

The continuous theory of dislocations for a material containing dislocations to one Burgers vector only

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Abstract

We review the continuous theory of dislocations from a mathematical point of view using mathematical tools, which were only partly available when the theory was developed several decades ago. We define a space of dislocation measures, which includes Hausdorff measures representing the dislocation measures of single dislocation curves. The evolution equation for dislocation measures is defined on this space. It is derived from four basic conditions, which must be satisfied by the model.

1 Introduction

Plastic deformation of metallic bodies is caused by the creation and movement of dislocations in the crystal lattice of the metallic material. Therefore the plastic deformation depends on the number of dislocations and on the restrictions of the dislocation movement by the crystal structure and by the geometry of the body. Standard phenomenological models for viscoplastic material behavior do not reflect these material properties depending on the microstructure of the material. A modelling approach taking this microstructure into account is the continuous theory of dislocations, which was developed several decades ago and which is well understood in continuum mechanics. Under many articles in this field we only mention the classical expositions [7, 6, 4, 5] and the articles [1, 2] containing new developments. This theory has interesting and difficult mathematical aspects. We hope to make the mathematical aspects better accessible by reviewing the theory from a mathematical point of view using mathematical tools, which were only partly available when the theory was developed. We are convinced that the mathematical aspects deserve much more investigation and that the theory can be advanced and simulations based on the theory can be improved by such investigations.

We begin by stating the standard model for the deformation of a viscoplastic body consisting of a metallic material with dislocations moving only in one slip plane of the crystal structure. It consists of the equations

$$-\operatorname{div}_x T(x, t) = 0, \tag{1.1}$$

$$T(x, t) = D\left(\varepsilon(\nabla_x u(x, t)) - m\varepsilon_p(x, t)\right), \tag{1.2}$$

$$\partial_t \varepsilon_p = f(m : T(x, t)), \tag{1.3}$$

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which must hold for time $t \geq 0$ and for x varying in the open set $\Omega \subseteq \mathbb{R}^3$ representing the material points of the body. The unknowns are the displacement $u(x, t) \in \mathbb{R}^3$ of the material point x at time t , the Cauchy stress tensor $T(x, t) \in \mathcal{S}^3$, where \mathcal{S}^3 denotes the set of symmetric 3×3 -matrices, and the plastic strain $\varepsilon_p(x, t) \in \mathbb{R}$ along the slip plane. We use the standard notation

$$\operatorname{div}_x T = \left(\sum_{j=1}^3 \partial_{x_j} T_{ij} \right)_{i=1,2,3}.$$

$\nabla_x u$ denotes the 3×3 -matrix of first order partial derivations of u , and

$$\varepsilon(\nabla_x u) = \frac{1}{2}(\nabla_x u + (\nabla_x u)^T) \in \mathcal{S}^3$$

is the linear strain tensor. We write A^T for the transpose of a matrix A . The elasticity tensor $D : \mathcal{S}^3 \rightarrow \mathcal{S}^3$ is a linear, symmetric, positive definite mapping and the constant matrix m is given by

$$m = \varepsilon(\hat{b} \otimes g) = \frac{1}{2}(\hat{b} \otimes g + g \otimes \hat{b}) \in \mathcal{S}^3, \quad (1.4)$$

where $g \in \mathbb{R}^3$ is the unit vector normal to the slip plane, and $\hat{b} \in \mathbb{R}^3$ is a unit vector in the direction of plastic slip; it is therefore a vector in the slip plane. For vectors $a, b \in \mathbb{R}^3$ we write $a \otimes b$ to denote the matrix $(a_i b_j)_{i,j=1,2,3}$. The scalar product of two 3×3 -matrices A, B is denoted by $A : B = \sum_{i,j=1}^3 a_{ij} b_{ij}$. Finally, $f : \mathcal{D}(f) \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a given function satisfying

$$s \cdot f(s) \geq 0,$$

for all $s \in \mathbb{R}$. We call f constitutive function. A typical choice for f is $f(s) = C|s|^{\gamma-1}s$, with constants $C > 0$ and $\gamma > 1$. It is well known that if boundary conditions for u or T are imposed and if f is a maximal monotone function with $0 \in \mathcal{D}(f)$ and $f(0) = 0$ satisfying suitable growth conditions, then the initial-boundary value problem to (1.1) – (1.3) has a unique solution. This is proved for example in [3].

The constitutive equation (1.3) is an ordinary differential equation in time and does not reflect material behavior caused by the dislocation microstructure. The field equations of the continuous theory of dislocations do reflect this microstructure. To derive these field equations we start in Section 2 by discussing the Volterra model for dislocation curves in linear elasticity and by defining the space of dislocation measures. The evolution equation for dislocation measures is derived in Section 3 from four basic principles. In Sections 3.3 we discuss the restrictions on the model equations following from the incompressibility constraint for the plastic part of the strain tensor and we obtain the final field equations. The simplification following from the assumption that dislocations move in slip planes is discussed in Section 4. At the end of that section we compare the field equations thus derived to the standard equations (1.1) – (1.3).

As usual in the derivation of model equations, we cannot define the function spaces, to which the solutions of the model equations belong, with the same level of rigour as in investigations of existence. This concerns in particular the space of dislocation measures. We must assume that the solutions have certain properties; the exact properties can be determined only after the model equations are known.

Several technical proofs are not included in this article to keep the length acceptable. These proofs will be published elsewhere.

2 The boundary value problem for the stress field of a dislocation curve

In this section we review the Volterra model for dislocation curves within the linear theory of elasticity, cf. [5]. This model suggests the definition of the space $\mathcal{M}_d(\Omega)$ of dislocation measures, which is given at the end of the section.

Let ℓ be a closed curve in the elastic body $\overline{\Omega}$, which represents a dislocation curve. It is allowed that a part of the curve belongs to the boundary $\partial\Omega$. We assume that an arc length parametrization $s \mapsto \ell(s)$ is given, which is continuously differentiable with the exception of at most finitely many points. This parametrization defines a unit tangent vector field $\tau = \frac{d}{ds}\ell$ along ℓ . Let Σ be a surface in Ω with $\partial\Sigma = \ell$ and let $n : \Sigma \rightarrow \mathbb{R}^3$ be a continuous normal vector field on Σ . We choose n such that at $x \in \partial\Sigma$ the vector $n(x) \times \tau(x)$ points into the surface Σ . For functions v defined on $\Omega \setminus (\Sigma \cup \ell)$ and for $x \in \Sigma$ we use the notations

$$v^\pm(x) = \lim_{\eta \searrow 0} v(x + \eta n(x)), \quad [v]_\Sigma(x) = v^+(x) - v^-(x). \quad (2.1)$$

With the dislocation curve ℓ we associate a fixed vector $b \in \mathbb{R}^3$, $b \neq 0$, the Burgers vector of the dislocation curve. We use the notation

$$\hat{b} = \frac{b}{|b|}$$

for the unit vector in direction of b .

