q,s,T'} \le c\|v\|_{X_{T'}}$ with a constant $c = c(\Omega,q) > 0$.

Then we obtain for any $0 < T' \le \min(1,T)$, using Hölder's inequality, the properties (2.3) - (2.5) and (2.10) the following estimates for $(J_m v + E)\otimes(v+E)$ with some constant $c = c(\Omega) > 0$:

$$\begin{aligned}
\|J_m v \otimes v\|_{2,2;T'} &\le \|J_m v\|_{6,4;T'}\,\|v\|_{3,4;T'} \le c\,\|A^{1/2} J_m v\|_{2,4;T'}\,\|v\|_{X_{T'}} \\
&\le c\,m\,\|v\|_{2,4;T'}\,\|v\|_{X_{T'}} \le c\,m\,(T')^{1/4}\,\|v\|^2_{X_{T'}}, \\
\|E\otimes v\|_{2,2;T'} &\le \|E\|_{q_0,s_0;T'}\,\|v\|_{(\frac{1}{2}-\frac{1}{q_0})^{-1},(\frac{1}{2}-\frac{1}{s_0})^{-1},T'} \quad \le c\,\|E\|_{q_0,s_0;T'}\,\|v\|_{X_{T'}} \\
\|J_m v\otimes E\|_{2,2;T'} &\le \|E\|_{q_0,s_0;T'}\,\|J_m v\|_{(\frac{1}{2}-\frac{1}{q_0})^{-1},(\frac{1}{2}-\frac{1}{s_0})^{-1},T'} \le c\,\|E\|_{q_0,s_0;T'}\,\|v\|_{X_{T'}}.
\end{aligned}$$

Since $\|E\otimes E\|_{2,2;T'} \le \|E\|^2_{4,4;T'}$, we proved the estimate

$$\begin{aligned}
&\|(J_m v + E)\otimes(v+E)\|_{2,2;T'} \\
&\qquad \le c\,m(T')^{1/4}\,\|v\|^2_{X_{T'}} + \|E\|^2_{4,4;T'} + c\,\|E\|_{q_0,s_0;T'}\,\|v\|_{X_{T'}}.
\end{aligned} \tag{3.5}$$

With the definition

$$\hat{F}_1(v) = F_1 - (J_m v + E)\otimes(v+E)$$

we write the system (3.1) in the form

$$\begin{aligned}
v_t - \Delta v + \nabla p = \operatorname{div}\hat{F}_1(v), &\quad \operatorname{div} v = 0 \\
v = 0 \text{ on } \partial\Omega, &\quad v(0) = v_0
\end{aligned}.$$

Since $v_0 \in L^2_\sigma(\Omega)$ and $\hat{F}_1(v) \in L^2\big(0,T';L^2(\Omega)\big)$, we apply classical L^2-results [25, Ch. IV] on weak solutions of the instationary Stokes system to get that $v \in C^0\big([0,T');L^2_\sigma(\Omega)\big)$ and satisfies the fixed point relation $v = \mathcal{F}_{T'}(v)$ in $X_{T'}$; here

$$\big(\mathcal{F}_{T'}(v)\big)(t) = e^{-tA}v_0 + \int_0^t A^{\frac{1}{2}}\,e^{-(t-\tau)A}\,A^{-\frac{1}{2}}P_2\operatorname{div}\,\hat{F}_1(v)(\tau)\,d\tau\,; \tag{3.6}$$

see [25, III.2.6] concerning the operator $A^{-\frac{1}{2}}P_2\operatorname{div}$. Moreover, v satisfies even an energy equality for $t \in [0,T')$ instead of the energy inequality (3.4), and, by (3.5), the energy estimate

$$\|\mathcal{F}_{T'}(v)\|_{X_{T'}} \le a\,\|v\|^2_{X_{T'}} + b\,\|v\|_{X_{T'}} + d \tag{3.7}$$

where

$$a = c\, m(T')^{1/4},\ b = c\,\|E\|_{q_0,s_0;T'},\ d = c\big(\|v_0\|_2 + \|E\|^2_{4,4;T'} + \|F_1\|_{2,2;T'}\big) \qquad (3.8)$$

with constants $c > 0$ independent of v, m and T'.

By analogy, we get for two elements $v_1, v_2 \in X_{T'}$ the estimate

$$\begin{aligned}
&\|\mathcal{F}_{T'}(v_1) - \mathcal{F}_{T'}(v_2)\|_{X_{T'}} \\
&\qquad \le c\,\|v_1 - v_2\|_{X_{T'}}\big(m(T')^{1/4}\big(\|v_1\|_{X_{T'}} + \|v_2\|_{X_{T'}}\big) + \|E\|_{q_0,s_0;T'}\big) \qquad (3.9)\\
&\qquad \le \|v_1 - v_2\|_{X_{T'}}\big(a\,(\|v_1\|_{X_{T'}} + \|v_2\|_{X_{T'}}) + b\big).
\end{aligned}$$

Up to now, to derive (3.7), (3.9), we considered a given solution $v = v_m \in X_{T'}$ of (3.1).

In the next step, we solve the fixed point problem $v = \mathcal{F}_{T'}(v)$ in $X_{T'}$. Assuming the smallness condition

$$4\,a\,d + 2\,b < 1 \qquad (3.10)$$

we easily see that the quadratic equation $y = ay^2 + by + d$ has a minimal positive root y_1 which also satisfies $2ay_1 + b < 1$. Hence, under the assumption (3.10), $\mathcal{F}_{T'}$ maps the closed ball $B_{T'} = \{v \in X_{T'} : \|v\|_{X_{T'}} \le y_1\}$ into itself. Moreover, (3.9), (3.10) imply that $\mathcal{F}_{T'}$ is a strict contradiction on $B_{T'}$. Now Banach's Fixed Point Theorem yields the existence of a unique fixed point $v = v_m \in B_{T'}$ of $\mathcal{F}_{T'}$. This solution is a weak solution of the approximate perturbed Navier-Stokes system (3.1). Moreover, v satisfies an energy identity, cf. (3.15) below, and $v \in C^0\big([0,T'); L^2_\sigma(\Omega)\big)$.

To satisfy the smallness assumption (3.10) (for fixed $m \in \mathbb{N}$), it suffices in view of (3.8) to choose $T' \in (0, \min(1,T))$ sufficiently small.

Finally, we show that the solution just found, $v = v_m$, which is unique in $B_{T'}$, is even unique in $X_{T'}$. Indeed, consider any solution $\tilde v \in X_{T'}$ of (3.1). Then there exists $0 < T^* \le \min(1,T')$ such that $\|\tilde v\|_{X_{T^*}} \le y_1$, and the estimate (3.9) with T' replaced by $T^* \in (0, \min(1,T))$ implies that

$$\|v - \tilde v\|_{X_{T^*}} = \|\mathcal{F}_{T^*}(v) - \mathcal{F}_{T^*}(\tilde v)\|_{X_{T^*}} \le (2ay_1 + b)\|v - \tilde v\|_{X_{T^*}}.$$

Since $2ay_1 + b < 1$, we conclude that $v = \tilde v$ on $[0,T^*]$. When $T^* < T'$, we repeat this step finitely many times to see that $v = \tilde v$ on $[0,T']$. ■

To prove that the approximate solution $v = v_m$ does not exist only on an interval $[0,T')$ where $T' = T'(\nu, v_0, F_1, E, m)$, but on $[0,T)$, and to pass to the limit $m \to \infty$, we need a global (in time) and uniform (in $m \in \mathbb{N}$) energy estimate of v_m.

