Very Weak Solutions of the Stationary Stokes Equations in Unbounded Domains of Half Space Type

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Abstract

We consider the theory of very weak solutions of the stationary Stokes system with nonhomogeneous boundary data and divergence in domains of half space type, such as $\mathbb{R}^n_+$, bent half spaces the boundary of which can be written as the graph of a Lipschitz function, perturbed half spaces as local, but possibly large perturbations of $\mathbb{R}^n_+$, and in aperture domains. The proofs are based on duality arguments and corresponding results for strong solutions in these domains which have to be constructed in homogeneous Sobolev spaces. In addition to very weak solutions we also construct corresponding pressure functions in negative homogeneous Sobolev spaces.

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Key Words: Stokes equations; very weak solution; strong solutions; domains of half space type.

1 Introduction

Let us consider the stationary Stokes equations for an incompressible fluid

$$\begin{aligned} -\nu\Delta u + \nabla p &= f = \operatorname{div} F && \text{in } \Omega,\\ \operatorname{div} u &= k && \text{in } \Omega,\\ u &= g && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

with unknown velocity u and pressure p in a domain $\Omega \subset \mathbb{R}^n$, external force density f and viscosity $\nu = 1$. It will prove to be convenient later on to write the external force density in divergence form $f = \operatorname{div}\, F$. Note that we include nonzero divergence data k. The boundary condition $u|_{\partial\Omega} = g$ generalizes the well-known no-slip condition. The main goal of this paper is to set up a notion of a special class of solutions, the very weak solutions, for unbounded domains of half space type.

The concept of very weak solutions was first introduced by H. Amann [1], [2] for the nonstationary case and elaborately investigated by R. Farwig, G. P. Galdi, C. G. Simader, H. Sohr and H. Kozono [5, 6, 7, 8, 15] in the case of bounded and exterior domains, and F. Riechwald [19, 20] for arbitrary unbounded domains. Very weak solutions are solutions to (1) with data of low regularity, which are not differentiable except for the existence of the divergence and do not have finite kinetic energy in general. The main advantage of considering very weak solutions is the fact that this concept furnishes us with unique solvability even of nonlinear Navier-Stokes systems in a bounded or exterior domain Ω under Serrin's condition $\frac{2}{s} + \frac{3}{q} = 1$ for the exponents of the solution $u \in L^s(0,T;L^q(\Omega))$. One problem in the case of unbounded domains is to ensure the existence of a unique strong solution of an auxiliary Stokes problem, since there is a duality correspondence between strong and very weak solutions, as pointed out by K. Schumacher [21, 22]. Therefore, in this paper we extend a known result on strong solutions for the half space and prove an analogous

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result for the bent half space by using perturbation arguments and for the perturbed half space via a localization method. Moreover, we consider very weak solutions for aperture domains where the flux of the fluid through the aperture must be prescribed to ensure uniqueness.

We will use some common notation and terminology. The definitions of the different types of domains are as follows:

- $\mathbb{R}^n$ is the *whole space*, and $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ is the *(upper) half space*
- a *bent half space* is a domain of the form $H_\omega = \{x = (x', x_n) \in \mathbb{R}^n : x_n > \omega(x')\}$, where $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz continuous function in $W^{2,1}_{\text{loc}}(\mathbb{R}^{n-1})$, such that the gradient $\nabla'\omega = (\partial_1, \ldots, \partial_{n-1})\omega$ is bounded in $\mathbb{R}^{n-1}$.
- a *perturbed half space* is a domain of class $C^{1,1}$ such that $\Omega \backslash B = \mathbb{R}^n_+ \backslash B$ for some open ball B.
- an *aperture domain* is a domain of class $C^{1,1}$ such that $\Omega \cup B = \mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B$ for some open ball $B = B_R(0) \subset \mathbb{R}^n$ of radius R and center 0, where

  $$\mathbb{R}^n_- := \{x \in \mathbb{R}^n : x_n < -d\}$$

  for some $d > 0$. Since Ω is connected, we may choose a smooth $(n-1)$-dimensional manifold $S \in \Omega \cap B$ such that $\Omega \backslash S$ consists of two disjoint perturbed half spaces Ω_+ and Ω_- with $S = \partial\Omega_+ \cap \partial\Omega_-$ and $\Omega = \Omega_+ \cup S \cup \Omega_-$.

Let $\Omega \subset \mathbb{R}^n$ be one of the unbounded domains considered above. We define the space of test functions

$$C^2_{0,\sigma}(\overline{\Omega}) = \{w \in C^2(\overline{\Omega}) : \text{div}\, w = 0,\ \text{supp}\, w \text{ compact in } \overline{\Omega}, w|_{\partial\Omega} = 0\},$$

and formally test the Stokes system (1) with $w \in C^2_{0,\sigma}(\overline{\Omega})$ to get the identities

$$\begin{aligned} -(u, \Delta w) &= -\langle g, N \cdot \nabla w\rangle_{\partial\Omega} - (F, \nabla w) \text{ for all } w \in C^2_{0,\sigma}(\overline{\Omega}), \\ \text{div}\, u = k &\text{ in } \Omega, \quad u \cdot N = g \cdot N \text{ on } \partial\Omega. \end{aligned} \tag{2}$$

Here, N denotes the exterior normal vector on $\partial\Omega$.

This motivates the following definition, giving a precise meaning to all the terms in (2). Note that due to the unboundedness of the domains considered, we have to work with homogeneous Sobolev spaces and respective dual and trace spaces. In particular, the data on the boundary lie in the space $\dot{W}^{-\frac{1}{q},q}(\partial\Omega)$ with corresponding norm $\|\cdot\|_{-1/q,q,\partial\Omega}$. Moreover, in the main results we construct the pressure in the space $\hat{W}^{-1,q}(\Omega)$ with corresponding norm $\|\cdot\|_{-1,q}$. For an exact definition of the functions spaces we refer to Chapter 2.

Definition 1.1. *Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a half space, a bent half space, a perturbed half space or an aperture domain. Let furthermore $1 < r < q < \infty$, $r < n$, with $\frac{1}{n} + \frac{1}{q} = \frac{1}{r}$. Then for given data*

$$F \in L^r(\Omega), \quad k \in L^r(\Omega), \quad g \in \dot{W}^{-\frac{1}{q},q}(\partial\Omega), \tag{3}$$

we call a vector field $u \in L^q(\Omega)$ a very weak solution *to* (1) *if it satisfies the identities* (2).

Note that all terms are well defined in their respective sense. In particular, $N \cdot \nabla w \in \dot{W}^{\frac{1}{q},q'}(\partial\Omega)$ for every $q \in (1, \infty)$. The two last identities of (2) are obtained by testing the equation div $u = k$ with some scalar-valued $\psi \in C^1_0(\overline{\Omega})$, yielding the variational problem

$$-(u, \nabla\psi) = (k, \psi) - \langle g, \psi N\rangle_{\partial\Omega}. \tag{4}$$

Since $N \cdot \nabla w$ has a vanishing normal component on $\partial\Omega$ for functions w from the (solenoidal) test space $C^2_{0,\sigma}(\overline{\Omega})$, we cannot recover the information of the normal component of g via the term $\langle g, N \cdot \nabla w\rangle_{\partial\Omega}$.

With these definitions in mind, the main goal of this paper is to find sufficient conditions to prove existence and uniqueness of very weak solutions to (1) with data specified in Definition 1.1. The two main results read as follows.

Theorem 1.2. *Assume that one of the following conditions holds.*

(i) Half space: $n \geq 2$, $\frac{n}{n-1} < q < \infty$, *and* $\Omega = \mathbb{R}^n_+$.

(ii) Bent half space: $n \geq 3$, $\frac{n-1}{n-2} < q < \infty$, *and* $\Omega = H_\omega$ *such that* ω *satisfies the conditions*

$$\|\nabla'\omega\|_\infty \leq K \quad \text{and} \quad \|\nabla'^2\omega\|_{L^{n-1}(\mathbb{R}^{n-1})} \leq K \text{ or } \||\cdot|\nabla'^2\omega\|_\infty \leq K, \tag{5}$$

where the constant $K = K(n,q) > 0$ *is determined in Theorem 3.2.1.*

(iii) Perturbed half space: $n \geq 3$, $\frac{n}{n-2} < q < \infty$, *and* $\Omega \subset \mathbb{R}^n$ *is a perturbed half space.*

Let $1 < r < n$ *satisfy* $\frac{1}{n} + \frac{1}{q} = \frac{1}{r}$. *Then for given data* F, k *and* g *as in Definition 1.1, there exists a unique very weak solution* $u \in L^q(\Omega)$ *to* (1). *This solution satisfies the estimate*

$$\|u\|_q \leq c\left(\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega}\right) \tag{6}$$

with $c = c(n,\Omega,q) > 0$. *Moreover, there exists a pressure* $p \in \hat{W}^{-1,q}(\Omega)$ *such that* $-\Delta u + \nabla p = f$ *in the sense of distributions and such that* (u,p) *satisfy the estimate*

$$\|u\|_q + \|p\|_{-1,q} \leq c\left(\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega}\right) \tag{7}$$

with $c = c(n,\Omega,q) > 0$.

The second result deals with aperture domains. In such domains one observes the interesting effect that the usual boundary condition $u|_{\partial\Omega} = 0$ is not sufficient to guarantee uniqueness of the solution, but has to be completed by the additional flux condition $\hat{\phi}(u) = \alpha$, see (55) below for the definition of $\hat{\phi}(u)$.