To compute the stress field generated by the dislocation curve in the body Ω , consider the boundary and transmission problem for the displacement field $u : \overline{\Omega} \setminus (\Sigma \cup \ell) \rightarrow \mathbb{R}^3$ and the Cauchy stress tensor field $T : \overline{\Omega} \setminus (\Sigma \cup \ell) \rightarrow \mathcal{S}^3$:

$$-\operatorname{div} T = 0, \quad (2.2)$$

$$T = D\varepsilon(\nabla u), \quad (2.3)$$

$$[u]_\Sigma = -b, \quad (2.4)$$

$$[T]_\Sigma n = 0, \quad (2.5)$$

$$T|_{\partial\Omega} n_B = \gamma. \quad (2.6)$$

The elasticity equations (2.2) and (2.3) must hold on $\Omega \setminus (\Sigma \cup \ell)$, equations (2.4) and (2.5) are jump relations on Σ , and (2.6) is the boundary condition on $\partial\Omega$, where $n_B(x) \in \mathbb{R}^3$ denotes the unit normal to $\partial\Omega$ at $x \in \partial\Omega$ pointing to the exterior of Ω and $\gamma : \partial\Omega \rightarrow \mathbb{R}^3$ are the given boundary data. This problem describes the displacement and stress fields in an elastic body, which is cut along the surface Σ . After cutting, the two boundary parts created by the cutting are displaced elastically against one another by the vector b and glued together again. Since the length of b is approximately equal to the lattice constant of the crystal lattice, after this procedure the atoms on both sides of Σ are again in the right positions to form an elastically stressed, but otherwise perfect crystal at Σ . The crystal is disturbed only along the boundary ℓ of Σ , making (2.2) – (2.6) a model for the dislocation curve ℓ . The stress field generated by this dislocation curve is given by T . This stress field will have a singularity along ℓ .

Our goal is to start from this model and to generalize it to a model for bodies containing an array of dislocations described by a dislocation density. To do this rigorously, it would be necessary to solve the problem (2.2) – (2.6) in a suitable function space and

to study the singularity of T along ℓ . This difficult task is out of the scope of this article. Instead, we consider the dislocation problem in a different situation with a simpler geometry, where an explicit solution is known.

Namely, we assume that Ω is equal to \mathbb{R}^3 , that the dislocation curve ℓ is equal to the x_3 -axis, and that the material is isotropically elastic, which means that the elasticity tensor is given by

$$D_I \varepsilon = \lambda \operatorname{trace}(\varepsilon)I + 2\mu \varepsilon, \quad (2.7)$$

for all $\varepsilon \in \mathcal{S}^3$, where I is the identity matrix and where λ, μ are material constants satisfying $\mu > 0$ and $3\lambda + 2\mu > 0$.

To formulate the boundary value problem for the stress field in this situation we assume that the Burgers vector is of the form $b = (b_1, 0, b_3)$. This can always be achieved by rotation of the coordinate system around the x_3 -axis. We define the tangential vector field τ along the ℓ by $\tau(x) = e_3 = (0, 0, 1)$. We are free to choose for Σ any half plane with boundary ℓ . Therefore we take

$$\Sigma = \{(x_1, 0, x_3) \mid x_1 > 0, x_3 \in \mathbb{R}\}$$

and define the unit normal vector field n on Σ by $n(x) = e_2$. With this definition we obtain for $x \in \ell$ that the vector $n(x) \times \tau(x) = e_2 \times e_3 = e_1$ points into Σ , hence our requirement for the orientation of n is satisfied. Moreover, from (2.1) we obtain for functions v defined on $\mathbb{R}^3 \setminus (\Sigma \cup \ell)$ and $x \in \Sigma$ that

$$v^\pm(x) = \lim_{\eta \searrow 0} v(x_1, \pm\eta, x_3), \quad [v]_\Sigma(x) = \lim_{\eta \searrow 0} (v(x_1, \eta, x_3) - v(x_1, -\eta, x_3)).$$

The problem corresponding to (2.2) – (2.6) for the stress field generated by the dislocation line ℓ consists of the equations

$$-\operatorname{div} T(x) = 0, \quad (2.8)$$

$$T(x) = D_I \varepsilon(\nabla u(x)), \quad (2.9)$$

$$[u]_\Sigma(x) = -(b_1, 0, b_3), \quad (2.10)$$

$$([T]_\Sigma(x))e_3 = 0, \quad (2.11)$$

where the first two equations must hold for $x \in \mathbb{R}^3 \setminus (\Sigma \cup \ell)$ and the last two for $x \in \ell$. A solution of this problem is given in [5, pp. 49 – 53]. In the following theorem we state this solution and give some additional properties of it. We use the notation $x = (x', x_3) \in \mathbb{R}^3$ with $x' = (x_1, x_2)$.

Theorem 2.1 *Let $u = (u_1, u_2, u_3) : \mathbb{R}^3 \setminus \bar{\Sigma} \rightarrow \mathbb{R}^3$ be defined by*

$$u_1(x) = \frac{b_1}{2\pi} \left(\arctan \frac{x_2}{x_1} + \frac{1}{2(1-\nu)} \frac{x_1 x_2}{r^2} \right), \quad (2.12)$$

$$u_2(x) = -\frac{b_1}{4\pi(1-\nu)} \left((1-2\nu) \log r + \frac{x_1^2}{r^2} \right), \quad (2.13)$$

$$u_3(x) = \frac{b_3}{2\pi} \arctan \frac{x_2}{x_1}, \quad (2.14)$$

where $\nu = \frac{\lambda}{2(\lambda+\mu)}$ is Poisson's ratio and $r^2 = |x'|^2 = x_1^2 + x_2^2$. Also, let the symmetric tensor function $T = (T_{ij})_{i,j=1,\dots,3} : \mathbb{R}^3 \setminus \bar{\Sigma} \rightarrow \mathcal{S}^3$ be given by

$$T_{11} = -D_1 \frac{x_1(3x_1^2 + x_2^2)}{r^4}, \quad T_{12} = D_1 \frac{x_1(x_1^2 - x_2^2)}{r^4}, \quad T_{13} = -D_2 \frac{x_2}{r^2}, \quad (2.15)$$

$$T_{22} = D_1 \frac{x_2(x_1^2 - x_2^2)}{r^4}, \quad T_{23} = D_2 \frac{x_1}{r^2}, \quad T_{33} = -2\nu D_1 \frac{x_2}{r^2}, \quad (2.16)$$

with $D_1 = \frac{\mu b_1}{2\pi(1-\nu)}$, $D_2 = \frac{\mu b_3}{2\pi}$. Then the function (u, T) solves (2.8) – (2.11), has the asymptotic behavior

$$\nabla u(x) = O\left(\frac{1}{|x'|}\right), \quad T(x) = O\left(\frac{1}{|x'|}\right), \quad \text{for } |x'| \rightarrow 0, \quad (2.17)$$

and satisfies the condition

$$\lim_{r \searrow 0} \int_{\mathcal{C}_r} (T(x)n(x)) \cdot \varphi(x) dS = 0, \quad (2.18)$$

for every $\varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$, where $\mathcal{C}_r = \{(x', x_3) \in \mathbb{R}^3 \mid |x'| = r\}$.

The proof of this theorem is omitted. (2.15), (2.16) show that the stress tensor T is infinitely differentiable on $\mathbb{R}^3 \setminus \ell = \{(x', x_3) \in \mathbb{R}^3 \mid x' \neq 0\}$. In particular, it is infinitely differentiable at every point of Σ . A simple computation shows that this is also true for ∇u . Of course, this must be the case, since Σ is an artificially introduced surface: The crystal lattice is undisturbed at this surface.