Lemma 3.3 *Let $v = v_m$, $m \in \mathbb{N}$, be a weak solution of the approximate perturbed Navier-Stokes system* (3.1) *on some interval $[0,T') \subseteq [0,T)$ where $v_0 \in L^2_\sigma(\Omega)$,*

$f_1 = \operatorname{div} F_1,\ F_1 \in L^2\big(0,T;L^2(\Omega)\big)$, *and let* E *satisfy* (1.6). *Then* v *satisfies the energy estimate*

$$
\begin{aligned}
&\|v(t)\|_2^2 + \nu\|\nabla v\|_{2,2;t}^2 \\
&\qquad \le \Big(\|v_0\|_2^2 + \frac{4}{\nu}\,\|F_1\|_{2,2;t}^2 + \frac{4}{\nu}\,\|E\|_{4,4;t}^4\Big)\exp\Big(\frac{c}{\nu^{s_0-1}}\|E\|_{q_0,s_0;t}^{s_0}\Big)
\end{aligned}
\tag{3.11}
$$

for all $t \in [0,T')$ *where* $c = c(\Omega, q_0) > 0$ *is a constant.*

Proof In view of the energy inequality (3.4) we have to estimate the crucial term $\int_0^t \langle (J_m v + E)\otimes E, \nabla v\rangle_\Omega\, d\tau$. By Hölder's inequality, (2.3)–(2.5) and (2.10), we get

$$
\begin{aligned}
\Big|\int_0^t \langle (J_m v)\otimes E, \nabla v\rangle\, d\tau\Big| &\le \int_0^t \|J_m v\|_{(\frac12-\frac1{q_0})^{-1}}\|E\|_{q_0}\,\|\nabla v\|_2\, d\tau \\
&\le c\int_0^t \|J_m v\|_2^\alpha\,\|\nabla J_m v\|_2^{1-\alpha}\,\|E\|_{q_0}\,\|\nabla v\|_2\, d\tau \\
&\le c\int_0^t \|v\|_2^\alpha\,\|E\|_{q_0}\,\|\nabla v\|_2^{2-\alpha}\, d\tau
\end{aligned}
\tag{3.12}
$$

where $\alpha = 1 - \frac{3}{q_0} = \frac{2}{s_0}$, cf. (2.5). Hence, by Young's inequality,

$$
\Big|\int_0^t \langle (J_m v)\otimes E, \nabla v\rangle\, d\tau\Big| \le \frac{\nu}{8}\,\|\nabla v\|_{2,2;t}^2 + \frac{c}{\nu^{s_0-1}}\int_0^t \|v\|_2^2\,\|E\|_{q_0}^{s_0}\, d\tau \tag{3.13}
$$

with a constant $c = c(q_0, \Omega) > 0$. Moreover,

$$
\Big|\int_0^t \langle E\otimes E, \nabla v\rangle\, d\tau\Big| \le \int_0^t \|E\|_4^2\,\|\nabla v\|_2\, d\tau \le \frac{\nu}{8}\,\|\nabla v\|_{2,2;t}^2 + \frac{2}{\nu}\|E\|_{4,4;t}^4\,; \tag{3.14}
$$

the term $\int_0^t \langle F_1, \nabla v\rangle_\Omega\, d\tau$ is treated similarly. Inserting these estimates into (3.4) we are led to the estimate

$$
\|v(t)\|_2^2 + \nu\|\nabla v\|_{2,2;t}^2 \le \|v_0\|_2^2 + \frac{4}{\nu}\,\|E\|_{4,4;t}^4 + \frac{4}{\nu}\,\|F_1\|_{2,2;t}^2 + \frac{c}{\nu^{s_0-1}}\int_0^t \|v\|_2^2\,\|E\|_{q_0}^{s_0}\, d\tau.
$$

Then Gronwall's Lemma proves (3.11). ∎

Lemma 3.4 *Under the assumptions of Lemma 3.2 for every* $m \in \mathbb{N}$ *there exists a unique weak solution* $v = v_m$ *of* (3.1) *on* $[0,T)$. *This solution* $v \in C^0([0,T); L^2_\sigma(\Omega))$ *satisfies in addition to the energy inequality* (3.4) *the (strong) energy identity*

$$
\frac12\,\|v(t)\|_2^2 + \nu\int_{t_0}^t \|\nabla v\|_2^2\, d\tau = \frac12\,\|v(t_0)\|_2^2 - \int_{t_0}^t \langle F_1 - (J_m v + E)\otimes E, \nabla v\rangle\, d\tau \tag{3.15}
$$

for all $t_0 \in [0,T)$ *and* $t_0 < t < T$, *and it holds*

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_2^2 + \nu\|\nabla v(t)\|_2^2 = -\langle F_1 - (J_m v + E)\otimes E, \nabla v\rangle\,(t) \tag{3.16}$$

in the sense of distributions on $[0,T)$.

Proof Let $[0,T^*) \subseteq [0,T)$ be the largest interval of existence of $v = v_m$, and assume that $T^* < T$. Since $v \in C^0([0,T^*); L^2_\sigma(\Omega))$, we find $0 < T_0 < T^*$ arbitrarily close to T^* with $v(T_0) \in L^2_\sigma(\Omega)$ which will be taken as initial value at T_0 in (3.1) in order to extend v beyond T_0. Since the length δ of the interval of existence $[T_0, T_0+\delta)$ of this unique extension depends only on $\|v(T_0)\|_2$ and $\|F_1\|_{2,2;T}$, $\|E\|_{4,4;T}$, $\|E\|_{q_0,s_0;T}$ by Lemma 3.2, we see that v can be extended beyond T^* in contradiction with the assumption.

Since $v = v_m$ in Lemma 3.2 satisfies an energy identity instead of only an energy inequality, $v = v_m$ will satisfy the strong energy equality (3.15) on $[0,T)$. Since both integrands in (3.15) are L^1-functions, the corresponding integrals are absolutely continuous in t; hence we get the differential identity (3.16) in the sense of distributions. ■

4 Proofs of Theorems 1.2, 1.4 and of Corollary 1.7

Theorem 1.2 will be proved by extracting a suitable (weakly) convergent subsequence of (v_m), the sequence of approximate solutions on $[0,T)$ constructed above, and by passing to the limit. Let $0 < T' \le T$ be finite. By (3.11) we find a constant $c = c(T') > 0$ such that

$$\|v_m\|_{2,\infty;T'} + \|\nabla v_m\|_{2,2;T'} \le c \quad \text{for all } m \in \mathbb{N}. \tag{4.1}$$

Hence there exists a subsequence of (v_m) which for simplicity will again be denoted by (v_m) with the following properties:

There exists a vector field $v \in L^\infty(0,T'; L^2_\sigma(\Omega)) \cap L^2(0,T'; H_0'(\Omega))$ such that for each bounded subdomain $\Omega' \subset \Omega$ and for each $0 < T' < T$:

$$\begin{array}{rlll} v_m \overset{*}{\rightharpoonup} v & \text{in} & L^\infty(0,T'; L^2_\sigma(\Omega)) & \text{(weakly}^*\text{)} \\ v_m \rightharpoonup v & \text{in} & L^2(0,T'; H^1_0(\Omega)) & \text{(weakly)} \\ v_m \to v & \text{in} & L^2(0,T'; L^2(\Omega')) & \text{(strongly)} \\ v_m(t) \to v(t) & \text{in} & L^2(\Omega') \text{ for a.a. } t \in [0,T) & \text{(strongly)} \end{array} \tag{4.2}$$

The third property is based on compactness arguments just as for the classical Navier-Stokes system and needs the boundedness of the underlying domain. Indeed, concerning the crucial strong convergences $(4.2)_{3,4}$, we use a sequence of increasing bounded subdomains $\Omega_k \subset \Omega$ such that $\bigcup_k \Omega_k = \Omega$, find for each Ω_k

a suitable subsequence of (v_m) such that (4.2) holds for $\Omega' = \Omega_k$ and, finally, apply a typical diagonal argument. Property $(4.2)_4$ is a well-known consequence of the strong convergence in $L^2(0,T';L^2(\Omega'))$. A similar diagonal argument for subsequences is used to get $(4.2)_3$ for any T'.

Moreover, for all $t \in [0,T)$ and almost all $\tau \in (0,T)$

$$\begin{aligned} \|\nabla v\|_{2,2;t} &\leq \liminf_{m\to\infty} \|\nabla v_m\|_{2,2;t}, \\ \|v(\tau)\|_2 &\leq \liminf_{m\to\infty} \|v_m(\tau)\|_2. \end{aligned} \tag{4.3}$$

For the proof of $(4.3)_2$ we first get that $\|v(\tau)\|_{L^2(\Omega')} \leq \liminf_{m\to\infty} \|v_m(\tau)\|_{L^2(\Omega)}$ for each bounded subdomain $\Omega' \subset \Omega$ using $(4.2)_4$ and then apply Fatou's Lemma. By Hölder's inequality, (4.1) and (4.2) we also conclude (after extracting a further subsequence again denoted by (v_m)) that

$$\begin{aligned} v_m &\rightharpoonup v \quad \text{in} \quad L^{s_2}(0,T';L^{q_2}(\Omega)),\ \tfrac{2}{s_2}+\tfrac{3}{q_2}=\tfrac{3}{2},\ 2\leq s_2,q_2<\infty \\ v_m v_m &\rightharpoonup vv \quad \text{in} \quad L^{s_3}(0,T';L^{q_3}(\Omega)),\ \tfrac{2}{s_3}+\tfrac{3}{q_3}=3,\ 1\leq s_3,q_3<\infty \\ v_m\cdot\nabla v_m &\rightharpoonup v\cdot\nabla v \quad \text{in} \quad L^{s_4}(0,T';L^{q_4}(\Omega)),\ \tfrac{2}{s_4}+\tfrac{3}{q_4}=4,\ 1\leq s_4,q_4<\infty \end{aligned} \tag{4.4}$$

