A tour to compact type operators and sequences related to the finite sections projection

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Dedicated to Prof. António Ferreira dos Santos

Abstract

We prove that all compact operators acting on $L^p(\mathbb{R})$ belong to the algebra generated by the operator of multiplication by the characteristic function of the positive half-axis and by the convolution operators with continuous generating function. This result, together with the similar classical result on the algebra generated by the operators of multiplication and the singular integral operator, is then used to prove that certain ideals of compact-like operator sequences in infinite products of Banach algebras are included in the algebra generated by convolution and multiplication operators and the finite section projection sequence.

1 Introduction

In [8], we studied the finite sections method for operators which are composed by operators of multiplication by a piecewise continuous function, operators of (Fourier) convolution by a piecewise continuous Fourier multiplier, and by a certain flip operator. This class of operators is extremely large; some prominent members of this class are Toeplitz plus Hankel operators on Hardy spaces H^p, and Wiener-Hopf plus Hankel operators on Lebesgue spaces L^p. The techniques we used to attack the stability problem for these operators were of algebraic nature; for example the stability of a sequence is equivalent to the invertibility of an associated element in a suitably constructed Banach algebra (some details will be given below).

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At some point in [8] we needed that certain sequences with very special properties (the sequences of compact type, mentioned in the title of the paper) belong to our algebra and form a closed ideal there. Roughly speaking, the only reason why we needed these sequences was to be able to factor them out. That's why we decided not to spend much time with them; we just made our algebra a little bit larger by including all desired and needed sequences by hand. Although this practice was successful, we were not satisfied with it. The question remained if the enlargement of the algebra was really necessary, or if the needed sequences were already contained in the smaller original algebra.

Questions of this type occur frequently in operator theory and numerical analysis. For a concrete example, suppose we are interested in the Fredholm theory of singular integral operators $aI + bS$. Here I is the identity operator, a and b are operators of multiplication by (say, continuous) functions, and S is the singular integral operator

$$(S_\Gamma u)(x) := \frac{1}{\pi \mathbf{i}} \int_\Gamma \frac{u(y)}{y - x}\, dy, \quad x \in \Gamma, \tag{1.1}$$

with the integral understood in the sense of the Cauchy principal value. It is well-known that this operator is bounded on $L^p(\Gamma)$ if $1 < p < \infty$ and if Γ is the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$ or the real line $\mathbb{R}$, for instance.

Since the Fredholm property of a bounded operator A on $L^p(\Gamma)$ is equivalent to its invertibility modulo the ideal of the compact operators on $L^p(\Gamma)$, and since invertibility is typically studied in algebras which should not be too large, this leads naturally to the question: *Is the ideal of the compact operators contained in the smallest closed algebra which contains all operators $aI + bS$ we are interested in?* In this setting, the answer is well known and turns out to be YES, and the following is a prototype of results that we will meet in this paper.

Theorem 1.1. *The ideal of the compact operators on $L^p(\Gamma)$ is contained in the smallest closed algebra which contains all singular integral operators $aI + bS$ with a, b continuous on Γ if $\Gamma = \mathbb{T}$ and continuous on the one point compactification of Γ if $\Gamma = \mathbb{R}$.*

So we decided to attack the above mentioned problem again, and after some efforts we were indeed able to show that the original algebra was already large enough to include all needed sequences. On the way to this result we will encounter a lot of results in the same spirit, both in the context of operator theory and of numerical analysis.

Throughout this paper, we let $1 < p < \infty$. Moreover, for a Banach space X, we denote the Banach algebra of all bounded linear operators on X by $\mathcal{B}(X)$ and the set of the compact operators on X by $\mathcal{K}(X)$. If $\mathcal{A}$ is a non-empty subset of $\mathcal{B}(X)$ then $\mathrm{alg}\mathcal{A}$ and $\mathrm{clos\,alg}\mathcal{A}$ stand for the smallest subalgebra and for the smallest closed subalgebra of $\mathcal{B}(X)$ which contain all operators in $\mathcal{A}$, respectively.

2 On the unit circle $\mathbb{T}$

We start our tour on the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$. Let $\mathcal{P}_{\mathbb{T}}$ stand for the algebra of all trigonometric polynomials on $\mathbb{T}$. We write the elements of $\mathcal{P}_{\mathbb{T}}$ as

$$\sum_{r=-\infty}^{+\infty} f_r t^r, \quad f_r \in \mathbb{C},$$

where only a finite number of the f_r do not vanish. Throughout what follows we suppose that $\alpha \in \mathbb{R}$ is such that

$$0 < \frac{1}{p} + \alpha < 1. \tag{2.1}$$

Let $L^p(\mathbb{T}, \alpha)$ denote the space of all Lebesgue-integrable functions f on $\mathbb{T}$ with

$$\|f\|_{L^p(\mathbb{T},\alpha)} := \left(\int_{\mathbb{T}} |f(t)|^p |1-t|^{\alpha p} \, dt \right)^{1/p} < \infty.$$

Lemma 2.1. *The following statements hold:*

(i) *$\mathcal{P}_{\mathbb{T}}$ is dense in $L^p(\mathbb{T}, \alpha)$.*

(ii) *The operator*

$$P_{\mathbb{T}} : \mathcal{P}_{\mathbb{T}} \to \mathcal{P}_{\mathbb{T}}, \quad \sum_{r=-\infty}^{+\infty} f_r t^r \mapsto \sum_{r=0}^{+\infty} f_r t^r$$

extends to a bounded linear operator on $L^p(\mathbb{T}, \alpha)$.