Let $\Omega \subseteq \mathbb{R}^3$ be an open set. To every C^1 -curve ℓ in Ω representing a dislocation curve with unit tangent vector field τ and Burgers vector b we define vector and tensor valued Radon measure ρ_ℓ , $\hat{b} \otimes \rho_\ell$ respectively, by setting for all $\varphi \in C_0(\Omega, \mathbb{R}^3)$, $\tilde{\varphi} \in C_0(\Omega, \mathbb{R}^{3 \times 3})$,

$$\langle \rho_\ell, \varphi \rangle = |b| \int_\ell \tau(x) \cdot \varphi(x) ds_x, \quad (2.19)$$

$$\langle \hat{b} \otimes \rho_\ell, \tilde{\varphi} \rangle = \int_\ell (b \otimes \tau(x)) : \tilde{\varphi}(x) ds_x, \quad (2.20)$$

with the Nye dislocation tensor $b \otimes \tau(x)$. As usual, for a tensor valued distribution w and for $\varphi \in C_0^\infty(\Omega, \mathbb{R}^3)$, $\tilde{\varphi} \in C_0^\infty(\Omega, \mathbb{R}^{3 \times 3})$ we define

$$\langle \operatorname{div} w, \varphi \rangle = -\langle w, \nabla \varphi \rangle, \quad \langle \varepsilon(w), \varphi \rangle = \langle w, \varepsilon(\varphi) \rangle, \quad \langle \operatorname{rot} w, \tilde{\varphi} \rangle = \langle w, \operatorname{rot} \tilde{\varphi} \rangle.$$

Lemma 2.2 *Let ℓ be the x_3 -axis with $\tau(x) = e_3$, and let u, T be the functions given in (2.12) – (2.16) to the Burgers vector $b = (b_1, 0, b_3)$. Then the tensor valued distributions \tilde{h}_e and T defined by*

$$\langle \tilde{h}_e, \tilde{\varphi} \rangle = \int_{\mathbb{R}^3} \nabla u(x) : \tilde{\varphi}(x) dx, \quad (2.21)$$

$$\langle T, \tilde{\varphi} \rangle = \int_{\mathbb{R}^3} T(x) : \tilde{\varphi}(x) dx \quad (2.22)$$

satisfy the equations

$$-\operatorname{div} T = 0, \quad (2.23)$$

$$T = D_I \varepsilon(\tilde{h}_e), \quad (2.24)$$

$$\operatorname{rot} \tilde{h}_e = -\hat{b} \otimes \rho_\ell. \quad (2.25)$$

Note that the integrals in (2.21), (2.22) exist because of (2.17). D_I is defined in (2.7). We must omit also the proof of this lemma.

We call the two Radon measures ρ_ℓ and $\hat{b} \otimes \rho_\ell$ defined in (2.19) and (2.20) vector valued and tensor valued dislocation measures of the dislocation curve ℓ . The form of these measures and the equations (2.21), (2.25) provide the idea for the definition of general dislocation measures given now.

Let $\Omega \subseteq \mathbb{R}^3$ an open set and let μ be a scalar Radon measure on Ω . As usual we say that μ vanishes in a neighborhood \mathcal{U} of a point x , if $\langle \mu, \varphi \rangle = 0$ for all $\varphi \in C_0(\mathcal{U}, \mathbb{R})$. The support $\text{supp } \mu$ of μ is defined as the set of all points in Ω which have no neighborhood where μ vanishes. Let $\beta \in C(\text{supp } \mu, \mathbb{R}^3)$ be a given function. Since $\text{supp } \mu$ is a relatively closed set in Ω , the theorem of Tietze-Urysohn implies that we can extend β to a function $\hat{\beta} \in C(\Omega, \mathbb{R}^3)$ with $\|\hat{\beta}\|_{L^\infty(\Omega)} \leq \|\beta\|_{L^\infty(\text{supp } \mu)}$. The linear mapping

$$(x \mapsto \varphi(x)) \mapsto (x \mapsto \hat{\beta}(x) \cdot \varphi(x)) : C_0(\Omega, \mathbb{R}^3) \rightarrow C_0(\Omega, \mathbb{R})$$

satisfies $\|\hat{\beta} \cdot \varphi\|_{L^\infty(\Omega)} \leq \|\hat{\beta}\|_{L^\infty(\Omega)} \|\varphi\|_{L^\infty(\Omega)}$, hence it is continuous. Therefore a vector valued Radon measure $\beta\mu$ on Ω is defined by

$$\langle \beta\mu, \varphi \rangle = \langle \mu, \hat{\beta} \cdot \varphi \rangle, \quad \varphi \in C_0(\Omega, \mathbb{R}^3).$$

It is not difficult to see that this definition does not depend on the special continuation $\hat{\beta}$ chosen. Therefore the notation $\beta\mu$ is justified.

Definition 2.3 *The set of all measures $\beta\mu$ on Ω with a scalar Radon measure μ and $\beta \in C(\text{supp } \mu, \mathbb{R}^3)$ is denoted by $\mathcal{M}(\Omega)$. We call $\rho = \tau\mu \in \mathcal{M}(\Omega)$ a vector valued dislocation measure, if μ is a nonnegative measure, if $\tau \in C(\text{supp } \mu, \mathbb{R}^3)$ satisfies $|\tau(x)| = 1$ for all $x \in \text{supp } \mu$ and if a function $h \in L^{1,\text{loc}}(\Omega, \mathbb{R}^3)$ exists such that $\rho = \text{rot } h$. The set of vector valued dislocation measures is denoted by $\mathcal{M}_d(\Omega)$. For $\rho = \tau\mu \in \mathcal{M}_d(\Omega)$ and $\tilde{\varphi} \in C_0(\Omega, \mathbb{R}^{3 \times 3})$ we also use the notation and definition*

$$|\rho| = \mu, \quad \frac{\rho}{|\rho|} = \tau, \quad \langle \hat{b} \otimes \rho, \tilde{\varphi} \rangle = \langle |\rho|, (\hat{b} \otimes \tau) : \tilde{\varphi} \rangle.$$

$\hat{b} \otimes \rho$ is called tensor valued dislocation measure. If a dislocation measure has a density in $L^1(\Omega)$, we call it dislocation density and denote the measure and the density by the same symbol.

Remark The dislocation measure of a dislocation curve ℓ defined in (2.19) belongs to $\mathcal{M}_d(\Omega)$ and has the form $\rho_\ell = \tau|\rho|$ with $|\rho| = |b| \mathcal{H}_\ell$, where $\mathcal{H}_\ell = \mathcal{H}^1 \llcorner \ell$ is the one-dimensional Hausdorff measure restricted to the curve ℓ .