When passing to the limit in the weak formulation (3.3) only some terms in $\langle (J_m v_m + E)\otimes(v_m+E), \nabla w\rangle_{\Omega,T'}$ need a special consideration. Concerning the crucial term $J_m v_m \otimes v_m$ we first note that

$$J_m v_m - v_m = -J_m\Big(\frac{1}{m}A^{1/2}v_m\Big) \to 0 \quad \text{in} \quad L^2(0,T';L^2(\Omega));$$

hence, as $m\to\infty$, by $(4.4)_1$ and Hölder's inequality

$$\begin{aligned} J_m v_m\otimes v_m - v\otimes v &= (J_m v_m - v_m)\otimes v_m + v_m\otimes(v_m - v) + (v_m - v)\otimes v \\ &\to 0 \quad \text{in } L^2(0,T';L^1(\Omega'))\,. \end{aligned} \tag{4.5}$$

Proceeding similarly with all other terms we prove that v is a weak solution of the perturbed Navier-Stokes system satisfying Definition 1.1 (i), (ii).

It remains to show that v satisfies the energy inequality (1.8). To this aim we consider the energy equality (3.15) for v_m and $t_0 = 0$. By (4.3) and $(4.2)_4$ the first three terms in (3.15) pose no problems for $m\to\infty$ (for a.a. $t>0$). The same holds true for the terms $\langle F, \nabla v_m\rangle$ and $\langle E\otimes E, \nabla v_m\rangle$. To treat the remaining term we have to prove that

$$\int_0^t \langle (J_m v_m)\otimes E, \nabla v_m\rangle\, d\tau \to \int_0^t \langle v\otimes E, \nabla v\rangle\, d\tau \quad \text{as} \quad m\to\infty. \tag{4.6}$$

Since $C_0^\infty((0,t)\times\Omega)$ is dense in $L^{s_0}\big(0,t;L^{q_0}(\Omega)\big)$ and $E\in L^{s_0}\big(0,t;L^{q_0}(\Omega)\big)$, it suffices to show (4.6) for any smooth $\tilde{E}$ and that the sequence $((J_m v_m)\nabla v_m)$ is bounded in $L^{s_0'}(0,t;L^{q_0'}(\Omega))$. Indeed, for $\tilde E \in C_0^\infty((0,t)\times\Omega)$

$$\begin{aligned}&\int_0^t \big(\langle J_m v_m \otimes \tilde E, \nabla v_m\rangle - \langle v\otimes \tilde E, \nabla v\rangle\big)\,d\tau\\ &\quad = \; -\int_0^t \langle J_m v_m \otimes v_m - v\otimes v, \nabla \tilde E\rangle\,d\tau \;\to\; 0 \quad \text{as} \quad m\to\infty\end{aligned}$$

due to (4.5). Moreover,

$$\|(J_m v_m)\cdot\nabla v_m\|_{q_0',s_0';t} \le \|\nabla v_m\|_{2,2;t}\,\|J_m v_m\|_{(\frac{1}{q_0'}-\frac12)^{-1},(\frac{1}{s_0'}-\frac12)^{-1};t} \tag{4.7}$$

is uniformly bounded in $m\in\mathbb{N}$ by (4.1) and since $2(\frac{1}{s_0'}-\frac12)+3(\frac{1}{q_0'}-\frac12)=\frac32$.

Summarizing the previous ideas we proved (4.6) for a.a. $t>0$. However, since $v(t)$ is weakly continuous in $L^2(\Omega)$, the result easily extends to all $t>0$. Now the proof of Theorem 1.2 is complete. ∎

To show the *strong* energy estimate, we do need further *a priori* estimates of (v_m) and in particular of the corresponding pressure functions provided the data $f_1 = \operatorname{div} F_1$ and E has better properties. We follow the ideas of T. Miyakawa and H. Sohr [21] and decompose the solution into several parts $v_m = \sum_{j=1}^5 w_m^{(j)}$ where $w_m^{(1)}$ and a corresponding pressure $p_m^{(1)}$ is a solution of the instationary Stokes system

$$\begin{aligned}\partial_t w_m^{(1)} - \nu\Delta w_m^{(1)} + \nabla p_m^{(1)} = f_1, \quad \operatorname{div} w_m^{(1)} = 0,\\ w_m^{(1)}|_{\partial\Omega} = 0, \quad w_m^{(1)}(0) = v_0.\end{aligned} \tag{4.8}$$

Moreover, for given v_m, let $f_j = f_m^{(j)}$, $2\le j\le 5$, be defined by

$$f_2 = -J_m v_m\cdot\nabla E, \quad f_3 = -E\cdot\nabla E, \quad f_4 = -J_m v_m\cdot\nabla v_m, \quad f_5 = -E\cdot\nabla v_m, \tag{4.9}$$

and let $(w_j,p_j) = \big(w_m^{(j)}, p_m^{(j)}\big)$, $2\le j\le 5$, be the solution of the Stokes system

$$\begin{aligned}\partial_t w_j - \nu\Delta w_j + \nabla p_j = f_j, \quad \operatorname{div} w_j = 0,\\ w_j|_{\partial\Omega} = 0, \quad w_j(0) = 0.\end{aligned} \tag{4.10}$$

Lemma 4.1 *Assume* $f_1\in L^2(0,T;L^2(\Omega))$, $v_0\in L^2_\sigma(\Omega)$, *and let* E *satisfy* (1.6) *as well as*

$$\nabla E\in L^{s_1}(0,T;L^{q_1}(\Omega)) \quad \text{where} \quad \frac12 < \frac{1}{q_0}+\frac{1}{q_1} < \frac56,\ \frac{1}{s_0}+\frac{1}{s_1} < 1, \tag{4.11}$$

with exponents $1<s_1<\infty$, $2<q_1<\infty$. *Then for almost all* $t_0\in(0,t)$ *we get the following results on solutions* $(w_j,\,p_j)$, $1\le j\le 5$, *of* (4.8) *and* (4.10).

(i) *System* (4.8) *has a weak solution* $(w_1, p_1) = (w_m^{(1)}, p_m^{(1)})$ *satisfying for each* $\varepsilon \in (0, T)$

$$\begin{aligned} \int_\varepsilon^T \big(\|\partial_t w_1\|_2^2 + \|\nu A w_1\|_2^2 + \|\nabla p_1\|_2^2 + \|p_1\|_6^2\big)\, d\tau \\ \le C_\varepsilon\Big(\|v_0\|_2^2 + \int_0^T \|f_1\|_2^2\, d\tau\Big) \end{aligned} \tag{4.12}$$

with a constant $C_\varepsilon > 0$ *independent of* m, ν, T *and* f_1, v_0.

(ii) *For* $j = 2, \dots, 5$ *there are exponents* $1 < \gamma_j < \infty$, $1 < \rho_j < 2$ *with* $\frac{2}{\gamma_j} + \frac{3}{\rho_j} < 5$ *and* $\frac{1}{\rho_j^*} := \frac{1}{\rho_j} - \frac{1}{3}$ *to be described in the proof below such that system* (4.10) *has a solution* $(w_j, p_j) = \big(w_m^{(j)}, p_m^{(j)}\big)$ *satisfying the estimate*

$$\int_0^T \big(\|\partial_t w_j\|_{\rho_j}^{\gamma_j} + \|\nu A w_j\|_{\rho_j}^{\gamma_j} + \|\nabla p_j\|_{\rho_j}^{\gamma_j} + \|p_j\|_{\rho_j^*}^{\gamma_j}\big)\, d\tau \le C \int_0^T \|f_j\|_{\rho_j}^{\gamma_j}\, d\tau; \tag{4.13}$$

here $C > 0$ *is a constant independent of* m, ν, T *and* f_j.