(iii) *Let $m \in \mathbb{Z}$. The operator*

$$M_m^{\mathbb{T}} : \mathcal{P}_{\mathbb{T}} \to \mathcal{P}_{\mathbb{T}}, \quad \sum_{r=-\infty}^{+\infty} f_r t^r \mapsto \sum_{r=-\infty}^{+\infty} f_r t^{r+m}$$

extends to a bounded linear operator on $L^p(\mathbb{T}, \alpha)$, the operator of multiplication by t^m.

Assertions (i) and (ii) are taken from [1, 1.44 and 5.9], whereas (iii) is evident since $|t^m| = 1$. We denote the extensions of the operators in (ii) and (iii) by $P_{\mathbb{T}}$ and $M_m^{\mathbb{T}}$ again and remark that $\|M_m^{\mathbb{T}}\|_{L(L^p(\mathbb{T},\alpha))} = 1$ for $m \in \mathbb{Z}$.

For $u, v \in \mathcal{P}_{\mathbb{T}}$, consider the operator

$$K_{u,v} : \mathcal{P}_{\mathbb{T}} \to \mathcal{P}_{\mathbb{T}}, \quad f \mapsto \langle f, u \rangle \, v$$

where $\langle f, u \rangle := \int_{\mathbb{T}} f(t)\overline{u(t)}\, dt$.

Lemma 2.2. *$K_{u,v} \in \operatorname{alg}\left\{P_{\mathbb{T}}, M_{\pm 1}^{\mathbb{T}}\right\}$ for $u, v \in \mathcal{P}_{\mathbb{T}}$.*

Proof. It is sufficient to prove the assertion for $u(t) = t^k$ and $v(t) = t^l$, with $k, l \in \mathbb{Z}$. For these u,v and for $f \in \mathcal{P}_{\mathbb{T}}$, it is

$$K_{u,v} f = \langle f, u \rangle \, v = \left\langle f, M_k^{\mathbb{T}} \mathbf{1} \right\rangle M_l^{\mathbb{T}} \mathbf{1} = \left\langle M_{-k}^{\mathbb{T}} f, \mathbf{1} \right\rangle M_l^{\mathbb{T}} \mathbf{1}$$

which implies that

$$K_{u,v} = M_l^{\mathbb{T}} K_{\mathbf{1},\mathbf{1}} M_{-k}^{\mathbb{T}} = (M_1^{\mathbb{T}})^l K_{\mathbf{1},\mathbf{1}} (M_{-1}^{\mathbb{T}})^k, \tag{2.2}$$

where $\mathbf{1}$ refers to the constant function $t \mapsto 1$ on $\mathbb{T}$. Further,

$$K_{\mathbf{1},\mathbf{1}} = P_{\mathbb{T}} - M_1^{\mathbb{T}} P_{\mathbb{T}} M_{-1}^{\mathbb{T}}. \tag{2.3}$$

The identities (2.2) and (2.3) imply that $K_{u,v} \in \operatorname{alg}\left\{P_{\mathbb{T}}, M_{\pm 1}^{\mathbb{T}}\right\}$ for all $k, l \in \mathbb{Z}$, whence the assertion. $\square$

Since $\mathcal{P}_{\mathbb{T}}$ is dense in $L^p(\mathbb{T}, \alpha)$ and in $(L^p(\mathbb{T}, \alpha))^* = L^q(\mathbb{T}, -\alpha)$, with $1/p + 1/q = 1$, by Lemma 2.1, the operators $K_{u,v}$, with $u, v \in \mathcal{P}_{\mathbb{T}}$, span a dense subset of $K(L^p(\mathbb{T}, \alpha))$. So we conclude from Lemma 2.2 that

Theorem 2.3. $\mathcal{K}(L^p(\mathbb{T}, \alpha)) \subseteq \operatorname{clos} \operatorname{alg}\left\{P_{\mathbb{T}}, M_{\pm 1}^{\mathbb{T}}\right\} = \operatorname{clos} \operatorname{alg}\{P_{\mathbb{T}}, C(\mathbb{T}) I\}$.

3 From $\mathbb{T}$ to $\mathbb{R}$

Given $p \in (1, \infty)$, we now specify $\alpha := 1 - 2/p$. Note that then

$$1 < p < \infty \Leftrightarrow 0 < 1/p < 1 \Leftrightarrow 0 < 1 - 1/p < 1 \Leftrightarrow 0 < 1/p + \alpha < 1$$

for this special value of α. Hence, the pair (p, α) satisfies (2.1). The basic observation to pass from $\mathbb{T}$ to $\mathbb{R}$ is given by the following lemma, whose proof can be found in [3, Chapter 1, Theorem 5.1] and in [5, page 56].

Lemma 3.1. *The operator*

$$B : L^p(\mathbb{R}) \to L^p(\mathbb{T}, \alpha), \quad (B\varphi)(t) := \frac{1}{t-1}\varphi\left(\mathbf{i}\frac{t+1}{t-1}\right) \quad (t \in \mathbb{T})$$

is bounded and invertible. Its inverse is given by

$$B^{-1} : L^p(\mathbb{T}, \alpha) \to L^p(\mathbb{R}), \quad (B\psi)(s) := \frac{2\mathbf{i}}{s-\mathbf{i}}\psi\left(\frac{s+\mathbf{i}}{s-\mathbf{i}}\right) \quad (s \in \mathbb{R}).$$

Note that similar operators are used in [1, Section 9.1]. The transformation this last reference leaves the natural orientations of $\mathbb{R}$ and $\mathbb{T}$ invariant.

