Finite section approximation in an algebra of convolution, multiplication and flip operators on $L^p(\mathbb{R})$

Steffen Roch, Pedro A. Santos*

Abstract

This paper is concerned with the approximation of solutions of operator equations using the finite sections method with the operators belonging to the closed subalgebra of $\mathcal{B}(L^p(\mathbb{R}))$, $1 < p < \infty$, generated by operators of multiplication by piecewise continuous functions in $\dot{\mathbb{R}}$, operators of convolution by piecewise continuous Fourier multipliers and the flip operator. This algebra includes Wiener-Hopf and Hankel operators with piecewise continuous symbols. To prove the result, we use algebraic techniques and introduce a larger algebra of sequences, which contains the special sequences we are interested and the usual operator algebra generated by the operators of multiplication, convolution and flip. There is a direct relationship between the applicability of the finite section method for a given operator and invertibility of the corresponding sequence in this algebra. Exploring this relationship and using local principles, we construct locally equivalent representations that allow to derive invertibility criteria.

1 Introduction

Several problems in diffraction theory lead to Toeplitz or Wiener-Hopf plus Hankel operators. In [5], for instance, the authors consider classes of problems of wave diffraction by a plane angular screen occupying a 270 degrees wedge with combinations of Dirichlet, Neumann and impedance boundary conditions, and explicitly derive the corresponding operators. Other diffraction problems that result in operators equivalent to Wiener-Hopf plus Hankel operators with piecewise continuous generating functions are described for instance in [3, 6]. These operators are considered on L^p-spaces over the real line $\mathbb{R}$, or in related Bessel spaces.

In general, the exact inversion of such operators is extremely hard, and can only be found under specific circumstances (see for instance [4], [10]) which brings approximation methods for these problems into focus.

Consider then the operator equation $Au = v$, where the operator A is a general operator belonging to the algebra generated by the flip J ($(Ju)(x) := u(-x)$), convolution and multiplication operators. This class includes the Wiener-Hopf plus Hankel and Toeplitz plus Hankel operators. To solve the equation $Au = v$ numerically by a direct method, formally one specifies a sequence of simpler operators A_τ which converge strongly to A, and replaces the equation $Au = v$ by the sequence of the (simpler) equations $A_\tau u_\tau = v$. The crucial question is if this method *applies*, *i.e.* if the equations $A_\tau u_\tau = v$ possess unique solutions for every right-hand side v and for every sufficiently large τ, say for $\tau \geq \tau_0$, and if the sequence $(u_\tau)_{\tau\geq\tau_0}$

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converges to the solution u of the original equation $Au = v$. The applicability of the method is equivalent to the stability of the sequence (A_τ), *i.e.* to the invertibility of the operators A_τ for τ being large enough and to the uniform boundedness of the norms of their inverses.

By *simpler* it is meant in that context that we replace the operator A by its compressions to the compact intervals $[-\tau, \tau]$ with $\tau \in (0, \infty)$. These compressions are also called the finite sections of A, whence the name *finite sections method* for this kind of approximate solution.

In [19], Bernd Silbermann and the authors studied operators which are sums and products of operators of multiplication by piecewise continuous functions and operators of convolution by piecewise continuous Fourier multipliers. These operators were considered on $L^p(\mathbb{R})$. In order to identify the corresponding local algebras and, thus, to obtain invertibility conditions for the local representatives, we used homomorphisms which are defined by certain strong limits. More precisely, given a sequence (A_n) of approximation operators, one multiplies A_n by certain shift operators V_n (which have to be specified in each context), and then the homomorphism maps the sequence (A_n) to the strong limit of the sequence $V_n^{-1} A_n V_n$. Homomorphisms of this form are widely used (see for instance the monographs and textbooks [1, 2, 8, 12, 13, 16, 20] and the papers cited there).

The objective of the present work is to extend the above results with the addition of the flip operator to the possible operator building blocks. This extension allows the study of stability of finite sections sequences for Wiener-Hopf plus Hankel operators on $L^p(\mathbb{R}^+)$ and Toeplitz plus Hankel operators on $H^p(\mathbb{R})$, for instance. The main technical difficulty had to do with finding appropriate homomorphisms "V_n". The point is that the homomorphisms used for the algebras without the flip are not, in general, defined in the algebra containing the flip.

In [17] the authors managed to describe homomorphisms applicable to algebras generated by multiplication, convolution and flip operators. In this paper we use those homomorphisms in the larger algebra of sequences generated when joining finite section related projections. The developed techniques allow to study both the usual finite section method (FSM) on the real line or half-axes as well as its Fourier counterpart, which we will call Fourier finite section method (FFSM). The discrete variant of the FFSM corresponds to the classical Fourier approximation and has been extensively and since a long time used in applications (in Economy or Antennas, see for instance [7]). Its continuous version, which we treat here, was already referred to in [11, Chapter IV, Section 4] for a special case.

The paper is organized as follows. In Section 2 we generalize the well-known notion of (function) Fourier multipliers to operators. That notion facilitates the description of operators that appear in later sections. Section 3 is devoted to introducing and describing properties of the building blocks for the homomorphisms, as well as proving basic convergence results. In Section 4 is the main part of this paper, where the approximation problem is described as a invertibility problem in a suitable Banach algebra and then invertibility conditions are derived for elements of that algebra. That section end with the main result. Finally, a few examples of applications are provided in the concluding section.

2 Operator Fourier multipliers

The following notation is used throughout the paper. For a Banach space X, we denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X, and by $\mathcal{K}(X)$ the ideal of the compact operators on X. The identity operator is denoted by I. Further, for a given (bounded or unbounded) interval $\mathbb{I} \subseteq \mathbb{R}$, $L^p(\mathbb{I})$ refers to the standard Lebesgue space on $\mathbb{I}$ with norm $\|.\|_p$. Unless mentioned explicitly we will assume that $p \in (1, \infty)$ and write q for the conjugate exponent $p/(p-1)$. If we define the Fourier transform F on the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing infinite differentiable functions in the form

$$(Fu)(y) = \int_{-\infty}^{+\infty} e^{-2\pi \mathbf{i} yx} u(x)\, dx, \quad y \in \mathbb{R}, \tag{1}$$

then its inverse is given by

$$(F^{-1}v)(x) = \int_{-\infty}^{+\infty} e^{2\pi \mathbf{i} xy} v(y)\, dy, \quad x \in \mathbb{R}. \tag{2}$$

It is well known that F and F^{-1} extend continuously to bounded and unitary operators on the Hilbert space $L^2(\mathbb{R})$, which we denote by F and F^{-1} again. Thus, if A is a bounded operator on $L^2(\mathbb{R})$, then the composition $F^{-1}AF$ is also bounded on $L^2(\mathbb{R})$. This simple observation is of particular interest when $A = aI$ is the operator of multiplication by a bounded measurable function a. Then $F^{-1}AF$ becomes a (Fourier) convolution operator which we denote by $W^0(a)$; thus, multiplication and convolution operators on $L^2(\mathbb{R})$ are unitarily equivalent and can be treated "on the same level".

The study of convolution operators on $L^p(\mathbb{R})$, for $p \neq 2$, is more delicate. It is still true that F extends continuously to a bounded operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ if $1 < p \leq 2$ (see [23, Theorem 74]), but this property is not strong enough to give expressions like $F^{-1}AF$ a sense when A is an arbitrary operator in $\mathcal{B}(L^p(\mathbb{R}))$. This leads us to the following definition.

Definition 2.1. For $p \in (1, \infty)$, let $\mathcal{M}_p^\circ$ stand for the set of all operators $A \in \mathcal{B}(L^2(\mathbb{R}))$ with the property that $F^{-1}AFu \in L^p(\mathbb{R})$ whenever $u \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ and there is a constant c_p such that $\|F^{-1}AFu\|_p \leq c_p \|u\|_p$ for all $u \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$. If A is an operator in $\mathcal{M}_p^\circ$, then the composition $F^{-1}AF$ extends continuously to a bounded operator on $L^p(\mathbb{R})$. We denote this extension by A^F and call it the *Fourier image* of A.

If A, B are in $\mathcal{M}_p^\circ$, then so are $A+B$ and AB, with $(A+B)^F = A^F + B^F$ and $(AB)^F = A^F B^F$. It is also clear that the identity operator I is in $\mathcal{M}_p^\circ$ for every $p > 1$, with $I^F = I$. Thus, $\mathcal{M}_p^\circ$ is a unital algebra with respect to the natural operations, and the mapping $\mathcal{M}_p^\circ \to \mathcal{B}(L^p(\mathbb{R}))$, $A \mapsto A^F$ is a unital algebra homomorphism. The definition $\|A\|_{\mathcal{M}_p^\circ} := \|A^F\|_{\mathcal{B}(L^p(\mathbb{R}))}$ makes $\mathcal{M}_p^\circ$ to a normed algebra and the mapping $A \mapsto A^F$ to an isometry.

Definition 2.2. Let $\mathcal{M}_p := \mathcal{M}_p^\circ \cap \mathcal{M}_q^\circ$ and define $\|A\|_{\mathcal{M}_p} := \max\{\|A\|_{\mathcal{M}_p^\circ}, \|A\|_{\mathcal{M}_q^\circ}\}$ for $A \in \mathcal{M}_p$. We call the operators in $\mathcal{M}_p$ the *operator Fourier multipliers on* $L^p(\mathbb{R})$.

It is evident that $\mathcal{M}_p = \mathcal{M}_q$ and $\mathcal{M}_2 = \mathcal{B}(L^2(\mathbb{R}))$, and that $\mathcal{M}_p$ is a normed unital algebra.

Theorem 2.1. *The following statements hold:*

(i) *if* $A \in \mathcal{M}_p^\circ$, *then* $A^* \in \mathcal{M}_q^\circ$, *and* $\|A\|_{\mathcal{M}_p^\circ} = \|A^*\|_{\mathcal{M}_q^\circ}$. *Moreover,* $(A^*)^F = (A^F)^*$,

(ii) $\mathcal{M}_p$ *is an involutive algebra, and* $\|A\|_{\mathcal{M}_p} = \|A^*\|_{\mathcal{M}_p}$ *for* $A \in \mathcal{M}_p$,

(iii) $\|A\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq \|A\|_{\mathcal{M}_p}$ *for* $A \in \mathcal{M}_p$,

(iv) $\mathcal{M}_p$ *is a Banach algebra.*

Proof. (i) Let $A \in \mathcal{M}_p^\circ$, $u \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ and $v \in L^2(\mathbb{R}) \cap L^q(\mathbb{R})$. Then

$$\langle F^{-1}A^*Fv,\, u\rangle = \langle v,\, F^{-1}AFu\rangle = \overline{\langle F^{-1}AFu,\, v\rangle} \tag{3}$$

where $\langle u,\, v\rangle := \int_{\mathbb{R}} u\bar{v}dx$. Since $F^{-1}AFu = A^F u$, this implies

$$|\langle F^{-1}A^*Fv,\, u\rangle| = |\langle A^F u,\, v\rangle| \leq \|A^F\|_{\mathcal{B}(L^p(\mathbb{R}))}\|u\|_p\|v\|_q.$$

This estimate holds for all functions u in a dense subset of $L^p(\mathbb{R})$. Hence, $F^{-1}A^*Fv \in L^q(\mathbb{R})$ and

$$\|F^{-1}A^*Fv\|_q \leq \|A^F\|_{\mathcal{B}(L^p(\mathbb{R}))}\|v\|_q$$

for every function $v \in L^2(\mathbb{R}) \cap L^q(\mathbb{R})$. Consequently, $A^* \in \mathcal{M}_q^\circ$ and $\|A^*\|_{\mathcal{M}_q^\circ} \leq \|A\|_{\mathcal{M}_p^\circ}$. Applying this estimate to $A^* \in \mathcal{M}_q^\circ$ in place of A, we obtain the reverse norm inequality $\|A\|_{\mathcal{M}_p^\circ} = \|(A^*)^*\|_{\mathcal{M}_p^\circ} \leq \|A^*\|_{\mathcal{M}_q^\circ}$. From (3) we then conclude $\langle (A^*)^F v,\, u\rangle = \langle v,\, A^F u\rangle$ for all $u \in L^p(\mathbb{R})$ and $v \in L^q(\mathbb{R})$, whence $(A^*)^F = (A^F)^*$.

(ii) If $A \in \mathcal{M}_p^\circ \cap \mathcal{M}_q^\circ$, then $A^* \in \mathcal{M}_p^\circ \cap \mathcal{M}_q^\circ$ by assertion (i). The norm equality follows also easily from the norm equality in (i).

(iii) Let $A \in \mathcal{M}_p$. By the Riesz-Thorin interpolation theorem,

$$\begin{aligned}\|A\|^2_{\mathcal{B}(L^2(\mathbb{R}))} &= \|F^{-1}AF\|^2_{\mathcal{B}(L^2(\mathbb{R}))} \\ &\leq \|A^F\|_{\mathcal{B}(L^p(\mathbb{R}))}\|A^F\|_{\mathcal{B}(L^q(\mathbb{R}))} \\ &= \|A\|_{\mathcal{M}_p^\circ}\|A\|_{\mathcal{M}_q^\circ} \leq \|A\|^2_{\mathcal{M}_p}.\end{aligned}$$

(iv) Let (A_n) be a Cauchy sequence in $\mathcal{M}_p$. Then (A_n^F) is a Cauchy sequence in $\mathcal{B}(L^p(\mathbb{R}))$, hence convergent. We denote its limit by B. Further, by assertion (iii), (A_n) is a Cauchy sequence in $\mathcal{B}(L^2(\mathbb{R}))$, and we write A for its limit. Thus, if $u \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$, then $A_n^F u = F^{-1}A_nFu$ converges to Bu on $L^p(\mathbb{R})$ and to $F^{-1}AFu$ on $L^2(\mathbb{R})$. Consequently, $B = F^{-1}AF$ on $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$. But then $A \in \mathcal{M}_p^\circ$ and $B = A^F$. Repeating this argument for the conjugate exponent q, we get the assertion. □

Three examples of operator Fourier multipliers are basic in this paper. The first one is the flip, or reflection, operator $(Ju)(x) := u(-x)$, which clearly is an operator Fourier multiplier for every $p > 1$ with $J^F = J$. Note in this connection that $JF = FJ$ and $F^{-1} = FJ$ on $L^2(\mathbb{R})$.

The second example is given by convolution operators. For $a \in L^\infty(\mathbb{R})$, the convolution operator $W^0(a) = F^{-1}aF$ is bounded on $L^2(\mathbb{R})$, and this operator is an operator Fourier multiplier because $F^{-1}W^0(a)F = F^{-2}aF^2 = JaJ$ and JaJ is bounded on $L^p(\mathbb{R})$ for every $p > 1$. Since $JaJ = \tilde{a}I$ with $\tilde{a}(t) := a(-t)$, the Fourier image of a convolution operator is a multiplication operator.

The third basic example is multiplication operators. In contrast with convolution operators, not every operator aI of multiplication by a function $a \in L^\infty(\mathbb{R})$ is an operator Fourier multiplier. In fact, the intersection of the algebra of bounded multiplication operators with the algebra $\mathcal{M}_p$ is just the algebra of the classical Fourier multipliers. The Fourier image of a multiplication operator in $\mathcal{M}_p$ is a convolution operator. For multiplication operators aI, we will mainly use the standard notation $W^0(a)$ in place of $(aI)^F$. We will also write $a \in \mathcal{M}_p$ instead of $aI \in \mathcal{M}_p$, likewise for $\mathcal{M}_p^\circ$.

It turns out that if $a \in L^\infty(\mathbb{R})$ is in $\mathcal{M}_p^\circ$, then a is already in $\mathcal{M}_p$. To see this, note that $(aI)^* = \overline{a}I = CaC$, where C is the operator of complex conjugation, $(Cu)(x) := \overline{u(x)}$. Using $CF = F^{-1}C = FJC$ we obtain

$$F^{-1}(aI)^*F = F^{-1}CaCF = CFaF^{-1}C = CJF^{-1}aFJC.$$

Since C and J are bounded on $L^p(\mathbb{R})$ for every $p > 1$, this implies that $(aI)^*$ is in $\mathcal{M}_p^\circ$ whenever a is in $\mathcal{M}_p^\circ$ and that $((aI)^*)^F = CJ(aI)^F JC$ or, equivalently, $W^0(\overline{a}) = CJW^0(a)JC$. But $(aI)^* \in \mathcal{M}_p^\circ$ implies $a \in \mathcal{M}_q^\circ$ by Theorem 2.1 (i); so a is in $\mathcal{M}_p$.

We call a function $a \in L^\infty(\mathbb{R})$ *piecewise constant* (resp. *piecewise linear*) if there is a partition $-\infty = t_0 < t_1 < \ldots < t_n = +\infty$ of the real line such that a is constant (resp. linear) on each interval $[t_k, t_{k+1}]$. Stechkin's inequality (see for instance [9])

$$\|a\|_{\mathcal{M}_p} \le c_p(\|a\|_\infty + V(a)),$$

where c_p is a constant depending on p and $V(a)$ represents the total variation of a, entails that the multiplier algebra $\mathcal{M}_p$ contains the (non-closed) algebras C_0 of all continuous and piecewise linear functions on $\dot{\mathbb{R}}$ and PC_0 of all piecewise constant functions on $\mathbb{R}$. Let C_p and PC_p denote the closures of C_0 and PC_0 in $\mathcal{M}_p$, respectively. Further we write $a(s^+)$ resp. $a(s^-)$ for the limit of the function a at s from the right- resp. left-hand side.

We continue with two special instances of multiplication and convolution operators. The first example is P_τ $(\tau > 0)$, the operator of multiplication by $\chi_{[-\tau,\tau]}$, where χ_U stands for the characteristic function of the set U. These functions are in PC_p for every p. Clearly, both P_τ and the associated Fourier multiplier P_τ^F are projections. We will also need the complementary projections $Q_\tau := I - P_\tau$ and $Q_\tau^F := I - P_\tau^F$.

Lemma 2.2. *The operators P_τ and P_τ^F are uniformly bounded on $L^p(\mathbb{R})$, and they converge strongly to the identity as $\tau \to \infty$.*

The assertion is trivial for P_τ. The uniform boundedness of the P_τ^F is a consequence of the Stechkin inequality, and the strong convergence of P_τ^F will follow from Proposition 3.5 below.