Proof (i) The unique solution of (4.8) has the representation

$$w_1(t) = e^{-tA_2} v_0 + \int_0^t e^{-\nu(t-\tau)A_2} P_2 f_1(\tau)\, d\tau\,.$$

Then classical L^2-estimates easily yield the assertion; for a related estimate see [21, (3.17), (3.17')]. Moreover, using the Helmholtz projection P_2, there exists a pressure $p_m^{(1)}$, cf. [21, Lemma 3.2], defined by $\nabla p_m^{(1)} = -(I - P_2)\big(\partial_t w_m^{(1)} - \nu\Delta w_m^{(1)} - f_1\big)$. Hence $\nabla p_m^{(1)} \in L^2(\varepsilon, T; L^2(\Omega))$ and $\nabla p_m^{(1)}$ satisfies (4.12). Since $\nabla p_m^{(1)}(\tau) \in L^2(\Omega)$ for a.a. $\tau \in (0, T)$, we may even determine a unique function $p_m^{(1)}(\tau) \in L^6(\Omega)$ such that $p_m^{(1)} \in L^2(\varepsilon, T; L^6(\Omega))$ with norm bounded by the right hand-side of (4.12) uniformly in $m \in \mathbb{N}$.

(ii) Assume for a moment that $f_j \in L^{\gamma_j}\big(0, T; L^{\rho_j})\big)$ for any $1 < \gamma_j < \infty$, $1 < \rho_j < 3$. Then the maximal regularity estimate (2.7), (2.9) for the Stokes system in an exterior domain yields the existence of a unique solution (w_j, p_j) of (4.10) satisfying

$$\int_0^T \big(\|\partial_t w_j\|_{\rho_j}^{\gamma_j} + \|\nu A w_j\|_{\rho_j}^{\gamma_j} + \|\nabla p_j\|_{\rho_j}^{\gamma_j} + \|p_j\|_{\rho_j^*}^{\gamma_j}\big)\, d\tau \le C \int_0^T \|f_j\|_{\rho_j}^{\gamma_j}\, d\tau$$

with a constant $C > 0$ independent of the data. Hence (4.13) will be proved, but it remains to show that $f_j \in L^{\gamma_j}\big(0, T; L^{\rho_j})\big)$ for suitable exponents γ_j, ρ_j.

Consider $f_2 = -J_m v_m \cdot \nabla E$ and let $\frac{1}{\rho_2} = \frac{1}{2} + \frac{1}{q_1}$, $\gamma_2 = s_1$. Obviously $1 < \rho_2 < 2$ and $\frac{2}{\gamma_2} + \frac{3}{\rho_2} = \big(\frac{2}{s_1} + \frac{3}{q_1}\big) + \frac{3}{2} < 5$ since $\frac{2}{s_1} + \frac{3}{q_1} < \frac{7}{2}$. Then by Hölder's inequality

$$\|f_2\|_{\rho_2, \gamma_2; t} \le \|J_m v_m\|_{2, \infty; t} \|\nabla E\|_{q_1, s_1; t}\,.$$

For an estimate of $f_3 = -E \cdot \nabla E$ we need the assumption $\frac{1}{\rho_3} := \frac{1}{q_0} + \frac{1}{q_1} > \frac{1}{2}$ to get $\rho_3 < 2$. Then by Hölder's inequality

$$\|f_3\|_{\rho_3,\gamma_3;t} \le \|E\|_{q_0,s_0;t}\|\nabla E\|_{q_1,s_1;t}\,;$$

here $\frac{1}{\gamma_3} = \frac{1}{s_0}+\frac{1}{s_1} \in (0,1)$ by assumption and $\frac{2}{\gamma_3}+\frac{3}{\rho_3} = \big(\frac{2}{s_0}+\frac{3}{q_0}\big)+\big(\frac{2}{s_1}+\frac{3}{q_1}\big) < \frac{9}{2} < 5$.

Concerning f_4 we note that (4.7) may be rewritten in the form

$$\|f_4\|_{(\frac{1}{q}+\frac{1}{2})^{-1},(\frac{1}{s}+\frac{1}{2})^{-1};t} \le \|\nabla v_m\|_{2,2;t}\,\|J_m v_m\|_{q,s;t}$$

where $2 \le s \le \infty, 2 \le q \le 6$ satisfying $\frac{2}{s}+\frac{3}{q} = \frac{3}{2}$ are arbitrary. Choosing any $q \in (2,6)$ we find exponents γ_4, ρ_4 such that $1 < \rho_4 < 2$.

Since $E \in L^4\big(0,T;L^4(\Omega)\big)$ we get by analogy to the estimate of f_4 that

$$\|f_5\|_{4/3,8/7;t} \le \|\nabla v_m\|_{2,2;t}\,\|E\|_{4,8/3;t} \le C_t\|\nabla v_m\|_{2,2;t}\,\|E\|_{4,4;t}\,.$$

Now the proof of the Lemma is complete. ■

We note that the assumption $1 < \rho_j < 2$ in Lemma 4.1 is not needed to get (4.13) (here $1 < \rho_j < 3$ suffices), but in the proof of Theorem 1.4 below.

Proof of Theorem 1.4 To prove the strong energy inequality choose a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^3;[0,1])$ such that $\varphi(x) = 1$ for $|x| \le 1$, and let $\varphi_N(x) = \varphi(x/N)$, $N \in \mathbb{N}$. Then we write the equation for v_m in the form

$$\partial_t v_m - \nu\Delta v_m + \nabla P_m = \sum_{j=1}^{5} f_j, \quad \operatorname{div} v_m = 0$$

together with initial and boundary value conditions, where the f_j's are defined in (4.9), v_m, P_m can be written in the form

$$v_m = \sum_{j=1}^{5} w_j, \;\; w_j = w_m^{(j)}, \quad P_m = p_1 + \sum_{j=2}^{5} p_j, \;\; p_j = p_m^{(j)},$$

and where (v_j, p_j) have the properties as described in Lemma 4.1. Now we test the equation for v_m, see (3.3), with $v_m\varphi_N$ on $(t_0,t)\times\Omega$ to get

$$\begin{aligned}
&\frac{1}{2}\langle v_m, \varphi_N v_m\rangle(t) + \nu\int_{t_0}^{t}\langle\nabla v_m, \varphi_N\nabla v_m\rangle\,d\tau \\
&\quad= \frac{1}{2}\langle v_m, \varphi_N v_m\rangle(t_0) + \int_{t_0}^{t}\langle f_1, \varphi_N v_m\rangle\,d\tau \\
&\quad+ \int_{t_0}^{t}\langle J_m v_m\otimes E, \nabla(v_m\varphi_N)\rangle\,d\tau + \int_{t_0}^{t}\langle E\otimes E, \nabla(\varphi_N v_m)\rangle\,d\tau \\
&\quad-\nu\int_{t_0}^{t}\langle(\nabla\varphi_N)\cdot\nabla v_m, v_m\rangle\,d\tau + \frac{1}{2}\int_{t_0}^{t}\langle|v_m|^2, (J_m v_m + E)\cdot(\nabla\varphi_N)\rangle\,d\tau \\
&\quad+\sum_{j=2}^{5}\int_{t_0}^{t}\langle v_m\cdot(\nabla\varphi_N), p_j\rangle\,d\tau - \int_{t_0}^{t}\langle\varphi_N v_m, \nabla p_1\rangle\,d\tau.
\end{aligned} \tag{4.14}$$

Next we pass to the limit $m \to \infty$ and then $N \to \infty$ in each term.

Obviously, for almost all $t > t_0$, using $(4.2)_{3,4}$ and Fatou's Lemma, the left-hand side terms in (4.14) obey the estimate

$$\frac{1}{2}\|v(t)\|_2^2 + \int_{t_0}^t \|\nabla v\|_2^2 \, d\tau \\ \le \liminf_{N\to\infty} \liminf_{m\to\infty} \Big(\frac{1}{2}\langle v_m, \varphi_N v_m\rangle(t) + \int_{t_0}^t \langle \nabla v_m, \varphi_N \nabla v_m)\rangle \, d\tau \Big).$$

The next four terms will yield in the limit the missing terms in the energy inequality. By $(4.2)_4$, for almost all $t_0 \in (0,T)$, the norm $\frac{1}{2}\langle v_m, \varphi_N v_m\rangle(t_0)$ converges to $\frac{1}{2}\|v(t_0)\|_2^2$. Evidently, $\int_{t_0}^t (f_1, \varphi_N v_m)\, d\tau$ converges to $\int_{t_0}^t \langle f_1, v\rangle \, d\tau$ due to $(4.2)_1$. For the term involving $J_m v_m \otimes E$ we conclude as in the proof of (4.6) that

$$\int_{t_0}^t \langle J_m v_m \otimes E, \nabla(v_m \varphi_N)\rangle \, d\tau \to \int_{t_0}^t (v \otimes E, \nabla(v\varphi_N))\, d\tau \quad \text{as } m \to \infty.$$

Then, for $N \to \infty$, we get that the integral $\int_{t_0}^t \langle v \otimes E, (\nabla v)\varphi_N\rangle \, d\tau$ converges to $\int_{t_0}^t \langle v \otimes E, \nabla v\rangle \, d\tau$, cf. the proof of (4.7); the remaining integral involving $v(\nabla\varphi_N)$ we estimate as follows:

$$\begin{aligned} \Big| \int_{t_0}^t \langle v \otimes E, (\nabla\varphi_N) v\rangle \, d\tau \Big| &\le c\|\nabla\varphi_N\|_\infty \int_{t_0}^t \|E\|_4 \|v\|_{8/3}^2 \, d\tau \\ &\le c\|\nabla\varphi_N\|_\infty \int_{t_0}^t \|E\|_4 \|v\|_2^{5/4} \|\nabla v\|_2^{3/4} \, d\tau \qquad (4.15) \\ &\le \frac{C}{N}\|E\|_{4,4;t} \|\nabla v\|_{2,1;t}^{3/4} \to 0 \end{aligned}$$

as $N \to \infty$.