Assertion (i) of the following lemma is evident; assertion (ii) is proved in [5, page 56] and [5, page 56] again, with the difference that the authors of the first mentioned reference arrive at $B^{-1}S_{\mathbb{T}}B = +S_{\mathbb{R}}$ (with a plus sign). For that reason, we sketch the proof here.

Lemma 3.2. (i) $B^{-1}M_m^{\mathbb{T}}B =: M_m^{\mathbb{R}}$ *is the operator of multiplication by the function* $s \mapsto (\frac{s+\mathbf{i}}{s-\mathbf{i}})^m$ *for every* $m \in \mathbb{Z}$*;*

(ii) $B^{-1}S_{\mathbb{T}}B = -S_{\mathbb{R}}$*.*

Proof. As already mentioned, we only prove the second assertion. First note that

$$(B^{-1}S_{\mathbb{T}}B\varphi)(s) = \frac{2\mathbf{i}}{\pi\mathbf{i}(s-\mathbf{i})}\int_{\mathbb{T}} \frac{\varphi(\mathbf{i}\frac{x+1}{x-1})}{(x-1)(x-\frac{s+\mathbf{i}}{s-\mathbf{i}})}\,dx. \tag{3.1}$$

We substitute $\mathbf{i}\frac{x+1}{x-1} = t$, respective $x = \frac{t+\mathbf{i}}{t-\mathbf{i}}$, and

$$\frac{dx}{dt} = \frac{(t-\mathbf{i})-(t+\mathbf{i})}{(t-\mathbf{i})^2} = \frac{-2\mathbf{i}}{(t-\mathbf{i})^2}.$$

Note that if t moves on $\mathbb{R}$ from 0 to $+\infty$, then x moves on $\mathbb{T}$ in the clockwise direction. Since the standard orientation on $\mathbb{T}$ is the counter-clockwise one, this gives a minus sign. Thus, (3.1) becomes

$$\begin{aligned}
(B^{-1}S_{\mathbb{T}}B\varphi)(s) &= \frac{-2}{\pi(s-\mathbf{i})}\int_{\mathbb{R}} \frac{\varphi(t)}{(\frac{t+\mathbf{i}}{t-\mathbf{i}}-1)(\frac{t+\mathbf{i}}{t-\mathbf{i}}-\frac{s+\mathbf{i}}{s-\mathbf{i}})}\frac{-2\mathbf{i}}{(t-\mathbf{i})^2}\,dt \\
&= \frac{4\mathbf{i}}{\pi(s-\mathbf{i})}\int_{\mathbb{R}} \frac{\varphi(t)}{(t+\mathbf{i}-(t-\mathbf{i}))\big(t+\mathbf{i}-\frac{(s+\mathbf{i})(t-\mathbf{i})}{s-\mathbf{i}}\big)}\,dt \\
&= \frac{4\mathbf{i}}{\pi}\int_{\mathbb{R}} \frac{\varphi(t)}{(t+\mathbf{i}-(t-\mathbf{i}))((t+\mathbf{i})(s-\mathbf{i})-(s+\mathbf{i})(t-\mathbf{i}))}\,dt \\
&= \frac{4\mathbf{i}}{\pi}\int_{\mathbb{R}} \frac{\varphi(t)}{2\mathbf{i}(\mathbf{i}s-\mathbf{i}t-\mathbf{i}t+\mathbf{i}s)}\,dt \\
&= -\frac{1}{\mathbf{i}\pi}\int_{\mathbb{R}} \frac{\varphi(t)}{t-s}\,dt \;=\; -(S_{\mathbb{R}}\varphi)(s).
\end{aligned}$$

□

Corollary 3.3. *With $P_\Gamma := (I+S_\Gamma)/2$ and $Q_\Gamma := (I-S_\Gamma)/2$, one obtains*

$$B^{-1}P_{\mathbb{T}}B = Q_{\mathbb{R}}, \quad B^{-1}Q_{\mathbb{T}}B = P_{\mathbb{R}}.$$

The following is just a translation of the corresponding results on $\mathbb{T}$ stated in Lemmas 2.1 and 2.2 and in Theorem 2.3.

Lemma 3.4. (i) *The set $\mathcal{P}_{\mathbb{R}} := B^{-1}\mathcal{P}_{\mathbb{T}}$ is dense in $L^p(\mathbb{R})$.*

(ii) *For $u, v \in \mathcal{P}_{\mathbb{R}}$ the operator $K_{u,v} : \mathcal{P}_{\mathbb{R}} \to \mathcal{P}_{\mathbb{R}}, f \mapsto \langle f, u\rangle_{\mathbb{R}}\, v$ belongs to $\operatorname{alg}\{Q_{\mathbb{R}}, M^{\mathbb{R}}_{\pm 1}\}$.*

(iii) $K(L^p(\mathbb{R})) \subseteq \operatorname{clos}\operatorname{alg}\{Q_{\mathbb{R}}, M^{\mathbb{R}}_{\pm 1}\}$.