Theorem 1.3. *Let* $n \geq 3$, $\frac{n}{n-2} < q < \infty$, *and let* r *satisfy* $\frac{1}{n} + \frac{1}{q} = \frac{1}{r}$. *Let furthermore* $\Omega \subset \mathbb{R}^n$ *be an aperture domain. Then for all* $\alpha \in \mathbb{C}$ *and for given data* F, k *and* g *as in Definition 1.1, there exists a unique very weak solution* $u \in L^q(\Omega)$ *to* (1) *with* $\hat{\phi}(u) = \alpha$. *This solution satisfies the estimate*

$$\|u\|_q \leq c\left(\|F\|_r + \|k\|_r + \|g\|_{-1/q,q,\partial\Omega} + |\alpha|\right) \tag{8}$$

with $c = c(n,\Omega,q) > 0$. *Moreover, there exists a distribution* p *such that* $-\Delta u + \nabla p = f$ *in the sense of distributions.*

The proofs of Theorem 1.2 and 1.3 are based on duality arguments. Therefore, corresponding results for strong solutions in homogeneous Sobolev spaces have to be established.

This paper is organized as follows. The function spaces used in this paper are introduced in Chapter 2, alongside some of their properties. In particular we characterize the trace spaces of homogeneous Sobolev spaces in domains of half space type. The main results of this paper and the corresponding results for strong solutions are proven in the Sections 3.1, 3.2, 3.3 and 3.4 dealing with the half space, the bent half space, the perturbed half space and the aperture domain, respectively.

2 Preliminaries

2.1 Function Spaces

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain. As subspaces of $C^k(\Omega)$, $k \in \mathbb{N}_0$, we consider the space of k-times differentiable functions with compact support in Ω denoted by $C_0^k(\Omega)$ as well as $C_0^k(\overline{\Omega}) = \{u|_\Omega : u \in C_0^k(\mathbb{R}^n)\}$ and $C^k(\overline{\Omega}) = \{u|_\Omega : u \in C^k(\mathbb{R}^n)\}$. The dual space of the space of test functions $C_0^\infty(\Omega)$ is the space of distributions $C_0^\infty(\Omega)'$. Duality pairing will be denoted by $\langle \cdot, \cdot \rangle_\Omega$, where the index may be omitted if there is no danger of confusion.

Now let $1 \leq q \leq \infty$ and let $q' = \frac{q}{q-1}$ be its Hölder conjugate. Then $L^q(\Omega)$ and $W^{\alpha,q}(\Omega)$, $\alpha \geq 0$, are the usual Lebesgue and Sobolev(-Slobodeckiĭ) spaces with norms $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_{q,\Omega} = \|\cdot\|_q$ and $\|\cdot\|_{W^{\alpha,q}(\Omega)}$, respectively. For $1 \leq q < \infty$ and $\alpha > 0$, the spaces $W_0^{\alpha,q}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{\alpha,q}(\Omega)}$. The dual space of $W_0^{\alpha,q}(\Omega)$ will be denoted by $W^{-\alpha,q'}(\Omega)$. The pairing $\int_\Omega uv\,dx$ will be referred to by $(u,v)_\Omega$, if $uv \in L^1(\Omega)$.

Furthermore, $u \in L^q_{\mathrm{loc}}(\Omega)$ indicates that $u \in L^q(\Omega')$ for all bounded domains $\Omega' \subset\subset \Omega$, *i.e.*, for all $\Omega' \subset \overline{\Omega'} \subset \Omega$, and $u \in L^q_{\mathrm{loc}}(\overline{\Omega})$ specifies that $u \in L^q_{\mathrm{loc}}(\Omega \cap B)$ for any ball B. Finally, for a bounded domain Ω', we define

$$L_0^q(\Omega') = \{u \in L^q(\Omega') : \int_{\Omega'} u\,dx = 0\}.$$

In the context of the Navier-Stokes equations, the concept of homogeneous Sobolev spaces appears naturally when considering unbounded domains. For $m \geq 0$ and $1 \leq q < \infty$, they are defined as

$$\dot{W}^{m,q}(\Omega) = \{u \in L^q_{\mathrm{loc}}(\bar{\Omega}) : D^\alpha u \in L^q(\Omega), |\alpha| = m\}.$$

Note that by Ehrling's lemma [12], for each $u \in \dot{W}^{m,q}(\Omega)$ it holds that $u \in W^{m,q}_{\mathrm{loc}}(\Omega)$ and for locally Lipschitzian domains Ω even $u \in W^{m,q}_{\mathrm{loc}}(\bar{\Omega})$. We can turn $\dot{W}^{m,q}(\Omega)$ into a separable (and for $1 < q < \infty$ reflexive) Banach space [16], if we identify two functions differing at most by a polynomial of degree $m-1$ and endow the space with the norm

$$\|u\|_{\dot{W}^{m,q}(\Omega)} = \left(\sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^q\,dx \right)^{\frac{1}{q}}. \tag{9}$$

Consider now the space $\hat{W}^{m,q}(\Omega)$ defined as the completion of $C_0^\infty(\overline{\Omega})$ in the norm (9). Note that $\dot{W}^{m,q}(\Omega)$ and $\hat{W}^{m,q}(\Omega)$ do not coincide in general [11], see also Lemma 3.4.1 below for aperture domains. However, they do coincide for a large class of unbounded domains, including the whole space and the half space, as well as perturbed and bent half spaces [10].

The dual space of $\hat{W}^{1,q}(\Omega)$ is denoted by $\hat{W}^{-1,q'}(\Omega) = (\hat{W}^{1,q}(\Omega))^*$ and is endowed with the norm

$$\|\gamma\|_{\hat{W}^{-1,q'}(\Omega)} = \sup_{0 \neq w \in C_0^\infty(\overline{\Omega})} \frac{|\langle \gamma, w \rangle|}{\|\nabla w\|_q}.$$

We have the following lemma.

Lemma 2.1.1. *Let $1 < q < \infty$, $n \geq 2$ and let Ω be the half space, a perturbed half space, an aperture domain or the whole space. Then $\hat{W}^{1,q}(\Omega)$ is the closure of $W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega)$ with respect to the norm $\|\nabla \cdot\|_q$.*

Proof. First observe that $W_c^{1,q}(\Omega) = \{\gamma \in W^{1,q}(\Omega) : \operatorname{supp}\gamma \text{ compact in } \overline{\Omega}\}$ is a dense subset of $\hat{W}^{1,q}(\Omega)$, since $C_0^\infty(\overline{\Omega})$ is a dense subset of $\hat{W}^{1,q}(\Omega)$ and $C_0^\infty(\overline{\Omega}) \subset W_c^{1,q}(\Omega) \subset \hat{W}^{1,q}(\Omega)$. Unfortunately, $W_c^{1,q}(\Omega)$ is not a subset of the dual space $\hat{W}^{-1,q}(\Omega)$, where we identify $\gamma \in W_c^{1,q}(\Omega)$ with the functional

$$\langle \gamma, \cdot \rangle : v \mapsto \int_\Omega \gamma v\,dx, \qquad v \in C_0^\infty(\overline{\Omega}),$$

and extend it to all $v \in \hat{W}^{1,q'}(\Omega)$ if $\langle \gamma, \cdot \rangle$ is continuous with respect to $\|\nabla \cdot\|_{q'}$. Nevertheless, this continuity is guaranteed for all $\gamma \in W_c^{1,q}(\Omega)$ with $\int_\Omega \gamma \, dx = 0$ due to Poincaré's inequality.

Thus, it is left to show that the elements of $W_c^{1,q}(\Omega)$ with vanishing mean form a dense subspace of $W_c^{1,q}(\Omega)$ with respect to the gradient norm. It suffices to construct a sequence $(\tilde{\gamma}_k) \subset W_c^{1,q}(\Omega)$ with $\|\nabla \tilde{\gamma}_k\|_q \to 0$ and $\int_\Omega \tilde{\gamma}_k \, dx = 1$ for all $k \in \mathbb{N}$, since then for every $\gamma \in W_c^{1,q}(\Omega)$ with mean $\int_\Omega \gamma \, dx =: M_\gamma$, the sequence $(\gamma_k)_{k\in\mathbb{N}}$, $\gamma_k := \gamma - M_\gamma \tilde{\gamma}_k$, converges towards γ with respect to $\|\nabla \cdot\|_q$ and we have $\int_\Omega \gamma_k \, dx = 0$ for all $k \in \mathbb{N}$. In the case of the half space, a sequence of functions with the desired properties is given by the cone functions $\beta_k : \Omega \to \mathbb{R}$ defined via $\beta_k(x) = \frac{1}{k^n}\beta\big(\frac{x}{k}\big)$, where $\beta(r) = \frac{n(n+1)}{\kappa_n}(1-r)_+$ and $\kappa_n = \frac{1}{2}\int_{\partial B_1(0)} d\sigma$ is the surface of the half unit sphere $\partial B_1(0) \cap \Omega$. In fact, we get

$$\int_\Omega \beta_k \, dx = \int_\Omega \beta \, dx = n(n+1)\int_0^1 (1-r)r^{n-1} \, dr = n(n+1)\left[\frac{r^n}{n} - \frac{r^{n+1}}{n+1}\right]_0^1 = 1$$

and for the gradient norm $\|\nabla \beta_k\|_q = k^{-1-n+n/q}\|\nabla \beta\|_q \to 0$ as $k \to \infty$, which proves the assertion. Similarly, one shows the assertion for domains of perturbed half space type, aperture domains and the whole space. □

2.2 Traces of Homogeneous Sobolev Spaces

If Ω is locally Lipschitzian and $\partial\Omega \cap B \neq \emptyset$ for an open ball B, then for every $u \in \dot{W}^{1,q}(\Omega) \subset W^{1,q}(\Omega \cap B)$, $1 < q < \infty$, there is a well-defined trace $\Gamma(u) \in W^{1-1/q,q}(\partial(\Omega \cap B))$ (modulo $\mathbb{R}$). However, if $\partial\Omega$ is noncompact, finiteness of the norm of $\Gamma(u)$ on the whole of the boundary cannot be concluded, though. For an unbounded domain of half space type Ω we introduce the notion

$$\|\Gamma(u)\|_{\dot{W}^{1/q',q}(\partial\Omega)} := \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|\Gamma(u)(x) - \Gamma(u)(y)|^q}{|x-y|^{n-2+q}} \, d\sigma_x \, d\sigma_y\right)^{\frac{1}{q}}, \tag{10}$$

where $d\sigma_x$ and $d\sigma_y$ are the surface measures with respect to x and y, respectively, and where we integrate only over those $x, y \in \partial\Omega$ with $|x-y| < \frac{d}{2}$ whenever $x \in \partial\Omega_\pm$ and $y \in \partial\Omega_\mp$ in the case of an aperture domain. Moreover, we introduce the space $\dot{W}^{1-1/q,q}(\partial\Omega) = \dot{W}^{1/q',q}(\partial\Omega)$ consisting of all functions for which (10) is finite. Identifying two functions that differ only by a constant, (10) even defines a norm on $\dot{W}^{1/q',q}(\partial\Omega)$ and turns this space into a Banach space [16]. Its dual space will be denoted by $\dot{W}^{-1/q',q'}(\partial\Omega)$, the corresponding norm by $\|\cdot\|_{-1/q',q',\partial\Omega}$.