For the second example, take $s \in \mathbb{R}$ and denote by U_s the operator of multiplication by the function $x \mapsto e^{-2\pi \mathbf{i} xs}$. This function has infinite variation; so we cannot use Stechkin's inequality to conclude that U_s is a Fourier multiplier. But if V_s denotes the shift operator $(V_s u)(x) = u(x - s)$, then a direct computation gives $FV_s = U_sF$ and $FU_s = V_{-s}F$ if $p = 2$. Thus, U_s and V_s are operator Fourier multipliers for every $p > 1$ with $U_s^F = V_s$ and $V_s^F = U_{-s}$. Clearly, $U_s^{-1} = U_{-s}$, $V_s^{-1} = V_{-s}$ and, moreover, $V_sU_t = e^{2\pi \mathbf{i} st}U_tV_s$.

3 Auxiliary operators and convergence results

Our next goal is to introduce further types of shift, projection and reflection operators on $L^p(\mathbb{R})$ and to collect some of their properties. These operators are used later to define strong limits, which will provide us with the key tools to identify local algebras. The reader can skip the results of this section on a first reading, and refer to them when checking a proof in the continuation.

For $\tau > 0$, define

$$(R_\tau u)(x) = \begin{cases} u(\tau - x) & \text{if} \quad 0 < x < \tau \\ u(-\tau - x) & \text{if} \quad -\tau < x < 0 \\ 0 & \text{if} \quad |x| > \tau \end{cases}, \tag{4}$$

$$(S_\tau u)(x) = \begin{cases} 0 & \text{if} \quad |x| < \tau \\ u(x - \tau) & \text{if} \quad x > \tau \\ u(x + \tau) & \text{if} \quad x < -\tau \end{cases}, \tag{5}$$

$$(S_{-\tau} u)(x) = \begin{cases} u(x + \tau) & \text{if} \quad x > 0 \\ u(x - \tau) & \text{if} \quad x < 0 \end{cases}, \tag{6}$$

which are bounded and have norm 1 on every $L^p(\mathbb{R})$. We proceed with a couple of lemmata which collect some elementary properties of these operators which will be used often without reference in what follows.

We first observe that the operators P_τ, R_τ and $S_{\pm\tau}$ can be written in terms of the shifts $V_{\mp\tau}$ and the multiplication operators $\chi_\pm I$. Let $s \in \mathbb{R}$ and $\tau > 0$ in the forthcoming lemmata.

Lemma 3.1. *The following equalities hold.*

(i) $\chi_\pm P_\tau = \chi_\pm V_{\pm\tau} \chi_\mp V_{\mp\tau} \chi_\pm = \chi_\pm V_{\pm\tau} \chi_\mp V_{\mp\tau} = V_{\pm\tau} \chi_\mp V_{\mp\tau} \chi_\pm,$

(ii) $\chi_\pm Q_\tau = V_{\pm\tau} \chi_\pm V_{\mp\tau}$ *and* $\chi_\pm R_\tau = J \chi_\mp V_{\mp\tau} \chi_\pm I = \chi_\pm V_{\pm\tau} \chi_\mp J,$

(iii) $\chi_\pm S_\tau = V_{\pm\tau} \chi_\pm I$ *and* $\chi_\pm S_{-\tau} = \chi_\pm V_{\mp\tau},$

(iv) P_τ, R_τ *and* $S_{\pm\tau}$ *are operator Fourier multipliers for every* $p > 1$*, and the Fourier images of these operators are uniformly bounded with respect to* $\tau > 0$ *for every fixed* $p > 1$*.*

Proof. The proof of assertions (i), (ii) and (iii) is straightforward. For (iv) note that the operators P_τ, R_τ and $S_{\pm\tau}$ commute with the operator of multiplication by the characteristic function $\chi_\pm$ of the positive and negative half-axis, respectively. It is thus sufficient to prove that the restriction of each of these operators to the positive and negative half-axis is an operator Fourier multiplier. In assertions (i), (ii) we observed that these restrictions are composed by operators of shift V_τ, multiplication by $\chi_\pm$, and reflection J, which are operator Fourier multipliers as already noticed. The uniform boundedness assertion follows also from this observation, together with the uniform boundedness of the operators $V_s^F = U_{-s}$. □

Note that the relations (i)-(iii) above hold with the elements $\chi_\pm$, P_τ, R_τ and $S_{\pm\tau}$ substituted by their $F-$counterparts.

Lemma 3.2. (i) $R_\tau P_\tau = P_\tau R_\tau = R_\tau$, $P_\tau = R_\tau^2$, $R_\tau^* = R_\tau$,

(ii) $S_\tau S_{-\tau} = Q_\tau$, $S_{-\tau} S_\tau = I$, $(S_\tau)^* = S_{-\tau}$,

(iii) $P_\tau S_\tau = S_{-\tau} P_\tau = 0$,

(iv) $J S_{\pm\tau} = S_{\pm\tau} J$, $J P_\tau = P_\tau J$, $J R_\tau = R_\tau J$, $J U_s = U_{-s} J$ *and* $J V_s = V_{-s} J$

For $\tau > 0$, we further introduce the operators $(Z_\tau u)(x) := \tau^{-1/p} u(x/\tau)$. Clearly, $Z_\tau^{-1} = Z_{\tau^{-1}}$, and these operators are bounded and have norm 1 on every $L^p(\mathbb{R})$. The adjoint operator acts on $L^q(\mathbb{R})$ and is given by $Z_\tau^* = Z_{\tau^{-1}}$. The latter identity needs an explanation, because Z_τ depends on p by definition. If we write $Z_{\tau,p}$ instead of Z_τ then, more precisely, $Z_{\tau,p}^* = Z_{\tau^{-1},q}$. We will nevertheless often omit the p in the notation, because the Z_τ will typically appear in products of the form $Z_{\tau^{-1}} A Z_\tau$, which are independent of p.

Lemma 3.3. *We have* $JZ_\tau = Z_\tau J$, $Z_\tau P_s Z_\tau^{-1} = P_{s\tau}$, $Z_\tau R_s Z_\tau^{-1} = R_{s\tau}$ *and* $Z_\tau S_{\pm s} Z_\tau^{-1} = S_{\pm s\tau}$.

Next we turn to convergence issues. We start with a general observation, which allows us in most situations to be able to work in the L^2-setting, the convergence results being valid in L^p due to the uniform boundedness of the operators involved. In what follows, the arrow $\to$ will be used to indicate strong convergence, whereas $\rightharpoondown$ is reserved for weak convergence. Moreover, a sequence (A_τ) of operators on a Banach space X is said to converge **-strongly* if it converges strongly on X and if the adjoint sequence (A_τ^*) converges strongly on the dual space X^*. In this case,

$$\operatorname*{s-lim}_{\tau\to\infty} A_\tau^* = (\operatorname*{s-lim}_{\tau\to\infty} A_\tau)^*.$$

The following is also well-known.

Lemma 3.4. *If* $A_\tau^* \to A$, $B_\tau \to B$ *strongly and* $C_\tau \rightharpoondown C$ *weakly, then* $A_\tau C_\tau B_\tau \rightharpoondown ACB$ *weakly.*

Proposition 3.5. *Let* $\mathbb{I}$ *be a real interval and* (A_τ) *be uniformly bounded on* $L^p(\mathbb{I})$. *Then*

(i) *if* $A \in \mathcal{B}(L^p(\mathbb{I})) \cap \mathcal{B}(L^2(\mathbb{I}))$ *and* $A_\tau \rightharpoondown A$ *weakly on* $L^2(\mathbb{I})$, *then* $A_\tau \rightharpoondown A$ *weakly on* $L^p(\mathbb{I})$;

(ii) *if, moreover,* (A_τ) *is uniformly bounded on* $L^r(\mathbb{I})$ *for all* r *in a neighborhood of* p *and if* $A_\tau \to A$ *strongly on* $L^2(\mathbb{I})$, *then* $A \in \mathcal{B}(L^p(\mathbb{I}))$ *and* $A_\tau \to A$ *strongly on* $L^p(\mathbb{I})$.

Proof. If $A_\tau \rightharpoondown A$, then $\langle (A_\tau - A)f, g\rangle \to 0$ for any piecewise constant functions f, g (which vanish at infinity in case $\mathbb{I}$ is unbounded). Because the set of those functions is dense in both $L^p(\mathbb{I})$ and $L^q(\mathbb{I})$ and (A_τ) is uniformly bounded on $L^p(\mathbb{I})$, a standard density argument gives the result.

For the second assertion, notice that if $A_\tau \to A$ on $L^2(\mathbb{I})$, then $A_\tau f$ converges for every f in a dense subset of $L^p(\mathbb{I})$ by the interpolation theorem. Thus it converges for any $u \in L^p(\mathbb{I})$. By the Banach-Steinhaus theorem, the operator A' defined by $A'u = \operatorname{s-lim} A_\tau u$ is in $\mathcal{B}(L^p(\mathbb{I}))$. This operator must coincide with A on the intersection space $L^p(\mathbb{I}) \cap L^2(\mathbb{I})$. □

Lemma 3.6. *The following limits hold on* $L^p(\mathbb{R})$ *as* $\tau \to \infty$.

(i) $S_{-\tau} \to 0$ *strongly;*

(ii) R_τ, S_τ, $V_{\pm\tau}$, $U_{\pm\tau}$, R_τ^F, S_τ^F, $V_{\pm\tau}^F$, $U_{\pm\tau}^F$, $Z_\tau^{\pm 1} \rightharpoondown 0$ *weakly;*

(iii) *for every* $s \in \mathbb{R}$, $R_{\tau-s} Z_\tau^{-1} \rightharpoondown 0$ *weakly and* $S_{s-\tau} Z_\tau^{-1} \to 0$ *strongly;*

(iv) *for every* $s \in \mathbb{R}$, $R_\tau U_s Z_\tau \rightharpoondown 0$ *and* $S_{-\tau} U_s Z_\tau \rightharpoondown 0$ *weakly.*

Proof. The proof of assertions (i) and (ii) is straightforward for the operators P_τ, R_τ, $S_{\pm\tau}$, $V_{\pm\tau}$ and $U_{\pm\tau}$. The convergence of their $F-$counterparts is immediate on $L^2(\mathbb{R})$ and then on $L^p(\mathbb{R})$ via Proposition 3.5. For $Z_\tau^{\pm 1}$, the assertion is proved in [20, Lemma 4.2.12]. The first assertion of (iii) can be proved as [20, Lemma 4.2.12] by a straightforward calculation again. The second assertion in (iii) follows from $S_{s-\tau}Z_\tau^{-1}\chi_{[a,b]} = 0$ for τ sufficiently large. In assertion (iv), we first let $s = 0$. Then the identities $R_\tau Z_\tau = Z_\tau R_1$ and $S_{-\tau}Z_\tau = Z_\tau S_{-1}$ together with assertion (ii) imply the weak convergence to 0. For $s \neq 0$ we write

$$R_\tau U_s Z_\tau = R_\tau U_s P_\tau Z_\tau = (R_\tau U_s R_\tau)(R_\tau Z_\tau).$$

The second factor tends weakly to zero, as we already know. For the first factor, consider its action on $\mathbb{R}^+$ and $\mathbb{R}^-$ separately. On $\mathbb{R}^+$, one easily checks that $R_\tau U_s R_\tau = e^{-2\pi i s\tau}U_{-s}$. Since U_{-s} is independent of τ and the scalars $e^{-2\pi i s\tau}$ are uniformly bounded, the first assertion of (iv) follows. For the second assertion of (iv) we employ the identity

$$S_{-\tau}U_s Z_\tau = (S_{-\tau}U_s S_\tau)(S_{-\tau}Z_\tau) = e^{-2\pi i s\tau}U_s(S_{-\tau}Z_\tau)$$

on $\mathbb{R}^+$ and argue as before. □

Lemma 3.7. *Each of the following sequences tends weakly to zero on $L^p(\mathbb{R})$ as $\tau \to \infty$:*

$$R_\tau^F R_\tau, \quad S_{-\tau}^F S_\tau, \quad S_{-\tau}S_\tau^F, \quad R_\tau R_\tau^F, \quad S_{-\tau}^F R_\tau, \quad R_\tau S_\tau^F$$

Proof. By Proposition 3.5 it is just necessary to prove the results for $p = 2$. We first check that $U_{\pm\tau}V_\tau \rightharpoonup 0$ weakly. Let $a, b, c, d \in \mathbb{R}$ such that $a < b$ and $c < d$. Then,

$$\langle U_{\pm\tau}V_\tau\chi_{[a,b]}, \chi_{[c,d]}\rangle = \langle U_{\pm\tau}\chi_{[a+\tau,b+\tau]}, \chi_{[c,d]}\rangle = \int_{[a+\tau,b+\tau]\cap[c,d]} e^{\mp 2\pi \mathbf{i} x\tau}\, dx \to 0$$

since the integration set is empty for sufficiently large τ. A density argument and the uniform boundedness of the involved operators give the result. Analogously, $U_{\pm\tau}V_{-\tau} \rightharpoonup 0$.

We now use the above results to check that if $\chi_{1,2} \in \{\chi_+, \chi_-\}$, then $V_{\pm\tau}\chi_1 F\chi_2 V_\tau$ and $V_{\pm\tau}\chi_1 F\chi_2 V_{-\tau}$ tend weakly to 0. Note that $V_{\pm\tau}\chi_1 V_{\mp\tau}$ converges *-strongly (to 0 or I) as $\tau \to \infty$. For instance, by writing

$$V_{\pm\tau}\chi_1 F\chi_2 V_\tau = (V_{\pm\tau}\chi_1 V_{\mp\tau})\,(V_{\pm\tau}FV_\tau)\,(V_{-\tau}\chi_2 V_\tau),$$

noting that the middle term equals $FU_{\mp\tau}V_\tau$, and applying Lemma 3.4, we get the result.

Now we turn to $R_\tau^F R_\tau = F^{-1}R_\tau F R_\tau$. It is sufficient to show that $R_\tau F R_\tau \rightharpoonup 0$ as $\tau \to \infty$. The result comes by writing

$$R_\tau F R_\tau = R_\tau\chi_+ F\chi_+ R_\tau + R_\tau\chi_+ F\chi_- R_\tau + R_\tau\chi_- F\chi_+ R_\tau + R_\tau\chi_- F\chi_- R_\tau,$$

noting that each of the summands on the right-hand side can be decomposed with the help of Lemma 3.1(ii), and applying the observations above. For instance,

$$R_\tau\chi_\pm F\chi_\pm R_\tau = J\chi_\mp V_{\mp\tau}\chi_\pm F\chi_\pm V_{\pm\tau}\chi_\mp J$$

and $V_{\mp\tau}\chi_\pm F\chi_\pm V_{\pm\tau}$ was seen to tend weakly to 0. The weak convergence of $S_{-\tau}^F S_\tau$ and $S_{-\tau}S_\tau^F$ can be proved in the same way, and the result for the remaining three sequences follows by taking adjoints. □

Lemma 3.8. *Let $s > 0$. Then*

(i) $\|S_{-\tau}R_sZ_\tau^{-1}\| \to 0$, $S_{-\tau}S_sZ_\tau^{-1} \to 0$ *strongly, and* $R_\tau R_sZ_\tau^{-1} \rightharpoondown 0$ *and* $R_\tau S_sZ_\tau^{-1} \rightharpoondown 0$ *weakly as* $\tau \to \infty$.

(ii) $R_\tau R_s^F Z_\tau \rightharpoondown 0$, $S_{-\tau}R_s^F Z_\tau \rightharpoondown 0$, $R_\tau S_s^F Z_\tau \rightharpoondown 0$, *and* $S_{-\tau}S_s^F Z_\tau \rightharpoondown 0$ *weakly as* $\tau \to \infty$.

Proof. (i) We use that, for $\tau > s$,

$$R_\tau R_s = S_{\tau-s}P_s, \quad S_{-\tau}R_s = 0, \quad R_\tau S_s = R_{\tau-s}, \quad S_{-\tau}S_s = S_{\tau-s}. \tag{7}$$

Then, by Lemma 3.6 (i), $(R_\tau R_sZ_\tau^{-1})^* = (S_{\tau-s}P_sZ_\tau^{-1})^* = Z_\tau P_sS_{\tau-s} \to 0$ strongly; hence, $R_\tau R_sZ_\tau^{-1} \rightharpoondown 0$ weakly. Further, $\|S_{-\tau}R_sZ_\tau^{-1}\| = 0$ for large τ by the second identity in (7). Finally, the last identities in (7) give $R_\tau S_sZ_\tau^{-1} = R_{\tau-s}Z_\tau^{-1}$ and $S_{-\tau}S_sZ_\tau^{-1} = S_{\tau-s}Z_\tau^{-1}$, and the weak resp. strong convergence of these sequences to zero follows from Lemma 3.6 (iv).
(ii) For the first assertion, we write $R_s = J\chi_-V_{-s}\chi_+I + J\chi_+V_s\chi_-I$ to obtain

$$R_\tau R_s^F Z_\tau = R_\tau(J\chi_-^F U_s\chi_+^F + J\chi_+^F U_{-s}\chi_-^F)Z_\tau.$$

Since R_τ commutes with J and the operators $\chi_\pm^F$ are homogeneous, this implies

$$R_\tau R_s^F Z_\tau = J(R_\tau\chi_-^F U_sZ_\tau)\chi_+^F + J(R_\tau\chi_+^F U_{-s}Z_\tau)\chi_-^F;$$

so it remains to show that

$$R_\tau\chi_\mp^F U_{\pm s}Z_\tau \rightharpoondown 0 \quad \text{weakly as } \tau \to \infty. \tag{8}$$

We use Lemma 3.4. For that purpose, we write the operators in (8) as

$$R_\tau\chi_\mp^F(R_\tau^2 + S_\tau S_{-\tau})U_{\pm s}Z_\tau = (R_\tau\chi_\mp^F R_\tau)(R_\tau U_{\pm s}Z_\tau) + (R_\tau\chi_\mp^F S_\tau)(S_{-\tau}U_{\pm s}Z_\tau)$$

and note that the sequences $R_\tau U_{\pm s}Z_\tau$ and $S_{-\tau}U_{\pm s}Z_\tau$ converge weakly to 0 as $\tau \to \infty$ by Lemma 3.6 (v), whereas the sequences $(R_\tau\chi_\mp^F R_\tau)^*$ and $(R_\tau\chi_\mp^F S_\tau)^*$ converge strongly by Proposition 3.11 below (take into account that $\chi_\pm \in PC_p(\dot{\mathbb{R}})$). Hence, (8) follows, which implies the weak convergence of $R_\tau R_s^F Z_\tau$ to zero.