Let $J : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ denote the flip operator $(Jf)(t) := f(-t)$. It is well known that

$$JP_{\mathbb{R}}J = Q_{\mathbb{R}}, \quad JQ_{\mathbb{R}}J = P_{\mathbb{R}} \tag{3.2}$$

and easy to check that

$$(JM^{\mathbb{R}}_m Jf)(s) = \left(\frac{-s+\mathbf{i}}{-s-\mathbf{i}}\right)^m f(s) = \left(\frac{s+\mathbf{i}}{s-\mathbf{i}}\right)^{-m} f(s),$$

whence

$$JM^{\mathbb{R}}_m J = M^{\mathbb{R}}_{-m} \quad \text{for } m \in \mathbb{Z}. \tag{3.3}$$

Summarizing Lemma 3.4 (iii) and (3.2)-(3.3) we arrive at the next stop of our tour.

Theorem 3.5. $\mathcal{K}(L^p(\mathbb{R})) \subseteq \operatorname{clos}\operatorname{alg}\{P_{\mathbb{R}}, M^{\mathbb{R}}_{\pm 1}\} = \operatorname{clos}\operatorname{alg}\Big\{P_{\mathbb{R}}, C(\dot{\mathbb{R}})\Big\}$.

Here, $\dot{\mathbb{R}}$ stands for the compactification of the real line by one point ∞.

4 From $\mathbb{R}$ to $\mathbb{R}$ by Fourier transform

The next step will lead us to a statement which can be viewed as the Fourier-symmetric version of Theorem 3.5. We define the Fourier transform for functions in the Schwartz space by

$$(Fu)(y) = \int_{-\infty}^{+\infty} e^{-2\pi \mathbf{i} yx} u(x)\, dx, \quad y \in \mathbb{R}, \tag{4.1}$$

then its inverse is given by

$$(F^{-1}v)(x) = \int_{-\infty}^{+\infty} e^{2\pi \mathbf{i} xy} v(y)\, dy, \quad x \in \mathbb{R}. \tag{4.2}$$

It is well known that F and F^{-1} extend continuously to bounded and unitary operators on the Hilbert space $L^2(\mathbb{R})$, which we denote by F and F^{-1} again. Thus, if A is a bounded operator on $L^2(\mathbb{R})$, then the composition $F^{-1}AF$ is well defined, and it is a bounded on $L^2(\mathbb{R})$ again.

We call an operator $A \in \mathcal{B}(L^2(\mathbb{R}))$ a *p-Fourier multiplier* if $F^{-1}AFu \in L^p(\mathbb{R})$ whenever $u \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ and there is a constant c_p such that $\|F^{-1}AFu\|_p \le c_p \|u\|_p$ for all $u \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$. If A owns this property, then the composition $F^{-1}AF$ extends continuously to a bounded operator on $L^p(\mathbb{R})$. We denote this extension by A^F and call it the *Fourier image* of A. For some general facts on these operators, see [8].

It is well known that $P_{\mathbb{R}}$ and $M^{\mathbb{R}}_{\pm 1}$ are p-Fourier multipliers for every $p \in (1, \infty)$ (note that the functions $s \mapsto (\frac{s+i}{s-i})^m$ have a bounded total variation on $\mathbb{R}$; so they are Fourier multipliers by Stechkin'n inequality, see [1, 9.3 (e)]) and that $P^F_{\mathbb{R}}$ is the operator of multiplication by the characteristic function of $[0, \infty)$. It makes thus sense to consider

$$\operatorname{alg}\left\{P^F_{\mathbb{R}},\, (M^{\mathbb{R}}_{\pm 1})^F\right\} = \operatorname{alg}\left\{\chi_+ I,\, (M^{\mathbb{R}}_{\pm 1})^F\right\}.$$

By Lemma 3.4(ii), this algebra contains all operators $K^F_{u,v}$ with $u,\, v \in \mathcal{P}_{\mathbb{R}}$ (here we only use the algebraic properties of the mapping $A \mapsto A^F$). Since

$$K^F_{u,v}\varphi = \langle F\varphi,\, u\rangle\, F^{-1}v = \left\langle \varphi,\, F^{-1}u\right\rangle F^{-1}v = K_{F^{-1}u,\, F^{-1}v}\varphi,$$

it is $K^F_{u,v} = K_{F^{-1}u, F^{-1}v}$, and we conclude that

$$K_{F^{-1}u, F^{-1}v} \in \operatorname{alg}\left\{\chi_+ I,\, (M^{\mathbb{R}}_{\pm 1})^F\right\} \quad \text{for } u, v \in \mathcal{P}_{\mathbb{R}}. \tag{4.3}$$

It would follow from this line that $\operatorname{clos}\operatorname{alg}\left\{\chi_+ I, (M^{\mathbb{R}}_{\pm 1})^F\right\}$ contains *all* compact operators, if we would know that the span of $\left\{K_{F^{-1}u,F^{-1}v} : u, v \in \mathcal{P}_{\mathbb{R}}\right\}$ is dense in $\mathcal{K}(L^p(\mathbb{R}))$. This, on its hand, would be clear if we would know that $F^{-1}\mathcal{P}_{\mathbb{R}}$ is dense in $L^p(\mathbb{R})$, for every $p \in (1, +\infty)$. We are going to show this now.