The following theorem largely due to Kudryavtsev [17], [18] characterizes the space $\dot{W}^{1/q',q}(\partial\Omega)$ as the desired trace space.

Theorem 2.2.1. *Let $n \geq 2$, $1 < q < \infty$ and let $\Omega \subset \mathbb{R}^n$ be the half space, a bent half space, a perturbed half space or an aperture domain.*

(i) For every $u \in \dot{W}^{1,q}(\Omega)$ the trace $\Gamma(u)$ is well defined and belongs to $\dot{W}^{1/q',q}(\partial\Omega)$. Furthermore, the trace estimate

$$\|\Gamma(u)\|_{\dot{W}^{1/q',q}(\partial\Omega)} \leq c\|u\|_{\dot{W}^{1,q}(\Omega)} \tag{11}$$

holds true for a constant $c = c(\Omega, q) > 0$.

(ii) For every $\bar{u} \in \dot{W}^{1/q',q}(\partial\Omega)$, there exists $u \in \dot{W}^{1,q}(\Omega)$ such that $\Gamma(u) = \bar{u}$ and

$$\|u\|_{\dot{W}^{1,q}(\Omega)} \leq c\|\bar{u}\|_{\dot{W}^{1/q',q}(\partial\Omega)} \tag{12}$$

with a constant $c = c(\Omega, q) > 0$.

Proof. For the assertion concerning the half space, see [17, 18].

For a bent half space H_ω where $\|\nabla'\omega\|_\infty < \infty$ it holds for every measurable function v on ∂H_ω

$$\begin{aligned}\int_{\partial H_\omega} |v|\, d\sigma_x &= \int_{\partial\mathbb{R}^n_+} v(x',\omega(x'))\sqrt{1+|\nabla'\omega|^2}\, dx' \\ &\le c\int_{\partial\mathbb{R}^n_+} |v(x',\omega(x'))|\, dx' = c\int_{\partial\mathbb{R}^n_+} |\tilde v(\tilde x',0)|\, d\tilde x',\end{aligned} \tag{13}$$

for some constant $c = c(n,q,\omega) > 0$; here we used the transformation $(x', x_n - \omega(x')) := \tilde x = (\tilde x', \tilde x_n)$ and the definition $\tilde v(\tilde x) = v(x)$. Furthermore, since ω is a Lipschitz function, we have that

$$|x'-y'|^{n-2+q} \le |x'-y'|^{n-2+q} + |\omega(x')-\omega(y')|^{n-2+q} \le c|x'-y'|^{n-2+q},$$

with $c > 0$ depending on the Lipschitz constant of ω. This implies for $u \in \dot W^{1,q}(H_\omega)$ that

$$c_1\|\widetilde{\Gamma(u)}\|_{\dot W^{1/q',q}(\partial\mathbb{R}^n_+)} \le \|\Gamma(u)\|_{\dot W^{1/q',q}(\partial H_\omega)} \le c_2\|\widetilde{\Gamma(u)}\|_{\dot W^{1/q',q}(\partial\mathbb{R}^n_+)}$$

for some constants $c_1, c_2 > 0$. Using the notation $\tilde\nabla = (\tilde\nabla', \tilde\partial_n)$ for the differential operator acting on the variable $\tilde x \in \mathbb{R}^n_+$, we obtain $\nabla w = (\tilde\nabla - (\tilde\nabla'\omega, 0)\tilde\partial_n)\tilde w$ and the estimate

$$c_1\|\tilde\nabla\tilde u\|_{L^q(\mathbb{R}^n_+)} \le \|\nabla u\|_{L^q(H_\omega)} \le c_2\|\tilde\nabla\tilde u\|_{L^q(\mathbb{R}^n_+)},$$

cf. also (34), (35) below. Hence (11) and (12) hold due to the half space result and $\widetilde{\Gamma(u)} = \Gamma(\tilde u)$, and

$$\begin{aligned}\|\Gamma(u)\|_{\dot W^{1/q',q}(\partial H_\omega)} &\le c\|\widetilde{\Gamma(u)}\|_{\dot W^{1/q',q}(\partial\mathbb{R}^n_+)} \\ &= c\|\Gamma(\tilde u)\|_{\dot W^{1/q',q}(\partial\mathbb{R}^n_+)} \le c\|\tilde u\|_{\dot W^{1,q}(\mathbb{R}^n_+)} \\ &\le c\|u\|_{\dot W^{1,q}(H_\omega)}.\end{aligned} \tag{14}$$

By analogy, for $\bar u \in \dot W^{1,q}(H_\omega)$

$$\begin{aligned}\|u\|_{\dot W^{1,q}(H_\omega)} &\le c\|\tilde u\|_{\dot W^{1,q}(\mathbb{R}^n_+)} \\ &\le c\|\Gamma(\tilde u)\|_{\dot W^{1/q',q}(\partial\mathbb{R}^n_+)} = c\|\overline{\tilde u}\|_{\dot W^{1/q',q}(\partial\mathbb{R}^n_+)} \\ &= c\|\tilde{\bar u}\|_{\dot W^{1/q',q}(\partial\mathbb{R}^n_+)} \le c\|\bar u\|_{\dot W^{1/q',q}(\partial H_\omega)}.\end{aligned} \tag{15}$$

To show the assertion concerning the perturbed and the aperture domain let us first sharpen the result in the case of a bent half space H_ω. Let $\bar u \in \dot W^{1/q',q}(\partial H_\omega)$ have compact support in ∂H_ω. Then for every $\delta > 0$ there is an extension $u_\delta \in \dot W^{1,q}(H_\omega)$ with $\Gamma(u_\delta) = \bar u$ that vanishes outside of a layer of width δ and satisfies the estimate (12) with a constant $c = c(H_\omega, q, \delta) > 0$. This may be seen by the following consideration. Take a cut-off function $\varphi_\delta \in C_0^\infty(\mathbb{R}^n)$ with $\varphi_\delta|_\Sigma = 1$ for $\Sigma \subset \partial H_\omega$ containing $\operatorname{supp}\bar u$ and $\varphi_\delta(x) = 0$ for $\operatorname{dist}(x, \partial H_\omega) \ge \delta$. Denote by $u \in \dot W^{1,q}(H_\omega)$ an extension of $\bar u$ as in (ii) satisfying (12). Then $u_\delta = u\varphi_\delta$ has compact support, fulfils $\Gamma(u_\delta) = \bar u$ and satisfies the estimate

$$\begin{aligned}\|u_\delta\|_{\dot W^{1,q}(H_\omega)} &\le \|\varphi_\delta\nabla u\|_{q,H_\omega} + \|u\nabla\varphi_\delta\|_{q,G} \\ &\le c(\|u\|_{\dot W^{1,q}(H_\omega)} + \|u\|_{q,G}),\end{aligned} \tag{16}$$

where $G \subset H_\omega$ is an open, bounded domain containing $\operatorname{supp} u_\delta$. But since $\bar u = 0$ on a subset $\Lambda \subset \partial H_\omega \cap \overline{G}$ with positive measure, $\|u\|_{q,G} \le c\|u\|_{\dot W^{1,q}(H_\omega)}$ by the Poincaré inequality and thus

$$\|u_\delta\|_{\dot W^{1,q}(H_\omega)} \le c\|u\|_{\dot W^{1,q}(H_\omega)} \le c\|\bar u\|_{\dot W^{1/q',q}(\partial H_\omega)}. \tag{17}$$

Now we turn our focus to the perturbed half space Ω. Let $B = B_0$ be a ball with center 0 such that $\Omega \backslash B = \mathbb{R}^n_+ \setminus B$. Then choose open balls $B_1, \dots, B_m \subset \mathbb{R}^n$ satisfying

$$\overline{\Omega} \subset (\mathbb{R}^n_+ \backslash \overline{B}) \cup \bigcup_{j=1}^{m} B_j, \tag{18}$$

and cut-off functions $\varphi_0, \dots, \varphi_m \in C^\infty(\mathbb{R}^n)$ defining a partition of unity such that $\varphi_0 = 1$ outside of some open ball B' with $\overline{B} \subset B'$, $\varphi_0 = 0$ in a neighbourhood of $\overline{B}$, $\operatorname{supp} \varphi_j \subset B_j$ for $1 \le j \le m$ and $\sum_{j=0}^m \varphi_j = 1$ in Ω. Since $\Gamma(\varphi_j u) = \varphi_j|_{\partial\Omega} \cdot \Gamma(u)$ for $u \in \dot{W}^{1,q}(\Omega)$, we have to control only those φ_j with $B_j \cap \partial\Omega \neq \emptyset$, say, for $j = 1, \dots, m'$. Furthermore, due to the regularity of the boundary of Ω, we find for each $1 \le j \le m'$ with $B_j \cap \partial\Omega \neq \emptyset$ a function $\omega_j \in C^{1,1}(\mathbb{R}^{n-1})$ of compact support such that with the bent half space $H_j = H_{\omega_j}$