The weak convergence of $S_{-\tau}R_s^F Z_\tau$ to zero follows by exactly the same arguments. For the third sequence in (ii) we write

$$R_\tau S_s^F Z_\tau = R_\tau(V_s\chi_+ + V_{-s}\chi_-)^F Z_\tau = R_\tau U_{-s}Z_\tau\chi_+^F + R_\tau U_sZ_\tau\chi_-^F.$$

So the weak convergence of this sequence to zero is an immediate consequence of Lemma 3.6 (v). The weak convergence of fourth sequence in (ii) follows in exactly the same way. □

For $s \in \mathbb{R}$ and every function $a : \mathbb{R} \to \mathbb{C}$, set $\tilde{a}(x) := a(-x)$ and $a_{\{s\}}(x) := a(x - s)$. The following lemma is straightforward. Lemma 3.10 is proved in [20, Lemma 5.4.2 (i), (ii)].

Lemma 3.9. *If $a \in L^\infty(\mathbb{R})$, $b \in \mathcal{M}_p$ and $s \in \mathbb{R}$, then*

(i) $U_{-s}aU_s = aI$, $V_saV_{-s} = a_{\{s\}}I$ *and* $JaJ = \tilde{a}I$,

(ii) $U_{-s}W^0(b)U_s = W^0(b_s)$, $V_sW^0(b)V_{-s} = W^0(b)$ *and* $JW^0(b)J = W^0(\tilde{b})$.

Lemma 3.10. *If $a \in PC(\dot{\mathbb{R}})$ and $b \in PC_p$, then*

$$V_{-s}aV_s \to a(\pm\infty)I \quad \text{and} \quad U_sW^0(b)U_{-s} \to b(\pm\infty)I \quad \text{as } s \to \pm\infty.$$

The following proposition is proved in [17], but we give here a short proof whose idea will be further used.

Proposition 3.11. *Let $a \in PC_p(\dot{\mathbb{R}})$ and $\tau > 0$. Then*

$$R_\tau\chi_\pm W^0(a)\chi_\pm R_\tau = P_\tau\chi_\pm W^0(\tilde{a})\chi_\pm P_\tau, \quad R_\tau\chi_\pm W^0(a)\chi_\pm S_\tau = P_\tau\chi_\pm JW^0(a)\chi_\pm I,$$
$$S_{-\tau}\chi_\pm W^0(a)\chi_\pm R_\tau = \chi_\pm W^0(a)J\chi_\pm P_\tau, \quad S_{-\tau}\chi_\pm W^0(a)\chi_\pm S_\tau = \chi_\pm W^0(a)\chi_\pm I$$

and

$$R_\tau\chi_\pm W^0(a)\chi_\mp R_\tau \to 0, \quad R_\tau\chi_\pm W^0(a)\chi_\mp S_\tau \to 0,$$
$$S_{-\tau}\chi_\pm W^0(a)\chi_\mp R_\tau \to 0, \quad S_{-\tau}\chi_\pm W^0(a)\chi_\mp S_\tau \to 0$$

strongly as $\tau \to \infty$.

Proof. We will only check the first identity. The others can be proved in the same way. By Lemma 3.1, $R_\tau\chi_\pm = \chi_\pm R_\tau = J\chi_\mp V_{\mp\tau}\chi_\pm$. Then, because $V_{\pm\tau}W^0(a)V_{\mp\tau} = W^0(a)$ and $JaJ = \tilde{a}I$,

$$\begin{aligned} R_\tau\chi_\pm W^0(a)\chi_\pm R_\tau &= J\chi_\mp V_{\mp\tau}\chi_\pm W^0(a)J\chi_\mp V_{\mp\tau}\chi_\pm \\ &= \chi_\pm V_{\pm\tau}\chi_\mp JW^0(a)J\chi_\mp V_{\mp\tau}\chi_\pm \\ &= \chi_\pm V_{\pm\tau}\chi_\mp V_{\mp\tau}W^0(\tilde{a})V_{\pm\tau}\chi_\mp V_{\mp\tau}\chi_\pm \;=\; P_\tau\chi_\pm W^0(\tilde{a})\chi_\pm P_\tau, \end{aligned}$$

again using Lemma 3.1. The strong limits can be found as is shown below, again for the first one.

$$\begin{aligned} R_\tau\chi_\pm W^0(a)\chi_\mp R_\tau &= J\chi_\mp V_{\mp\tau}\chi_\pm W^0(a)J\chi_\pm V_{\pm\tau}\chi_\mp \\ &= \chi_\pm V_{\pm\tau}\chi_\mp V_{\pm\tau}W^0(\tilde{a})V_{\mp\tau}\chi_\pm V_{\pm\tau}\chi_\mp \\ &= \chi_\pm V_{\pm\tau}\chi_\mp S_{-\tau}W^0(\tilde{a})P_\tau\chi_\mp. \end{aligned}$$

The assertion follows form the fact that $S_{-\tau}W^0(\tilde{a})P_\tau\chi_\mp$ tend strongly to zero and $\chi_\pm V_{\pm\tau}\chi_\mp$ is uniformly bounded. □

The $F-$dual of Proposition 3.11 for operator Fourier multipliers reads as follows and is proved in the same way.

Proposition 3.12. *Let $a \in PC(\dot{\mathbb{R}})$ and $\tau > 0$. Then*

$$R^F_\tau\chi^F_\pm a\chi^F_\pm R^F_\tau = P^F_\tau\chi^F_\pm\tilde{a}\chi^F_\pm P^F_\tau, \quad R^F_\tau\chi^F_\pm a\chi^F_\pm S^F_\tau = P^F_\tau\chi^F_\pm Ja\chi^F_\pm,$$
$$S^F_{-\tau}\chi^F_\pm a\chi^F_\pm R^F_\tau = \chi^F_\pm aJ\chi^F_\pm P^F_\tau, \quad S^F_{-\tau}\chi^F_\pm a\chi^F_\pm S^F_\tau = \chi^F_\pm a\chi^F_\pm.$$

and

$$R^F_\tau\chi^F_\pm a\chi^F_\mp R^F_\tau \to 0, \quad R^F_\tau\chi^F_\pm a\chi^F_\mp S^F_\tau \to 0,$$
$$S^F_{-\tau}\chi^F_\pm a\chi^F_\mp R^F_\tau \to 0, \quad S^F_{-\tau}\chi^F_\pm a\chi^F_\mp S^F_\tau \to 0$$

strongly as $\tau \to \infty$.

4 Approximate invertibility

Besides the standard finite sections $P_\tau A P_\tau$ of an operator $A \in \mathcal{B}(L^p(\mathbb{R}))$ with respect to the projections P_τ, we also consider its Fourier finite sections $P_\tau^F A P_\tau^F$. While the standard finite sections are extensively studied, their F-counterparts attracted less attention in the literature. The physical meaning of the projection P_τ^F is to cut the higher values in the frequency domain. This cutting is the continuous analogue of the usual (discrete) Fourier analysis for periodic functions.

In both settings we will actually work with the *extended* finite sections $P_\tau A P_\tau + Q_\tau$ (with $Q_\tau := I - P_\tau$) and $P_\tau^F A P_\tau^F + Q_\tau^F$. The passage from standard to extended finite sections does not involve any complications since the sequences and the extended sequences are simultaneously stable or not. One technical advantage of using extended finite sections is that the operator A and its extended finite sections act on the same space.

The stability of a bounded sequence of operators is equivalent to the invertibility of a certain associated element in a suitable Banach algebra. To make this precise, let $\mathcal{E}$ be set of all bounded sequences $(A_\tau)_{\tau>0}$ of operators $A_\tau \in \mathcal{B}(L^p(\mathbb{R}))$ and $\mathcal{G}$ the subset of $\mathcal{E}$ which consists of all sequences tending to zero in the norm. With respect to pointwise defined operations and the supremum norm, the set $\mathcal{E}$ becomes a Banach algebra, and $\mathcal{G}$ is a closed ideal of that algebra. A Neumann series argument shows then that a sequence in $\mathcal{E}$ is stable if and only if its coset modulo $\mathcal{G}$ is invertible in the quotient algebra $\mathcal{E}/\mathcal{G}$.

In what follows we will have to pay particular attention to the inverse closedness of subalgebras (of $\mathcal{E}/\mathcal{G}$ and of other "super-algebras"). The following elementary observations are often useful. The proof of the second one is evident.

Lemma 4.1. *Let X be a Banach space, $\mathcal{Z}$ the Banach algebra of all bounded sequences $\mathbf{A} = (A_\tau)_{\tau>0}$ of operators on X, with pointwise defined operations and the supremum norm, and $\mathcal{G}$ the closed ideal of $\mathcal{Z}$ of the sequences tending in the norm to zero. Suppose that we are given a uniformly bounded family $(\mathsf{H}_\tau)_{\tau>0}$ of homomorphisms $\mathsf{H}_\tau : \mathcal{B}(X) \to \mathcal{B}(X)$ such that $\mathsf{H}_\tau(I) \to I$ *-strongly. Let finally $\mathcal{Z}_0$ stand for the set of all sequences $\mathbf{A} = (A_\tau) \in \mathcal{Z}$ such that the *-strong limit $\mathsf{H}(\mathbf{A}) := \text{s-lim}_{\tau\to\infty} \mathsf{H}_\tau(A_\tau)$ exists. Then*

(i) *$\mathcal{Z}_0$ is a closed unital subalgebra of $\mathcal{Z}$, and $\mathcal{G}$ is a closed ideal of $\mathcal{Z}_0$;*

(ii) *$\mathsf{H} : \mathcal{Z}_0 \to \mathcal{B}(X)$ is a bounded unital homomorphism, and $\mathcal{G}$ is in the kernel of H;*

(iii) *the quotient algebra $\mathcal{Z}_0/\mathcal{G}$ is inverse closed in $\mathcal{Z}/\mathcal{G}$, and $\mathcal{Z}_0$ is inverse closed in $\mathcal{Z}$.*

Proof. We concentrate on the proof of the inverse closedness assertions. Let $\mathbf{A} = (A_\tau) \in \mathcal{Z}_0$, and let $\mathbf{B} = (B_\tau) \in \mathcal{Z}$ be an inverse modulo $\mathcal{G}$ of $\mathbf{A}$. Then $(B_\tau)(A_\tau) - (I) =: (G_\tau) \in \mathcal{G}$. For every $u \in X$, there is a positive constant c such that

$$\begin{aligned} \|\mathsf{H}_\tau(I)u\|_X &= \| \left(\mathsf{H}_\tau(B_\tau)\mathsf{H}_\tau(A_\tau) - \mathsf{H}_\tau(G_\tau)\right) u\|_X \\ &\le c\|\mathsf{H}_\tau(A_\tau)u\|_X + \|\mathsf{H}_\tau(G_\tau)u\|_X. \end{aligned}$$

Taking the limit as $\tau \to \infty$ we obtain $\|u\|_X \le c\|\mathsf{H}(\mathbf{A})u\|_X$ which implies that the operator $\mathsf{H}(\mathbf{A})$ has a trivial kernel and a closed range. Applying H_τ to $(A_\tau)(B_\tau) - (I) \in \mathcal{G}$, taking adjoints, and passing then to the strong limit as $\tau \to \infty$, we obtain similarly that the kernel of $\mathsf{H}(\mathbf{A})^*$ is also trivial. Hence, $\mathsf{H}(\mathbf{A})$ is invertible in $\mathcal{B}(X)$. Now we estimate

$$\begin{aligned}
&\| \left(\mathsf{H}_\tau(B_\tau) - \mathsf{H}(\mathbf{A})^{-1}\right) u\|_X \\
&\quad = \|\mathsf{H}_\tau(B_\tau)u - (\mathsf{H}_\tau(I) + I - \mathsf{H}_\tau(I))\, \mathsf{H}(\mathbf{A})^{-1}u\|_X \\
&\quad = \|\mathsf{H}_\tau(B_\tau)u - (\mathsf{H}_\tau(B_\tau)\mathsf{H}_\tau(A_\tau) - \mathsf{H}_\tau(G_\tau) + I - \mathsf{H}_\tau(I))\, \mathsf{H}(\mathbf{A})^{-1}u\|_X \\
&\quad \le c\,\|u - \mathsf{H}_\tau(A_\tau)\mathsf{H}(\mathbf{A})^{-1}u\| + \|\,(-\mathsf{H}_\tau(G_\tau) + I - \mathsf{H}_\tau(I))\, \mathsf{H}(\mathbf{A})^{-1}u\|_X \\
&\quad = c\,\|\mathsf{H}(\mathbf{A})v - \mathsf{H}_\tau(A_\tau)v\| + \|\,(-\mathsf{H}_\tau(G_\tau) + I - \mathsf{H}_\tau(I))\, u\|_X,
\end{aligned}$$

where $v := \mathsf{H}(\mathbf{A})^{-1}u$. It is easy to see that the latter expression goes to zero as $\tau \to \infty$. Thus the strong limit of $\mathsf{H}_\tau(B_\tau)$ exists and is $\mathsf{H}(\mathbf{A})^{-1}$. The same arguments imply the existence of the adjoint limit. So $\mathcal{B}$ is indeed in $\mathcal{Z}_0$, which settles the inverse closedness of $\mathcal{Z}_0/\mathcal{G}$ in $\mathcal{Z}/\mathcal{G}$. The inverse closedness of $\mathcal{Z}_0$ in $\mathcal{Z}$ follows then from [20, Lemma 1.2.33]. □

Lemma 4.2. *For a Banach algebra $\mathcal{A}$ and a non-empty subset $M \subset \mathcal{A}$, the commutant $\{a \in \mathcal{A} : am = ma \text{ for all } m \in M\}$ of M is a closed and inverse closed subalgebra of $\mathcal{A}$.*

4.1 Essentialization

Let $\mathcal{F}$ denote the set of all sequences $\mathbf{A} = (A_\tau)_{\tau>0} \in \mathcal{E}$ such that the following limits exist in the *-strong sense:

- $\mathsf{W}_0(\mathbf{A}) := \text{s-lim}_{\tau\to\infty} A_\tau$;
- $\mathsf{W}_1(\mathbf{A}) := \text{s-lim}_{\tau\to\infty} \begin{bmatrix} R_\tau \\ S_{-\tau} \end{bmatrix} A_\tau \begin{bmatrix} R_\tau & S_\tau \end{bmatrix}$;
- $\mathsf{W}_1^F(\mathbf{A}) := \text{s-lim}_{\tau\to\infty} \begin{bmatrix} R_\tau^F \\ S_{-\tau}^F \end{bmatrix} A_\tau \begin{bmatrix} R_\tau^F & S_\tau^F \end{bmatrix}$;
- $\mathsf{Y}_0(\mathbf{A}) := \text{s-lim}_{\tau\to\infty} Z_\tau A_\tau Z_\tau^{-1}$; $\qquad \mathsf{Y}_0^F(\mathbf{A}) := \text{s-lim}_{\tau\to\infty} Z_\tau^{-1} A_\tau Z_\tau$;
- $\mathsf{Y}_s(\mathbf{A}) := \text{s-lim}_{\tau\to\infty} \begin{bmatrix} Z_\tau R_s \\ Z_\tau S_{-s} \end{bmatrix} A_\tau \begin{bmatrix} R_s Z_\tau^{-1} & S_s Z_\tau^{-1} \end{bmatrix}$ for every $s > 0$;
- $\mathsf{Y}_s^F(\mathbf{A}) := \text{s-lim}_{\tau\to\infty} \begin{bmatrix} Z_\tau^{-1} R_s^F \\ Z_\tau^{-1} S_{-s}^F \end{bmatrix} A_\tau \begin{bmatrix} R_s^F Z_\tau & S_s^F Z_\tau \end{bmatrix}$ for every $s > 0$.

Remark. Note that the $F-$homomorphisms are related to the others formally by $\mathsf{W}^F(\mathbf{A}) = F^{-1}\mathsf{W}(F\mathbf{A}F^{-1})F$ whenever that makes sense. W_0 is its own "F-dual", and $\mathsf{Y}_0^F(\mathbf{A})$ could be written as $\text{s-lim}_{\tau\to\infty} Z_\tau^F A_\tau (Z_\tau^F)^{-1}$. In [17], Y_0, Y_0^F, Y_s and Y_s^F are denoted by $\mathsf{Y}_{0,\infty}$, $\mathsf{Y}_{\infty,0}$, $\mathsf{Y}_{s,\infty}$ and $\mathsf{Y}_{\infty,s}$, respectively. With this change of notation we want to emphasize the symmetry between those homomorphisms (and the related ideals defined below).

Lemma 4.3. (i) *$\mathcal{F}$ is a closed unital subalgebra of $\mathcal{E}$, and $\mathcal{G}$ is a closed ideal of $\mathcal{F}$.*

(ii) *W_0, Y_0 and Y_0^F are bounded unital homomorphisms from $\mathcal{F}$ to $\mathcal{B}(L^p(\mathbb{R}))$ with norm 1, and W_1, W_1^F, Y_s and Y_s^F $(s > 0)$ are bounded unital homomorphisms from $\mathcal{F}$ to $(\mathcal{B}(L^p(\mathbb{R})))^{2\times 2}$. The ideal $\mathcal{G}$ lies in the kernel of each of these homomorphisms.*

(iii) *The quotient algebra $\mathcal{F}/\mathcal{G}$ is inverse closed in $\mathcal{E}/\mathcal{G}$, and $\mathcal{F}$ is inverse closed in $\mathcal{E}$.*

Proof. Using the lemmata in the previous section and simple identities like

$$\begin{bmatrix} R_\tau & S_\tau \end{bmatrix} \begin{bmatrix} R_\tau \\ S_{-\tau} \end{bmatrix} = R_\tau R_\tau + S_\tau S_{-\tau} = I,$$

the assertions follow easily from Lemma 4.1. □

Let $\mathcal{K} = \mathcal{K}(L^p(\mathbb{R}))$ denote the closed ideal of the compact operators in $\mathcal{B}(L^p(\mathbb{R}))$ and set

$$\begin{aligned} \mathcal{J}_0 &:= \{(K + G_\tau)_{\tau>0} : K \in \mathcal{K},\ (G_\tau) \in \mathcal{G}\}, \\ \mathcal{J}_1 &:= \{(R_\tau K_1 R_\tau + R_\tau K_2 S_{-\tau} + S_\tau K_3 R_\tau + S_\tau K_4 S_{-\tau} + G_\tau)_{\tau>0} : K_k \in \mathcal{K},\ (G_\tau) \in \mathcal{G}\}. \\ \mathcal{J}_1^F &:= \{(R_\tau^F K_1 R_\tau^F + R_\tau^F K_2 S_{-\tau}^F + S_\tau^F K_3 R_\tau^F + S_\tau^F K_4 S_{-\tau}^F + G_\tau)_{\tau>0} : K_k \in \mathcal{K},\ (G_\tau) \in \mathcal{G}\}. \end{aligned}$$

Proposition 4.4. *$\mathcal{J}_0$, $\mathcal{J}_1$ and $\mathcal{J}_1^F$ are closed ideals of $\mathcal{F}$.*

Proof. First one has to show that $\mathcal{J}_0$, $\mathcal{J}_1$ and $\mathcal{J}_1^F$ are contained in $\mathcal{F}$. We postpone this proof to Propositions 4.7 - 4.12 where these facts are collected together with some closely related assertions. In anticipation of these results it is clear that $\mathcal{J}_0$ is even a linear subspace of $\mathcal{F}$. Now let $\mathbf{A} = (A_\tau) \in \mathcal{F}$ and K a compact operator, and set $A := \mathsf{W}_0(\mathbf{A})$. Then $(A_\tau)(K) = (AK) - ((A - A_\tau)K)$ is in $\mathcal{J}_0$ since the operator AK is compact and the sequence $((A - A_\tau)K)$ is in $\mathcal{G}$ because $A_\tau \to A$ strongly. Hence, $\mathcal{J}_0$ is a left ideal. The right ideal property of $\mathcal{J}_0$ follows similarly, using that $A_\tau^* \to A^*$ strongly.