Since $E \in L^4(0,t;L^4(\Omega))$ and $v_m \rightharpoonup v$, $\nabla v_m \rightharpoonup \nabla v$ in $L^2\big(0,t;L^2(\Omega)\big)$,

$$\int_{t_0}^t \langle E \otimes E, \nabla(\varphi_N v_m)\rangle \, d\tau \to \int_{t_0}^t \langle E \otimes E, \nabla(\varphi_N v)\rangle \, d\tau \quad \text{as } m \to \infty;$$

the latter integral converges to $\int_{t_0}^t \langle E \otimes E, \nabla v\rangle \, d\tau$ as $N \to \infty$, since the term $\int_{t_0}^t \langle E \otimes E, v\nabla\varphi_N\rangle \, d\tau$ is easily seen to vanish for $N \to \infty$.

Now it suffices to show that the remaining terms in (4.14) converge to 0 as $m \to \infty$ and then $N \to \infty$. Indeed,

$$\Big| \int_{t_0}^t \langle (\nabla\varphi_N) \cdot \nabla v_m, v_m\rangle \, d\tau \Big| \le \|\nabla\varphi_N\|_\infty \|v_m\|_{2,2;t} \|\nabla v_m\|_{2,2;t} \le \frac{c}{N} \to 0.$$

Furthermore, since $\|J_m v_m\|_3 \le c\|v_m\|_3 \le c\|v_m\|_2^{1/2}\|\nabla v_m\|_2^{1/2}$, we see that

$$\Big| \int_{t_0}^t \langle |v_m|^2, J_m v_m \cdot (\nabla\varphi_N)\rangle \, d\tau \Big| \le c\|\nabla\varphi_N\|_\infty \int_{t_0}^t \|v_m\|_2^{3/2} \|\nabla v_m\|_2^{3/2} \, d\tau \to 0.$$

The integral $\int_{t_0}^t \langle |v_m|^2, E\cdot(\nabla\varphi_N)\rangle\, d\tau$ is easily seen to converge to 0 using an estimate as in (4.15)

For the next terms $I_j := \int_{t_0}^t \langle v_m\cdot(\nabla\varphi_N)p_j\rangle\, d\tau$, $p_j = p_m^{(j)}$, $2\le j\le 5$, we use (4.13), i.e., that

$$\int_0^T \|p_m^{(j)}\|_{\rho_j^*}^{\gamma_j}\, d\tau \le c\int_0^T \|f_j\|_{\rho_j}^{\gamma_j}\, d\tau \le C$$

is bounded uniformly in $m\in\mathbb{N}$ by a constant $C\in(0,\infty)$; here $1<\rho_j^*<6$ since $1<\rho_j<2$, $2\le j\le 5$. By Hölder's inequality, with an $r=r_j\in[2,6]$,

$$\begin{aligned}
|\langle v_m\cdot(\nabla\varphi_N), p_m^{(j)}\rangle| &\le \|v_m\|_r\|p_m^{(j)}\|_{\rho_j^*}\|\nabla\varphi_N\|_{(1-\frac{1}{r}-\frac{1}{\rho_j^*})^{-1}}\\
&\le \|v_m\|_2^{\frac{3}{r}-\frac{1}{2}}\|\nabla v_m\|_2^{\frac{3}{2}-\frac{3}{r}}\|p_m^{(j)}\|_{\rho_j^*}\|\nabla\varphi_N\|_{(\frac{4}{3}-\frac{1}{r}-\frac{1}{\rho_j})^{-1}}\,.
\end{aligned}$$

Here $\frac{4}{3}-\frac{1}{r}-\frac{1}{\rho_j}<\frac{1}{3}$ provided $\frac{1}{r}+\frac{1}{\rho_j}>1$; since $1<\rho_j<2$, this can be achieved for an adequate $r=r_j\in[2,6]$ to be chosen below. This condition is needed to conclude that $\Phi_N := \|\nabla\varphi_N\|_{(\frac{4}{3}-\frac{1}{r}-\frac{1}{\rho_j})^{-1}} \to 0$ as $N\to\infty$. Now we proceed with the estimate

$$\begin{aligned}
\big|I_j\big| &\le C\Phi_N\int_{t_0}^t \|\nabla v_m\|_2^{\frac{3}{2}-\frac{3}{r}}\|p_m^{(j)}\|_{\rho_j^*}\, d\tau\\
&\le \Phi_N\Big(\int_{t_0}^t \|\nabla v_m\|_2^{\gamma_j'(\frac{3}{2}-\frac{3}{r})}\, d\tau\Big)^{1/\gamma_j'}\|p_m^{(j)}\|_{\rho_j^*,\gamma_j;t}
\end{aligned}$$

where $\gamma_j'>1$ is the conjugate exponent to γ_j. By Lemma 4.1 the sequence $\big(\|p_m^{(j)}\|_{\rho_j^*,\gamma_j;t}\big)$ is bounded uniformly in $m\in\mathbb{N}$. Moreover, $\gamma_j'(\frac{3}{2}-\frac{3}{r})<2$ which is equivalent to $\frac{2}{\gamma_j}\le\frac{1}{2}+\frac{3}{r}$; for an $r>2$ sufficiently close to 2 this can be achieved for any $\gamma_j>1$. This proves that I_j converges to 0 as $N\to\infty$.

Finally, concerning the term $\int_{t_0}^t \langle\varphi_N v_m, \nabla p_1)\rangle\, d\tau$, $p_1 = p_m^{(1)}$, we conclude from (4.12) that we may extract a (diagonal) subsequence from $\big(\nabla p_m^{(1)}\big)$ converging weakly in $L^2_{\mathrm{loc}}\big(0,T;L^2(\Omega)\big)$ to some gradient field ∇P_1. Then $(4.2)_3$ implies that

$$\int_{t_0}^t \langle\varphi_N v_m, \nabla p_m^{(1)}\rangle\, d\tau \to \int_{t_0}^t \langle\varphi_N v, \nabla P_1\rangle\, d\tau \quad \text{as} \quad m\to\infty.$$

The latter term converges to $\int_{t_0}^t (v,\nabla P_1)\, d\tau$ as $N\to\infty$; this integral vanishes since v is solenoidal and a formal integration by parts can be justified.