$$B_j \cap \Omega \subset H_j, \quad B_j \cap \partial\Omega \subset \partial H_j; \tag{19}$$

we have tacitly rotated and translated the coordinate system depending on j. Finally, let $H_0 = \mathbb{R}^n_+$. It should be understood, that if $B_j \cap \partial\Omega$ is empty, then we may assume $\overline{B}_j \subset \Omega$. Given $\bar{u} \in \dot{W}^{1/q',q}(\partial\Omega)$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that either φ or $1 - \varphi$ has compact support we have

$$\|\varphi\bar{u}\|_{\dot{W}^{1/q',q}(\partial\Omega)} \le c(\|\bar{u}\|_{\dot{W}^{1/q',q}(\partial\Omega)} + \|\bar{u}\|_{q,\Sigma}) \tag{20}$$

with an open and bounded $\Sigma \subset \partial\Omega$ containing $\operatorname{supp} \varphi\bar{u}$ or $\operatorname{supp}(\bar{u} - \varphi\bar{u})$, respectively. This follows easily from $\|\bar{u}\|_{\dot{W}^{1/q',q}(\partial\Omega)} \le c\|\bar{u}\|_{\dot{W}^{1/q',q}(\Sigma)}$ for functions with compact support in Σ and a truncation lemma, see [3, Lemmata 5.1., 5.3.], as well as from the triangle inequality applied to $\varphi\bar{u} = \bar{u} - (1 - \varphi)\bar{u}$ in $\dot{W}^{1/q',q}(\partial\Omega)$. But then, with the partition of unity considered above, denote for $\bar{u} \in \dot{W}^{1/q',q}(\partial\Omega)$ the extension of $\varphi_j\bar{u}$ to the bent half space H_j by u_j. Now, if one chooses the extensions to vanish in a δ-neighborhood of H_j, it follows due to the smoothness of the boundary of Ω that $u = \sum_{j=0}^m u_j$ is an extension of $\bar{u}$ satisfying

$$\begin{aligned}
\|u\|_{\dot{W}^{1,q}(\Omega)} &\le \sum_{j=0}^{m'} \|u_j\|_{\dot{W}^{1,q}(H_j)} \\
&\overset{(17)}{\le} c \sum_{j=0}^{m'} \|\Gamma(u_j)\|_{\dot{W}^{1/q',q}(\partial H_j)} = c \sum_{j=0}^{m'} \|\varphi_j\bar{u}\|_{\dot{W}^{1/q',q}(\partial H_j)} \\
&\overset{(20)}{\le} c(\|\bar{u}\|_{\dot{W}^{1/q',q}(\partial\Omega)} + \|\bar{u}\|_{q,\partial G}),
\end{aligned}$$

where $G \subset \Omega$ is a bounded domain as above such that $\varphi_0 = 1$ outside of G. Since $\bar{u} \in \dot{W}^{1/q',q}(\partial\Omega)$ is defined only up to a constant, we assume that $\int_{\partial G} \bar{u}\, d\sigma = 0$. Then, by the Poincaré inequality for Sobolev-Slobodeckiĭ spaces (see e.g. [13, Theorem 2.6]) we have $\|\bar{u}\|_{q,\partial G} \le c\|\bar{u}\|_{\dot{W}^{1/q',q}(\partial G)} \le c\|\bar{u}\|_{\dot{W}^{1/q',q}(\partial\Omega)}$ with a constant $c = c(G, \Omega, q) > 0$.

For the converse direction, we immediately see by (20) and the Poincaré inequality that

$$\|\varphi_j\Gamma(u)\|_{\dot{W}^{1/q',q}(\partial\Omega)} \le c\|\Gamma(\varphi_j u)\|_{\dot{W}^{1/q',q}(\partial G)} \le c\|\Gamma(\varphi_j u)\|_{\dot{W}^{1/q',q}(\partial H_j)}$$

and thus by (14)

$$\begin{aligned}
\|\Gamma(u)\|_{\dot{W}^{1/q',q}(\partial\Omega)} &\le c \sum_{j=0}^{m'} \|\Gamma(\varphi_j u)\|_{\dot{W}^{1/q',q}(\partial H_j)} \\
&\le c \sum_{j=0}^{m'} \|\varphi_j u\|_{\dot{W}^{1,q}(H_j)} \\
&\le c(\|u\|_{\dot{W}^{1,q}(\Omega)} + \|u\|_{q,B'\cap\Omega})
\end{aligned} \tag{21}$$

with $c = c(n,q,\Omega) > 0$. Since $u \in \dot W^{1,q}(\Omega)$ is defined only up to a constant, we can assume that $\int_{B'\cap\Omega} u\,dx = 0$. This gives $\|u\|_{q,B'\cap\Omega} \le c\|u\|_{\dot W^{1,q}(\Omega)}$ for some constant $c > 0$.

A similar procedure as the one used for the perturbed half space yields the assertion for the aperture domain, if one chooses open balls $B = B_0, B_1, \dots, B_m \subset \mathbb{R}$ such that

$$\Omega \cup B = \mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B, \qquad \overline{\Omega} \subset ((\mathbb{R}^n_+ \cup \mathbb{R}^n_-)\backslash\overline{B}) \cup \bigcup_{j=1}^m B_j,$$

and cut-off functions $\varphi_+, \varphi_-, \varphi_1, \dots, \varphi_m \in C^\infty(\mathbb{R}^n)$ defining a partition of unity with supp $\varphi_j \subset B_j$ for $1 \le j \le m$ and $\varphi_\pm = 1$ in $\Omega_\pm\backslash B'$ for some open ball B' with $\overline{B} \subset B'$, $\varphi_\pm = 0$ in a neighborhood of $\overline{B}$ and in $\Omega_\mp$. The condition $|x-y| < \frac{d}{2}$ in the integral norm is crucial in our context to exclude mixed terms coming from the upper and lower part of the boundary far away from the origin. In fact, without this condition one cannot expect (20) to be valid for $\varphi_\pm$, since neither $\varphi_\pm$ nor $1-\varphi_\pm$ has compact support. But if we do impose the condition, we may write

$$\|\varphi_\pm\bar u\|_{\dot W^{1/q',q}(\partial\Omega)} = \|\varphi_\pm\bar u\|_{\dot W^{1/q',q}(\Sigma)} + \int_\Sigma \int_{(\partial\Omega\backslash\Sigma)\cap\{|x-y|<\frac{d}{2}\}} \frac{|\varphi_\pm(x)\bar u(x)|^q}{|x-y|^{n-2+q}}\,\mathrm{d}y\,\mathrm{d}x,$$

with $\Sigma \subset \partial\Omega$ containing the support of $\varphi_\pm$. Because $0 < \delta \le |x-y| < \frac{d}{2}$, the second term of the right hand side can be estimated by $c\|\varphi_\pm\bar u\|_{q,\Sigma}$. Since $\varphi_\pm\bar u = 0$ on a subset $\Lambda \subset \Sigma$ of positive measure, we get by Poincaré's inequality $\|\varphi_\pm\bar u\|_{\dot W^{1/q',q}(\partial\Omega)} \le \|\varphi_\pm\bar u\|_{\dot W^{1/q',q}(\Sigma)}$ and a cut-off argument on Σ yields (20).

It should be noted, that these results do not depend on the choice of the partition of unity. This is because the respective norms ${}_\varphi\|\cdot\|_{\dot W^{1/q',q}(\partial\Omega)}$ and ${}_\psi\|\cdot\|_{\dot W^{1/q',q}(\partial\Omega)}$ corresponding to different partitions of unity $(\varphi_j)_{0\le j\le m}$ and $(\psi_k)_{0\le k\le \ell}$ are equivalent, which can be seen by considering for every j

$$\begin{aligned} \|\varphi_j\bar u\|_{\dot W^{1/q',q}(\partial\Omega)} &= \Big\|\sum_{k=0}^{\ell}\psi_k\varphi_j\bar u\Big\|_{\dot W^{1/q',q}(\partial\Omega)} \\ &\le \sum_{k=0}^{\ell}\|\psi_k\varphi_j\bar u\|_{\dot W^{1/q',q}(\partial\Omega)} = \sum_{k=0}^{\ell}\|\varphi_j(\psi_k\bar u)\|_{\dot W^{1/q',q}(\partial\Omega)} \\ &\le c\sum_{k=0}^{\ell}\|\psi_k\bar u\|_{\dot W^{1/q',q}(\partial\Omega)}. \end{aligned} \tag{22}$$

This proves Theorem 2.2.1. □

The considerations above motivate the notation $u|_{\partial\Omega} := \Gamma(u)$, $\|u\|_{\dot W^{1/q',q}(\partial\Omega)} := \|\Gamma(u)\|_{\dot W^{1/q',q}(\partial\Omega)}$ and $\dot W^{1,q}_0(\Omega) := \{u \in \dot W^{1,q}(\Omega) : u|_{\partial\Omega} = 0\}$.

3 Very Weak Solutions

3.1 Very Weak Solutions in the Half Space

In order to prove Theorem 1.2 in the case of a half space, we introduce a generalization of Definition 1.1 of very weak solutions. Therefore, we generalize a result by Farwig and Sohr [10] concerning the strong Stokes system

$$\begin{aligned} -\Delta w - \nabla\psi &= v &&\text{in } \Omega \\ \nabla \mathrm{div}\, w &= \nabla\gamma &&\text{in } \Omega \\ w|_{\partial\Omega} &= 0 &&\text{on } \partial\Omega, \end{aligned} \tag{23}$$

where $\Omega = \mathbb{R}^n_+$ or $\Omega = \mathbb{R}^n$, and the boundary condition is not needed when $\Omega = \mathbb{R}^n$. The latter case will be of interest in the proof of Theorem 3.3.1.