To prove that $\mathcal{J}_0$ is closed, note first that

$$\|K\| = \lim_{\tau\to\infty} \|K + G_\tau\| \le \|(K + G_\tau)\|_{\mathcal{E}} \tag{9}$$

for $(K + G_\tau) \in \mathcal{J}_0$. Now consider a Cauchy sequence $((K^{(k)} + G_\tau^{(k)}))_{k\ge1}$ in $\mathcal{J}_0$. Then, by (9), the sequence $(K^{(k)})_{k\ge1}$ is also a Cauchy sequence. Thus, there exists a compact operator K such that $\|K - K^{(k)}\| \to 0$. But then, $((G_\tau^{(k)}))_{k\ge1}$ is also a Cauchy sequence. Since $\mathcal{G}$ is closed in $\mathcal{E}$, there exists a $(G_\tau) \in \mathcal{G}$ such that $\|G_\tau - G_\tau^{(k)}\| \to 0$. We conclude that the sequence $(K + G_\tau)$ is the limit of the sequences $(K^{(k)} + G_\tau^{(k)})$ as $k \to \infty$. This finishes the proof of the assertion for the ideal $\mathcal{J}_0$.

The proof for $\mathcal{J}_1$ and $\mathcal{J}_1^F$ proceeds in a similar way. We will only check the left ideal property of $\mathcal{J}_1$. For $(A_\tau) \in \mathcal{F}$ and K_1 a compact operator,

$$\begin{aligned} A_\tau\, R_\tau K_1 R_\tau &= R_\tau R_\tau A_\tau R_\tau K_1 R_\tau + S_\tau S_{-\tau} A_\tau R_\tau K_1 R_\tau \\ &= R_\tau A_{11} K_1 R_\tau + R_\tau (R_\tau A_\tau R_\tau - A_{11}) K_1 R_\tau \\ &\qquad + S_\tau A_{21} K_1 R_\tau + S_\tau (S_{-\tau} A_\tau R_\tau - A_{21}) K_1 R_\tau. \end{aligned}$$

Since K_1 is compact and the sequences $(R_\tau A_\tau R_\tau - A_{11})$ and $(S_{-\tau} A_\tau R_\tau - A_{21})$ tend strongly to zero, the sequence $(A_\tau\, R_\tau K_1 R_\tau)$ belongs to $\mathcal{J}_1$. Similarly,

$$\begin{aligned} A_\tau\, R_\tau K_2 S_{-\tau} &= R_\tau A_{11} K_2 S_{-\tau} + S_\tau A_{21} K_2 S_{-\tau} + G_\tau^{(1)}, \\ A_\tau\, S_\tau K_3 R_\tau &= R_\tau A_{12} K_3 R_\tau + S_\tau A_{22} K_3 R_\tau + G_\tau^{(2)}, \\ A_\tau\, S_\tau K_4 S_{-\tau} &= R_\tau A_{12} K_4 S_{-\tau} + S_\tau A_{22} K_4 S_{-\tau} + G_\tau^{(3)} \end{aligned}$$

with sequences $(G_\tau^{(k)})$ in $\mathcal{G}$. Hence, $\mathcal{J}_1$ is a left ideal. □

Let $\mathcal{J}$ denote the smallest closed ideal of $\mathcal{F}$ which contains $\mathcal{J}_0$, $\mathcal{J}_1$ and $\mathcal{J}_1^F$. One easily checks that $\mathcal{J} = \mathcal{J}_0 + \mathcal{J}_1 + \mathcal{J}_1^F$. Specifying the lifting theorem [20, Theorem 6.3.8] to the present context, we obtain the following result (which also has a simple direct proof). The result indicates that a main task in the stability analysis of a sequence $\mathbf{A}$ in $\mathcal{F}$ is to study the invertibility of its coset in $\mathcal{F}/\mathcal{J}$.

Theorem 4.5. *Let $\mathbf{A} \in \mathcal{F}$. The coset $\mathbf{A} + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{G}$ if and only if the operator $\mathsf{W}_0(\mathbf{A})$ is invertible in $\mathcal{B}(L^p(\mathbb{R}))$, the operators $\mathsf{W}_1(\mathbf{A})$ and $\mathsf{W}_1^F(\mathbf{A})$ are invertible in $(\mathcal{B}(L^p(\mathbb{R})))^{2\times 2}$, and the coset $\mathbf{A} + \mathcal{J}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{J}$.*

4.2 The sequences we are interested in

The sequences we are interested in belong to the smallest closed subalgebra of $\mathcal{E}$ which contains all constant sequences (aI) of operators of multiplication by a function $a \in PC(\dot{\mathbb{R}})$, all constant sequences $(W^0(b))$ of operators of convolution by a multiplier $b \in PC_p$, the constant sequence (J), the sequences $(P_\tau)_{\tau>0}$ and $(P_\tau^F)_{\tau>0}$, and the sequences in the ideal $\mathcal{J}$. We denote this subalgebra by $\mathcal{A} = \mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, J, (P_\tau), (P_\tau^F)\big)$. This algebra can be seen as an extension of the algebra $\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, J\big)$ studied in [17, 20] and of the algebra $\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$ studied in [19]. The goal of this section is to prove the following basic fact.

Theorem 4.6. $\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, J, (P_\tau), (P_\tau^F)\big)$ *is a subalgebra of* $\mathcal{F}$.

To get this assertion, it is sufficient to show that the generating sequences of the algebra $\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, J, (P_\tau), (P_\tau^F)\big)$ belong to $\mathcal{F}$, i.e., that the strong limit homomorphisms which specify $\mathcal{F}$ exist for the generating sequences. The existence and computation of these specific limits will be the subject of Propositions 4.7 - 4.12 below. So Theorem 4.6 will follow once these propositions are verified. Note that also the proof of Proposition 4.4 is completed by the last assertion in each of these propositions.

Proposition 4.7. *The strong limit $\mathsf{W}_0(\mathbf{A})$ exists for the following sequences:*

(i) $\mathsf{W}_0((P_\tau)) = I$; $\quad \mathsf{W}_0((P_\tau^F)) = I$; $\quad \mathsf{W}_0(J) = J$;

(ii) $\mathsf{W}_0(aI) = aI$ *for* $a \in PC(\dot{\mathbb{R}})$;

(iii) $\mathsf{W}_0(W^0(b)) = W^0(b)$ *for* $b \in PC_p$;

(iv) $\mathsf{W}_0(K) = K$ *for a compact operator K, and* $\mathsf{W}_0(\mathbf{J}) = 0$ *for* $\mathbf{J} \in \mathcal{J}_1 \cup \mathcal{J}_1^F$.

These assertions are evident for constant sequences and for the sequence (P_τ). For (P_τ^F) the result is in Proposition 2.2. The assertion for sequences in $\mathcal{J}_1$ and $\mathcal{J}_1^F$ is a consequence of Lemma 3.6.

In what follows, we set $a_s := a(s^-)\chi_- + a(s^+)\chi_+$ and $a_\infty := a(-\infty)\chi_- + a(+\infty)\chi_+$ for every piecewise continuous function a and every $s \in \mathbb{R}$.

Proposition 4.8. *The strong limit $\mathsf{W}_1(\mathbf{A})$ exists for the following sequences:*

(i) $\mathsf{W}_1((P_\tau)) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; $\quad \mathsf{W}_1((P_\tau^F)) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$; $\quad \mathsf{W}_1(J) = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$;

(ii) $\mathsf{W}_1(aI) = \begin{bmatrix} a_\infty I & 0 \\ 0 & a_\infty I \end{bmatrix}$ *for* $a \in PC(\dot{\mathbb{R}})$;

(iii) $\mathsf{W}_1(W^0(b)) =$

$$\begin{bmatrix} \chi_+ W^0(\tilde{b})\chi_+ I + \chi_- W^0(\tilde{b})\chi_- I & \chi_+ JW^0(b)\chi_+ I + \chi_- JW^0(b)\chi_- I \\ \chi_+ W^0(b)J\chi_+ I + \chi_- W^0(b)J\chi_- I & \chi_+ W^0(b)\chi_+ I + \chi_- W^0(b)\chi_- I \end{bmatrix}$$

for $b \in PC_p$;

(iv) $\mathsf{W}_1((R_\tau K_1 R_\tau + R_\tau K_2 S_{-\tau} + S_\tau K_3 R_\tau + S_\tau K_4 S_{-\tau})) = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}$ *for compact operators* K_j, *and* $\mathsf{W}_1(\mathbf{J}) = 0$ *for* $\mathbf{J} \in \mathcal{J}_0 \cup \mathcal{J}_1^F$.

Proof. The computation of the limits for (P_τ) and the constant sequence (J) are straightforward (see Lemma 3.2). For (P_τ^F), decompose each entry of the matrix in a sum with regards to the characteristic functions of the half-lines. For instance the first entry decomposes into

$$R_\tau P_\tau^F R_\tau = R_\tau \chi_+ P_\tau^F \chi_+ R_\tau + R_\tau \chi_+ P_\tau^F \chi_- R_\tau + R_\tau \chi_- P_\tau^F \chi_+ R_\tau + R_\tau \chi_- P_\tau^F \chi_- R_\tau.$$

Now, Proposition 3.11 gives

$$R_\tau \chi_\pm P_\tau^F \chi_\pm R_\tau = P_\tau \chi_\pm P_\tau^F \chi_\pm P_\tau \to \chi_\pm.$$

Also as in the proof of that proposition, one obtains that the crossed elements $R_\tau \chi_\pm P_\tau^F \chi_\mp R_\tau$ tend strongly to zero. Thus $(R_\tau P_\tau^F R_\tau) \to I$.

One does the same decomposition for $R_\tau P_\tau^F S_\tau$. Proposition 3.11 then gives

$$R_\tau \chi_\pm P_\tau^F \chi_\pm S_\tau = P_\tau \chi_\pm J P_\tau^F \chi_\pm \to \chi_\pm J \chi_\pm = 0.$$

The elements $R_\tau \chi_\pm P_\tau^F \chi_\mp S_\tau$ tend also to zero and thus $(R_\tau P_\tau^F S_\tau) \to 0$. Likewise, one obtains that $(S_{-\tau} P_\tau^F R_\tau) \to 0$ and $(S_{-\tau} P_\tau^F S_\tau) \to I$.

The W_1-limits of the constant sequences (aI) and $(W^0(b))$ were calculated in [17, Propositions 4.6-4.8]. The computation of the limits in (iv) is straightforward for sequences in $\mathcal{J}_1$. For sequences in $\mathcal{J}_0$ and $\mathcal{J}_1^F$ use Lemmas 3.6 and 3.7, respectively. □

The next proposition is the dual of the previous one and can be easily proved for operators acting on $L^2(\mathbb{R})$ and then on $L^p(\mathbb{R})$ via Proposition 3.5.

Proposition 4.9. *The strong limit* $\mathsf{W}_1^F(\mathbf{A})$ *exists for the following sequences:*

(i) $\mathsf{W}_1^F((P_\tau)) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$; $\quad \mathsf{W}_1^F((P_\tau^F)) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; $\quad \mathsf{W}_1^F(J) = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$;

(ii) $\mathsf{W}_1^F(aI) = \begin{bmatrix} \chi_+^F \tilde{a} \chi_+^F + \chi_-^F \tilde{a} \chi_-^F & \chi_+^F J a \chi_+^F + \chi_-^F J a \chi_-^F \\ \chi_+^F a J \chi_+^F + \chi_-^F a J \chi_-^F & \chi_+^F a \chi_+^F + \chi_-^F a \chi_-^F \end{bmatrix}$ *for* $a \in PC(\dot{\mathbb{R}})$;

(iii) $\mathsf{W}_1^F(W^0(b)) = \begin{bmatrix} W^0(b_\infty) & 0 \\ 0 & W^0(b_\infty) \end{bmatrix}$ *for* $b \in PC_p$;

(iv) $\mathsf{W}_1^F((R_\tau^F K_1 R_\tau^F + R_\tau^F K_2 S_{-\tau}^F + S_\tau^F K_3 R_\tau^F + S_\tau^F K_4 S_{-\tau}^F)) = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}$ *for compact operators* K_j, *and* $\mathsf{W}_1(\mathbf{J}) = 0$ *for* $\mathbf{J} \in \mathcal{J}_0 \cup \mathcal{J}_1$.

Proposition 4.10. *The strong limit* $\mathsf{Y}_0(\mathbf{A})$ *exists for the following sequences:*

(i) $\mathsf{Y}_0((P_\tau)) = I$; $\quad \mathsf{Y}_0((P_\tau^F)) = P_1^F$; $\quad \mathsf{Y}_0(J) = J$;

(ii) $\mathsf{Y}_0(aI) = a_0 I$ *for* $a \in PC(\dot{\mathbb{R}})$;

(iii) $\mathsf{Y}_0(W^0(b)) = W^0(b_\infty)$ *for* $b \in PC_p$;

(iv) $\mathsf{Y}_0(\mathbf{J}) = 0$ *for* $\mathbf{J} \in \mathcal{J}$.

Proposition 4.11. *The strong limit* $\mathsf{Y}_0^F(\mathbf{A})$ *exists for the following sequences:*

(i) $\mathsf{Y}_0^F((P_\tau)) = P_1$; $\quad \mathsf{Y}_0^F((P_\tau^F)) = I$; $\quad \mathsf{Y}_0^F(J) = J$;

(ii) $\mathsf{Y}_0^F(aI) = a_\infty I$ *for* $a \in PC(\dot{\mathbb{R}})$;

(iii) $\mathsf{Y}_0^F(W^0(b)) = W^0(b_0)$ *for* $b \in PC_p$;

(iv) $\mathsf{Y}_0^F(\mathbf{J}) = 0$ *for* $\mathbf{J} \in \mathcal{J}$.

For constant sequences of multiplication and convolution operators, the proof of the preceding propositions can be found in [20, Propositions 5.4.1, 5.4.3]. The proof for the sequences (P_τ), (P_τ^F) and (J) is evident. The assertions (iv) follow from Lemma 3.6 (iii) (for Proposition 4.10) and (v) (for Proposition 4.11) by choosing $s = 0$. □

Proposition 4.12. *Let* $s > 0$*. The strong limit* $\mathsf{Y}_s^F(\mathbf{A})$ *exists for the following sequences:*

(i) $\mathsf{Y}_s^F((P_\tau^F)) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$; $\quad \mathsf{Y}_s^F(J) = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$;

(ii) $\mathsf{Y}_s^F((P_\tau)) = \begin{bmatrix} \chi_+^F P_1 \chi_+^F + \chi_-^F P_1 \chi_-^F & \chi_+^F P_1 \chi_-^F J + \chi_-^F P_1 \chi_+^F J \\ \chi_+^F P_1 \chi_-^F J + \chi_-^F P_1 \chi_+^F J & \chi_+^F P_1 \chi_+^F + \chi_-^F P_1 \chi_-^F \end{bmatrix}$;

(iii) $\mathsf{Y}_s^F(aI) = \begin{bmatrix} \chi_+^F \tilde{a}_\infty \chi_+^F + \chi_-^F \tilde{a}_\infty \chi_-^F & \chi_+^F \tilde{a}_\infty \chi_-^F J + \chi_-^F \tilde{a}_\infty \chi_+^F J \\ \chi_+^F a_\infty \chi_-^F J + \chi_-^F a_\infty \chi_+^F J & \chi_+^F a_\infty \chi_+^F + \chi_-^F a_\infty \chi_-^F \end{bmatrix}$
for $a \in PC(\dot{\mathbb{R}})$;

(iv) $\mathsf{Y}_s^F(W^0(b)) = \begin{bmatrix} b((-s)^+)\chi_-^F + b(s^-)\chi_+^F & 0 \\ 0 & b((-s)^-)\chi_-^F + b(s^+)\chi_+^F \end{bmatrix}$
for $b \in PC_p(\dot{\mathbb{R}})$;

(v) $\mathsf{Y}_s^F(\mathbf{J}) = 0$ *for* $\mathbf{J} \in \mathcal{J}$.