This definition suggests to generalize the problem (2.23) – (2.25) to a boundary value problem in a domain $\Omega \subseteq \mathbb{R}^3$ with an arbitrarily given tensor valued dislocation measure of the form $-\hat{b} \otimes \rho$ on the right hand side of (2.25). Before we formulate this general problem we discuss the meaning of \tilde{h}_e in the context of the theory of viscoplasticity at small strains. In this theory one uses the additive decomposition

$$\nabla u = (\nabla u - \tilde{h}_p) + \tilde{h}_p$$

of the deformation gradient ∇u into a plastic part \tilde{h}_p and an elastic part $\nabla u - \tilde{h}_p$, where only the elastic part generates the stress field:

$$T = D\varepsilon(\nabla u - \tilde{h}_p).$$

Comparing this equation with (2.24) we see that \tilde{h}_e is the elastic part of the deformation gradient:

$$\tilde{h}_e = \nabla u - \tilde{h}_p. \quad (2.26)$$

In the following we work with \tilde{h}_p instead of \tilde{h}_e . We eliminate \tilde{h}_e in (2.25) by using (2.26). If we also replace ρ_ℓ on the right hand side of (2.25) by an arbitrary dislocation measure $\rho \in \mathcal{M}_d(\Omega)$ we obtain

$$\operatorname{rot} \tilde{h}_p = -\operatorname{rot} \tilde{h}_e = \hat{b} \otimes \rho. \quad (2.27)$$

This equation can be simplified slightly by noting that if the function $h_p \in L^{1,\text{loc}}(\Omega, \mathbb{R}^3)$ satisfies

$$\operatorname{rot} h_p = \rho, \quad (2.28)$$

then (2.27) is fulfilled with $\tilde{h}_p = \hat{b} \otimes h_p$. Taking this expression for \tilde{h}_p , inserting (2.26) into (2.24) and replacing (2.25) by (2.28), we obtain the boundary value problem for the displacement field u and the stress field T in an open set $\Omega \subseteq \mathbb{R}^3$ representing the material points of a viscoplastic body:

$$-\operatorname{div} T = 0, \quad (2.29)$$

$$T = D(\varepsilon(\nabla u) - \varepsilon(\hat{b} \otimes h_p)), \quad (2.30)$$

$$\operatorname{rot} h_p = \rho, \quad (2.31)$$

$$T|_{\partial\Omega} n_B = \gamma. \quad (2.32)$$

The elasticity tensor $D : \mathcal{S}^3 \rightarrow \mathcal{S}^3$ can be any linear, symmetric, positive definite mapping, The dislocation measure $\rho \in \mathcal{M}_d(\Omega)$ and the boundary data γ are given, \hat{b} is the unit vector in direction of the Burgers vector.

We note that the splitting (2.26) of \tilde{h}_e into a gradient field and a field \tilde{h}_p satisfying $\operatorname{rot} \tilde{h}_p = -\operatorname{rot} \tilde{h}_e$ is not unique, since we can add the same gradient field to ∇u and \tilde{h}_p and obtain a new splitting. This means in particular, that if (u, T, h_p) is a solution of (2.29) – (2.32) and if $\Gamma \in L^{1,\text{loc}}(\Omega, \mathbb{R})$, then we obtain another solution (u', T', h'_p) by setting

$$u' = u + \hat{b}\Gamma, \quad T' = T, \quad h'_p = h_p + \nabla\Gamma,$$

since these equations imply $\operatorname{rot} h'_p = \operatorname{rot} h_p = \rho$ and

$$\varepsilon(\nabla u' - \hat{b} \otimes h'_p) = \varepsilon(\nabla u + \hat{b} \otimes \nabla\Gamma - \hat{b} \otimes h_p - \hat{b} \otimes \nabla\Gamma) = \varepsilon(\nabla u - \hat{b} \otimes h_p).$$

3 The evolution equation for dislocation measures

Let $[0, T_e)$ be a time interval with $T_e > 0$. In the following we formulate an evolution equation for dislocation measures

$$\rho : [0, T_e) \rightarrow \mathcal{M}_d(\Omega)$$

depending on the time. We base this formulation on four principles:

(P1) By Definition 2.3, the dislocation measure $\rho(t)$ and the time derivative $\partial_t \rho(t)$ must be a rotation field. Therefore the evolution equation must be of the form

$$\partial_t \rho = \text{rot}_x \alpha[T, \rho, b], \quad (3.1)$$

with a function $\alpha : C(\bar{\Omega}, \mathcal{S}^3) \times \mathcal{M}_d(\Omega) \times \mathbb{R}^3 \rightarrow \mathcal{M}(\Omega)$ to be determined.

(P2) There must exist a free energy $\psi(\varepsilon, h_p)$ and a flux $q(\varepsilon, u_t, h_p)$ of the free energy such that the Clausius-Duhem inequality holds:

$$\partial_t \psi + \text{div}_x q \leq 0. \quad (3.2)$$

(P3) The evolution equation (3.1) must allow for solutions $t \rightarrow \rho_{\ell(t)}$, which are dislocation measures of dislocation curves, which move with driving force given by the Peach-Koehler force

$$F = \tau \times T b. \quad (3.3)$$

(P4) Plastic deformation is volume conserving. This means that we must have

$$\text{trace } \tilde{h}_p = \text{trace}(\hat{b} \otimes h_p) = 0. \quad (3.4)$$

3.1 Conditions (P1) and (P2)

We first discuss the consequences of (P1) and (P2). Combination of (2.29) – (2.32) with (3.1) yields the closed system of partial differential equations governing the evolution of the dislocation measure and the stress field in a viscoplastic body:

$$-\text{div}_x T(x, t) = 0, \quad (3.5)$$

$$T(x, t) = D\left(\varepsilon(\nabla_x u(x, t)) - \varepsilon(\hat{b} \otimes h_p(x, t))\right), \quad (3.6)$$

$$\text{rot}_x h_p(x, t) = \rho(x, t), \quad (3.7)$$

$$\partial_t \rho(x, t) = \text{rot}_x \alpha[T, \rho, b](x, t), \quad (3.8)$$

$$T(x, t) n_B(x, t) = \gamma(x, t), \quad (x, t) \in \partial\Omega \times [0, T_e], \quad (3.9)$$

where the first four equations must hold for $(x, t) \in \Omega \times [0, T_e]$.

Lemma 3.1 (i) *Let $\Omega \subseteq \mathbb{R}^3$ be an open, bounded, simply connected set. Then (u, T, h_p, ρ) satisfies the equations (3.5) – (3.8) if and only if there is a function $\Gamma : \Omega \times [0, T_e] \rightarrow \mathbb{R}$ such that (u, T, h_p, Γ) solves the equations*

$$-\text{div}_x T = 0, \quad (3.10)$$

$$T = D(\varepsilon(\nabla_x u) - \varepsilon(\hat{b} \otimes h_p)), \quad (3.11)$$

$$\partial_t h_p = \alpha[T, \text{rot}_x h_p, b] + \nabla_x \Gamma. \quad (3.12)$$

On the other hand, if (u, T, h_p, Γ) solves (3.10) – (3.12) define $\rho = \text{rot}_x h_p$. Then (u, T, h_p, ρ) satisfies (3.5) – (3.8).