Theorem 3.1.1. *Let $n \geq 2$, $1 < q < \infty$ and let $\Omega = \mathbb{R}^n_+$ or $\Omega = \mathbb{R}^n$. Then for every $v \in L^q(\Omega)$, $\gamma \in \hat{W}^{1,q}(\Omega)$, there exists a solution $(w, \psi) \in \hat{W}^{2,q}(\Omega) \times \hat{W}^{1,q}(\Omega)$ of (23) satisfying*

$$\|\nabla^2 w\|_q + \|\nabla \psi\|_q \leq c\,(\|v\|_q + \|\nabla \gamma\|_q) \tag{24}$$

with $c = c(n, q) > 0$. The pressure ψ is unique up to a constant and the velocity field w is

(i) unique up to a linear polynomial $a + Ax$, where $a \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n,n}$, if $\Omega = \mathbb{R}^n$, and

(ii) unique up to a linear term bx_n, where $b \in \mathbb{C}^n$, if $\Omega = \mathbb{R}^n_+$.

If $1 < q < n$ and $\frac{1}{n} + \frac{1}{r} = \frac{1}{q}$ then we may single out a special solution by the condition $\nabla w \in L^r(\Omega)$ (and up to the additive constant $a \in \mathbb{C}^n$ if $\Omega = \mathbb{R}^n$).

Proof. The proof for data $v \in L^q(\Omega)$ and $\gamma \in W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega)$ in [10] uses an approximation procedure of the generalized resolvent problem where the equation $-\Delta w - \nabla \psi = v$ in (23) is replaced by $\lambda w - \Delta w - \nabla \psi = v$ with $\lambda \to 0+$. Now let $v \in L^q(\Omega)$ and $\gamma \in \hat{W}^{1,q}(\Omega)$. In view of Lemma 2.1.1 there exists a sequence $(\gamma_i) \subset W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega)$ such that $\|\nabla(\gamma - \gamma_i)\|_q \to 0$ as $i \to \infty$. To each (v, γ_i) corresponds a solution $(w_i, \psi_i) \in \hat{W}^{2,q}(\Omega) \times \hat{W}^{1,q}(\Omega)$ satisfying the estimate (24). This ensures, that both (w_i) and (ψ_i) are Cauchy sequences in their respective spaces and hence converge to some $w \in \hat{W}^{2,q}(\Omega)$ and $\psi \in \hat{W}^{1,q}(\Omega)$. The pair (w, ψ) actually solves the system (23), because

$$\begin{aligned} \| -\Delta w + \nabla \psi - v\|_q &\leq \| -\Delta(w - w_i)\|_q + \|\nabla(\psi - \psi_i)\|_q + \| -\Delta w_i + \nabla \psi_i - v\|_q \\ &\leq \|\nabla^2(w - w_i)\|_q + \|\nabla(\psi - \psi_i)\|_q \to 0 \end{aligned}$$

as $i \to \infty$; by analogy, we get $\nabla \mathrm{div}\, w = \nabla \gamma$. Furthermore (w, ψ) is easily seen to satisfy the *a priori* estimate (24).

Concerning uniqueness, it suffices to consider a solution $(w, \psi) \in \hat{W}^{2,q}(\Omega) \times \hat{W}^{1,q}(\Omega)$ to (23) with data $v = 0 \in L^q(\Omega)$ and $\gamma = 0 \in W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega)$. Since this case has already been investigated in [10], the proof is complete. □

Now we introduce homogeneous Sobolev spaces $\hat{Y}^{2,q}(\Omega)$ and $\hat{Y}^{2,q}_\sigma(\Omega)$ related to the domain of the Laplacian and the Stokes operator for the system (23), respectively, *i.e.*, the space of solutions

$$\hat{Y}^{2,q}(\Omega) = \{w \in \hat{W}^{2,q}(\Omega) : w = 0 \text{ on } \partial\Omega\} \tag{25}$$

and

$$\hat{Y}^{2,q}_\sigma(\Omega) := \{w \in \hat{W}^{2,q}(\Omega) : \nabla \mathrm{div}\, w = 0, w = 0 \text{ on } \partial\Omega\}. \tag{26}$$

If $1 < q < n$, we include the condition $\nabla u \in L^r(\Omega)$, $\frac{1}{n} + \frac{1}{r} = \frac{1}{q}$, to single out a unique function; e.g.,

$$\hat{Y}^{2,q}(\Omega) = \{w \in \hat{W}^{2,q}(\Omega) : \nabla w \in L^r(\Omega),\ w = 0 \text{ on } \partial\Omega\}. \tag{27}$$

All spaces are endowed with the norm $\|\nabla^2 \cdot\|_q$. Their dual spaces will be denoted by

$$\hat{Y}^{-2,q'}(\Omega) := \hat{Y}^{2,q}(\Omega)^* \text{ and } \hat{Y}^{-2,q'}_\sigma(\Omega) := \hat{Y}^{2,q}_\sigma(\Omega)^*,$$

respectively. Evidently, analogous spaces are well defined Banach spaces when $\Omega \subset \mathbb{R}^n$ is a bent or perturbed half space. In the case of an aperture domain Ω and $1 < q < n$, we will have to add the no-flux condition $\hat{\phi}(w) = 0$ in the definition of $\hat{Y}^{2,q}_\sigma(\Omega)$ (see Section 3.4 below).

The following definition follows a generalization of the concept of very weak solutions due to Schumacher [21, 22], see also [9].

Definition 3.1.2. *Let $n \geq 2$, $1 < q < \infty$, let $\Omega = \mathbb{R}^n_+$ be the half space and let $\mathcal{F} \in \hat{Y}^{-2,q}_\sigma(\Omega), \mathcal{K} \in \hat{W}^{-1,q}(\Omega)$ be given. Then $u \in L^q(\Omega)$ is called a* very weak solution of the Stokes problem with data *$\mathcal{F}, \mathcal{K}$ if*

$$\begin{aligned} -(u, \Delta w) &= \langle \mathcal{F}, w \rangle, \quad w \in \hat{Y}^{2,q'}_\sigma(\Omega), \\ -(u, \nabla\psi) &= \langle \mathcal{K}, \psi \rangle, \quad \psi \in \hat{W}^{1,q'}(\Omega). \end{aligned} \tag{28}$$

Remark 3.1.3. *(i) Given $u \in L^q(\Omega)$ and setting $\langle \mathcal{F}, w \rangle := -(u, \Delta w)$ and $\langle \mathcal{K}, \psi \rangle := -(u, \nabla\psi)$, one readily sees that any vector field $u \in L^q(\Omega)$ is a very weak solution of the Stokes problem with suitable data. Thus, one cannot define boundary values of solutions in this abstract setting.*

(ii) For $\frac{n}{n-1} < q < \infty$ and given data F, k and g as in Definition 1.1, $\mathcal{F}$ and $\mathcal{K}$ defined via

$$\begin{aligned} \langle \mathcal{F}, w \rangle &:= -(F, \nabla w) - \langle g, N \cdot \nabla w \rangle_{\partial\Omega}, \quad & w \in \hat{Y}^{2,q'}_\sigma(\Omega), \\ \langle \mathcal{K}, \psi \rangle &:= (k, \psi) - \langle g, \psi N \rangle_{\partial\Omega}, & \psi \in \hat{W}^{1,q'}(\Omega) \end{aligned} \tag{29}$$

yield elements in $\hat{Y}^{-2,q}_\sigma(\Omega)$ and $\hat{W}^{-1,q}(\Omega)$, respectively. This can be seen easily by the embeddings $\hat{W}^{1,q'}(\Omega) \subset L^{r'}(\Omega)$, $\hat{Y}^{2,q'}_\sigma(\Omega) \subset \hat{W}^{1,r'}(\Omega)$ and the estimate

$$\|\psi N\|_{\hat{W}^{1/q,q'}(\partial\Omega)} \leq c\|\psi\|_{\hat{W}^{1/q,q'}(\partial\Omega)} \leq c\|\nabla\psi\|_{q'},$$

which is due to Theorem 2.2.1; note that $1 < q' < n$ and $\frac{1}{n} + \frac{1}{r'} = \frac{1}{q'}$. Consequently

$$\begin{aligned} |\langle \mathcal{F}, w \rangle| &\leq \|F\|_r \|\nabla w\|_{r'} + \|g\|_{\hat{W}^{-1/q,q}(\partial\Omega)} \|N \cdot \nabla w\|_{\hat{W}^{1/q,q'}(\partial\Omega)} \\ &\leq c(\|F\|_r + \|g\|_{\hat{W}^{-1/q,q}(\partial\Omega)}) \|\nabla^2 w\|_{q'} \end{aligned}$$

and

$$\begin{aligned} |\langle \mathcal{K}, \psi \rangle| &\leq \|k\|_r \|\psi\|_{r'} + \|g\|_{\hat{W}^{-1/q,q}(\partial\Omega)} \|\psi N\|_{\hat{W}^{1/q,q'}(\partial\Omega)} \\ &\leq c(\|k\|_r + \|g\|_{\hat{W}^{-1/q,q}(\partial\Omega)}) \|\nabla\psi\|_{q'} \end{aligned}$$

with constants $c = c(n, q) > 0$, respectively. Thus, the norms may be estimated by

$$\|\mathcal{F}\|_{\hat{Y}^{-2,q}_\sigma(\Omega)} + \|\mathcal{K}\|_{\hat{W}^{-1,q}(\Omega)} \leq c(\|F\|_r + \|k\|_r + \|g\|_{\hat{W}^{-1/q,q}(\partial\Omega)}). \tag{30}$$

Theorem 3.1.4. *Let $\Omega = \mathbb{R}^n_+$ be the half space, $1 < q < \infty$ and $\mathcal{F} \in \hat{Y}^{-2,q}_\sigma(\Omega)$, $\mathcal{K} \in \hat{W}^{-1,q}(\Omega)$. Then the problem (28) has a unique very weak solution $u \in L^q(\Omega)$ satisfying*

$$\|u\|_q \leq c(\|\mathcal{F}\|_{\hat{Y}^{-2,q}_\sigma(\Omega)} + \|\mathcal{K}\|_{\hat{W}^{-1,q}(\Omega)}), \tag{31}$$

where $c = c(n, q) > 0$ is a constant.