Proof. The strong limits of the constant sequences were calculated in [17] again. Next we determine the first entry of the matrix $\mathsf{Y}_s^F((P_\tau))$. The calculation of the other entries runs similar. It is easy to check that $Z_\tau^{-1} F^{-1} = F^{-1} Z_\tau \tau^{1-2/p}$ and $F Z_\tau = \tau^{-1+2/p} Z_\tau^{-1} F$ (remember that we are working on $L^p(\mathbb{R})$ and that Z_τ depends on p by definition). Thus,

$$\begin{aligned} Z_\tau^{-1} R_s^F P_\tau R_s^F Z_\tau &= Z_\tau^{-1} F^{-1} R_s F P_\tau F^{-1} R_s F Z_\tau \\ &= F^{-1} R_{s\tau} F Z_\tau^{-1} P_\tau Z_\tau F^{-1} R_{s\tau} F \\ &= F^{-1} R_{s\tau} F P_1 F^{-1} R_{s\tau} F \ = \ R_{s\tau}^F P_1 R_{s\tau}^F. \end{aligned}$$

Writing $R_{s\tau}$ as $R_{s\tau}\chi_+ + R_{s\tau}\chi_-$, we express $R_{s\tau}^F P_1 R_{s\tau}^F$ as the sum

$$(\chi_+ R_{s\tau})^F P_1 (R_{s\tau}\chi_+)^F + (\chi_+ R_{s\tau})^F P_1 (R_{s\tau}\chi_-)^F \\ + (\chi_- R_{s\tau})^F P_1 (R_{s\tau}\chi_+)^F + (\chi_- R_{s\tau})^F P_1 (R_{s\tau}\chi_-)^F. \tag{10}$$

Using Lemma 3.1, we find for the first and last item in this sum

$$\begin{aligned}(\chi_\pm R_{s\tau})^F P_1 (R_{s\tau}\chi_\pm)^F &= J\chi_\mp^F V_{\mp s\tau}^F \chi_\pm^F P_1 J \chi_\mp^F V_{\mp s\tau}^F \chi_\pm^F \\ &= \chi_\pm^F V_{\pm s\tau}^F \chi_\mp^F P_1 \chi_\mp^F V_{\mp s\tau}^F \chi_\pm^F\end{aligned}$$

Note that these terms are uniformly bounded on $L^p(\mathbb{R})$. Since $V_{\pm s\tau}^F = U_{\mp s\tau}$ and U_t commutes with P_1, these terms are further equal to

$$\chi_\pm^F U_{\mp s\tau} \chi_\mp^F U_{\pm s\tau} P_1 U_{\mp s\tau} \chi_\mp^F U_{\pm s\tau} \chi_\pm^F. \tag{11}$$

Taking into account that $U_{\mp\tau}(aI)^F U_{\pm\tau} \to a(\mp\infty)$ as $\tau \to \infty$ by [20, Lemma 5.4.2 (ii)]) we conclude that the operators (11) converge strongly to $\chi_\pm^F P_1 \chi_\pm^F$ as $\tau \to \infty$.

For the terms $(\chi_\pm R_{s\tau})^F P_1 (R_{s\tau}\chi_\mp)^F$ in the sum (10) we obtain

$$\begin{aligned}(\chi_\pm R_{s\tau})^F P_1 (R_{s\tau}\chi_\mp)^F &= J(\chi_\mp V_{\mp s\tau}\chi_\pm)^F P_1 J (\chi_\pm V_{\pm s\tau}\chi_\mp)^F \\ &= (\chi_\pm V_{\pm s\tau}\chi_\mp)^F P_1 (\chi_\pm V_{\pm s\tau}\chi_\mp)^F \\ &= (\chi_\pm V_{\pm s\tau}\chi_\mp V_{\pm s\tau})^F P_1 (V_{\mp s\tau}\chi_\pm V_{\pm s\tau}\chi_\mp)^F \\ &= (\chi_\pm V_{\pm s\tau}\chi_\mp V_{\pm s\tau} Q_{s\tau})^F P_1 (\chi_\mp P_{s\tau})^F.\end{aligned}$$

Since $(\chi_\pm V_{\pm s\tau}\chi_\mp V_{\pm s\tau})^F$ is uniformly bounded, the asserted strong convergence follows from $Q_{s\tau}^F \to 0$ and $P_{s\tau}^F \to I$ strongly as $\tau \to \infty$ by Lemma 3.6 (iii). Finally, assertion (v) follows from Lemma 3.8 (ii) and from the weak convergence of Z_τ^{-1} to 0 by Lemma 3.6 (iii). □

Proposition 4.13. *Let $s > 0$. The strong limit $\mathsf{Y}_s(\mathbf{A})$ exists for the following sequences:*

(i) $\mathsf{Y}_s((P_\tau)) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$; $\qquad \mathsf{Y}_s(J) = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$;

(ii) $\mathsf{Y}_s((P_\tau^F)) = \begin{bmatrix} \chi_+ P_1^F \chi_+ I + \chi_- P_1^F \chi_- I & \chi_+ P_1^F \chi_- J + \chi_- P_1^F \chi_+ J \\ \chi_+ P_1^F \chi_- J + \chi_- P_1^F \chi_+ J & \chi_+ P_1^F \chi_+ I + \chi_- P_1^F \chi_- I \end{bmatrix}$;

(iii) $\mathsf{Y}_s(aI) = \begin{bmatrix} a((-s)^+)\chi_- I + a(s^-)\chi_+ I & 0 \\ 0 & a((-s)^-)\chi_- I + a(s^+)\chi_+ I \end{bmatrix}$ *for $a \in PC(\dot{\mathbb{R}})$;*

(iv) $\mathsf{Y}_s(W^0(b)) = \begin{bmatrix} \chi_+ \tilde{b}_\infty^F \chi_+ I + \chi_- \tilde{b}_\infty^F \chi_- I & \chi_+ \tilde{b}_\infty^F \chi_- J + \chi_- \tilde{b}_\infty^F \chi_+ J \\ \chi_+ b_\infty^F \chi_- J + \chi_- b_\infty^F \chi_+ J & \chi_+ b_\infty^F \chi_+ I + \chi_- b_\infty^F \chi_- I \end{bmatrix}$ *for $b \in PC_p(\dot{\mathbb{R}})$;*

(v) $\mathsf{Y}_s(\mathbf{J}) = 0$ *for $\mathbf{J} \in \mathcal{J}$.*

The proof runs similar to that of the previous (F-symmetric) lemma.

4.3 Localization

Let $\tilde{C}(\dot{\mathbb{R}})$ and $\tilde{C}_p$ denote the sets of all even functions in $C(\dot{\mathbb{R}})$ and C_p, respectively. We say that a sequence $(A_\tau) \in \mathcal{F}$ is of *local type* if

$$(A_\tau)(fI) - (fI)(A_\tau) \in \mathcal{J}, \quad (A_\tau)(W^0(g)) - (W^0(g))(A_\tau) \in \mathcal{J}$$

for all $f \in \tilde{C}(\dot{\mathbb{R}})$ and all $g \in \tilde{C}_p$. Let $\mathcal{L}$ denote the set of all sequences of local type. The proof of the next lemma is very similar to that of [14, Lemma 6.2]. Assertion (ii) follows from Lemma 4.2.

Lemma 4.14. (i) *$\mathcal{L}$ is a closed unital subalgebra of $\mathcal{F}$, and $\mathcal{J}$ is a closed two-sided ideal of $\mathcal{L}$.*

(ii) *The quotient algebra $\mathcal{L}/\mathcal{J}$ is inverse closed in $\mathcal{F}/\mathcal{J}$, and $\mathcal{L}$ is inverse closed in $\mathcal{F}$.*

The algebra $\mathcal{L}$ is still large enough to contain all sequences that interest us.

Proposition 4.15. *$\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, J, (P_\tau), (P_\tau^F)\big)$ is a (closed, unital) subalgebra of $\mathcal{L}$.*

Proof. We have to show that the generators (aI) with $a \in PC(\dot{\mathbb{R}})$, $(W^0(b))$ with $b \in PC_p$, (J), (P_τ) and (P_τ^F) of $\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, J, (P_\tau), (P_\tau^F)\big)$ commute modulo sequences in $\mathcal{J}$ with the constant sequences (fI) and $(W^0(g))$ where $f \in \tilde{C}(\dot{\mathbb{R}})$ and $g \in \tilde{C}_p$. This is trivial for (J). For the other constant generating sequences, this property follows from [20, Proposition 5.3.1]. In fact, it was shown there that these sequences commute modulo $\mathcal{J}$ already with the "non-symmetric" sequences (fI) and $(W^0(g))$, where $f \in C(\dot{\mathbb{R}})$ and $g \in C_p$.

It is further evident that (P_τ) commutes with every (fI) and (P_τ^F) commutes with every $(W^0(g))$. Next we verify that the commutator

$$(P_\tau)(W^0(g)) - (W^0(g))(P_\tau) = (P_\tau W^0(g) - W^0(g) P_\tau)$$

belongs to $\mathcal{J}$ for every multiplier $g \in C_p$. Write this commutator as

$$\begin{aligned}(P_\tau W^0(g) Q_\tau - Q_\tau W^0(g) P_\tau) = {} & (P_\tau \chi_+ W^0(g) \chi_+ Q_\tau) - (Q_\tau \chi_+ W^0(g) \chi_+ P_\tau) \\ & + (P_\tau \chi_+ W^0(g) \chi_- Q_\tau) - (Q_\tau \chi_+ W^0(g) \chi_- P_\tau) \\ & + (P_\tau \chi_- W^0(g) \chi_+ Q_\tau) - (Q_\tau \chi_- W^0(g) \chi_+ P_\tau) \\ & + (P_\tau \chi_- W^0(g) \chi_- Q_\tau) - (Q_\tau \chi_- W^0(g) \chi_- P_\tau).\end{aligned}$$

By Proposition 3.11, the first sequence in the first line of the right-hand side of this equation is equal to

$$(P_\tau \chi_+ W^0(g) \chi_+ Q_\tau) = (R_\tau (R_\tau \chi_+ W^0(g) \chi_+ S_\tau) S_{-\tau}) = (R_\tau (J \chi_- W^0(g) \chi_+) S_{-\tau}).$$

Since the operator $\chi_- W^0(g) \chi_+ I$ is compact by [20, Proposition 5.3.1], this sequence is in $\mathcal{J}_1$. Similarly, the second sequence in the first line and the sequences in the last line belong to $\mathcal{J}_1$. The sequences in the second and third line belong to the ideal $\mathcal{G}$, which follows from the compactness of the operators $\chi_\pm W^0(g) \chi_\mp I$ by [20, Proposition 5.3.1] again and from the strong convergence of the Q_τ to zero. Likewise, one proves that (P_τ^F) commutes with (fI) for $f \in C(\dot{\mathbb{R}})$ modulo sequences in $\mathcal{J}$, as $\mathcal{J}_1^F$ is part of it. □

Let $\bar{\mathbb{R}}^+$ denote the compactification of $\mathbb{R}^+$ by the point $\{\infty\}$, i.e., $\bar{\mathbb{R}}^+$ is homeomorphic to the interval [0, 1]. The maximal ideal space $M_{\mathcal{C}}$ of the algebra $\mathcal{C}$ generated by all cosets $\Phi(fW^0(g))$ with $f \in \tilde{C}(\dot{\mathbb{R}})$ and $g \in \tilde{C}_p$ is homeomorphic to the subset $(\bar{\mathbb{R}}^+ \times \{\infty\}) \cup (\{\infty\} \times \bar{\mathbb{R}}^+)$ of the square $\bar{\mathbb{R}}^+ \times \bar{\mathbb{R}}^+$, and the value of the Gelfand transform of an element $\Phi(fW^0(g)) \in \mathcal{C}$ at the point $(s,t) \in M_{\mathcal{C}}$ is $f(s)g(t)$; see [20, Section 5.7].

Given $(s,t) \in (\bar{\mathbb{R}}^+ \times \{\infty\}) \cup (\{\infty\} \times \bar{\mathbb{R}}^+)$, let $\mathcal{I}_{s,t}$ denote the smallest closed two-sided ideal of the quotient algebra $\mathcal{L}/\mathcal{J}$ which contains the maximal ideal corresponding to the point (s,t), and let $\Phi^{\mathcal{J}}_{s,t}$ refer to the canonical homomorphism from $\mathcal{L}/\mathcal{J}$ onto the quotient algebra $\mathcal{L}^{\mathcal{J}}_{s,t} := (\mathcal{L}/\mathcal{J})/\mathcal{I}_{s,t}$. In order not to burden the notation, we write $\Phi^{\mathcal{J}}_{s,t}(\mathbf{A})$ instead of $\Phi^{\mathcal{J}}_{s,t}(\mathbf{A}+\mathcal{J})$ for every sequence $\mathbf{A} \in \mathcal{L}$. Then Allan's local principle (see, for instance, [20, Section 2.2]) states that the coset $\mathbf{A}+\mathcal{J}$ of a sequence $\mathbf{A} \in \mathcal{L}$ is invertible in $\mathcal{L}/\mathcal{J}$ if and only if all "local" cosets $\Phi^{\mathcal{J}}_{s,t}(\mathbf{A})$ are invertible in the corresponding "local" algebras $\mathcal{L}^{\mathcal{J}}_{s,t}$.

One cannot hope to find a complete description of the local algebra $\mathcal{L}^{\mathcal{J}}_{s,t}$. But we will be able to identify its smallest closed subalgebra $\mathcal{A}^{\mathcal{J}}_{s,t}$ of $\mathcal{L}^{\mathcal{J}}_{s,t}$ which contains all cosets $(P_\tau)+\mathcal{I}_{s,t}$, $(P^F_\tau)+\mathcal{I}_{s,t}$, $(aI)+\mathcal{I}_{s,t}$ with $a \in PC(\dot{\mathbb{R}})$, $(W^0(b))+\mathcal{I}_{s,t}$ with $b \in PC_p$ and $(J)+\mathcal{I}_{s,t}$, and this identification will be sufficient for our purposes. For the algebras $\mathcal{A}^{\mathcal{J}}_{s,t}$ with $(s,t) \neq (\infty,\infty)$, we achieve this description by means of the family of the Y-homomorphisms introduced above. The algebra $\mathcal{A}^{\mathcal{J}}_{\infty,\infty}$ will require a modification of these mappings.

4.4 The local algebras $\mathcal{A}^{\mathcal{J}}_{0,\infty}$ and $\mathcal{A}^{\mathcal{J}}_{\infty,0}$

In the case of the local algebras with $s=0$ or $t=0$, we have localization at "a single point of" $\mathbb{R}$, which implies that these algebras can be described by the techniques in [19]. With Lemmas 4.10 and 4.11 it is easy to see that the homomorphisms Y_0 and Y^F_0 are well defined as quotient homomorphisms on the local algebras $\mathcal{A}^{\mathcal{J}}_{0,\infty}$ and $\mathcal{A}^{\mathcal{J}}_{\infty,0}$, respectively. We use the same notation for a homomorphism and its quotient. Further, given a set M of operators on a Banach space X, we write $\operatorname{alg} M$ for the smallest closed subalgebra of $\mathcal{B}(X)$ which contains all operators in M. The proofs of the following results proceed then as those of [20, Theorems 6.6.13 and 6.6.15].

Theorem 4.16. *The mapping $\Phi^{\mathcal{J}}_{0,\infty}(\mathbf{A}) \mapsto \mathsf{Y}_0(\mathbf{A})$ is an isometric isomorphism from the local algebra $\mathcal{A}^{\mathcal{J}}_{0,\infty}$ to the closed subalgebra $\operatorname{alg}\{I, \chi_+ I, P^F_1, W^0(\chi_+), J\}$ of $\mathcal{B}(L^p(\mathbb{R}))$.*

Theorem 4.17. *The mapping $\Phi^{\mathcal{J}}_{\infty,0}(\mathbf{A}) \mapsto \mathsf{Y}^F_0(\mathbf{A})$ is an isometric isomorphism from the local algebra $\mathcal{A}^{\mathcal{J}}_{\infty,0}$ to the closed subalgebra $\operatorname{alg}\{I, \chi_+ I, P_1, W^0(\chi_+), J\}$ of $\mathcal{B}(L^p(\mathbb{R}))$.*

4.5 The local algebras $\mathcal{A}^{\mathcal{J}}_{\infty,s}$ for $s > 0$

We start with describing a generating system of $\mathcal{A}^{\mathcal{J}}_{\infty,s}$.

Proposition 4.18. *Let $s>0$. The algebra $\mathcal{A}^{\mathcal{J}}_{\infty,s}$ is generated by the identity $e := \Phi^{\mathcal{J}}_{\infty,s}(I)$, by the projections $p_1 := \Phi^{\mathcal{J}}_{\infty,s}(W^0(\chi_{(0,s)}))$, $p_2 := \Phi^{\mathcal{J}}_{\infty,s}(W^0(\chi_{(s,\infty)}))$, $p_3 := \Phi^{\mathcal{J}}_{\infty,s}(\chi_+ I)$ and $p_4 := \Phi^{\mathcal{J}}_{\infty,s}((P_\tau))$, and by the flip $\Phi^{\mathcal{J}}_{\infty,s}(J)$.*

Proof. We prove that $\Phi^{\mathcal{J}}_{\infty,s}((P^F_\tau)) = e$. Let g be an even continuous function with bounded support such that $g=1$ in a neighborhood of s. Then $\Phi^{\mathcal{J}}_{\infty,s}(W^0(g))$ is the identity element of the local algebra $\mathcal{A}^{\mathcal{J}}_{\infty,s}$, and the asserted identity follows from

$$e - \Phi^{\mathcal{J}}_{\infty,s}(P^F_\tau) = \Phi^{\mathcal{J}}_{\infty,s}(Q^F_\tau) = \Phi^{\mathcal{J}}_{\infty,s}(W^0(g))\Phi^{\mathcal{J}}_{\infty,s}(Q^F_\tau) = \Phi^{\mathcal{J}}_{\infty,s}(W^0(g)Q^F_\tau) = 0,$$

because $gQ_\tau = 0$ for large τ. Since $jp_1j = \Phi^{\mathcal{J}}_{\infty,s}(\chi_{(-s,0)}I)$ and $jp_2j = \Phi^{\mathcal{J}}_{\infty,s}(\chi_{(-\infty,-s)}I)$, the remainder of the proof runs analogously to that of Proposition 6.4 in [17]. □

We conclude from Proposition 4.12 that the homomorphism Y^F_s is well defined on the quotient algebra $\mathcal{A}^{\mathcal{J}}_{\infty,s}$ and that its range $\mathcal{Y}^F$ is independent of s. The structure of the algebra $\mathcal{Y}^F$ seems to be different from and more involved than that of its analogue described in [18, Proposition 2.32]. We will see now that both algebras coincide. For that, set $P := \chi^F_+$ and $Q := \chi^F_-$ and consider the matrix U_F and its inverse, defined by

$$U_F := \begin{bmatrix} JP & P \\ Q & JQ \end{bmatrix}, \qquad U_F^{-1} = \begin{bmatrix} PJ & Q \\ P & QJ \end{bmatrix} \tag{12}$$

Thus the mapping

$$Y \mapsto U_F Y U_F^{-1} \tag{13}$$

is an isomorphism of $(\mathcal{L}(L^p(\mathbb{R})))^{2\times 2}$ onto itself. The images of the generators of $\mathcal{Y}^F$ under this mapping can be easily calculated and are given in the following proposition.