Summarizing the previous results we get the energy estimate for v, for almost $t_0\in(0,T)$ and for almost all $t\in(t_0,T)$. However, due to the weak continuity of $v(\cdot)$ in $L^2_\sigma(\Omega)$ the energy estimate even holds for all $t\in(t_0,T)$. ■

Proof of Corollary 1.7 Since the proof will be based on a differential inequality rather than on Gronwall's Lemma applied to an integral inequality we have to consider the sequence of approximate solutions (v_m) first of all. By the differential equation (3.16) for $v = v_m$ we get the estimate

$$\frac{1}{2}\frac{d}{dt}\|v_m(t)\|_2^2 + \nu\|\nabla v_m(t)\|_2^2 \le (\|F_1\|_2 + \|E\|_4^2)\|\nabla v_m\|_2 + \langle (J_m v_m)E, \nabla v_m\rangle$$

where the last term due to Assumption 1.6 can be estimated as follows:

$$|\langle (J_m v_m)E, \nabla v_m\rangle| \le \frac{\nu}{4}\|\nabla(J_m v_m)\|_2\|\nabla v_m\|_2 \le \frac{\nu}{4}\|\nabla v_m\|_2^2.$$

Then Young's inequality and an absorption argument lead to the estimate

$$\frac{d}{dt}\|v_m(t)\|_2^2 + \nu\|\nabla v_m(t)\|_2^2 \le \frac{2}{\nu}(\|F_1\|_2^2 + \|E\|_4^4)(t) \tag{4.16}$$

for a.a. $t \ge 0$ yielding

$$\|v_m(t)\|_2^2 + \nu\int_0^t \|\nabla v_m\|_2\, d\tau \le \|v_0\|_2^2 + \frac{2}{\nu}(\|F_1\|_{2,2;\infty}^2 + \|E\|_{4,4;\infty}^4). \tag{4.17}$$

Hence v_m is uniformly bounded on $(0,T)$ for all $T < \infty$ with a bound independent of $m \in \mathbb{N}$. By the pointwise convergence property $(4.2)_4$ and Fatou's Lemma $v(t)$ satisfies the same bound, first of all for a.a. $t \ge 0$, but due to its weak continuity property in $L^2(\Omega)$ even for all $t \ge 0$. ■

5 Construction of the vector field E

To apply Theorem 1.2 and Corollary 1.7 and to find solutions u of the Navier-Stokes system (1.1) in the form $u = v + E$ we have to construct a suitable vector field E solving (1.3); the solution should satisfy the assumptions (1.6) to apply Theorem 1.2 and (1.14) to apply Corollary 1.7, respectively.

First we consider very weak solutions E of (1.3), see [7], for suitable data g, E_0 and f_0. For their definition we introduce the space of initial values, $\mathcal{J}_\sigma^{q,s}(\Omega)$, by

$$\mathcal{J}_\sigma^{q,s}(\Omega) = \Big\{u_0 \in \mathcal{D}(A_{q'})' : \|u_0\|_{\mathcal{J}_\sigma^{q,s}} = \Big(\int_0^\infty \|A_q e^{-\tau A_q}(A_q^{-1}P_q u_0)\|_q^s\, d\tau\Big)^{\frac{1}{s}} < \infty\Big\}. \tag{5.1}$$

Here, $\mathcal{D}(A_{q'})$ is equipped with the homogeneous norm $\|A_q u\|_q$, $u \in \mathcal{D}(A_{q'})$, and the term $A_q^{-1}P_q u_0$ for $u_0 \in \mathcal{D}(A_{q'})'$ denotes the unique element $u^* \in L_\sigma^q(\Omega)$ such that $\langle u^*, \varphi\rangle = \langle A_q^{-1}P_q u_0, \varphi\rangle = \langle u_0, P_{q'}A_{q'}^{-1}\varphi\rangle$ for all $\varphi \in \mathcal{R}(A_{q'})$.

Proposition 5.1 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{1,1}$, let $0 < T \le \infty$ and let $1 < q, r, s < \infty$ satisfy $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$. Assume that $f_0 = \operatorname{div} F_0$,*

$$F_0 \in L^s\big(0,T; L^r(\Omega)\big),\ g \in L^s\big(0,T; W^{-\frac{1}{q},q}(\partial\Omega)\big) \tag{5.2}$$

and $E_0 \in \mathcal{J}_\sigma^{q,s}(\Omega)$. *Then the Stokes system* (1.3) *has a unique very weak solution*

$$E \in L^s\big(0,T;L^q(\Omega)\big) \tag{5.3}$$

in the sense that for all test functions $w \in C_0^1\big([0,T);C_{0,\sigma}^2(\overline{\Omega})\big)$

$$\begin{aligned} &-\langle E, w_t\rangle_{\Omega,T} - \nu\langle E, \Delta w\rangle_{\Omega,T} = \langle E_0, w(0)\rangle - \langle F_0, \nabla w\rangle_{\Omega,T} - \langle g, N\cdot\nabla w\rangle_{\partial\Omega,T} \\ &\quad \operatorname{div} E = 0 \ \text{ in } \ \Omega\times(0,T), \ \ E\cdot N = g\cdot N \ \text{ on } \ \partial\Omega\times(0,T). \end{aligned} \tag{5.4}$$

This solution satisfies the a priori estimate

$$\|\nu E\|_{q,s;T} \le c\,(\|F_0\|_{r,s;T} + \|\nu g\|_{L^s(0,T;W^{-1/q,q}(\partial\Omega))} + \|\nu^{1-1/s}u_0\|_{\mathcal{J}_\sigma^{q,s}}) \tag{5.5}$$

with a constant $c = c(s,q,\Omega) > 0$ *independent of* T, ν *and of the data.*

Proof The result is proved in [7, Theorem 1.4] where $\partial\Omega \in C^{2,1}$ was assumed. However, the result also holds when $\partial\Omega \in C^{1,1}$ only; see a remark in [6, §1.3] on the extension of results in [3], [4] to this case when Ω is bounded. ■

For more details on very weak solutions we refer to [1]–[6] and [23], [24]. Note that Serrin's condition $\frac{2}{s} + \frac{3}{q} = 1$ is not needed in the linear theory.

Corollary 5.2 *Let* $\Omega \subset \mathbb{R}^3$ *be an exterior domain with* $\partial\Omega \in C^{1,1}$, *let* $0 < T \le \infty$ *and let* $1 < q_0, r_0, s_0 < \infty$ *satisfy* $\frac{2}{s_0} + \frac{3}{q_0} = 1$, $\frac{1}{3} + \frac{1}{q_0} = \frac{1}{r_0}$. *Assume that* $f_0 = \operatorname{div} F_0$,

$$\begin{aligned} &F_0 \in L^{s_0}\big(0,T;L^{r_0}(\Omega)\big) \cap L^4\big(0,T;L^{\frac{12}{7}}(\Omega)\big), \\ &g \in L^{s_0}\big(0,T;W^{-\frac{1}{q_0},q_0}(\partial\Omega)\big) \cap L^4\big(0,T;W^{-\frac{1}{4},4}(\partial\Omega)\big), \\ &E_0 \in \mathcal{J}_\sigma^{q_0,s_0}(\Omega) \cap \mathcal{J}_\sigma^{4,4}(\Omega)\,. \end{aligned} \tag{5.6}$$

Then the inhomogeneous Stokes system (1.3) *has a unique very weak solution* E *satisfying* (1.6), *i.e.*

$$E \in L^{s_0}\big(0,T;L^{q_0}(\Omega)\big) \cap L^4\big(0,T;L^4(\Omega)\big), \tag{5.7}$$

and the a priori estimate

$$\begin{aligned} \|\nu E\|_{q_0,s_0;T} + \|\nu E\|_{4,4;T} \le c\,\Big(&\|F_0\|_{r_0,s_0;T} + \|F_0\|_{\frac{12}{7},4;T} \\ &+ \|\nu g\|_{L^{s_0}(0,T;W^{-\frac{1}{q_0},q_0}(\partial\Omega))} + \|\nu g\|_{L^4(0,T;W^{-\frac{1}{4},4}(\partial\Omega))} \\ &+ \|\nu^{1-1/s_0}u_0\|_{\mathcal{J}_\sigma^{q_0,s_0}} + \|\nu^{3/4}u_0\|_{\mathcal{J}_\sigma^{4,4}}\Big) \end{aligned} \tag{5.8}$$

with a constant $c = c(q_0, r_0, s_0, \Omega) > 0$ *independent of* T, ν *and of the data.*

Proof We apply Proposition 5.1 with the exponents s_0, q_0, r_0 and $4, 4, \frac{12}{7}$. Since the very weak solution E of (1.3) in [7] is constructed in a finite number of steps where each of them yields the same result for s_0, q_0, r_0 and for $4, 4, \frac{12}{7}$, it is easily seen that the unique solution E satisfies (5.7), (5.8). ∎

Remark 5.3 *(i) In the case* $s_0 = 8$, $q_0 = 4$ *and* T *finite the* $L^{s_0}(L^{q_0})$*-conditions in* (5.6) *imply the* $L^4(L^4)$*-conditions; then* (5.6)-(5.8) *simplify considerably.*

(ii) For the system (1.1) *consider data* $f = \operatorname{div} F$, $F \in L^2(0,T; L^2(\Omega))$, $u_0 \in L^2_\sigma(\Omega)$ *and boundary data* g *as in* $(5.6)_2$. *Then solve* (1.3) *with data* $f_0 = 0$, $E_0 = 0$ *and* g *to get a (unique) very weak solution* E *satisfying* (5.7) *and the a priori estimate*

$$\|E\|_{q_0,s_0;T} + \|E\|_{4,4;T} \le c \left(\|g\|_{L^{s_0}(0,T;W^{-1/q_0,q_0}(\partial\Omega))} + \|g\|_{L^4(0,T;W^{-1/4,4}(\partial\Omega))} \right).$$

Next, by Theorem 1.2, we find a weak solution v *of the perturbed Navier-Stokes system* (1.9) *with data* $f_1 = f = \operatorname{div} F$, $F_1 = F$, *and* $v_0 = u_0$ *satisfying* (1.8), (1.10). *Then* $u = v + E$ *is a weak solution of* (1.1) *split into a weak and a very weak part,* v *and* E.