Proof. Let $v \in L^{q'}(\Omega)$ be a vector field. Then in view of Theorem 3.1.1 there exists a unique solution $w \in \hat{Y}^{2,q'}_\sigma(\Omega)$, $\psi \in \hat{W}^{1,q'}(\Omega)$ of the system

$$-\Delta w - \nabla\psi = v, \quad \text{div } w = 0 \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega \tag{32}$$

depending linearly on v and satisfying the estimate

$$\|\nabla^2 w\|_{q'} + \|\nabla\psi\|_{q'} \leq c\, \|v\|_{q'},$$

where $c = c(n, q) > 0$ is a constant. Therefore, and due to the duality of Lebesgue spaces, an element $u \in L^q(\Omega)$ is uniquely defined via the relation

$$(u, v) = \langle \mathcal{F}, w \rangle + \langle \mathcal{K}, \psi \rangle \qquad \forall v \in L^{q'}(\Omega),$$

satisfying

$$\begin{aligned} |(u, v)| &\le \|\mathcal{F}\|_{\hat{Y}_\sigma^{-2,q}(\Omega)} \left\|\nabla^2 w\right\|_{q'} + \|\mathcal{K}\|_{\hat{W}^{-1,q}(\Omega)} \|\nabla\psi\|_{q'} \\ &\le c(\|\mathcal{F}\|_{\hat{Y}_\sigma^{-2,q}(\Omega)} + \|\mathcal{K}\|_{\hat{W}^{-1,q}(\Omega)}) \|v\|_{q'}, \end{aligned}$$

and thus verifying the estimate (31). Indeed, u is a very weak solution of the system (28): Let $w \in \hat{Y}_\sigma^{2,q'}(\Omega)$ and $\psi \in \hat{W}^{1,q'}(\Omega)$ be arbitrary test functions and define $v = -\Delta w - \nabla\psi$. Then, by definition of u,

$$-(u, \Delta w) - (u, \nabla\psi) = (u, v) = \langle \mathcal{F}, w \rangle + \langle \mathcal{K}, \psi \rangle .$$

Thus, u satisfies (28).

Now let $u \in L^q(\Omega)$ be a very weak solution to the data $\mathcal{F} = 0$, $\mathcal{K} = 0$. Then for all $v \in L^{q'}(\Omega)$, $v = -\Delta w - \nabla\psi$, where $w \in \hat{Y}_\sigma^{2,q'}(\Omega)$ and $\psi \in \hat{W}^{1,q'}(\Omega)$ are the unique solution of (32),

$$(u, v) = -(u, \Delta w) - (u, \nabla\psi) = \langle \mathcal{F}, w \rangle + \langle \mathcal{K}, \psi \rangle = 0.$$

Consequently $u = 0$. This completes the proof. □

We are now in the position to prove Theorem 1.2.

Proof of Theorem 1.2 (i). Given data as in Definition 1.1 and functionals $\mathcal{F}$ and $\mathcal{K}$ as in Remark 3.1.3 (ii), one may apply Theorem 3.1.4 to receive a unique very weak solution $u \in L^q(\Omega)$. The estimate (6) follows from (31) and (30).

In order to show that there exists a pressure $p \in \hat{W}^{-1,q}(\Omega)$, let $\gamma \in \hat{W}^{1,q'}(\Omega)$. Then by Theorem 3.1.1, there exists $w \in \hat{Y}^{2,q'}(\Omega)$ with $\operatorname{div} w = \gamma$ in $\hat{W}^{1,q'}(\Omega)$ and we have the estimate $\|\nabla^2 w\|_{q'} \le c\|\nabla\gamma\|_{q'}$. Define $p \in \hat{W}^{-1,q}(\Omega)$ via

$$\langle p, \gamma \rangle = \langle p, \operatorname{div} w \rangle := -(u, \Delta w) + (F, \nabla w) + \langle g, N \cdot \nabla w \rangle_{\partial\Omega}, \qquad \gamma \in \hat{W}^{1,q'}(\Omega).$$

Note that p is well-defined: For $w_1, w_2 \in \hat{Y}^{2,q'}(\Omega)$ with $\operatorname{div} w_1 = \operatorname{div} w_1$ we have that $w_1 - w_2 \in \hat{Y}_\sigma^{2,q'}(\Omega)$ is solenoidal and consequently $\langle p, \operatorname{div} w_1 \rangle = \langle p, \operatorname{div} w_2 \rangle$, because u is a very weak solution to the data F and g. Obviously, estimate (7) is fulfilled and $-\Delta u + \nabla p = \operatorname{div} F$ in the sense of distributions. □

3.2 Very Weak Solutions in the Bent Half Space

The main result concerning very weak solutions in bent half spaces follows as in the case of a half space, once one is able to ensure existence and uniqueness of strong solutions.

Theorem 3.2.1. *Let $n \ge 3$, $1 < q < n-1$ and $\omega \in C^{0,1}(\mathbb{R}^{n-1}) \cap W^{2,1}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ such that for simplicity $\omega(0') = 0$. Then there exists a constant $K = K(n, q) > 0$ such that if ω satisfies* (5)*, then for all $v \in L^q(H_\omega)$ and $\gamma \in \hat{W}^{1,q}(H_\omega)$ there exists a strong solution $(w, \psi) \in \hat{Y}^{2,q}(H_\omega) \times \hat{W}^{1,q}(H_\omega)$ of* (23) *satisfying the estimate*

$$\|\nabla^2 w\|_q + \|\nabla\psi\|_q \le c\,(\|v\|_q + \|\nabla\gamma\|_q) \tag{33}$$

with a constant $c = c(\omega, n, q) > 0$.

The pressure is unique up to a constant. Furthermore, the velocity field w is unique up to a linear term Ax, where $A \in \mathbb{C}^{n,n}$ and $A(x', \omega(x')) = 0$. In particular, if ω is nonlinear, the velocity field w is unique. If however ω is a linear transformation, say, $\omega(x') = d'^T \cdot x'$ with $d' \in \mathbb{R}^{n-1}$, then w is unique up to a vector field of the form Ax where $A = a_n \otimes (-d'^T, 1)$ with a column vector $a_n \in C^n$. By the condition $\|\nabla w\|_r < \infty$, where $\frac{1}{n} + \frac{1}{r} = \frac{1}{q}$, we may single out a special solution.

Proof. We will transform the problem into a problem on $\mathbb{R}^n_+$ and use a classical perturbation argument. Let the transformation $\phi : H_\omega \to \mathbb{R}^n_+$ be defined via $x = (x', x_n) \mapsto \tilde{x} = (\tilde{x}', \tilde{x}_n) := \phi(x) = (x', x_n - \omega(x'))$. Note that ϕ is a bijection and that

$$D\phi = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial \omega}{\partial x_1} & -\frac{\partial \omega}{\partial x_2} & \cdots & 1 \end{pmatrix},$$

ensuring that the Jacobian of ϕ is equal to 1. Define for a function w on H_ω a function $\tilde{w}$ on $\mathbb{R}^n_+$ via $\tilde{w}(\tilde{x}) = w(x)$. Using the notations $\tilde{\partial}_i$, $\tilde{\nabla} = (\tilde{\nabla}', \tilde{\partial}_n)$, $\tilde{\Delta}$ and $\widetilde{\mathrm{div}}$ for the respective differential operators acting on the variables $\tilde{x} \in \mathbb{R}^n_+$, we obtain the relations

$$\begin{aligned} \partial_i w &= \big(\tilde{\partial}_i - (\partial_i\omega)\tilde{\partial}_n\big)\tilde{w}, \qquad i = 1, \dots, n-1 \\ \Delta w(x) &= \big(\tilde{\Delta} + |\nabla'\omega|^2\tilde{\partial}_n^2 - 2(\nabla'\omega, 0)\cdot(\tilde{\nabla}\tilde{\partial}_n) - (\Delta'\omega)\tilde{\partial}_n\big)\tilde{w}(\tilde{x}) \\ \nabla\psi(x) &= \big(\tilde{\nabla} - (\nabla'\omega, 0)\tilde{\partial}_n\big)\tilde{\psi}(\tilde{x}) \\ \mathrm{div}\, w(x) &= \big(\widetilde{\mathrm{div}} - (\nabla'\omega, 0)\cdot\tilde{\partial}_n\big)\tilde{w}(\tilde{x}). \end{aligned} \tag{34}$$