Recall that $\mathcal{P}_{\mathbb{R}} = B^{-1}\mathcal{P}_{\mathbb{R}}$ is generated by the functions

$$s \mapsto \frac{1}{s - \mathbf{i}} \left(\frac{s + \mathbf{i}}{s - \mathbf{i}}\right)^m, \quad m \in \mathbb{Z}.$$

The inverse Fourier transforms of these functions can be calculated using residue calculus (see the theorem in [7, Section 14.2.1]). What results is that $F^{-1}\mathcal{P}_{\mathbb{R}}$ consists of all functions of the form

$$r(t) = \begin{cases} e^{2\pi t} p_1(t) & \text{if} \quad t < 0 \\ e^{-2\pi t} p_2(t) & \text{if} \quad t \geq 0 \end{cases} \tag{4.4}$$

where p_1 and p_2 are (algebraic) polynomials. The functions in (4.4) are dense in $L^1(\mathbb{R})$ (see [2, Section I.8]). We need the same property for $L^p(\mathbb{R})$ with $p > 1$. It is clearly sufficient to prove this for the semi-axes considered separately.

Lemma 4.1. *$\left\{e^{-at} f(t) : f \text{ a polynomial}\right\}$ is dense in $L^p(\mathbb{R}^+)$ for $p > 1$ and $a > 0$.*

Proof. The result is essentially stated in [4]. The argument runs as follows. Rescaling we can assume that $a = 1$. Because C_0^∞ is dense in $L^p(\mathbb{R}^+)$, it suffices to show that every function in C_0^∞ can be approximated in the L^p norm by functions of the form $e^{-t} f(t)$ with f a polynomial. So let $u \in C_0^\infty$. Then $e^t u$ is still in C_0^∞. If now Π_n denotes the set of all polynomials of degree less than or equal to n then, by [4, 2.5.32],

$$\inf_{p \in \Pi_n} \left\| e^t u - f \right\|_{L^p(\mathbb{R}^+, e^{-t})} \leq C w(e^t u, 1/\sqrt{n}), \tag{4.5}$$

where w is a (certain) module of continuity introduced in [4]. Since

$$\left\| e^t u - f \right\|_{L^p(\mathbb{R}^+, e^{-t})} = \left\| (e^t u - f) e^{-t} \right\|_{L^p(\mathbb{R}^+)} = \left\| u - e^{-t} p \right\|_{L^p(\mathbb{R}^+)}$$

and $w(e^t u, 1/\sqrt{n}) \to 0$ as $n \to \infty$, the estimate (4.5) indeed implies the desired density result. □

Corollary 4.2. *The following holds for every $p \in (1, \infty)$:*

(i) $F^{-1}\mathcal{P}_\mathbb{R}$ *is dense in* $L^p(\mathbb{R})$*;*

(ii) $\operatorname{span}\left\{K_{F^{-1}u,F^{-1}v} : u, v \in \mathcal{P}_\mathbb{R}\right\}$ *is dense in* $\mathcal{K}(L^p(\mathbb{R}))$*.*

We already mentioned that every operator of multiplication by a continuous function a with bounded total variation on $\mathbb{R}$ is a Fourier multiplier. We denote the closure in the norm of $\mathcal{B}(L^p(\mathbb{R}))$ of the set of all operators $(aI)^F$ with a of this form by $W^0(C_p)$, in accordance with the notation in [8]. Thus, $W^0(C_p)$ is a closed subalgebra of $\mathcal{B}(L^p(\mathbb{R}))$. The following is then an immediate consequence of the preceding corollary.

Theorem 4.3. *It is*

$$\mathcal{K}(L^p(\mathbb{R})) \subseteq \operatorname{clos\,alg}\left\{\chi_+ I,\ (M^{\mathbb{R}}_{\pm 1})^F\right\} = \operatorname{clos\,alg}\left\{\chi_+ I,\ W^0(C_p)\right\}.$$

This is the end point on the operator theory side of our tour. We would not like to stop without mentioning that there is a lot of results of the same spirit in the literature; see, e.g., [1, 9.9] and [6, Proposition 3.3.1].

5 On the side of numerical analysis

Now we turn to the side of numerical analysis. First we introduce an algebra the role of which is comparable with that of the algebra $\mathcal{B}(L^p(\mathbb{R}))$ in operator theory. Let $\mathcal{E}$ denote the set of all bounded functions $\mathbf{A} : (0, +\infty) \to \mathcal{B}(L^p(\mathbb{R}))$, and write A_τ for the value of $\mathbf{A} \in \mathcal{E}$ at $\tau \in (0, +\infty)$. Sometimes we will also use the notation $(A_\tau)_{\tau>0}$ in place of $\mathbf{A}$. Provided with pointwise defined operations and the norm

$$\|\mathbf{A}\|_\mathcal{E} := \sup_{\tau\in(0,+\infty)} \|A_\tau\|_{\mathcal{B}(L^p(\Gamma))},$$

$\mathcal{E}$ becomes a Banach algebra, and the set $\mathcal{G}$ of all functions $\mathbf{G} \in \mathcal{E}$ for which $\lim_{\tau\to\infty} \|G_\tau\| = 0$ forms a closed two-sided ideal of $\mathcal{E}$. Every operator $A \in \mathcal{B}(L^p(\mathbb{R}))$ gives rise to a constant function $\tau \mapsto A$ in $\mathcal{E}$ which we denote by A again. The importance of the quotient algebra $\mathcal{E}/\mathcal{G}$ stems from the following elementary, but basic, observation: a function $\mathbf{A} = (A_\tau) \in \mathcal{E}$ is stable in the sense of numerical analysis if and only if the coset $\mathbf{A} + \mathcal{G}$ is invertible in $\mathcal{E}/\mathcal{G}$ (see, for instance, [10, Section 6.2]).