(ii) *Let the free energy and the flux be given by*

$$\psi(\varepsilon(\nabla_x u), h_p) = \frac{1}{2} (D\varepsilon(\nabla_x u - \hat{b} \otimes h_p)) : \varepsilon(\nabla_x u - \hat{b} \otimes h_p), \quad (3.13)$$

$$q(\varepsilon, u_t, h_p, \Gamma) = -T(u_t - \hat{b}\Gamma). \quad (3.14)$$

Then the Clausius-Duhem inequality (3.2) holds for every solution (u, T, h_p, Γ) of (3.10) – (3.12) if α satisfies for all points $(T, \rho) \in \mathcal{S}^3 \times \mathbb{R}^3$ the inequality

$$(T\hat{b}) \cdot \alpha[T, \rho, b] \geq 0. \quad (3.15)$$

Proof: Let (u, T, h_p, ρ) be a solution of (3.5) – (3.8). Combination of (3.7) and (3.8) yields

$$\text{rot}_x(\partial_t h_p - \alpha) = \partial_t \rho - \text{rot}_x \alpha = 0.$$

Since Ω is simply connected, this equation implies that $\partial_t h_p - \alpha$ is a gradient field. Consequently there is a function $\Gamma : \Omega \times [0, T_e] \rightarrow \mathbb{R}$ such that $\partial_t h_p - \alpha[T, \rho, b] = \nabla_x \Gamma$ holds. From this equation we obtain (3.12) if we use (3.7) to eliminate ρ in the argument of α . On the other hand, if (u, T, h_p, Γ) solves (3.10) – (3.12) then we obtain from (3.12) for $\rho = \text{rot}_x h_p$ that

$$\partial_t \rho = \text{rot}_x \partial_t h_p = \text{rot}_x \alpha[T, \rho, b],$$

since $\text{rot}_x \nabla_x \Gamma = 0$. This proves (i). To prove (ii) we infer from (3.13) and (3.11) that

$$\begin{aligned} \partial_t \psi(\varepsilon(\nabla_x u), h_p) &= \nabla_\varepsilon \psi : \varepsilon(\nabla_x u_t - \hat{b} \otimes \partial_t h_p) = T : \varepsilon(\nabla_x u_t - \hat{b} \otimes \partial_t h_p) \\ &= T : (\nabla_x u_t - \hat{b} \otimes \partial_t h_p) = T : \nabla_x u_t - (T\hat{b}) \cdot (\partial_t h_p), \end{aligned}$$

where we used several times that $T(x, t)$ is a symmetric matrix. (3.14) yields

$$\begin{aligned} \text{div}_x q &= -\text{div}_x (T(u_t - \hat{b}\Gamma)) \\ &= -(\text{div}_x T^T) \cdot (u_t - \hat{b}\Gamma) - T^T : \nabla_x u_t + T^T : (\hat{b} \otimes \nabla_x \Gamma) = -T : \nabla_x u_t + (T\hat{b}) \cdot \nabla_x \Gamma, \end{aligned}$$

where we employed that $\text{div}_x T^T = \text{div}_x T = 0$, by (3.10). Combination of the last two equations with (3.12) and (3.15) results in

$$\frac{\partial}{\partial t} \psi + \text{div}_x q = -(T\hat{b}) \cdot (\partial_t h_p - \nabla_x \Gamma) = -(T\hat{b}) \cdot \alpha \leq 0.$$

■

Remark This lemma shows that we are free to choose any function Γ in the evolution equation (3.12). This freedom is used in [1] to introduce an additional field variable to include dislocation nucleation. In our investigation we choose for simplicity $\Gamma = 0$, which avoids the unusual term $(T\hat{b})\Gamma$ in the free energy flux (3.14).

3.2 Condition (P3)

Next we construct a function α , for which condition (P3) is satisfied. Let \mathbb{S}^2 denote the unit sphere in \mathbb{R}^3 and let $\tilde{\alpha} \in C(\mathbb{S}^2 \times \mathbb{R}^3, \mathbb{R}^3)$ be a given function. For $T \in C(\overline{\Omega}, \mathcal{S}^3)$, $\rho = \tau|\rho| \in \mathcal{M}_d(\Omega)$ and $b \in \mathbb{R}^3$ we set

$$\alpha[T, \rho, b] = \tilde{\alpha}(\tau, \tau \times Tb) \times \tau |\rho|. \quad (3.16)$$

This defines a function $\alpha : C(\overline{\Omega}, \mathcal{S}^3) \times \mathcal{M}_d(\Omega) \times \mathbb{R}^3 \rightarrow \mathcal{M}(\Omega)$, since by Definition 2.3 we have $\tau = \frac{\rho}{|\rho|} \in C(\text{supp } |\rho|, \mathbb{R}^3)$, whence $\tilde{\alpha}(\tau, \tau \times Tb) \times \tau \in C(\text{supp } |\rho|, \mathbb{R}^3)$, which implies that the right hand side of (3.16) is in $\mathcal{M}(\Omega)$, again by Definition 2.3. With the notation introduced in Definition 2.3 we write the right hand side of (3.16) as $\tilde{\alpha}(\frac{\rho}{|\rho|}, \frac{\rho}{|\rho|} \times Tb) \times \rho$.

Definition 3.2 Let α be defined by (3.16), let $b \in \mathbb{R}^3$, and let $T : \Omega \times [0, T_e] \rightarrow \mathcal{S}^3$ with $T(t) \in C(\bar{\Omega}, \mathcal{S}^3)$ be given. The time dependent dislocation measure $\rho : [0, T_e] \rightarrow \mathcal{M}_d(\Omega)$ is a solution of

$$\partial_t \rho = \text{rot}_x \left(\tilde{\alpha} \left(\frac{\rho}{|\rho|}, \frac{\rho}{|\rho|} \times Tb \right) \times \rho \right), \quad (3.17)$$

if for all $\varphi \in C_0^\infty((0, T_e) \times \Omega, \mathbb{R}^3)$ the integrals in the following equation exist and satisfy

$$- \int_0^{T_e} \langle \rho(t), \partial_t \varphi(t) \rangle dt = \int_0^{T_e} \left\langle \tilde{\alpha} \left(\frac{\rho}{|\rho|}, \frac{\rho}{|\rho|} \times Tb \right) \times \rho, \text{rot}_x \varphi(t) \right\rangle dt.$$

As a consequence of the next theorem we see that condition (P3) is satisfied for the evolution equation (3.17). We need two definitions to state this theorem. Let the family $t \mapsto \ell(t)$ represent a moving dislocation curve with tangent vector $\tau(x, t)$ at the point $x \in \ell(t)$. By $\text{proj}_{\tau(x, t)}$ we denote the orthogonal projection

$$\text{proj}_{\tau(x, t)} : \mathbb{R}^3 \rightarrow H(x, t) = \{ \xi \in \mathbb{R}^3 \mid \xi \cdot \tau(x, t) = 0 \} \subseteq \mathbb{R}^3 \quad (3.18)$$

to the orthogonal space of $\tau(x, t)$. To define the normal velocity $v(x_0, t_0)$ of the dislocation curve at time t_0 at $x_0 \in \ell(t_0)$, let $x(t)$ be the intersection point of $\ell(t)$ with $H(x_0, t_0)$. Set

$$v(x_0, t_0) = \left. \frac{d}{dt} x(t) \right|_{t=t_0}.$$

Theorem 3.3 Let α be defined by (3.16) with a given function $\tilde{\alpha} \in C(\mathbb{S}^2 \times \mathbb{R}^3, \mathbb{R}^3)$.