Therefore, the norm estimates

$$\begin{aligned} \|w\|_{L^q(H_\omega)} &= \|\tilde{w}\|_{L^q(\mathbb{R}^n_+)} \\ \|\nabla w\|_{L^q(H_\omega)} &\le (1 + \|\nabla'\omega\|_\infty)\|\tilde{\nabla}\tilde{w}\|_{L^q(\mathbb{R}^n_+)} \\ \|\nabla^2 w\|_{L^q(H_\omega)} &\le c(1 + \|\nabla'\omega\|_\infty)^2\|\tilde{\nabla}^2\tilde{w}\|_{L^q(\mathbb{R}^n_+)} + c\|(\nabla'^2\omega)\tilde{\partial}_n\tilde{w}\|_{L^q(\mathbb{R}^n_+)} \end{aligned} \tag{35}$$

hold with a constant $c > 0$. In $(35)_3$ we still need an estimate of the term $\|(\nabla'^2\omega)\tilde{\partial}_n\tilde{w}\|_q$ by second order derivatives of $\tilde{w}$. For simplicity, let $u = \partial_n\tilde{w}$ so that $u \in \hat{W}^{1,q}(\mathbb{R}^n_+)$. Since $C_0^\infty(\overline{\mathbb{R}^n_+})$ is dense in $\hat{W}^{1,q}(\mathbb{R}^n_+)$, it suffices to consider $u \in C_0^\infty(\overline{\mathbb{R}^n_+})$. By the Sobolev embedding theorem, there exists a constant $c > 0$, such that for all $\tilde{x}_n > 0$

$$\|u(\cdot, \tilde{x}_n)\|_{L^s(\mathbb{R}^{n-1})} \le c\|\tilde{\nabla}'u(\cdot, \tilde{x}_n)\|_{L^q(\mathbb{R}^{n-1})},$$

where $s > q$ is defined via $\frac{1}{n-1} + \frac{1}{s} = \frac{1}{q}$. If $\|\nabla'^2\omega\|_{L^{n-1}(\mathbb{R}^{n-1})} \le K$, then we get by Hölder's inequality

$$\begin{aligned} \|(\nabla'^2\omega)u\|^q_{L^q(\mathbb{R}^n_+)} &\le c\int_0^\infty d\tilde{x}_n \int_{\mathbb{R}^{n-1}} |\nabla'^2\omega|^q|u(\cdot, \tilde{x}_n)|^q\, dx' \\ &\le c\|\nabla'^2\omega\|^q_{L^{n-1}(\mathbb{R}^{n-1})}\int_0^\infty \|u(\cdot, \tilde{x}_n)\|^q_{L^s(\mathbb{R}^{n-1})}\, d\tilde{x}_n \\ &\le cK\|\tilde{\nabla}'u\|^q_{L^q(\mathbb{R}^n_+)}. \end{aligned} \tag{36}$$

On the other hand, consider the weighted inequality $\||\cdot|^{-1}\varphi\|_q \le \frac{q}{(n-1)-q}\|\nabla\varphi\|_q$ for $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$, see [16], Section II.5, formula (5.3). Then, with the second condition $\||\cdot|\nabla'^2\omega\|_\infty \le K$, we get for $u \in C_0^\infty(\overline{\mathbb{R}^n_+})$ and for each $\tilde{x}_n > 0$

$$\||\cdot|^{-1}u(\cdot, \tilde{x}_n)\|_{L^q(\mathbb{R}^{n-1})} \le c\|\tilde{\nabla}'u(\cdot, \tilde{x}_n)\|_{L^q(\mathbb{R}^{n-1})},$$

which yields the estimate

$$\begin{aligned}\|(\nabla'^2\omega)u\|^q_{L^q(\mathbb{R}^n_+)} &\le c\int\limits_0^\infty d\tilde{x}_n \int_{\mathbb{R}^{n-1}} |(|\cdot|\nabla'^2\omega)|^q \,|\,|\cdot|^{-1}u(\cdot,\tilde{x}_n)|^q\, dx' \\ &\le c\||\cdot|\tilde{\nabla}'^2\omega\|^q_\infty \int\limits_0^\infty \|\tilde{\nabla}'u(\cdot,\tilde{x}_n)\|^q_{L^q(\mathbb{R}^{n-1})}\, d\tilde{x}_n \\ &\le cK\|\tilde{\nabla}'u\|^q_{L^q(\mathbb{R}^n_+)}.\end{aligned} \tag{37}$$

Hence in both cases we get in $(35)_3$ the estimate

$$\|(\nabla'^2\omega)\partial_n\tilde{w}\|^q_{L^q(\mathbb{R}^n_+)} \le cK\|\tilde{\nabla}'\partial_n\tilde{w}\|^q_{L^q(\mathbb{R}^n_+)}. \tag{38}$$

Consider now the spaces

$$\begin{aligned}\mathcal{X} &:= \hat{Y}^{2,q}(H_\omega)\times\hat{W}^{1,q}(H_\omega), & \tilde{\mathcal{X}} &:= \hat{Y}^{2,q}(\mathbb{R}^n_+)\times\hat{W}^{1,q}(\mathbb{R}^n_+),\\ \mathcal{Y} &:= L^q(H_\omega)\times\hat{W}^{1,q}(H_\omega), & \tilde{\mathcal{Y}} &:= L^q(\mathbb{R}^n_+)\times\hat{W}^{1,q}(\mathbb{R}^n_+).\end{aligned}$$

These spaces, if equipped with the norms

$$\|(w,\psi)\|_{\mathcal{X}} = \|\nabla^2 w\|_q + \|\nabla\psi\|_q, \qquad \|(v,\gamma)\|_{\mathcal{Y}} = \|(v,\nabla\gamma)\|_q,$$

and by analogy, $\tilde{\mathcal{X}},\tilde{\mathcal{Y}}$ equipped with similar norms $\|\cdot\|_{\tilde{\mathcal{X}}}$, $\|\cdot\|_{\tilde{\mathcal{Y}}}$, are obviously Banach spaces. In view of Theorem 3.1.1, we know that the operator

$$\tilde{S}_q:\tilde{\mathcal{X}}\to\tilde{\mathcal{Y}},\;\; \tilde{S}_q(\tilde{w},\tilde{\psi}) = (-\tilde{\Delta}\tilde{w}-\tilde{\nabla}\tilde{\psi}, -\widetilde{\operatorname{div}}\,\tilde{w})$$

is an isomorphism. Consider now the analogously defined operator $S_q:\mathcal{X}\to\mathcal{Y}$. By the relations (34), this operator decomposes into

$$S_q(w,\psi)(x) = \tilde{S}_q(\tilde{w},\tilde{\psi})(\tilde{x}) + \tilde{R}_q(\tilde{w},\tilde{\psi})(\tilde{x})$$

with a remainder $\tilde{R}_q:\tilde{\mathcal{X}}\to\tilde{\mathcal{Y}}$ defined via

$$\tilde{R}_q(\tilde{w},\tilde{\psi}) = \big(-|\nabla'\omega|^2\tilde{\partial}^2_n\tilde{w} + 2\nabla'\omega\cdot\tilde{\nabla}'\tilde{\partial}_n\tilde{w}) + (\Delta'\omega)\tilde{\partial}_n\tilde{w} + (\nabla'\omega,0)\tilde{\partial}_n\tilde{\psi}\,,\;\nabla'\omega\cdot\tilde{\partial}_n\tilde{w'}\big).$$

Employing the estimate (38) and the isomorphism property of $\tilde{S}_q$ we get that

$$\|\tilde{R}_q(\tilde{w},\tilde{\psi})\|_{\tilde{\mathcal{Y}}} \le k\|\tilde{S}_q(\tilde{w},\tilde{\psi})\|_{\tilde{\mathcal{Y}}}$$

with a constant $k=k(n,q,K)$, where we can choose the bound K of $\nabla'\omega$ and of $\nabla'^2\omega$ or $|\cdot|\nabla'^2\omega$ for given n and q in (36) or (37), respectively, small enough such that

$$k < \frac{1}{\|\tilde{S}^{-1}\|_{L(\tilde{\mathcal{Y}},\tilde{\mathcal{X}})}\|\tilde{S}\|_{L(\tilde{\mathcal{X}},\tilde{\mathcal{Y}})}}.$$

Now, for arbitrary $\tilde{v}\in L^q(\mathbb{R}^n_+)$ and $\tilde{\gamma}\in\hat{W}^{1,q}(\mathbb{R}^n_+)$, we want to apply the Banach Fixed Point Theorem to the map $N:\tilde{\mathcal{X}}\to\tilde{\mathcal{X}}$ defined via

$$N(\tilde{w},\tilde{\psi}) = \tilde{S}_q^{-1}(\tilde{v},\tilde{\gamma}) - \tilde{S}_q^{-1}(\tilde{R}_q(\tilde{w},\tilde{\psi})),$$

because a fixed point $(\tilde{w},\tilde{\psi})$ of the map N satisfies $\tilde{S}_q(\tilde{w},\tilde{\psi})+\tilde{R}(\tilde{w},\tilde{\psi}) = (\tilde{v},\tilde{\gamma})$, *i.e.*, $(\tilde{w},\tilde{\psi})$ is a solution to (23). Actually, N is a contraction map, since

$$\begin{aligned}\|N(\tilde{w}_1,\tilde{\psi}_1)-N(\tilde{w}_2,\tilde{\psi}_2)\|_{\tilde{\mathcal{X}}} &= \|\tilde{S}_q^{-1}(\tilde{R}_q(\tilde{w}_1-\tilde{w}_2,\tilde{\psi}_1-\tilde{\psi}_2))\|_{\tilde{\mathcal{X}}}\\ &\le \|\tilde{S}_q^{-1}\|_{L(\tilde{\mathcal{Y}},\tilde{\mathcal{X}})}\|\tilde{R}_q(\tilde{w}_1-\tilde{w}_2,\tilde{\psi}_1-\tilde{\psi}_2)\|_{\tilde{\mathcal{Y}}}\\ &\le k\|\tilde{S}_q^{-1}\|_{L(\tilde{\mathcal{Y}}),\tilde{\mathcal{X}}}\|\tilde{S}_q(\tilde{w}_1-\tilde{w}_2,\tilde{\psi}_1-\tilde{\psi}_2)\|_{\tilde{\mathcal{Y}}}\\ &\le k\|\tilde{S}_q^{-1}\|_{L(\tilde{\mathcal{Y}},\tilde{\mathcal{X}})}\|\tilde{S}_q\|_{L(\tilde{\mathcal{X}},\tilde{\mathcal{Y}})}\|(\tilde{w}_1-\tilde{w}_2,\tilde{\psi}_1-\tilde{\psi}_2)\|_{\tilde{\mathcal{X}}}\end{aligned}$$

and $k\|\tilde{S}_q^{-1}\|_{L(\tilde{\mathcal{Y}},\tilde{\mathcal{X}})}\|\tilde{S}_q\|_{L(\tilde{\mathcal{X}},\tilde{\mathcal{Y}})} < 1$. Thus, $\tilde{S}_q + \tilde{R}_q$ is an isomorphism from $\tilde{\mathcal{X}}$ to $\tilde{\mathcal{Y}}$ and hence S_q is an isomorphism from $\mathcal{X}$ to $\mathcal{Y}$. Finally,

$$\begin{aligned}\|(w,\psi)\|_{\mathcal{X}} &\le c_1\|(\tilde{w},\tilde{\psi})\|_{\tilde{\mathcal{X}}} \le c_2\|\tilde{S}_q(\tilde{w},\tilde{\psi})\|_{\tilde{\mathcal{Y}}} \\ &\le c_3\|(\tilde{S}_q+\tilde{R}_q)(\tilde{w},\tilde{\psi})\|_{\tilde{\mathcal{Y}}} \le c_4\|S_q(w,\psi)\|_{\mathcal{Y}}, \end{aligned} \tag{39}$$

with constants $c_1, c_2, c_3, c_4 > 0$ depending on ω, n, q.