(iii) To apply Theorem 1.4 we need E *(as in (ii) above) to satisfy* (1.11), *i.e.,* $\nabla E \in L^{s_1}\big(0,T; L^{q_1}(\Omega)\big)$ *where* $\frac{1}{s_0} + \frac{1}{s_1} < 1$ *and* $\frac{1}{2} < \frac{1}{q_0} + \frac{1}{q_1} < \frac{5}{6}$. *Since by assumption also* $q_1 > 2$, *these conditions imply that* $\frac{2}{s_1} + \frac{3}{q_1} < \frac{7}{2}$. *In view of scaling properties, Sobolev embedding estimates and the assumptions* (1.6) *on* E *this condition is relatively weak; actually,* (1.6) *would lead to the much stronger integrability condition of* ∇E *with exponents satisfying* $\frac{2}{s_1} + \frac{3}{q_1} = 2$ *or* $= \frac{9}{4}$.

In the second part of this Section we consider the Assumption 1.6. Suppose that the domain $\Omega \subset \mathbb{R}^3$ is exterior to $L \in \mathbb{N}$ bounded domains Ω_j with boundary components $\Gamma_j \in C^{1,1}$, $1 \le j \le L$, such that $\Omega = \mathbb{R}^3 \setminus \bigcup_{j=1}^L \overline{\Omega}_j$ and $\partial\Omega = \bigcup_{j=1}^L \Gamma_j \in C^{1,1}$. Further, let the boundary data g with $g(t) \in W^{\frac{1}{2},2}(\partial\Omega)$ for a.a. $t \in (0,T)$ satisfy the restricted flux condition

$$\int_{\Gamma_j} g(t) \cdot N do = 0, \ 1 \le j \le L. \tag{5.9}$$

Then, due to a construction in [18], there exists a compactly supported solenoidal extension $E = E_\varepsilon \in W^{1,2}(\Omega)$ of g for a.a. $t \in (0,T)$ satisfying (1.15) (for arbitrary but fixed $\varepsilon > 0$ and for a.a. t). However, we do need also an estimate of E and E_t in terms of g and g_t, respectively.

Proposition 5.4 *Let* $\Omega \subset \mathbb{R}^3$ *be an exterior domain as above and let the boundary function*

$$g \in L^\infty\big(0,\infty; W^{1/2,2}(\partial\Omega)\big),\ g_t \in L^\infty\big(0,\infty; W^{-1/2,2}(\partial\Omega)\big) \tag{5.10}$$

satisfy the restricted flux condition (5.9). *Then there exists an extension*

$$E \in L^\infty\big(0,\infty; W^{1,2}(\Omega)\big),\ E_t \in L^\infty\big(0,\infty; W^{-1,2}(\Omega)\big) \tag{5.11}$$

of g supported in a neighborhood of $\partial\Omega$, satisfying inequality (1.14), *and the a priori estimate*

$$\begin{aligned} \|E\|_{L^\infty(0,\infty;W^{1,2}(\Omega))} &\le c\,\|g\|_{L^\infty(0,\infty;W^{\frac{1}{2},2}(\partial\Omega))} \\ \|E_t\|_{L^\infty(0,\infty;W^{-1,2}(\Omega))} &\le c\,\|g_t\|_{L^\infty(0,\infty;W^{-\frac{1}{2},2}(\partial\Omega))} \end{aligned} \tag{5.12}$$

with a constant $c = c(\Omega) > 0$.

Proof We follow the ideas of E. Hopf [18] as described in [13, 16] to find an extension E of g written as the curl of a suitable vector potential and defined by a bounded linear operator $g \mapsto E$.

Ignoring $t \in (0,T)$ for a moment we consider $g \in W^{1/2,2}(\partial\Omega)$ satisfying the restricted flux condition as in (5.9). Then we use the theory of very weak solutions, see [3]–[6], to find a solution $u_j \in L^2(\Omega_j)$, $1 \le j \le L$, of the stationary Stokes system

$$-\Delta u_j + \nabla p_j = 0,\ \operatorname{div} u_j = 0 \text{ in } \Omega_j,\ u = g \text{ on } \partial\Omega_j \tag{5.13}$$

for each hole Ω_j, $1 \le j \le L$. By definition

$$\begin{aligned} -\langle u_j, \Delta w\rangle_{\Omega_j} + \langle g, N\cdot\nabla w\rangle_{\partial\Omega_j} &= 0 \quad \text{for all } w \in C^2_{0,\sigma}(\overline{\Omega}_j) \\ \operatorname{div} u_j &= 0 \text{ in } \Omega_j, \quad u_j\cdot N = g\cdot N \text{ on } \partial\Omega_j, \end{aligned}$$

and [4, Theorem 3] yields the existence of a unique very weak solution u_j satisfying the a priori estimate

$$\|u_j\|_{2,\Omega_j} \le c\,\|g\|_{W^{-1/2,2}(\partial\Omega_j)}\,;$$

here the necessary compatibility condition $\langle g, N\rangle_{\Gamma_j} = 0$ is fulfilled due to (5.9) for each j. Finally, we choose a ball of radius R such that $\bigcup_{j=1}^L \overline{\Omega}_j \subset B_R$ and consider the annular domain $A = B_R \setminus \bigcup_{j=1}^L \overline{\Omega}_j$. We find a unique very weak solution $u_A \in W^{1,2}(A)$ of the Stokes system

$$-\Delta u_A + \nabla p_A = 0, \operatorname{div} u_A = 0 \text{ in } A,\ u|_{\Gamma_j} = g,\ 1 \le j \le L,\ u|_{\partial B_R(0)} = 0;$$

note that the compatibility condition $\langle g, N\rangle_{\bigcup_j \Gamma_j} = 0$ is satisfied by (5.9) and that $\|u_A\|_{2,A} \le c\,\|g\|_{W^{-1/2,2}(\partial\Omega)}$.

Since $g \in W^{1/2,2}(\partial\Omega) \subset W^{-1/2,2}(\partial\Omega)$, the very weak solutions u_j, u_A constructed so far are also weak solutions, and, in particular, $u_j \in W^{1,2}(\Omega_j)$ and $\|u_j\|_{W^{1,2}(\Omega_j)} \le c\,\|g\|_{W^{1/2,2}(\partial\Omega_j)}$; for this regularity argument see [4, Remarks 2(1)].

Next we define u on $\mathbb{R}^3$ by $u = u_j$ in Ω_j, $1 \le j \le L$, $u = u_A$ in A and $u = 0$ in $\mathbb{R}^3 \backslash \overline{A}$. Obviously, $\operatorname{supp} u \subset \overline{B}_R$, $u \in W^{1,2}(\mathbb{R}^3)$, $\operatorname{div} u = 0$ in $\mathbb{R}^3$, and u satisfies the estimates

$$\begin{aligned} \|\nabla u\|_2 &\le c\, \|g\|_{W^{1/2,2}(\partial\Omega)}, \\ \|u\|_2 &\le c\, \|g\|_{W^{-1/2,2}(\partial\Omega)}. \end{aligned} \tag{5.14}$$

Let us construct a vector potential $\psi \in W^{2,2}(\mathbb{R}^3)$ of u satisfying

$$u = \operatorname{rot} \psi,\ \ \|\psi\|_2 \le c\, \|u\|_{L^2(B_R)},\ \ \|\nabla\psi\|_2 + \|\nabla^2\psi\|_2 \le c\, \|u\|_{W^{1,2}(\mathbb{R}^3)}. \tag{5.15}$$

Indeed, we consider the equation

$$\operatorname{rot}(\operatorname{rot}\psi) = \operatorname{rot} u, \ \ \operatorname{div}\psi = 0 \ \text{ in } \mathbb{R}^3.$$

Such a solution can be explicitly represented as

$$\psi(x) = \int_{\mathbb{R}^3} \Gamma(x-y)\ \operatorname{rot}_y u(y)\, dy = \int_{\mathbb{R}^3} K(x-y) \times u(y)\, dy,$$

where $\Gamma(x) = \frac{1}{4\pi}|x|^{-1}$ and $K(x) = \nabla\Gamma(x)$. Since $\operatorname{div} u = 0$, we easily see that $u = \operatorname{rot}\psi$. Moreover, since $K(x) \le c|x|^{-2}$, the Hardy-Littlewood-Sobolev inequality implies that

$$\|\psi\|_2 \le c\|u\|_{L^{6/5}(\mathbb{R}^3)} = c\|u\|_{L^{6/5}(B_R)} \le c\|u\|_{L^2(B_R)}.$$

Finally, since ∇K defines a Calderón-Zygmund kernel, we get the last two estimates in (5.15).