Proposition 4.19. *The images of the generators $\mathcal{Y}^F$ via the mapping* (13) *are given by*

$$U_F\,\mathsf{Y}^F_s((P_\tau))U_F^{-1} = \begin{bmatrix} P_1 & 0 \\ 0 & P_1 \end{bmatrix}, \qquad U_F\,\mathsf{Y}^F_s(J)U_F^{-1} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

$$U_F\,\mathsf{Y}^F_s(aI)U_F^{-1} = \begin{bmatrix} a_\infty I & 0 \\ 0 & \widetilde{a_\infty} I \end{bmatrix},$$

$$U_F\,\mathsf{Y}^F_s(W^0(b))U_F^{-1} = \begin{bmatrix} b(s^-)Q + b(s^+)P & 0 \\ 0 & b((-s)^+)Q + b((-s)^-)P \end{bmatrix} = \begin{bmatrix} W^0(b_s) & 0 \\ 0 & W^0(\widetilde{b_{-s}}) \end{bmatrix}.$$

It turns out that the above matrices are exactly the matrices obtained in [18, Proposition 2.32]. The image $\hat{\mathcal{Y}}^F$ of the algebra $\mathcal{Y}^F$ under the mapping (13) is generated by the matrices given in the above proposition.

Corollary 4.20. *The algebra $\mathcal{Y}^F$ is topologically isomorphic via the mapping* (13) *to the algebra*

$$\hat{\mathcal{Y}}^F = \left(\operatorname{alg}\{I, P_1, \chi_+, \chi^F_+\}\right)^{2\times 2} \subseteq (\mathcal{B}(L^p(\mathbb{R})))^{2\times 2}.$$

We now turn to the inverse-closedness of the occurring algebras. It was in order to be able to solve this (in general) complicated technical problem that it was necessary to change the procedure used in [18] and introduce the new homomorphisms Y_s and Y^F_s. Note that for non-commutative algebras $\mathcal{A}$ and $\mathcal{B}$ it is not known in general if $\mathcal{A}^{2\times 2}$ is inverse-closed in $\mathcal{B}^{2\times 2}$ when $\mathcal{A}$ is inverse-closed in $\mathcal{B}$. Such a result would make Lemma 4.21 an immediate consequence of lemmas 4.3(iii) and 4.14(ii). Instead, we again refer to Lemma 4.1 for the proof of the first part of the following result, and to Lemma 4.2 for its second part.

Lemma 4.21. *The algebra $\mathcal{L}^{2\times 2}$ is inverse-closed in $\mathcal{F}^{2\times 2}$, and the algebra $\mathcal{F}^{2\times 2}$ is inverse-closed in $\mathcal{E}^{2\times 2}$.*

For $s > 0$, define the mapping $\mathsf{X}_{s,1}^F : (\mathcal{B}(L^p(\mathbb{R})))^{2\times 2} \to \mathcal{E}^{2\times 2}$ by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} (U_s Z_\tau A Z_\tau^{-1} U_{-s})_{\tau>0} & (U_s Z_\tau B Z_\tau^{-1} U_{-s})_{\tau>0} \\ (U_s Z_\tau C Z_\tau^{-1} U_{-s})_{\tau>0} & (U_s Z_\tau D Z_\tau^{-1} U_{-s})_{\tau>0} \end{bmatrix}, \tag{14}$$

and write $\mathcal{D}_s$ for the subset of $(\mathcal{B}(L^p(\mathbb{R})))^{2\times 2}$ of all operators L such that $\mathsf{X}_{s,1}^F(L) \in \mathcal{L}^{2\times 2}$.

Lemma 4.22. *$\mathcal{D}_s$ is a closed and inverse-closed subalgebra of $(\mathcal{B}(L^p(\mathbb{R})))^{2\times 2}$ which contains $\hat{\mathcal{Y}}^F$. The mapping $\mathsf{X}_{s,1}^F$ is a continuous homomorphism.*

Proof. We will only prove the inverse-closedness of $\mathcal{D}_s$ in $(\mathcal{B}(L^p(\mathbb{R})))^{2\times 2}$. The other assertions are clear. Let $L \in \mathcal{D}_s$ be invertible in $(\mathcal{B}(L^p(\mathbb{R})))^{2\times 2}$. Then $\mathsf{X}_{s,1}^F(L)\mathsf{X}_{s,1}^F(L^{-1}) = \mathsf{X}_{s,1}^F(L^{-1})\mathsf{X}_{s,1}^F(L) = \mathsf{X}_{s,1}^F(I) = I$ in $\mathcal{E}^{2\times 2}$. Since $\mathcal{L}^{2\times 2}$ is inverse-closed in $\mathcal{E}^{2\times 2}$ by Lemma 4.21, this implies $\mathsf{X}_{s,1}^F(L^{-1}) \in \mathcal{L}^{2\times 2}$. Hence, $L^{-1} \in \mathcal{D}_s$. □

As a consequence of the previous result, the mapping $\mathsf{X}_{s,2}^F := \Phi_{\infty,s}^{\mathcal{J}} \circ \mathsf{X}_{s,1}^F$ (with applying $\Phi_{\infty,s}^{\mathcal{J}}$ to each entry of the matrix (14)) is a continuous homomorphism from $\mathcal{D}_s$ into $(\mathcal{L}_{\infty,s}^{\mathcal{J}})^{2\times 2}$, which maps the operators in $\hat{\mathcal{Y}}^F$ into $(\mathcal{A}_{\infty,s}^{\mathcal{J}})^{2\times 2}$.

Let f be a continuous function with value 1 at s and 0 outside the interval $[\frac{s}{2}, \frac{3s}{2}]$. Then $p := \Phi_{\infty,s}^{\mathcal{J}}(W^0(f))$ is a projection in $\mathcal{L}_{\infty,s}^{\mathcal{J}}$ and $\mathcal{A}_{\infty,s}^{\mathcal{J}}$. Define $\widehat{\mathcal{D}}_s$ as the set of all operators L in $\mathcal{D}_s$ such that $\mathsf{X}_{s,2}^F(L)$ and $pI_{2\times 2}$ commute. It is not difficult to see that $\hat{\mathcal{Y}}^F \subset \widehat{\mathcal{D}}_s$ (recall the proof of Proposition 4.15). We have then the following result the proof of which is standard.

Lemma 4.23. *The set $\widehat{\mathcal{D}}_s$ is a closed and inverse-closed subalgebra of $\mathcal{D}_s$, and the mapping $\mathsf{X}_{s,3}^F : L \mapsto p\mathsf{X}_{s,2}^F(L)$ is a continuous homomorphism from $\widehat{\mathcal{D}}_s$ to $(\mathcal{L}_{\infty,s}^{\mathcal{J}})^{2\times 2}$.*

Set $j := \Phi_{\infty,s}^{\mathcal{J}}(J)$. It is easy to check that the mapping

$$\mathsf{X}_{s,4}^F : \begin{bmatrix} \mathbf{A}^\Phi & \mathbf{B}^\Phi \\ \mathbf{C}^\Phi & \mathbf{D}^\Phi \end{bmatrix} \mapsto p\mathbf{A}^\Phi p + p\mathbf{B}^\Phi j(e-p) + (e-p)j\mathbf{C}^\Phi p + (e-p)j\mathbf{D}^\Phi j(e-p).$$

defined on $\mathsf{X}_{s,3}^F(\widehat{\mathcal{D}}_s)$ is multiplicative.

Theorem 4.24. *Let $\mathsf{X}_s^F := \mathsf{X}_{s,4}^F \circ \mathsf{X}_{s,3}^F$. Then*

(i) *X_s^F is a continuous homomorphism from $\widehat{\mathcal{D}}_s$ to $\mathcal{L}_{\infty,s}^{\mathcal{J}}$ which maps $\hat{\mathcal{Y}}^F$ to $\mathcal{A}_{\infty,s}^{\mathcal{J}}$;*

(ii) *$\mathsf{X}_s^F\left(U_F\,\mathsf{Y}_s^F(\cdot)U_F^{-1}\right)$ is the identity map on $\mathcal{A}_{\infty,s}^{\mathcal{J}}$;*

(iii) *if $\mathsf{Y}_s^F(\mathbf{A})$ is invertible in $(\mathcal{L}(L^p(\mathbb{R})))^{2\times 2}$, then $\Phi_{\infty,s}^{\mathcal{J}}(\mathbf{A})$ is invertible in $\mathcal{L}_{\infty,s}^{\mathcal{J}}$.*

Proof. Assertion (i) is clear, since $\mathsf{X}_{s,4}^F$ is a homomorphism. Assertion (ii) can then be checked for the generators of the algebra. For assertion (iii), note that if $\mathsf{Y}_s^F(\mathcal{A})$ is invertible as an operator, then $U_F\mathsf{Y}_s^F(\mathbf{A})U_F^{-1}$ is invertible as an operator. By Lemma 4.22, $U_F\mathsf{Y}_s^F(\mathbf{A})U_F^{-1}$ is invertible in $\widehat{\mathcal{D}}_s$. So one can apply the homomorphism X_s^F to obtain the invertibility of $\Phi_{\infty,s}^{\mathcal{J}}(\mathbf{A}) = \mathsf{X}_s^F\left(U_F\,\mathsf{Y}_s^F(\mathbf{A})U_F^{-1}\right)$ in $\mathcal{L}_{\infty,s}^{\mathcal{J}}$. □

4.6 The local algebras $\mathcal{A}^{\mathcal{J}}_{s,\infty}$ for $s > 0$

The local algebras $\mathcal{A}^{\mathcal{J}}_{s,\infty}$ are closely related to the algebras $\mathcal{A}^{\mathcal{J}}_{\infty,s}$ due to the F-symmetry of their generators. In particular, we will obtain an algebra $\mathcal{Y}$, which is F-symmetric to its counterpart $\mathcal{Y}^F$. Since the proofs are close to the ones given in the previous subsection, we will omit most of them. Again we start by describing a generating system for the algebra $\mathcal{A}^{\mathcal{J}}_{s,\infty}$.

Proposition 4.25. *Let $s > 0$. The algebra $\mathcal{A}^{\mathcal{J}}_{s,\infty}$ is generated by the identity element $e := \Phi^{\mathcal{J}}_{s,\infty}(I)$, by the projections $p_1 := \Phi^{\mathcal{J}}_{s,\infty}(\chi_{(0,s)}I)$, $p_2 := \Phi^{\mathcal{J}}_{s,\infty}(\chi_{(s,\infty)}I)$ and $r := \Phi^{\mathcal{J}}_{s,\infty}(W^0(\chi_+))$, and by the flip $j := \Phi^{\mathcal{J}}_{s,\infty}(J)$.*

By Proposition 4.13, the homomorphism Y_s is well defined on the local algebra $\mathcal{A}^{\mathcal{J}}_{s,\infty}$, and its range $\mathcal{Y}$ is independent of s. To get a simpler form of the generators of $\mathcal{Y}$ we again use a special automorphism of $(\mathcal{B}(L^p(\mathbb{R})))^{2\times 2}$. For that goal, we define a matrix U and its inverse as the F-counterpart of the matrix U_F introduced above by

$$U := \begin{bmatrix} J\chi_+ I & \chi_+ I \\ \chi_- I & J\chi_- I \end{bmatrix}, \qquad U^{-1} = \begin{bmatrix} \chi_+ J & \chi_- I \\ \chi_+ I & \chi_- J \end{bmatrix}. \tag{15}$$

Then the mapping

$$Y \mapsto UYU^{-1} \tag{16}$$

is an isomorphism of $\big(\mathcal{B}(L^2(\mathbb{R}))\big)^{2\times 2}$ onto itself. Next we calculate the images of the generators of $\mathcal{Y}$ under this mapping.

Proposition 4.26. *The images of the generators $\mathcal{Y}$ via the mapping* (16) *are given by*

$$U\,\mathsf{Y}_s((P^F_\tau))U^{-1} = \begin{bmatrix} P^F_1 & 0 \\ 0 & P^F_1 \end{bmatrix}, \qquad U\,\mathsf{Y}_s(J)U^{-1} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

$$U\,\mathsf{Y}_s(aI)U^{-1} = \begin{bmatrix} a(s^-)\chi_- I + a(s^+)\chi_+ I & 0 \\ 0 & a((-s)^+)\chi_- I + a((-s)^-)\chi_+ I \end{bmatrix} = \begin{bmatrix} a_s I & 0 \\ 0 & \widetilde{a_{-s}} I \end{bmatrix},$$

$$U\,\mathsf{Y}_s(W^0(b))U^{-1} = \begin{bmatrix} W^0(b_\infty) & 0 \\ 0 & W^0(\widetilde{b_\infty}) \end{bmatrix}.$$

Corollary 4.27. *The algebra $\mathcal{Y}$ is topologically isomorphic via the mapping* (16) *to the algebra*

$$\hat{\mathcal{Y}} := \Big(\mathrm{alg}\,\{I, P^F_1, \chi_+, \chi^F_+\}\Big)^{2\times 2} \subseteq \big(\mathcal{B}(L^2(\mathbb{R}))\big)^{2\times 2}.$$

For $s > 0$, define the mapping $\mathsf{X}_{s,1} : (\mathcal{B}(L^p(\mathbb{R})))^{2\times 2} \to \mathcal{E}^{2\times 2}$ by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} (V_{-s}Z^{-1}_\tau A Z_\tau V_s)_{\tau>0} & (V_{-s}Z^{-1}_\tau B Z_\tau V_s)_{\tau>0} \\ (V_{-s}Z^{-1}_\tau C Z_\tau V_s)_{\tau>0} & (V_{-s}Z^{-1}_\tau D Z_\tau V_s)_{\tau>0} \end{bmatrix}$$

and let $p := \Phi^{\mathcal{J}}_{s,\infty}(fI)$ with f as defined in the previous section after Lemma 4.22. Then we can introduce an inverse-closed algebra $\widehat{\mathcal{D}}_s$ and mappings $\mathsf{X}_{s,3} := p \circ \Phi^{\mathcal{J}}_{s,\infty} \circ \mathsf{X}_{s,1}$, and $\mathsf{X}_{s,4}$ in a similar way as before to obtain the following.

Theorem 4.28. *Let $\mathsf{X}_s := \mathsf{X}_{s,4} \circ \mathsf{X}_{s,3}$. Then*

- (i) *X_s is a continuous homomorphism from $\widehat{\mathcal{D}}_s$ to $\mathcal{L}^{\mathcal{J}}_{s,\infty}$ which maps $\hat{\mathcal{Y}}^F$ to $\mathcal{A}^{\mathcal{J}}_{s,\infty}$;*
- (ii) *$\mathsf{X}_s\left(U\,\mathsf{Y}_s(\cdot)U^{-1}\right)$ is the identity map on $\mathcal{A}^{\mathcal{J}}_{s,\infty}$;*
- (iii) *if $\mathsf{Y}_s(\mathbf{A})$ is invertible in $(\mathcal{B}(L^p(\mathbb{R})))^{2\times 2}$, then $\Phi^{\mathcal{J}}_{s,\infty}(\mathbf{A})$ is invertible in $\mathcal{L}^{\mathcal{J}}_{s,\infty}$.*

4.7 The local algebras $\mathcal{A}^{\mathcal{J}}_{\infty,\infty}$

At the point (∞,∞) we have again a single point localization. The generators of the local algebra are described by the following result.

Proposition 4.29. *The algebra $\mathcal{A}^{\mathcal{J}}_{\infty,\infty}$ is generated by the identity e, by the projections $p := \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ I)$, $r := \Phi^{\mathcal{J}}_{\infty,\infty}(W^0(\chi_+))$, $p_1 := \Phi^{\mathcal{J}}_{\infty,\infty}((P_\tau))$, $p_2 := \Phi^{\mathcal{J}}_{\infty,\infty}((P^F_\tau))$ and by the flip $j := \Phi^{\mathcal{J}}_{\infty,\infty}(J)$.*

The structure of this algebra seems to be too involved to be amenable to the analysis done in the other cases, at least to our present knowledge. In particular, the interaction between p_1 and p_2 is not known. What we do know about the generators of the local algebra is summarized in the next result. The results are evident, with the exception of the first one, which can be proved in a similar way to [18, Proposition 2.22].

Proposition 4.30. *The following relations hold in $\mathcal{A}^{\mathcal{J}}_{\infty,\infty}$:*

(i) $pr = rp$, $pp_1 = p_1 p$, $rp_2 = p_2 r$;

(ii) $jpj = e - p$, $jrj = e - r$, $jp_1 j = p_1$, $jp_2 j = p_2$.

Using these relations we are at least able to analyze some interesting subalgebras of $\mathcal{A}^{\mathcal{J}}_{\infty,\infty}$, where not all of the projections p, r, p_1 and p_2 are present. Specifically, we will do this for the algebras

- $\mathcal{A}^1 := \mathcal{A}\{PC(\dot{\mathbb{R}}), PC_p, J, (P_\tau)\}$,
- $\mathcal{A}^2 := \mathcal{A}\{PC(\dot{\mathbb{R}}), PC_p, J, (P^F_\tau)\}$,
- $\mathcal{A}^3 := \mathcal{A}\{PC_\infty(\dot{\mathbb{R}}), PC_{p,\infty}, J, (P_\tau), (P^F_\tau)\}$,

where $PC_\infty(\dot{\mathbb{R}})$ (resp. $PC_{p,\infty}$) stands for the algebra of all functions in $PC(\dot{\mathbb{R}})$ (resp. in PC_p) which are continuous at infinity. The corresponding local algebras at (∞, ∞) are then

- $\mathcal{A}^{1,\mathcal{J}}_{\infty,\infty} = \operatorname{alg}\{p, r, p_1, j\}$,
- $\mathcal{A}^{2,\mathcal{J}}_{\infty,\infty} = \operatorname{alg}\{p, r, p_2, j\}$,
- $\mathcal{A}^{3,\mathcal{J}}_{\infty,\infty} = \operatorname{alg}\{p_1, p_2, j\}$.

Let us start with $\mathcal{A}^{1,\mathcal{J}}_{\infty,\infty}$. The following result, which is immediate from Propositions 4.8 and 4.10, describes the action of the composition $\mathsf{Y}_0 \circ \mathsf{W}_1$ on the generating sequences of the algebra $\mathcal{A}^1$.

Proposition 4.31. (i) $(\mathsf{Y}_0 \circ \mathsf{W}_1)((P_\tau)) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$; $\quad (\mathsf{Y}_0 \circ \mathsf{W}_1)(J) = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$;

(ii) $(\mathsf{Y}_0 \circ \mathsf{W}_1)(aI) = \begin{bmatrix} a_\infty I & 0 \\ 0 & a_\infty I \end{bmatrix}$ *for $a \in PC(\dot{\mathbb{R}})$;*

(iii) $(\mathsf{Y}_0 \circ \mathsf{W}_1)(W^0(b)) = \begin{bmatrix} \chi_+ J b^F_\infty J \chi_+ I + \chi_- J b^F_\infty J \chi_- I & \chi_+ J b^F_\infty \chi_+ I + \chi_- J b^F_\infty \chi_- I \\ \chi_+ b^F_\infty J \chi_+ I + \chi_- J b^F_\infty \chi_- I & \chi_+ b^F_\infty \chi_+ I + \chi_- b^F_\infty \chi_- I \end{bmatrix}$
for $b \in PC_p$;

(iv) $(\mathsf{Y}_0 \circ \mathsf{W}_1)(\mathbf{J}) = 0$ *for* $\mathbf{J} \in \mathcal{J}$.