To state our results we need some more notation. For $s \in \mathbb{R}$, let $(V_s u)(x) := u(x - s)$ be the operator of shift by s on $L^p(\mathbb{R})$, and let U_s be the operator of multiplication by the function $x \mapsto e^{-2\pi \mathbf{i} xs}$. For $\tau > 0$

let P_τ denote the operator of multiplication by the characteristic function of the interval $[-\tau,\tau]$, set $Q_\tau := I - P_\tau$, and define R_τ, S_τ and $S_{-\tau}$ by

$$(R_\tau u)(x) = \begin{cases} u(\tau - x) & \text{if } \ 0 < x < \tau \\ u(-\tau - x) & \text{if } \ -\tau < x < 0 \\ 0 & \text{if } \ |x| > \tau \end{cases}, \tag{5.1}$$

$$(S_\tau u)(x) = \begin{cases} 0 & \text{if } \ |x| < \tau \\ u(x-\tau) & \text{if } \ x > \tau \\ u(x+\tau) & \text{if } \ x < -\tau \end{cases}, \tag{5.2}$$

$$(S_{-\tau} u)(x) = \begin{cases} u(x+\tau) & \text{if } \ x > 0 \\ u(x-\tau) & \text{if } \ x < 0 \end{cases}. \tag{5.3}$$

These operators are bounded and have norm 1 on every $L^p(\mathbb{R})$. If $\chi_\pm$ denotes the characteristic function of the positive (negative) semi-axis of $\mathbb{R}$, then

$$\chi_\pm P_\tau = \chi_\pm V_{\pm\tau}\chi_\mp V_{\mp\tau}\chi_\pm = \chi_\pm V_{\pm\tau}\chi_\mp V_{\mp\tau} = V_{\pm\tau}\chi_\mp V_{\mp\tau}\chi_\pm, \tag{5.4}$$

$$\chi_\pm R_\tau = J\chi_\mp V_{\mp\tau}\chi_\pm I = \chi_\pm V_{\pm\tau}\chi_\mp J, \tag{5.5}$$

$$\chi_\pm S_\tau = V_{\pm\tau}\chi_\pm I, \tag{5.6}$$

$$\chi_\pm S_{-\tau} = \chi_\pm V_{\mp\tau}. \tag{5.7}$$

Further we adopt our earlier notation and let now $\operatorname{clos\,alg} M$ stand for the smallest closed subalgebra of $\mathcal{E}$ which contains all sequences in the subset M of $\mathcal{E}$. (There will be no confusion because if M consists of constant sequences only, then $\operatorname{clos\,alg} M$ also consists of constant sequences and can, hence, be identified with a subalgebra of $\mathcal{B}(L^p(\mathbb{R}))$.)

The sequences in Theorems 5.1, 5.2 and 5.3 below are the "compact type sequences" addressed to in the title of the paper.

Theorem 5.1. *The sequence* $(K_1 + V_{-\tau}K_2V_\tau + V_\tau K_3 V_{-\tau})_{\tau>0}$ *belongs to the algebra* $\operatorname{clos\,alg}\{\chi_+, W^0(C_p), (P_\tau)_{\tau>0}\}$ *for* $K_1,\ K_2,\ K_3 \in \mathcal{K}(L^p(\mathbb{R}))$.

Proof. Let $K \in \mathcal{K}(L^p(\mathbb{R}))$. Then $K \in \operatorname{clos\,alg}\{\chi_+ I, W^0(C_p)\}$ by Theorem 4.3. Hence, and because the operators in $W^0(C_p)$ are shift invariant,

$$\begin{aligned}(V_{-\tau}KV_\tau)_{\tau>0} &\in \operatorname{clos\,alg}\{(V_{-\tau}\chi_+V_\tau)_{\tau>0},\ W^0(C_p)\} \\ &= \operatorname{clos\,alg}\{(\chi_{[-t,+\infty)})_{\tau>0},\ W^0(C_p)\} \\ &= \operatorname{clos\,alg}\{(P_\tau + \chi_+Q_\tau)_{\tau>0},\ W^0(C_p)\}.\end{aligned}$$

Similarly,

$$\begin{aligned}(V_\tau KV_{-\tau})_{\tau>0} &\in \operatorname{clos\,alg}\{(V_\tau\chi_+V_{-\tau})_{\tau>0},\ W^0(C_p)\} \\ &= \operatorname{clos\,alg}\{(\chi_{[t,+\infty)})_{\tau>0},\ W^0(C_p)\} \\ &= \operatorname{clos\,alg}\{(\chi_+Q_\tau)_{\tau>0},\ W^0(C_p)\},\end{aligned}$$

which implies the assertion. □

Theorem 5.2. *Let K_1, K_2, $K_3, K_4 \in \mathcal{K}(L^p(\mathbb{R}))$. Then the sequence*

$$(R_\tau K_1 R_\tau + R_\tau K_2 S_{-\tau} + S_\tau K_3 R_\tau + S_\tau K_4 S_{-\tau})_{\tau>0}$$

belongs to the algebra $\operatorname{clos\,alg}\left\{J,\, \chi_+,\, W^0(C_p),\, (P_\tau)_{\tau>0}\right\}$.