Summarizing (5.14), (5.15) we get that

$$\begin{aligned} \|\nabla\psi\|_2 + \|\nabla^2\psi\|_2 &\le c\, \|g\|_{W^{1/2,2}(\partial\Omega)} \\ \|\psi\|_2 &\le c\, \|g\|_{W^{-1/2,2}(\partial\Omega)} \end{aligned} \tag{5.16}$$

with a constant $c = c(\Omega) > 0$. Moreover, the map $g \mapsto \psi$ is linear.

In the next step we define the vector field $E = E_\varepsilon$ by $E = \operatorname{rot}(\theta_\varepsilon\psi)$ where $\theta = \theta_\varepsilon \in W^{1,\infty}(\mathbb{R}^3)$ is a carefully chosen cut-off function with support in an ε-neighborhood of $\partial\Omega$ and $\varepsilon > 0$ will be chosen below. Following [16, pp. 288-290] or [26, Ch. II, §1] for pointwise estimates of θ_ε and E we get for all $w_1, w_2 \in W^{1,2}_{0,\sigma}(\Omega)$ the estimates $|\langle w_1 \otimes E, \nabla w_2\rangle_\Omega| \le \|w_1 \otimes E\|_2 \|\nabla w_2\|_2$ and with $\chi_\varepsilon = \chi_{\operatorname{supp}\theta_\varepsilon}$ and $d(x) = \operatorname{dist}(x, \partial\Omega)$

$$\begin{aligned} \|w_1 \otimes E\|_2^2 &\le c \int_\Omega |w_1|^2 \Big(\frac{\varepsilon}{d(\cdot)}|\psi| + |\nabla\psi|\chi_\varepsilon\Big)^2 dx \\ &\le c\,\varepsilon^2 \left(\int_\Omega \Big|\frac{w_1}{d(\cdot)}\Big|^2 dx\right) \|\psi\|_\infty^2 + c\,\|w_1\|_6^2 \|\nabla\psi\|_6^2 \|\chi_\varepsilon\|_6^2 \\ &\le c\,\|\nabla w_1\|_2^2\ \|\psi\|_{H^2(\mathbb{R}^3)}^2\ (\varepsilon^2 + \|\chi_\varepsilon\|_6^2); \end{aligned}$$

here $\varepsilon > 0$ may be chosen arbitrarily small and is related to the size of $\operatorname{supp}\theta_\varepsilon$ which shrinks when $\varepsilon \to 0$. Hence (1.14) can be fulfilled for a.a. fixed $t > 0$ in the sense that

$$|\langle w_1 \otimes E, \nabla w_2\rangle| \le \frac{\nu}{4}\|\nabla w_1\|_2\,\|\nabla w_2\|_2, \quad w_1, w_2 \in W_0^{1,2}(\Omega) \cap L^2_\sigma(\Omega)\,. \tag{5.17}$$

Furthermore, since $\langle E, \varphi\rangle_\Omega = \langle \theta_\varepsilon\psi, \operatorname{rot}\varphi)\rangle_\Omega$ for all $\varphi \in W_0^{1,2}(\Omega)$ we get by $(5.16)_2$ the estimate

$$\|E\|_{W^{-1,2}(\Omega)} \le c\,\|g\|_{W^{-\frac{1}{2},2}(\partial\Omega)}. \tag{5.18}$$

In the final step we define $E(\cdot)$ satisfying (1.14) and (5.11). Given $g \in L^\infty(0,\infty; W^{\frac{1}{2},2}(\partial\Omega))$ fulfilling (5.9) for a.a. $t > 0$ we find by the previous arguments for a.a. $t > 0$ a vector field $E(t) = \operatorname{rot}(\theta\psi(t))$ satisfying (5.17) and, due to (5.16),

$$\|E(t)\|_{W^{1,2}(\Omega)} \le c\,\|g(t)\|_{W^{\frac{1}{2},2}(\partial\Omega)} \quad \text{for a.a. } \ t \in (0,T).$$

Hence $E \in L^\infty\big(0,\infty; W^{1,2}(\Omega)\big)$, and $(5.12)_1$, (1.14) are easy consequences. Since the map $g \mapsto E$ is linear and $g_t \in L^\infty\big(0,\infty; W^{-1/2,2}(\partial\Omega)\big)$, the previous arguments, the method of difference quotients and (5.18) also imply that $E_t \in L^\infty\big(0,\infty; W^{-1/2,2}(\partial\Omega)\big)$ and that $(5.12)_2$ holds.

Now Proposition 5.4 is completely proved. ∎

To apply Proposition 5.4 to the Navier-Stokes system (1.1) via Theorem 1.2 we have to consider the Stokes system (1.3) for E more closely. In this setting where E has already been defined by the boundary data g we have to determine f_0 and E_0 in (1.3). Let $h \equiv 0$ so that by the construction in the proof of Proposition 5.4

$$f_0 = E_t - \nu\Delta E, \quad E = \operatorname{rot}(\theta\psi),$$

which may be written also in the form $f_0 = \operatorname{div} F_0$. By (5.12) we easily get that $F_0 \in L^\infty\big(0,\infty; L^2(\Omega)\big)$ and that

$$\begin{aligned}\|F_0\|_{2,\infty;\infty} &\le c\,\big(\|E_t\|_{L^\infty(0,\infty;W^{-1,2}(\Omega))} + \|\nu E\|_{L^\infty(0,\infty;W^{1,2}(\Omega))}\big)\\ &\le c\,\big(\|g_t\|_{L^\infty(0,\infty;W^{-\frac{1}{2},2}(\partial\Omega))} + \|\nu g\|_{L^\infty(0,\infty;W^{\frac{1}{2},2}(\partial\Omega))}\big).\end{aligned} \tag{5.19}$$

Moreover, the properties of E, E_t and a classical interpolation result imply that $E \in C^0\big([0,\infty); L^2(\Omega)\big)$, the initial value $E_0 = E(0) \in L^2(\Omega)$ is well-defined and there exists a constant $c > 0$ such that

$$\begin{aligned}\|E_0\|_2 &\le c\,\big(\|E\|_{L^\infty(0,\infty;W^{1,2}(\Omega))} + \|E_t\|_{L^\infty(0,\infty;W^{-1,2}(\Omega))}\big)\\ &\le c\,\big(\|g\|_{L^\infty(0,\infty;W^{\frac{1}{2},2}(\partial\Omega))} + \|g_t\|_{L^\infty(0,\infty;W^{-\frac{1}{2},2}(\partial\Omega))}\big).\end{aligned} \tag{5.20}$$

Furthermore, $\operatorname{div} E_0 = 0$ and $E_0|_{\partial\Omega} = g(0)$ where $g(0)$ is well-defined in $L^2(\partial\Omega)$.

Now we are ready to state our final result.

Corollary 5.5 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with boundary $\partial\Omega \in C^{1,1}$ and boundary components Γ_j, $1 \le j \le L$. Assume that $f = \operatorname{div} F$, $F \in L^2\big(0,\infty; L^2(\Omega)\big)$, $u_0 \in L^2_\sigma(\Omega)$ and that g satisfies* (5.10) *and the restricted flux condition* (5.9). *Then the Navier-Stokes system* (1.1) *has a global weak solution $u = v + E$ where E satisfies* (5.11), (5.12) *and*

$$\|v(t)\|_2^2 + \nu \int_0^t \|\nabla v\|_2^2 \, d\tau \le \|v_0\|_2^2 + \frac{c}{\nu} \int_0^t \big(\|F_1\|_2^2 + \|E\|_4^4\big)\, d\tau \,.$$

Here $F_1 = F + F_0$ satisfies (5.19), *$v_0 = u_0 - E_0$ where E_0 is subject to* (5.20).

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