Proposition 4.31 implies that $(\mathsf{Y}_0 \circ \mathsf{W}_1)(\mathbf{A})$ only depends on the coset $\Phi^{\mathcal{J}}_{\infty,\infty}(\mathbf{A})$ of $\mathbf{A} \in \mathcal{A}^1$ in the local algebra $\mathcal{A}^{1,\mathcal{J}}_{\infty,\infty}$. Thus, we get a homomorphism

$$\mathsf{Y}_0 \circ \mathsf{W}_1 : \mathcal{A}^{1,\mathcal{J}}_{\infty,\infty} \to \left(\operatorname{alg}\{I, \chi_+, \chi_+^F, J\}\right)^{2\times 2}. \tag{17}$$

Again we use the matrices U and U^{-1} from (15) and apply the isomorphism (16) to give the generators of the algebra a more transparent structure. Let Y_∞ be the mapping $\mathbf{A} \mapsto U(\mathsf{Y}_0(\mathsf{W}_1(\mathbf{A})))U^{-1}$.

Proposition 4.32. *The homomorphism* Y_∞ *maps*

(i) $\mathsf{Y}_\infty((P_\tau)) = \begin{bmatrix} \chi_- & 0 \\ 0 & \chi_- \end{bmatrix}$; $\qquad \mathsf{Y}_\infty(J) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$;

(ii) $\mathsf{Y}_\infty(aI) = \begin{bmatrix} a(+\infty)I & 0 \\ 0 & a(-\infty)I \end{bmatrix}$ *for* $a \in PC(\dot{\mathbb{R}})$;

(iii) $\mathsf{Y}_\infty(W^0(b)) = \begin{bmatrix} b^F_\infty & 0 \\ 0 & \tilde{b}^F_\infty \end{bmatrix}$ *for* $b \in PC_p$;

(iv) $\mathsf{Y}_\infty(\mathbf{J}) = 0$ *for* $\mathbf{J} \in \mathcal{J}$.

Combining the previous proposition with (17) we easily obtain that the homomorphism Y_∞ is onto. We will now see that this mapping is in fact an isomorphism. For that purpose, notice that we are in the conditions to apply the flip elimination scheme described in [20, Section 1.1.5] due to the relations in Proposition 4.30. We thus get by [20, Corollary 1.1.20] an isomorphism L, as follows.

Lemma 4.33. *There is an isomorphism* $\mathsf{L} : \mathcal{A}^{1,\mathcal{J}}_{\infty,\infty} \to \left(p\mathcal{A}^{1,\mathcal{J}}_{\infty,\infty}p\right)^{2\times 2}$ *that maps:*

(i) $\Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ I) \mapsto \begin{bmatrix} \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ I) & 0 \\ 0 & 0 \end{bmatrix}$;

(ii) $\Phi^{\mathcal{J}}_{\infty,\infty}(J) \mapsto \begin{bmatrix} 0 & \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ I) \\ \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ I) & 0 \end{bmatrix}$;

(iii) $\Phi^{\mathcal{J}}_{\infty,\infty}(W^0(\chi_+)) \mapsto \begin{bmatrix} \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ W^0(\chi_+)\chi_+) & 0 \\ 0 & \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ W^0(\chi_-)\chi_+) \end{bmatrix}$;

(iv) $\Phi^{\mathcal{J}}_{\infty,\infty}((P_\tau)) \mapsto \begin{bmatrix} \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ P_\tau) & 0 \\ 0 & \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ P_\tau) \end{bmatrix}$.

For $[A_{ij}]_{i,j=1}^2 \in \left(\operatorname{alg}\{I, \chi_+, \chi_+^F, J\}\right)^{2\times 2}$, define

$$\mathsf{X}_\infty\left([A_{ij}]_{i,j=1}^2\right) := \left[\Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ V_\tau A_{ij} V_{-\tau}\chi_+)\right]_{i,j=1}^2. \tag{18}$$

It easy to see that $(V_\tau A V_{-\tau}) \in \mathcal{A}^1$ if $A \in \{I, \chi_+, \chi_+^F\}$ (for $A = \chi_+ I$ note that $V_{-\tau}\chi_+ V_\tau = \chi_{[\tau,\infty} = \chi_+(I - P_\tau)$). Thus, $A \mapsto (V_\tau A V_{-\tau})$ is a homomorphism from $\operatorname{alg}\{I, \chi_+, \chi_+^F\}$ to $\mathcal{A}^1$.

Taking into account that $\Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ I)$ commutes with the other elements of $\mathcal{A}^{1,\mathcal{J}}_{\infty,\infty}$ except $\Phi^{\mathcal{J}}_{\infty,\infty}(J)$, it is easy to conclude that

$$\mathsf{X}_\infty : \left(\operatorname{alg}\{I, \chi_+, \chi_+^F, J\}\right)^{2\times 2} \to \left(p\mathcal{A}^{1,\mathcal{J}}_{\infty,\infty}p\right)^{2\times 2} \tag{19}$$

is a continuous homomorphism. The action of this homomorphism on the generators of the algebra can be computed straightforwardly.

Lemma 4.34. *The homomorphism X_∞ maps the generators of $\left(\operatorname{alg}\{I, \chi_+, \chi_+^F, J\}\right)^{2\times 2}$ as follows:*

(i) $\begin{bmatrix} \chi_- & 0 \\ 0 & \chi_- \end{bmatrix} \mapsto \begin{bmatrix} \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ P_\tau) & 0 \\ 0 & \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ P_\tau) \end{bmatrix}$;

(ii) $\begin{bmatrix} a(+\infty)I & 0 \\ 0 & a(-\infty)I \end{bmatrix} \mapsto \begin{bmatrix} a(+\infty)\Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ I) & 0 \\ 0 & a(-\infty)\Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ I) \end{bmatrix}$;

(iii) $\begin{bmatrix} b^F_\infty & 0 \\ 0 & \tilde{b}^F_\infty \end{bmatrix} \mapsto \begin{bmatrix} \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ W^0(b_\infty)\chi_+) & 0 \\ 0 & \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ W^0(\tilde{b}_\infty)\chi_+) \end{bmatrix}$;

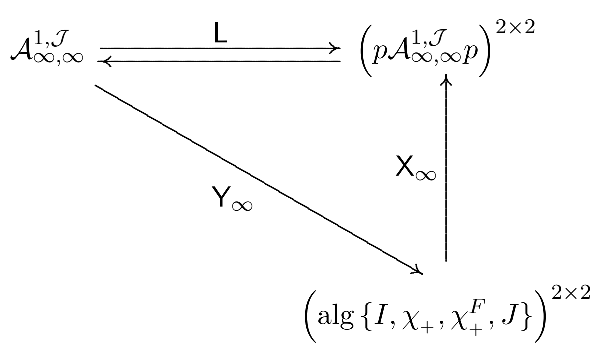
(iv) $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mapsto \begin{bmatrix} \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ P_\tau) & 0 \\ 0 & \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ P_\tau) \end{bmatrix}$.

Proof. For the first assertion note that $V_\tau \chi_- V_{-\tau} = \chi_{[-\infty,\tau]} I$ whence $\chi_+(V_\tau \chi_- V_{-\tau}) = \chi_+ P_\tau$. The other assertions are even more trivial. □

Combining Proposition 4.32 with Lemmas 4.33 and 4.34 we get that the mappings

$$\mathsf{X}_\infty \circ \mathsf{Y}_\infty : \mathcal{A}^{1,\mathcal{J}}_{\infty,\infty} \to \left(p\mathcal{A}^{1,\mathcal{J}}_{\infty,\infty}p\right)^{2\times 2} \quad \text{and} \quad \mathsf{L} : \mathcal{A}^{1,\mathcal{J}}_{\infty,\infty} \to \left(p\mathcal{A}^{1,\mathcal{J}}_{\infty,\infty}p\right)^{2\times 2}$$

coincide. The isomorphism between the algebras then is true because L is an isomorphism itself. The following diagram illustrates the relations between the algebras:



Summarizing we get

Theorem 4.35. *Let $\mathcal{A}^1 := \mathcal{A}\{PC(\dot{\mathbb{R}}), PC_p, J, (P_\tau)\}$. Then the local algebra $\mathcal{A}^{1,\mathcal{J}}_{\infty,\infty}$ is isomorphic to $\left(\operatorname{alg}\{I, \chi_+, \chi_+^F, J\}\right)^{2\times 2}$, with the isomorphism given by Y_∞. Further,*

(i) *if $\mathsf{W}_1(\mathbf{A})$ is invertible for a sequence $\mathbf{A} \in \mathcal{A}^1$, then $\Phi^{\mathcal{J}}_{\infty,\infty}(\mathbf{A})$ is also invertible;*

(ii) *there is a* 4×4*-matrix-valued symbol for* $\mathcal{A}^{1,\mathcal{J}}_{\infty,\infty}$.

Proof. The assertions are clear from the preceding discussion. For (ii), note that the algebra $\operatorname{alg}\{I, \chi_+, \chi_+^F\}$ is generated by two projections, having thus a 2×2-matrix-valued symbol by [20, Section 3.1]. □

A similar (F-symmetric) description holds for the local algebra $\mathcal{A}^{2,\mathcal{J}}_{\infty,\infty}$ of the algebra $\mathcal{A}^2$ at (∞, ∞). We omit the details. For the algebras $\mathcal{A}^1$ and $\mathcal{A}^2$, we then have the following result.

Theorem 4.36. *A sequence* $\mathbf{A}$ *belonging to one of the algebras* $\mathcal{A}^1$*, or* $\mathcal{A}^2$ *is stable if and only if the following operators are invertible in* $L^p(\mathbb{R})$ *or* $[L^p(\mathbb{R})]^{2\times 2}$*, as appropriate:*

(i) $\mathsf{W}_0(\mathbf{A})$, $\mathsf{W}_1(\mathbf{A})$ *and* $\mathsf{W}_1^F(\mathbf{A})$;

(ii) $\mathsf{Y}_s(\mathbf{A})$, $\mathsf{Y}_s^F(\mathbf{A})$ *for* $s \geq 0$;

Proof. If $\mathbf{A}$ belongs to one of the algebras $\mathcal{A}^{1,2,3}$, then $\mathbf{A}$ is in $\mathcal{L}$ by Proposition 4.15. Stability is equivalent to invertibility of the coset $\mathbf{A} + \mathcal{J}$ in $\mathcal{L}/\mathcal{J}$ and invertibility of the operators in (i), by Theorem 4.5 and Lemma 4.14. Applying Allan's local principle and Theorems 4.16, 4.17, 4.24 and 4.28, we see that invertibility of the coset $\mathbf{A} + \mathcal{J}$ in $\mathcal{L}/\mathcal{J}$ is equivalent the invertibility of the operators in (ii) and invertibility in the local algebra indexed by (∞, ∞). The invertibility of $\mathsf{W}_1(\mathbf{A})$ (resp. $\mathsf{W}_1^F(\mathbf{A})$) already implies invertibility in the local algebra (see Theorem 4.35). □

Finally, we turn our attention to the algebra $\mathcal{A}^{3,\mathcal{J}}_{\infty,\infty}$. This algebra is generated by the idempotents p_1 and p_2, the flip j and by the identity element. By Proposition 4.30, j is in the center of the algebra. So it is possible to define central projections $j_\pm := \Phi^{\mathcal{J}}_{\infty,\infty}(J_\pm)$, where $J_\pm := (I \pm J)/2$ is the projection onto the subspace of even (odd) functions, respectively, satisfying $J_+ + J_- = I$. The projections $j_\pm$ allow the decomposition of the algebra $\mathcal{A}^{3,\mathcal{J}}_{\infty,\infty}$ into the subalgebras

$$\mathcal{A}^{3,\mathcal{J}}_{\infty,\infty,\pm} := j_\pm \mathcal{A}^{3,\mathcal{J}}_{\infty,\infty} j_\pm$$

with identity elements j_+ and j_-, respectively, in the sense that an element a is invertible in $\mathcal{A}^{3,\mathcal{J}}_{\infty,\infty}$ if and only if aj_+ and aj_- are invertible in the respective subalgebras. Each of the algebras $\mathcal{A}^{3,\mathcal{J}}_{\infty,\infty,\pm}$ is generated by two projections and the identity, and it is thus subject to the two projections theorem (Theorem 3.1.4 in [20]).

To employ the theorem, we need the spectrum of $j_\pm p_1 p_2 p_1 + j_\pm(e - p_1)(e - p_2)(e - p_1)$ in $\mathcal{A}^{3,\mathcal{J}}_{\infty,\infty,\pm}$.

Proposition 4.37. *The spectrum of* $j_\pm p_1 p_2 p_1 + j_\pm(e - p_1)(e - p_2)(e - p_1)$ *in* $\mathcal{A}^{3,\mathcal{J}}_{\infty,\infty,\pm}$ *is the lentiform set* $\mathfrak{L}_p := \cup_{t \in I} \mathfrak{A}_t$ *where* I *is the interval between* $1/p$ *and* $1/q$ *and* $\mathfrak{A}_t$ *is the circular arc*

$$\{(1 + \coth((y + \mathbf{i}t)\pi))/2 : -\infty < y < \infty\} \cup \{-1, 1\}.$$

In order to prove the above proposition, we need some information on the stability of the related sequence

$$\big(J_\pm(P_\tau P_\tau^F P_\tau - \lambda I)\big)_{\tau > 0}, \qquad \lambda \in \mathbb{C}. \tag{20}$$

The sequence (20) is stable if and only if the sequence

$$\big(Z_\tau(J_\pm(P_\tau P_\tau^F P_\tau - \lambda I)Z_\tau^{-1})\big)_{\tau>0} = \big(J_\pm(P_{\tau^2} P_1^F P_{\tau^2} - \lambda I)\big)_{\tau>0}$$

is stable, which holds if and only if the sequence

$$\big(J_\pm(P_\tau P_1^F P_\tau - \lambda I)\big)_{\tau>0}$$

is stable. Since $J_\pm P_1^F = J_\pm W^0(\chi_{[-1,1]}) \in \mathcal{A}^1$, the stability of this sequence can be derived from Theorem 4.36. The result is described in the following lemma, where the notion *stability spectrum of a sequence* $\mathbf{A}$ is used to refer to the spectrum of the coset $\mathbf{A}+\mathcal{G}$ in $\mathcal{E}/\mathcal{G}$.

Lemma 4.38. *Let* $\mathbf{A}_\pm := (P_\tau J_\pm W^0(\chi_{[-1,1]})P_\tau + (I-P_\tau))_{\tau>0}$ *and* $\mathbf{A}_0 := (P_\tau J W^0(\chi_{[-1,1]})P_\tau + (I-P_\tau)))_{\tau>0}$. *Then the stability spectrum of*

(i) $\mathbf{A}_\pm$ *is equal to the lens* $\mathfrak{L}_p$*;*

(ii) $\mathbf{A}_0$ *is equal to the double lens* $\mathfrak{L}_p \cup -\mathfrak{L}_p$.

Proof. Set $J_{\pm 1} := J_\pm$ and $J_0 := J$. By Theorem 4.36, the stability spectrum is the union of the spectra of the operators mentioned in conditions (i) and (ii) of that theorem. It is easy to check that, among theose operators, the only ones for which the spectrum is not a subset of $\{-1,0,1\}$ are

$$J_k(\chi_+ W^0(\chi_{[-1,1]})\chi_+ + \chi_- W^0(\chi_{[-1,1]})\chi_-) \quad \text{and} \quad J_k(P\chi_{[-1,1]}P + Q\chi_{[-1,1]}Q). \tag{21}$$

The operator $J_k(\chi_+ W^0(\chi_{[-1,1]})\chi_+ + \chi_- W^0(\chi_{[-1,1]})\chi_-)$ can be considered an element of the algebra $\operatorname{alg}\{e,p,r,j\}$ with $e := I$, $p := W^0(\chi_{[-1,1]})$, $r = \chi_+$ and $j := J$. Then these elements satisfy

$$jpj = p, \qquad jrj = e - r.$$

Algebras with these properties were described by Krupnik and Spigel [15] (see also [20, Section 3.4]). We determine the 4×4-representations Φ_x for $x \in \sigma(prp)\setminus\{0\}$ according to [20, Theorem 3.4.7]. It is known that $\sigma(prp) = \sigma(rpr) = \sigma(W(\chi_{[-1,1]}))$ is the lens $\mathfrak{L}_p$ and so the two-dimensional representations do not play a role by Corollary 3.4.8. The symbol of the operator

$$\chi_+ W^0(\chi_{[-1,1]})\chi_+ + \chi_- W^0(\chi_{[-1,1]})\chi_- = rpr + (e-r)p(e-r),$$

which does not depend on k, is equal to[1]

$$\begin{bmatrix} A_x & 0 \\ 0 & A_x \end{bmatrix} \qquad \text{with} \quad A_x := \begin{bmatrix} x^2 + (1-x)^2 & (2x-1)\sqrt{x(1-x)} \\ (2x-1)\sqrt{x(1-x)} & 2x(1-x) \end{bmatrix}. \tag{22}$$

Then the symbol of $j_\pm(rpr + (e-r)p(e-r))$ is

$$\frac{1}{2}\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \pm \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\right)\begin{bmatrix} A_x & 0 \\ 0 & A_x \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A_x & \pm A_x \\ \pm A_x & A_x \end{bmatrix}. \tag{23}$$

Because all blocks of the 2×2 matrix

$$\frac{1}{2}\begin{bmatrix} A_x & \pm A_x \\ \pm A_x & A_x \end{bmatrix} - \lambda \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

[1] Here $\sqrt{x(1-x)}$ refers to any complex number the square of which is $x(1-x)$.

commute, this matrix is invertible if and only if its determinant is invertible (see [20, Lemma 1.2.34]), that is, if and only if

$$(\frac{1}{2}A_x - \lambda I)(\frac{1}{2}A_x - \lambda I) - \frac{1}{2}A_x\frac{1}{2}A_x = \lambda(\lambda I - A)$$

is invertible, which happens if and only if $\lambda \notin \sigma(A_x) \cup \{0\}$. A simple calculation shows that the eigenvalues of the matrix A_x are x and $1 - x$. Thus $\sigma(A_x) = \mathfrak{L}_p$ and, consequently, $\sigma(j_\pm(rpr + (e - r)p(e - r))) = \mathfrak{L}_p$.