Proof. First consider $(R_\tau K R_\tau)_{\tau>0}$ with K compact. Write this sequence as

$$(R_\tau \chi_+ K \chi_+ R_\tau) + (R_\tau \chi_+ K \chi_- R_\tau) + (R_\tau \chi_- K \chi_+ R_\tau) + (R_\tau \chi_- K \chi_- R_\tau).$$

By (5.5)-(5.7), the latter is equal to

$$\begin{aligned}(\chi_+ V_\tau \chi_- JKJ \chi_- V_{-\tau} \chi_+ I) + (\chi_+ V_\tau \chi_- JKJ \chi_+ V_\tau \chi_- I)\\ + (\chi_- V_{-\tau} \chi_+ JKJ \chi_- V_{-\tau} \chi_+ I) + (\chi_- V_{-\tau} \chi_+ JKJ \chi_+ V_\tau \chi_- I)\end{aligned} \tag{5.8}$$

The first and the last sequence in (5.8) are of the form

$$(\chi_+ V_\tau K_1 V_{-\tau} \chi_+ I) \quad \text{and} \quad (\chi_- V_{-\tau} K_2 V_\tau \chi_- I), \tag{5.9}$$

with $K_1 := \chi_- JKJ \chi_-$ and $K_2 := \chi_+ JKJ \chi_+$ compact. These sequences are in

$$\operatorname{clos\,alg}\left\{\chi_+, W^0(C_p), (P_\tau)_{\tau>0}\right\}.$$

by Theorem 5.1. The second sequence in (5.8) can be written as

$$(J\chi_- V_{-\tau} \chi_+ KJ \chi_+ V_\tau \chi_- I) = (J \chi_- V_{-\tau} K_3 V_\tau \chi_- I)$$

with $K_3 := \chi_+ KJ \chi_+$ compact. Again by Theorem 5.1, this sequence is in

$$\operatorname{clos\,alg}\left\{J, \chi_+, W^0(C_p), (P_\tau)_{\tau>0}\right\}.$$

Similarly, the third sequence in (5.8) is in this algebra. Thus, the assertion is proved for the sequences $(R_\tau K R_\tau)$. The other sequences can be treated similarly. □

Theorem 5.3. *Let K_1, K_2, $K_3, K_4 \in \mathcal{K}(L^p(\mathbb{R}))$. Then the sequence*

$$(R_\tau^F K_1 R_\tau^F + R_\tau^F K_2 S_{-\tau}^F + S_\tau^F K_3 R_\tau^F + S_\tau^F K_4 S_{-\tau}^F)_{\tau>0}$$

belongs to the algebra $\operatorname{clos\,alg}\left\{J,\, P_\mathbb{R},\, C(\dot{\mathbb{R}}),\, (P_\tau^F)\right\}$.

Proof. Let K be a compact operator. Again starting from Theorem 3.5, we get $K \in \operatorname{clos\,alg}\left\{P_{\mathbb{R}},\, C(\dot{\mathbb{R}})\right\}$ and, since the operators in $C(\dot{\mathbb{R}})$ commute with the U_s,

$$(U_{-s}KU_s)_{s>0} \in \operatorname{clos\,alg}\left\{(U_{-s}P_{\mathbb{R}}U_s)_{s>0},\, C(\dot{\mathbb{R}})\right\}.$$

Now, from

$$\begin{aligned} U_{-s}P_{\mathbb{R}}U_s &= U_{-s}W^0(\chi_+)U_s \;=\; F^{-1}V_s\chi_+V_{-s}F \\ &= W^0(\chi_{[s,+\infty)}) \\ &= W^0(\chi_+)W^0(\chi_{(-\infty,-s]} + \chi_{[s,+\infty)}) \\ &= P_{\mathbb{R}}Q_s^F = P_{\mathbb{R}}(1 - P_s^F), \end{aligned}$$

we conclude that $(U_{-s}KU_s)_{s>0} \in \operatorname{clos\,alg}\left\{P_{\mathbb{R}},\, C(\dot{\mathbb{R}}),\, (P_\tau^F)\right\}$. Similarly, the sequence $(U_sKU_{-s})_{s>0}$ belongs to this algebra. We now continue as in the proof of the previous theorem to get the assertion. □

6 Why we need these results

We will now briefly indicate where and why the results of Theorems 5.1, 5.2 and 5.3 are useful.

We say that a bounded function $\mathbf{A} : (0,+\infty) \to \mathcal{B}(L^p(\mathbb{R}))$ converges **-strongly* if it converges strongly as $t \to \infty$ and if the adjoint function $\mathbf{A}^*$ (which takes the value A_t^* at the point t) converges strongly on the dual space as $\tau \to \infty$. The *-strong limit of $\mathbf{A}$ is denoted by s-lim* $\mathbf{A}$.

Let $\{\mathsf{W}_{t,\bullet}\}_{t\in\mathsf{T}}$ be a family of algebra automorphisms with the following properties:

1. $0 \in \mathsf{T}$, and $\mathsf{W}_{0,\bullet}$ is the identity automorphism;
2. $\|\mathsf{W}_{t,\bullet}\| = 1$ for every $t \in \mathsf{T}$;
3. $\mathsf{W}_{t,\bullet}(\mathbf{A})^* = \mathsf{W}_{t,\bullet}(\mathbf{A}^*)$ for every $\mathbf{A} \in \mathcal{E}$ and $t \in \mathsf{T}$;
4. $\text{s-lim}^*\, \mathsf{W}_{t,\bullet}(\mathsf{W}_{s,\bullet}^{-1}(\mathbf{A})) = 0$ for every $\mathbf{A} \in \mathcal{E}$ and $t \neq s$.