(i) α satisfies the inequality (3.15) if

$$\xi \cdot \tilde{\alpha}(\tau, \xi) \geq 0, \quad \text{for all } (\tau, \xi) \in \mathbb{S}^2 \times \mathbb{R}^3. \quad (3.19)$$

(ii) Assume that $T \in C([0, T_e] \times \bar{\Omega}, \mathcal{S}^3)$ is a given stress field and that $\rho_{\ell(t)} = \tau |\rho_{\ell(t)}| \in \mathcal{M}_d(\Omega)$ is the dislocation measure of a dislocation curve $\ell(t)$ to the Burgers vector $b \in \mathbb{R}^3$. For $x \in \ell(t)$ let $F(x, t) = \tau(x, t) \times T(x, t)b$ be the Peach-Koehler force. Then $\rho_{\ell(t)}$ solves the evolution equation (3.17), if and only if for every $x \in \ell(t)$ the normal speed is

$$v(x, t) = \text{proj}_{\tau(x, t)} \tilde{\alpha} \left(\tau(x, t), F(x, t) \right). \quad (3.20)$$

We must omit the proof of this theorem. (3.20) shows that $\tilde{\alpha}$ can be considered to be a constitutive function determining the relation between the normal velocity and the Peach-Koehler force F . Since the continuity of $\tilde{\alpha}$ and the inequality (3.19) imply $\tilde{\alpha}(\tau, 0) = 0$, the normal velocity is equal to zero if $F = 0$. Therefore we can consider F to be the driving force for the movement of $\ell(t)$. In this sense, condition (P3) is satisfied by the evolution equation (3.17). The evolution equation (3.17) was in principal derived in [6], following ideas in [7], cf. also [4].

With the function α determined in (3.16), with $\Gamma = 0$ and with $\text{rot}_x h_p = \tau |\text{rot}_x h_p| \in \mathcal{M}_d(\Omega)$ the system (3.10) – (3.12) combined with the boundary condition (3.9) takes the form

$$-\text{div}_x T = 0, \quad (3.21)$$

$$T = D(\varepsilon(\nabla_x u) - \varepsilon(\hat{b} \otimes h_p)), \quad (3.22)$$

$$\partial_t h_p = \tilde{\alpha}(\tau, \tau \times Tb) \times \text{rot}_x h_p, \quad (3.23)$$

$$T|_{\partial\Omega} n_B = \gamma, \quad \text{on } \partial\Omega \times [0, T_e]. \quad (3.24)$$

3.3 Condition (P4)

To determine the consequences of condition (P4) we differentiate (3.4) with respect to t and use (3.23) to compute

$$0 = \text{trace}(\hat{b} \otimes \partial_t h_p) = \hat{b} \cdot \partial_t h_p = \hat{b} \cdot (\tilde{\alpha} \times \tau) |\text{rot} h_p| = 0.$$

Here we used that $\text{rot}_x h_p = \tau |\text{rot} h_p|$. From this equation we infer that

$$b \cdot \left(\tilde{\alpha}(\tau(x, t), \tau(x, t) \times T(x, t)b) \times \tau(x, t) \right) = 0. \quad (3.25)$$

must hold for all $(x, t) \in \Omega \times [0, T_e)$ if condition (P4) is satisfied. This equation has the following consequence for the movement of dislocation curves:

Corollary 3.4 *Let the stress field $T : \Omega \times [0, T_e) \rightarrow \mathcal{S}^3$ with $T(t) \in C(\bar{\Omega}, \mathcal{S}^3)$ be given, and let $\rho_{\ell(t)} = \tau |\rho_{\ell(t)}|$ be the dislocation measure of a moving dislocation curve $\ell(t)$. If $\rho_{\ell(t)}$ solves the evolution equation (3.17) and if the condition (3.25) holds for a point $x \in \ell(t)$ at which $\tau(x, t)$ is not parallel to b , then the normal speed $v(x, t)$ of the dislocation curve is parallel to the vector $\text{proj}_{\tau(x, t)} b$.*

Proof: τ and b are parallel if and only if $b \times \tau = 0$. If this product is not zero, then (3.25) implies that the vector $\tilde{\alpha}(\tau, \tau \times T b)$ must lie in the plane spanned by τ and b . In this case the vectors $\text{proj}_{\tau(x, t)} b$ and $\text{proj}_{\tau(x, t)} \tilde{\alpha}(\tau, \tau \times T b)$ lie on the same line, so they are parallel. Since by Theorem 3.3(ii) we have $v = \text{proj}_{\tau(x, t)} \tilde{\alpha}(\tau, F)$, the corollary follows. ■

Remark If the tangent vector $\tau(x, t)$ is parallel to the Burgers vector b , then $\ell(t)$ is called a screw dislocation at $x \in \ell(t)$. The condition that the volume is not changed by plastic deformation thus requires that if $\ell(t)$ is not a screw dislocation at $x \in \ell(t)$, then the normal speed $v(x, t)$ must be a linear combination of b and $\tau(x, t)$. Only for screw dislocations the direction of the normal speed is not determined by the Burgers vector. However, a dislocation curve can move freely in any direction only if it is a screw dislocation at every point, which means that it must be a straight dislocation line in direction of the Burgers vector, and if the movement is a parallel shift. As soon as the movement ceases to be a parallel shift, points appear on $\ell(t)$, at which τ and b are not parallel, restricting the direction of the movement of $\ell(t)$ at these points.

From (3.25) we see that in order to guarantee that condition (P4) is satisfied, the function $\tilde{\alpha}$ must be such that

$$b \cdot (\tilde{\alpha}(\tau, \xi) \times \tau) = 0,$$

for all $(\tau, \xi) \in \mathbb{S}^2 \times \mathbb{R}^3$. If b and τ are linearly independent, this holds if and only if $\tilde{\alpha}(\tau, \xi)$ belongs to the linear span of b and τ , hence there are real valued functions f_1 and f_2 such that $\tilde{\alpha}(\tau, \xi) = b f_1(\tau, \xi) + \tau f_2(\tau, \xi)$. This implies $\tilde{\alpha}(\tau, \xi) \times \tau = (b \times \tau) f_1(\tau, \xi)$, hence τf_2 does not contribute to the evolution equation and we can omit this term and construct $\tilde{\alpha}$ in the form

$$\tilde{\alpha}(\tau, \xi) = b f_1(\tau, \xi),$$

where the function f_1 must be chosen such that the inequality (3.19) holds. Under many possibilities a simple choice is

$$\tilde{\alpha}(\tau, \xi) = \frac{b}{|b \times \tau|} f\left(\frac{b}{|b \times \tau|} \cdot \xi\right), \quad (3.26)$$

with a function $f \in C(\mathbb{R}, \mathbb{R})$ satisfying $r \cdot f(r) \geq 0$ for all $r \in \mathbb{R}$. We insert (3.26) into (3.17) and obtain the evolution equation

$$\partial_t \rho = \operatorname{rot}_x \left(f \left(\frac{b \times \tau}{|b \times \tau|} \cdot Tb \right) \frac{b \times \tau}{|b \times \tau|} |\rho| \right), \quad (3.27)$$

for $\rho = \tau |\rho| \in \mathcal{M}_d(\Omega)$. Here we used that $b \cdot (\tau \times Tb) = (b \times \tau) \cdot Tb$ to rewrite the argument of f . The choice of the $\tilde{\alpha}$ in (3.26) is justified by the following result:

Theorem 3.5 (i) For the function $\tilde{\alpha}$ defined in (3.26) the inequality (3.19) holds.
(ii) Assume that $T \in C([0, T_e] \times \bar{\Omega}, \mathcal{S}^3)$ is a given stress field and that $\rho_{\ell(t)} = \tau |\rho_{\ell(t)}| \in \mathcal{M}_d(\Omega)$ is the dislocation measure of a dislocation curve $\ell(t)$ to the Burgers vector $b \in \mathbb{R}^3$. Let $F = \tau \times Tb$ be the Peach-Koehler force, and for $x \in \ell(t)$ denote the unit vector in the direction of $\operatorname{proj}_{\tau(x,t)} b$ by

$$b^{\perp \tau}(x, t) = \frac{\operatorname{proj}_{\tau(x,t)} b}{|\operatorname{proj}_{\tau(x,t)} b|}.$$

Then $\rho_{\ell(t)}$ solves the evolution equation (3.27), if and only if $\ell(t)$ moves with the normal speed

$$v(x, t) = f(b^{\perp \tau}(x, t) \cdot F(x, t)) b^{\perp \tau}(x, t). \quad (3.28)$$

(iii) Let (u, T, h_p) be a solution of the system

$$-\operatorname{div}_x T = 0, \quad (3.29)$$

$$T = D(\varepsilon(\nabla_x u) - \varepsilon(\hat{b} \otimes h_p)), \quad (3.30)$$

$$\partial_t h_p = f \left(\frac{b \times \tau}{|b \times \tau|} \cdot Tb \right) \frac{b \times \tau}{|b \times \tau|} |\operatorname{rot}_x h_p| \quad (3.31)$$

in $\Omega \times [0, T_e]$, where $\operatorname{rot}_x h_p = \tau |\operatorname{rot}_x h_p|$. If at time $t = 0$ we have $b \cdot h_p(x, 0) = 0$ for all $x \in \Omega$, then condition (3.4) holds for all $(x, t) \in \Omega \times [0, T_e]$.

Remarks (3.28) shows that under the choice of $\tilde{\alpha}$ given in (3.26) the driving force for the movement of dislocation curves is the component $b^{\perp \tau} \cdot F$ of the Peach-Koehler force in the direction of the vector $\operatorname{proj}_{\tau} b$. This justifies (3.26). The equation (3.31) is obtained by insertion of the function $\tilde{\alpha}$ from (3.26) into (3.23), using again that $b \cdot (\tau \times Tb) = (b \times \tau) \cdot Tb$.

Proof: For the proof of (i) note that the assumption $r \cdot f(r) \geq 0$ implies

$$\xi \cdot \tilde{\alpha}(\tau, \xi) = \frac{\xi \cdot b}{|b \times \tau|} f \left(\frac{b \cdot \xi}{|b \times \tau|} \right) \geq 0, \quad (\tau, \xi) \in \mathbb{S}^2 \times \mathbb{R}^3.$$

To verify (ii) note that since the evolution equation (3.27) is obtained from (3.17) by insertion of the function $\tilde{\alpha}$ defined in (3.26), we obtain from Theorem 3.3(ii) that $\rho_{\ell(t)}$ is a solution of (3.27) if and only if the dislocation curve moves with normal velocity

$$v = \operatorname{proj}_{\tau} \left(\frac{b}{|b \times \tau|} f \left(\frac{b}{|b \times \tau|} \cdot F \right) \right) = f \left(\frac{b}{|b \times \tau|} \cdot F \right) \frac{\operatorname{proj}_{\tau} b}{|b \times \tau|}. \quad (3.32)$$

Since $|b \times \tau| = |\operatorname{proj}_{\tau} b|$, we have $\frac{\operatorname{proj}_{\tau} b}{|b \times \tau|} = b^{\perp \tau}$. Noting that $F = \tau \times Tb$ is orthogonal to τ , we thus conclude

$$\frac{1}{|b \times \tau|} b \cdot F = \frac{1}{|b \times \tau|} (\operatorname{proj}_{\tau} b) \cdot F = b^{\perp \tau} \cdot F.$$

(3.28) is obtained by insertion of this equation into (3.32).

To prove (iii) observe that the right hand side of (3.31) is orthogonal to the vector b , which implies that $\partial_t(\hat{b} \cdot h_p) = \hat{b} \cdot \partial_t h_p = 0$. This equation and the assumption $b \cdot h_p(x, 0) = 0$ together imply $\hat{b} \cdot h_p(x, t) = 0$ for all $(x, t) \in \Omega \times [0, T_e]$, from which we conclude that $\text{trace}(\hat{b} \otimes h_p) = \hat{b} \cdot h_p = 0$. This is (3.4). \blacksquare

It is possible to simplify (3.31) slightly by working with a suitable coordinate system. To show this we assume that T is a given stress field and that h_p is a solution of (3.31) satisfying $\hat{b} \cdot h_p = \text{trace}(\hat{b} \otimes h_p) = 0$. Choose the cartesian coordinates (x_1, x_2, x_3) such that the x_2 -axis points into the direction of the Burgers vector b and let $h_p = (h_{p1}, h_{p2}, h_{p3})$ be the components of h_p in this coordinate system. Since $\hat{b} = (0, 1, 0)$, the equation $\hat{b} \cdot h_p = 0$ is equivalent to $h_{p2} = 0$. Using this property of h_p and noting that $\tau = \frac{\text{rot}_x h_p}{|\text{rot}_x h_p|}$, we obtain by a computation that in these coordinates

$$\begin{aligned} \text{rot}_x h_p &= (\partial_{x_2} h_{p3}, \partial_{x_3} h_{p1} - \partial_{x_1} h_{p3}, -\partial_{x_2} h_{p1}), & \hat{b} \times \text{rot}_x h_p &= -\partial_{x_2} h_p. \\ \frac{b \times \tau}{|b \times \tau|} &= \frac{b \times \text{rot}_x h_p}{|b \times \text{rot}_x h_p|} = -\frac{\partial_{x_2} h_p}{|\partial_{x_2} h_p|}, & |\text{rot}_x h_p| &= \sqrt{|\partial_{x_2} h_p|^2 + (\text{rot}_2 h_p)^2}, \end{aligned}$$

where $\text{rot}_2 h_p = \partial_{x_3} h_{p1} - \partial_{x_1} h_{p3}$. With these expressions (3.31) takes the form

$$\partial_t h_p = -f \left(-\frac{\partial_{x_2} h_p}{|\partial_{x_2} h_p|} \cdot T b \right) \frac{\partial_{x_2} h_p}{|\partial_{x_2} h_p|} \sqrt{|\partial_{x_2} h_p|^2 + (\text{rot}_2 h_p)^2}. \quad (3.33)$$

This is a system of two equations, since h_{p2} vanishes identically.

4 Dislocations moving in slip planes

We finally consider the situation, where every dislocation curve is contained in a plane and moves within this plane. Different dislocation curves can be contained in the same plane or in parallel planes. The planes are called slip planes.