Regarding the second operator $j(rpr + (e - r)p(e - r))$ in (21), we get

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} A_x & 0 \\ 0 & A_x \end{bmatrix} = \begin{bmatrix} 0 & A_x \\ A_x & 0 \end{bmatrix}.$$

for its symbol. Its spectrum is the set of all $\lambda \in \mathbb{C}$ such that

$$\det \begin{bmatrix} -\lambda I & A_x \\ A_x & -\lambda I \end{bmatrix} = \lambda^2 I - A^2 = (\lambda I - A)(\lambda I + A)$$

is not invertible, that is,

$$\sigma(j(rpr + (e - r)p(e - r))) = \sigma(A) \cup \sigma(-A) = \mathfrak{L}_p \cup -\mathfrak{L}_p,$$

and the lemma is proved. □

Now we proceed with the proof of Proposition 4.37. We know from the previous lemma that the spectrum of the coset $(J_\pm P_\tau P_\tau^F P_\tau - \lambda I) + \mathcal{G}$ in $\mathcal{E}/\mathcal{G}$ is the lens $\mathfrak{L}_p$. Since $\mathfrak{L}_p$ has a connected complement, the spectrum of $(J_\pm P_\tau P_\tau^F P_\tau) + \mathcal{G}$ in $\mathcal{L}/\mathcal{G}$ is also $\mathfrak{L}_p$. Thus, by Allan's local principle,

$$\mathfrak{L}_p = \bigcup_{(s,t)\in(\bar{\mathbb{R}}^+\times\{\infty\})\cup(\{\infty\}\times\bar{\mathbb{R}}^+)} \sigma\left(\Phi_{s,t}^{\mathcal{J}}((J_\pm P_\tau P_\tau^F P_\tau))\right). \tag{24}$$

Now we use Theorems 4.16, 4.17, 4.24 and 4.28 to determine the spectrum of the local coset $\Phi_{s,t}^{\mathcal{J}}((J_\pm P_\tau P_\tau^F P_\tau))$ at points $(s, t) \neq (\infty, \infty)$. Taking into account Propositions 4.10 – 4.13, we see that the homomorphisms that describe the local algebras map one of the sequences (P_τ), (P_τ^F) to the identity operator (and the other one to a projection), and they map $J_\pm$ to a commuting projection. Consequently, $\Phi_{s,t}^{\mathcal{J}}((J_\pm P_\tau P_\tau^F P_\tau))$ is an idempotent for these (s, t). Since the spectrum of an idempotent is in $\{0, 1\}$ we conclude from (24) that

$$\mathfrak{L}_p \setminus \{0, 1\} \subseteq \sigma\left(\Phi_{\infty,\infty}^{\mathcal{J}}((J_\pm P_\tau P_\tau^F P_\tau))\right) \subseteq \mathfrak{L}_p.$$

Since spectra are closed, this implies that

$$\mathfrak{L}_p = \sigma\left(\Phi_{\infty,\infty}^{\mathcal{J}}((J_\pm P_\tau P_\tau^F P_\tau))\right) = \sigma(j_\pm p_1 p_2 p_1).$$

From Corollary 3.1.5 in [20] we finally obtain the assertion. □

Using the two projections theorem (Theorem 3.1.4 in [20]) we arrive at the following.

Theorem 4.39. *The mapping $\mathsf{Y}_{\infty,\infty,\pm}$ which associates with $j_\pm$, $j_\pm p_1$ and $j_\pm p_2$ the functions*

$$x \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad x \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad x \mapsto \begin{bmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{bmatrix}$$

on the lens $\mathfrak{L}_p$, respectively, extends to a continuous homomorphism from $\mathcal{A}_{\infty,\infty,\pm}^{3,\mathcal{J}}$ to the algebra of all bounded functions from $\mathfrak{L}_p$ to $\mathbb{C}^{2\times 2}$, and the following assertions are equivalent for a sequence $\mathbf{A} \in \mathcal{A}^3$:

(i) *the cosets* $\Phi^{\mathcal{J}}_{\infty,\infty,\pm}(\mathbf{A})$ *are invertible in* $\mathcal{A}^{3,\mathcal{J}}_{\infty,\infty,\pm}$*;*

(ii) *the coset* $\Phi^{\mathcal{J}}_{\infty,\infty}(\mathbf{A})$ *is invertible in* $\mathcal{L}_{\infty,\infty}$*;*

(iii) *the functions* $\mathsf{Y}_{\infty,\infty,\pm}(\Phi^{\mathcal{J}}_{\infty,\infty,\pm}(\mathbf{A}))$ *are invertible at every point of* $\mathfrak{L}_p$*.*

We condense the main result in the following theorem. Its proof is the same as that of Theorem 4.36, except that the invertibility in the local algebra at (∞, ∞) is now described in Theorem 4.39.

Theorem 4.40. *A sequence* $\mathbf{A}$ *belonging to the algebra* $\mathcal{A}^3$ *is stable if and only if the following operators and matrices are invertible:*

(i) $\mathsf{W}_0(\mathbf{A})$, $\mathsf{W}_1(\mathbf{A})$ *and* $\mathsf{W}_1^F(\mathbf{A})$*;*

(ii) $\mathsf{Y}_s(\mathbf{A})$, $\mathsf{Y}_s^F(\mathbf{A})$ *for* $s \geq 0$*;*

(iii) $\mathsf{Y}_{\infty,\infty,\pm}(\Phi^{\mathcal{J}}_{\infty,\infty,\pm}(\mathbf{A}))(x)$ *for* $x \in \mathfrak{L}_p$*.*

5 Results and Discussion

First we consider the Wiener-Hopf plus Hankel operator on $L^p(\mathbb{R}^+)$,

$$A := W(a) + H(b). \tag{25}$$

Remember we use the notation $a_s := a(s^-)\chi_- + a(s^+)\chi_+$ and $a_\infty := a(-\infty)\chi_- + a(+\infty)\chi_+$ for every piecewise continuous function a and every $s \in \mathbb{R}$.

Theorem 5.1. *Let* $a, b \in PC_p$*. The finite section method with respect to the projections* $\chi_{[0,\tau]}$*,* $\tau > 0$*, applies to* A *if and only if*

(i) A *is invertible on* $L^p(\mathbb{R}^+)$*,*

(ii) $W(\tilde{a})$ *is invertible on* $L^p(\mathbb{R}^+)$*,*

(iii) $\chi_{[0,1]}(W(a_0) + H(b_0))|_{L^p([0,1])}$ *is invertible on* $L^p([0,1])$*, and*

(iv) $\begin{bmatrix} \chi_{[0,1]} & 0 \\ 0 & \chi_{[0,1]} \end{bmatrix} \begin{bmatrix} W(a_s) & H(b_s) \\ H(b_{-s}) & W(a_{-s}) \end{bmatrix} |_{L^p_2([0,1])}$ *is invertible on* $L^p_2([0,1])$*.*

Proof. The operator A can be extended to functions living on the whole real line by

$$A_{ext} := \chi_+ W^0(a)\chi_+ I + \chi_+ W^0(b) J \chi_+ I + \chi_- I. \tag{26}$$

Clearly, the finite sections method for A with respect to the $\chi_{[0,\tau]}$ applies if and only if the finite sections method for the operator A_{ext} with respect to the projections P_τ converges, i.e., if and only if the sequence $\mathbf{A} := (P_\tau A_{ext} P_\tau + Q_\tau)_{\tau>0}$ is stable. The sequence $\mathbf{A}$ belongs to the algebra $\mathcal{A}^1$. Applying Theorem 4.36, we obtain that stability of $\mathbf{A}$ is equivalent to invertibility of the following operators:

- W The operator $A_{ext} = \mathsf{W}(\mathbf{A})$ is invertible if and only if A is invertible, which gives condition (i).

W_1 By Proposition 4.8,

$$\mathsf{W}_1(\mathbf{A}) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \mathsf{W}_1(A_{ext}) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

The left upper entry of $\mathsf{W}_1(A_{ext})$ is

$$\chi_+ W^0(\tilde{a})\chi_+ + \chi_+ W^0(\tilde{b})\chi_+ J\chi_+ + \chi_- = \chi_+ W^0(\tilde{a})\chi_+ + \chi_-,$$

which gives (ii).

W_1^F A simple calculation using Proposition 4.9 shows that $U_F \mathsf{W}_1^F(\mathbf{A}) U_F^{-1}$ is equal to

$$\begin{bmatrix} a(+\infty)\chi_+ I & 0 \\ 0 & a(-\infty)\chi_- I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \chi_- I & 0 \\ 0 & \chi_+ I \end{bmatrix},$$

which is invertible whenever $a(\pm\infty) \neq 0$. The latter property follows if A is invertible.

Y_0 We obtain directly from Proposition 4.10 that

$$\mathsf{Y}_0(\mathbf{A}) = \chi_+ W^0(a_\infty)\chi_+ + \chi_+ W^0(b_\infty) J \chi_+ + \chi_- I.$$

This operator is invertible if A is invertible.

Y_0^F We obtain directly from Proposition 4.11 that

$$\mathsf{Y}_0(\mathbf{A}) = P_1(\chi_+ W^0(a_0)\chi_+ + \chi_+ W^0(b_0) J\chi_+ + \chi_- I) P_1 + Q_1.$$

This operator is invertible if and only if the operator in condition (iii) is invertible.

Y_s A simple calculation using Proposition 4.26 shows that

$$U \mathsf{Y}_s(\mathbf{A}) U^{-1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W^0(a_\infty) & 0 \\ 0 & W^0(\widetilde{a_\infty}) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

for $s > 0$. This operator is invertible if and only if $W^0(a_\infty)$ is invertible, i.e., if $a(\pm\infty) \neq 0$. The latter condition is implied by (i).

Y_s^F From Proposition 4.19 one obtains that $U_F \mathsf{Y}_s^F(\mathbf{A}) U_F^{-1}$ is equal to

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_1 \end{bmatrix} \begin{bmatrix} \chi_+ W^0(a_s)\chi_+ + \chi_- & \chi_+ W^0(b_s)\chi_- \\ \chi_- W^0(\widetilde{b_{-s}})\chi_+ & \chi_- W^0(\widetilde{a_{-s}})\chi_- + \chi_+ \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_1 \end{bmatrix} + \begin{bmatrix} Q_1 & 0 \\ 0 & Q_1 \end{bmatrix}$$

where $Q_1 = I - P_1$. Multiplying this operator by the diagonal matrix $\operatorname{diag}(I, J)$ from both sides we see that the resulting operator is invertible if and only if the operator in condition (iv) is invertible.

Y_∞ Finally, according to Theorem 4.35, the invertibility of $\mathsf{Y}_\infty(\mathbf{A})$ is implied by that of $\mathsf{W}_1(\mathbf{A})$. □

Next we consider the Toeplitz plus Hankel operator on $H^p(\mathbb{R}) = PL^p(\mathbb{R})$,

$$B = T(a) + H(b). \tag{27}$$

This operator can be extended to an operator acting on all of $L^p(\mathbb{R})$ by

$$B_{ext} := PaP + PbJP + Q,$$

which can be identified with the operator $A^F_{ext} = F^{-1}A_{ext}F$, with A_{ext} as in (26), but with a and b substituted by $\tilde{a}$ and $\tilde{b}$, respectively.

Theorem 5.2. *Let a, $b \in PC$. The finite section method with respect to the projections P_τ, $\tau > 0$, applies to B_{ext} if and only if*

(i) *B is invertible on $H^p(\mathbb{R})$,*

(ii) *the functions $y \mapsto \frac{1+a(\pm\infty)}{2} \pm \frac{1-a(\pm\infty)}{2} \coth(\pi(y + i/p))$ do not vanish on the extended real line $\bar{\mathbb{R}}$,*

(iii) *the operator $P_1(T(a_\infty) + H(b_\infty))P_1$ is invertible on $L^p([-1, 1])$.*

Proof. The finite sections method applies to B_{ext} if and only if the sequence

$$\mathbf{B} := (P_\tau B_{ext} P_\tau + Q_\tau)_{\tau>0} = (P_\tau(PaP + PbJP + Q)P_\tau + Q_\tau)_{\tau>0}$$

is stable. The sequence $\mathbf{B}$ belongs to the algebra $\mathcal{A}^1$. Applying Theorem 4.36, we obtain that stability of $\mathbf{B}$ is equivalent to invertibility of the following operators:

W The operator $B_{ext} = \mathsf{W}(\mathbf{B})$ is invertible if and only if B is invertible, which gives condition (i).

W_1 A simple calculation using Proposition 4.8 gives that $U\mathsf{W}_1(\mathbf{B})U^{-1}$ is equal to

$$\begin{bmatrix} \chi_-(a(+\infty)P + Q)\chi_- + \chi_+ & 0 \\ 0 & \chi_-(a(-\infty)Q + P)\chi_- + \chi_+ \end{bmatrix},$$

which is invertible if and only if $\chi_+(a(+\infty)Q+P)\chi_+ + \chi_-$ and $\chi_+(a(-\infty)P+Q)\chi_+ + \chi_-$ are invertible on $L^p(\mathbb{R})$ or, equivalently, if $a(+\infty)Q_{\mathbb{R}^+} + P_{\mathbb{R}^+}$ and $a(-\infty)P_{\mathbb{R}^+} + Q_{\mathbb{R}^+}$ are invertible on $L^p(\mathbb{R}^+)$. The latter condition can be effectively checked by means of the Mellin calculus, which gives condition (ii) (see Section 4.2.2 in [20]).

W_1^F Since $\mathsf{W}_1^F((P_\tau)) = \text{diag}\,(I, I)$, the operator $\mathsf{W}_1^F(\mathbf{B})$ is equal to $\mathsf{W}_1^F(B_{ext})$, which is invertible if B is invertible.

Y_0 Since $\mathsf{Y}_0((P_\tau)) = I$, $\mathsf{Y}_0(\mathbf{B})$ is equal to $\mathsf{Y}_0(B_{ext})$, which is invertible if B is invertible.

Y_0^F By Proposition 4.11, $\mathsf{Y}_0^F(\mathbf{B})$ is equal to $P_1(Pa_\infty P + Pb_\infty JP + Q)P_1 + Q_1$. This operator can be formally written as $P_1(T(a_\infty) + H(b_\infty))P_1 + Q_1$, which gives condition (iii).

Y_s Since $\mathsf{Y}_s((P_\tau)) = \text{diag}\,(I, I)$ for $s > 0$, the operator $\mathsf{Y}_s(\mathbf{B})$ is invertible whenever B is.

Y_s^F A simple calculation using Proposition 4.12 shows that $U_F \mathsf{Y}_s^F(\mathbf{B}) U_F^{-1}$ is equal to

$$\begin{bmatrix} P_1 a_\infty P_1 + Q_1 & 0 \\ 0 & P_1 + Q_1 \end{bmatrix}$$

for $s > 0$. This operator is invertible if and only if $P_1 a_\infty P_1$ is invertible, i.e., if and only if $a(\pm\infty) \neq 0$. The latter condition holds if B is invertible.

Y_∞ Finally, by Theorem 4.35, the invertibility of $\mathsf{Y}_\infty(\mathbf{B})$ is implied by that of $\mathsf{W}_1(\mathbf{B})$. □

Using the F-symmetry, it is easy now to establish conditions for the convergence of the Fourier Finite Section Method with respect to the projections P_τ^F for the operators A and B in (25) and (27) (respective for their extensions).

Theorem 5.3. *Let a, $b \in PC_p$. The FFSM with respect to the projections P_τ^F, $\tau > 0$, applies to A if and only if*

(i) *A is invertible on $L^p(\mathbb{R}^+)$,*

(ii) *the functions $y \mapsto \frac{1+a(\pm\infty)}{2} \pm \frac{1-a(\pm\infty)}{2} \coth(\pi(y + i/p))$ do not vanish on the extended real line $\bar{\mathbb{R}}$,*

(iii) *The operator $P_1^F(W(a_\infty) + H(b_\infty) + \chi_-)P_1^F + Q_1^F$ is invertible.*

Theorem 5.4. *Let a, $b \in PC_p$. The FFSM with respect to the projections P_τ^F, $\tau > 0$, applies to B if and only if*

(i) *B is invertible on $H^p(\mathbb{R}^+)$,*

(ii) *$P\tilde{a}P + Q$ is invertible,*

(iii) *$P_1^F(T(a_0) + H(b_0) + Q)P_1^F + Q_1^F$ is invertible, and*

(iv) $\begin{bmatrix} \chi_{[0,1]}^F & 0 \\ 0 & \chi_{[0,1]}^F \end{bmatrix} \begin{bmatrix} T(a_s) & H(b_s) \\ H(b_{-s}) & T(a_{-s}) \end{bmatrix} |_{P\chi_{[0,1]}^F H_2^p(\mathbb{R}))}$ *is invertible on $P\chi_{[0,1]}^F H_2^p(\mathbb{R}))$.*

5.1 Discussion and Conclusions

Since 1997, when the authors, together with B. Silbermann, derived conditions for stability of operator Wiener-Hopf plus Hankel related sequences in the L^2-norm [18], the corresponding results for the more general L^p-norms remained an open question.

The main problem was related to the algebraic techniques used, because the passage to sub-algebras is a trivial matter in C^*-algebras (due to their inverse-closedness property), but not in the Banach algebras that have to be used in the L^p-case. It took much more time than expected to overcome what in the beginning appeared to be a detail, as considerable technical challenges surfaced. To solve those problems, a deeper understanding of the structure of the algebras was needed and new tools had to be developed.

In the end, we were able to obtain conditions for approximation of solutions of both Wiener-Hopf plus Hankel and Toeplitz plus Hankel equations in the L^p-norm. The results highlight the duality between the two types of operator that extends beyond the L^2 case isomorphism.

The approximation methods considered were the finite section method and its Fourier equivalent, but it should be possible now to adapt the techniques presented in this work to spline approximation methods, as previous examples indicate (see for instance [12, 21, 22]).

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Steffen Roch
Fachbereich Mathematik,
Technische Universität Darmstadt,
Schlossgartenstrasse 7, 64289 Darmstadt
Germany
roch@mathematik.tu-darmstadt.de

Pedro A. Santos
Departmento de Matemática,
Instituto Superior Técnico,
1049-001 Lisboa
Portugal
pasantos@math.ist.utl.pt