Define now $\mathcal{F}$ as the set of all functions $\mathbf{A} \in \mathcal{E}$ with the property that, for every $t \in \mathbb{T}$, the function $\mathsf{W}_{t,\bullet}(\mathbf{A})$ converges *-strongly, and set

$$\mathsf{W}_t(\mathbf{A}) := \text{s-lim}^*\, \mathsf{W}_{t,\bullet}(\mathbf{A}).$$

The set $\mathcal{F}$ is a closed and inverse-closed subalgebra of $\mathcal{E}$ that includes the ideal $\mathcal{G}$, the mappings W_t act as bounded homomorphisms on $\mathcal{F}$, and the ideal $\mathcal{G}$ is in the kernel of each of these homomorphisms [9, Proposition 4.1]. Moreover, the sets

$$\mathcal{J}_t := \mathsf{W}_{t,\bullet}^{-1}(\mathcal{K}) + \mathcal{G}, \tag{6.1}$$

where $\mathcal{K}$ is the ideal of compact operators, are closed two-sided ideals of $\mathcal{F}$. The relation between the ideals $\mathcal{J}_t$ and the algebra $\mathcal{F}$ can be seen as similar to the relation between $\mathcal{K}$ and $\mathcal{B}(L^p(\Gamma))$.

Given a family of operators in $\mathcal{B}(L^p(\Gamma))$ and a sequence of projections P_τ with complementary projections $Q_\tau := I - P_\tau$ such that $\text{s-lim}^* P_\tau = I$, one tries to find a suitable family of compatible automorphisms $\{\mathsf{W}_{t,\bullet}\}_{t\in\mathsf{T}}$ so that it is possible to characterize invertibility in $\mathcal{F}/\mathcal{G}$.

If the family of operators belongs to the subalgebra of multiplication and convolution operators on $L^p(\mathbb{R})$ generated by piecewise continuous functions, and if $P_\tau = \chi_{[-\tau,\tau]}I$, the operator of multiplication by the characteristic function of the interval $[-\tau, \tau]$, then the relevant automorphisms $\mathsf{W}_{t,\bullet}$ are

$$\begin{aligned} \mathsf{W}_{0,\bullet} &: (A_\tau) \mapsto (A_\tau); \\ \mathsf{W}_{-1,\bullet} &: (A_\tau) \mapsto (V_{-\tau} A_\tau V_\tau); \\ \mathsf{W}_{1,\bullet} &: (A_\tau) \mapsto (V_\tau A_\tau V_{-\tau}) \end{aligned}$$

(see [9]). This simple picture changes if one also wants to consider Hankel operators. Then it is necessary to include the flip operator $(Ju)(x) := u(-x)$ into the family of operators. But because the (constant) sequence (J) is not included in the algebra $\mathcal{F}$ defined by the above family of automorphisms, a more complex construction is necessary.

Instead of considering only automorphisms in $\mathcal{E}$ we consider now also a homomorphism between the algebras $\mathcal{E}$ and $\mathcal{E}^{2\times 2}$ given by as

$$\mathsf{W}_{1,\bullet} : (A_\tau) \mapsto \left(\begin{bmatrix} R_\tau \\ S_{-\tau} \end{bmatrix} A_\tau \begin{bmatrix} R_\tau & S_\tau \end{bmatrix} \right) \tag{6.2}$$

(see [8]). In this regard note that

$$\begin{bmatrix} R_\tau & S_\tau \end{bmatrix} \begin{bmatrix} R_\tau \\ S_{-\tau} \end{bmatrix} = R_\tau R_\tau + S_\tau S_{-\tau} = P_\tau + Q_\tau = I.$$

It is also possible to consider the projection $P_\tau^F := F^{-1} P_\tau F$ associated with the Fourier finite section method. In this case, the homomorphism is defined as [8]:

$$\mathsf{W}_{1,\bullet}^F : (A_\tau) \mapsto \left(\begin{bmatrix} R_\tau^F \\ S_{-\tau}^F \end{bmatrix} A_\tau \begin{bmatrix} R_\tau^F & S_\tau^F \end{bmatrix} \right). \tag{6.3}$$

Summarizing, the results in Section 5 show that the ideal $\mathcal{J}_1$ defined by (6.1) using the inverse of (6.2) applied to $\mathcal{K}^{2\times 2}$ belongs to the subalgebra of $\mathcal{F}$ generated by the constant sequences of the singular integral operator and the operators of multiplication by continuous functions, and the non-constant projection sequence (P_τ). The same holds in the Fourier-symmetric setting, that is the ideal $\mathcal{J}_1^F$ related to (6.3) is generated by convolution operators with continuous symbol by the operator of multiplication by the characteristic function of the positive half-axis, and by the projection sequence (P_τ^F).

Note that we have not proved that the ideal $\mathcal{G}$ belongs to the algebra $\operatorname{clos\,alg}\left\{PC(\mathbb{R}), W^0(PC_p), J, (P_\tau), (P_\tau^F)\right\}$. Thus, Theorems 5.2 and 5.3 do *not* imply that the ideals $\mathcal{J}_1$ and $\mathcal{J}_1^F$ belong to that algebra. But the ideal $\mathcal{G}$ can be explicitly introduced and then be factored out, because one is usually interested in invertibility on $\mathcal{E}/\mathcal{G}$. In any case, we have

$$\mathcal{J}_0/\mathcal{G},\ \mathcal{J}_1/\mathcal{G},\ \mathcal{J}_1^F/\mathcal{G} \subseteq \operatorname{clos\,alg}\left\{PC, PC_p, \mathcal{J}, (P_\tau), (P_\tau^F), \mathcal{G}\right\}/\mathcal{G}.$$

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