Let $g \in \mathbb{R}^3$ be a unit vector, which is normal to all slip planes. Every tangent vector τ to a dislocation curve in a slip plane is normal to the vector g . Since the dislocation curve moves in the slip plane, the normal velocity v of the dislocation curve must also be normal to g . By Theorem 3.5(ii), the vector v is parallel to the vector $\text{proj}_\tau b$, which is the orthogonal projection of the Burgers vector b to the orthogonal space of τ . This implies that b itself is orthogonal to g . Therefore the vectors τ and b span the two dimensional subspace parallel to the slip planes. The vector $b \times \tau$ is normal to this subspace, hence

$$\frac{b \times \tau}{|b \times \tau|} = \pm g.$$

We insert this equation into (3.27) and obtain the evolution equation

$$\partial_t \rho = \text{rot}_x \left(\pm g f(\pm g \cdot T b) |\rho| \right). \quad (4.1)$$

The change of sign can occur at points, where the dislocation curve is a screw dislocation, that is at points where τ and b are parallel. We have not accounted for this situation in the definition (3.26) of the constitutive function $\tilde{\alpha}$, which is used to obtain (3.31). From Corollary 3.4 we know that at these points the direction of the normal velocity

is not determined by τ and b , and a separate definition of $\tilde{\alpha}$ should be given for such points. However, under our present assumption that dislocations move in slip planes, we can obtain an evolution equation valid everywhere by simply taking the $+$ -signs in (4.1). We avoid to discuss the justification of this choice in general, but simply assume in the following that f is chosen as an odd function, in which case the \pm -signs in (4.1) can obviously be replaced by $+$ -signs.

Corollary 4.1 *Assume that $T \in C([0, T_e] \times \bar{\Omega}, \mathcal{S}^3)$ is a given stress field and that $\rho_{\ell(t)} = \tau |\rho_{\ell(t)}| \in \mathcal{M}_d(\Omega)$ is the dislocation measure of a dislocation curve $\ell(t)$ to the Burgers vector $b \in \mathbb{R}^3$, which for all $t \in [0, T_e]$ is contained in a plane normal to the vector g . Then $\rho_{\ell(t)}$ is a solution of the evolution equation*

$$\partial_t \rho = \operatorname{rot}_x \left(g f(g \cdot Tb) |\rho| \right), \quad (4.2)$$

if and only if it moves with the normal speed given in (3.28).

This corollary follows immediately from Theorem 3.5(ii) and the construction of (4.2) given above.

We can simplify (4.2) by introducing a cartesian coordinate system with the x_2 -axis pointing into the direction of b and the x_3 -axis pointing into the direction of g . For a scalar function w we then have $\operatorname{rot}_x(wg) = -(g \times \nabla_x)w = \nabla_g^\perp w$, with $\nabla_g^\perp = (\partial_{x_2}, -\partial_{x_1}, 0)^T$. Equation (4.2) thus becomes

$$\partial_t \rho = \nabla_g^\perp \left(f(g \cdot Tb) |\rho| \right). \quad (4.3)$$

The equation for h_p corresponding to the evolution equation (4.2) is obtained by insertion of the vector g for $\frac{b \times \tau}{|b \times \tau|}$ in (3.31). We obtain

$$\partial_t h_p = g f(g \cdot Tb) |\operatorname{rot}_x h_p|. \quad (4.4)$$

From this equation we see that $\partial_t h_p$ is a scalar multiple of the vector g . We can assume that this is also true for the initial data $h_p(x, 0)$, from which we conclude that there is a function $\varepsilon_p : \Omega \times [0, T_e) \rightarrow \mathbb{R}$ such that for all $(x, t) \in \Omega \times [0, T_e)$

$$h_p(x, t) = \varepsilon_p(x, t) g. \quad (4.5)$$

With the coordinate system chosen as above and with the tangential gradient $\nabla_g = (\partial_{x_1}, \partial_{x_2}, 0)^T$ of the slip plane we obtain from (4.5) that

$$\operatorname{rot}_x h_p = \operatorname{rot}_x(\varepsilon_p g) = \nabla_g^\perp \varepsilon_p, \quad |\operatorname{rot}_x(\varepsilon_p g)| = |\nabla_g^\perp \varepsilon_p| = |\nabla_g \varepsilon_p|. \quad (4.6)$$

We insert the second expression and (4.5) into (4.4) and obtain the evolution equation for ε_p :

$$\partial_t \varepsilon_p = f(g \cdot Tb) |\nabla_g \varepsilon_p|.$$

For dislocations moving in slip planes this equation replaces the third equation in the system (3.29) – (3.31). The model equations for plastic deformation of materials with dislocations moving in slip planes thus take the form

$$-\operatorname{div}_x T = 0, \quad (4.7)$$

$$T = D(\varepsilon(\nabla_x u) - m \varepsilon_p), \quad (4.8)$$

$$\partial_t \varepsilon_p = f(|b| m : T) |\nabla_g \varepsilon_p|, \quad (4.9)$$

$$T|_{\partial\Omega} n_B = \gamma, \quad \text{on } \partial\Omega \times [0, T_e), \quad (4.10)$$

where m is defined in (1.4) and where we used that $\varepsilon(\hat{b} \otimes h_p) = \varepsilon(\hat{b} \otimes \varepsilon_p g) = m\varepsilon_p$ and $g \cdot T\hat{b} = |b| m : T$. Of course, this system has to be supplemented by a boundary condition and an initial condition for the function ε_p . We do not discuss this here.

Remark Let (u, T, ε_p) be a solution of (4.7) – (4.9). By (3.7) the dislocation measure to this solution is given by $\rho = \text{rot}_x h_p \in \mathcal{M}_d(\Omega)$, from which we obtain by (4.6) that $\rho = \tau|\rho| = \nabla_g^\perp \varepsilon_p$. Since $\nabla_g^\perp \varepsilon_p$ is orthogonal to $g = (0, 0, 1)^T$, we conclude that also the tangential vector field τ of the dislocation measure ρ is orthogonal to g , and this implies that the dislocation curves corresponding to solutions of the system (4.7) – (4.9) lie in slip planes $x_3 = c$ and that the function $(x_1, x_2) \mapsto \varepsilon_p(x_1, x_2, c, t) : \mathbb{R}^2 \rightarrow \mathbb{R}$ plays the role of a stream function of the dislocation measure ρ on the slip plane.

If $\rho = \rho_\ell = \tau|\rho_\ell|$ is the dislocation measure of a dislocation curve, then ε_p has a jump along the dislocation curve. By definition of ρ_ℓ in (2.19), the height of the jump is equal to the absolute value $|b|$ of the Burgers vector. If ρ is a dislocation density, then ε_p is smooth and the level curves of $(x_1, x_2) \mapsto \varepsilon_p(x_1, x_2, c, t)$ are the averaged dislocation curves.

Summary Both of the systems (3.29), (3.30), (3.33) and (4.7) – (4.9) with the corresponding evolution equations (3.27) and (4.3), respectively, satisfy the conditions (P1) – (P4). The simpler system (4.7) – (4.9) models dislocation curves which lie in slip planes. If this restrictive assumption is not made, then the more complicated equations (3.29), (3.30), (3.33) must be used.

The system (4.7) – (4.9) differs from the standard plasticity model (1.1) – (1.3) only by the term $|\nabla_g \varepsilon_p|$. From (4.6) we see that $|\rho| = |\nabla_g \varepsilon_p|$, hence $|\nabla_g \varepsilon_p|$ is the total variation measure to the dislocation measure and represents the “absolute value” of the dislocation density. If this density is high throughout the material, then $|\nabla_g \varepsilon_p|$ can be replaced approximately by a constant. In this case the system (4.7) – (4.9) reduces to the standard plasticity model (1.1) – (1.3). However, if this assumption is not valid, then (4.7) – (4.9) should be used as model for plastic deformation.

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