To prove the assertion about the uniqueness, let $(w,\psi) \in \hat{Y}^{2,q}(H_\omega) \times \hat{W}^{1,q}(H_\omega)$ be a solution of (23) with data $v = 0$ and $\gamma = 0$. By the above consideration, $(w,\psi) = S_q^{-1}(0,0) = (0,0) \in \hat{Y}^{2,q}(H_\omega) \times \hat{W}^{1,q}(H_\omega)$ and thus $\nabla\psi = 0$, whereas $\nabla^2 w = 0$ and hence $w = Ax + b$ with $A = (a_{i,j}) \in \mathbb{C}^{n,n}$, $b \in \mathbb{C}^n$. By the boundary condition and $\omega(0') = 0$ we get $b = 0$ and $A(x', \omega(x')) = 0$ for all $x' \in \mathbb{R}^{n-1}$, which is equivalent to

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n-1} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = -\begin{pmatrix} a_{1,n} \\ \vdots \\ a_{n,n} \end{pmatrix} \omega(x'), \qquad x' \in \mathbb{R}^{n-1}. \tag{40}$$

If ω is a nonlinear map, so is $-a_{i,n}\omega$ for each $i = 1,\ldots,n$ with $a_{i,n} \neq 0$, which contradicts the linear left-hand side. Hence, for every $i = 1,\ldots,n$ we have $a_{i,n} = 0$, and the i-th line of (40) gives us $a_{i,j} = 0$ for all $j = 1,\ldots,n-1$. Thus $A = 0$.

Now let ω be a linear transformation, *i.e.*, $\omega(x') = d'^T \cdot x'$ with some $d' \in \mathbb{R}^{n-1}$. Then (40) implies that $A = a_n \otimes (-d'^T, 1)$ where $a_n \in \mathbb{C}^n$ denotes the nth column vector of A. Finally, since $1 < q < n-1$, we define $q < r < \infty$ via $\frac{1}{n} + \frac{1}{r} = \frac{1}{q}$, and by Sobolev's imbedding theorem there exists a constant matrix $W_0 \in \mathbb{R}^{n,n}$ such that $\|\nabla w - W_0\|_r < \infty$. This completes the proof. □

Remark 3.2.2. *(i) In Theorem 3.2.1 with a linear ω, there always exists a unique solution whose gradient has finite norm in L^r, $\frac{1}{n} + \frac{1}{r} = \frac{1}{q}$. This assertion also holds true for the unique solutions in the case of a nonlinear ω. Actually, revising the proof and substituting the spaces $\hat{Y}^{2,q}$ by $\hat{Y}^{2,q} \cap \dot{W}^{1,r}$ in the definition of $\mathcal{X}$ and $\tilde{\mathcal{X}}$, respectively, we obtain for given data $v \in L^q(H_\omega)$ and $\gamma \in \hat{W}^{1,q}(H_\omega)$ a unique solution $w_r \in \hat{Y}^{2,q}(H_\omega) \cap \dot{W}^{1,r}(H_\omega) \subset \hat{Y}^{2,q}(H_\omega)$. Then by uniqueness of w, we get $w_r = w$ and thus $\|\nabla w\|_r < \infty$.*

(ii) If in Theorem 3.2.1 additionally $f \in L^s(H_\omega)$ and $g \in \hat{W}^{1,s}(H_\omega)$ for some $1 < s < n-1$ and $K \le \min\{K(n,q), K(n,s)\}$, then $\|\nabla^2 w\|_s < \infty$ and $\|\nabla\psi\|_s < \infty$. Indeed, the same procedure as in the proof of the theorem yields a solution $(w_s, \psi_s) \in \big(\dot{W}^{2,q}(H_\omega) \cap \dot{W}^{2,s}(H_\omega)\big) \times \big(\hat{W}^{1,q}(H_\omega) \cap \hat{W}^{1,s}(H_\omega)\big)$ and by the uniqueness of the solution (w,ψ) we get $w = w_s$ up to a linear polynomial and $\psi = \psi_s$ up to a constant.

(iii) An interesting special case of a bent half space is a "smooth" cone. In this case $\omega(x') = \alpha(1 + |x'|^2)^{1/2} - \alpha$, and it is readily seen, that for $|\alpha| \le K$, i.e., for smooth cones with aperture angle close to π, Theorem 3.2.1 may be applied with the condition $\||\cdot|\nabla'^2\omega\|_\infty \le K$. Since ω is a nonlinear map, we get a unique solution in the case of a cone.

The results about the strong solutions on the bent half space enable us to prove existence and uniqueness of very weak solutions in an analogous way as in the half space case, that is, the same argument as in the proof of Theorem 3.1.4 furnishes us with the following result.

Theorem 3.2.3. *Let $n \ge 3$, $\frac{n-1}{n-2} < q < \infty$ and $\mathcal{F} \in \hat{Y}_\sigma^{-2,q}(\Omega)$, $\mathcal{K} \in \hat{W}^{-1,q}(\Omega)$, where $\Omega = H_\omega$ is a bent half space. Then there exists a constant $K = K(n,q) > 0$ such that if $\|\nabla'\omega\|_\infty \le K$ and if $\|\nabla'^2\omega\|_{L^{n-1}(\mathbb{R}^{n-1})} \le K$ or $\||\cdot|\nabla'^2\omega\|_\infty \le K$, the problem (28) has a unique very weak solution $u \in L^q(\Omega)$ satisfying*

$$\|u\|_q \le c(\|\mathcal{F}\|_{\hat{Y}_\sigma^{-2,q}(\Omega)} + \|\mathcal{K}\|_{\hat{W}^{-1,q}(\Omega)}), \tag{41}$$

where $c = c(\Omega, q) > 0$ is a constant.

The part of Theorem 1.2 concerning bent half spaces now follows easily.

3.3 Very Weak Solutions in the Perturbed Half Space

Again, due to the duality arguments already pointed out in the previous sections, it suffices to investigate the corresponding strong solutions. In order to prove the result on strong solutions in the perturbed half space, we want to use the localization method as described in Subsection 2.2. We choose open balls $B_1, \dots, B_m \subset \mathbb{R}^n$ satisfying (18) and nonnegative functions $\varphi_0, \dots, \varphi_m \in C^\infty(\mathbb{R}^n)$ such that $\varphi_0 = 1$ outside of some ball B' with $\overline{B} \subset B'$, $\varphi_0 = 0$ in a neighbourhood of $\overline{B}$, $\operatorname{supp} \varphi_j \subset B_j$ for $1 \le j \le m$ and $\sum_{j=0}^m \varphi_j = 1$ in Ω. Finally, due to the regularity of the boundary of Ω, we find for each $1 \le j \le m$ with $B_j \cap \partial\Omega \neq \emptyset$ a function $\omega_j \in C^{1,1}(\mathbb{R}^{n-1})$ of compact support and a corresponding bent half space $H_j = H_{\omega_j}$ satisfying (19). By choosing a sufficiently large number of balls B_j, such that the support of the corresponding ω_j is sufficiently small, we get that

$$\begin{aligned} \|\nabla'\omega_j\|_\infty &\le \min\{K(n,q), K(n,s_1), \dots, K(n,s_{k(q)})\}, \\ \||\cdot|\nabla'^2\omega\|_\infty &\le \min\{K(n,q), K(n,s_1), \dots, K(n,s_{k(q)})\}, \end{aligned} \tag{42}$$

for a finite number of parameters q, s_k, $1 \le k \le k(q)$, to be determined in the proof of Theorem 3.3.1 below.

Theorem 3.3.1. *Let $n \ge 3$, $1 < q < n/2$. Then for all $v \in L^q(\Omega)$ and $\gamma \in \hat{W}^{1,q}(\Omega)$ there exists a solution $(w, \psi) \in \dot{W}^{2,q}(\Omega) \times \hat{W}^{1,q}(\Omega)$ of (23) satisfying the estimate*

$$\|\nabla^2 w\|_q + \|\nabla\psi\|_q \le c\,(\|v\|_q + \|\nabla\gamma\|_q) \tag{43}$$

with a constant $c = c(\Omega, n, q) > 0$. The pressure is unique up to a constant. Furthermore, the velocity field w is unique up to a linear term Ax, where $A \in \mathbb{C}^{n,n}$. In particular, if $\Omega \neq \mathbb{R}^n_+$, the velocity field w is unique.

Proof. We will first prove the assertion about the uniqueness. Therefore, let $(w, \psi) \in \dot{W}^{2,q}(\Omega) \times \hat{W}^{1,q}(\Omega)$ solve (23) for homogeneous data $v = 0$, $\gamma = 0$. Now let $0 \le j \le m$, where φ_j is a cut-off functions of type H_j, cf. (19) (in the case of cut-off functions of type $\mathbb{R}^n$ and $\mathbb{R}^n_+$ we proceed in an analogous way). Then, with a suitable constant $c_j \in \mathbb{R}$ to be determined below, $(\varphi_j w, \varphi_j(\psi - c_j))$ satisfies the local equations

$$\begin{aligned} -\Delta(\varphi_j w) - \nabla(\varphi_j(\psi - c_j)) &= v_j \\ \nabla \operatorname{div}(\varphi_j w) &= \nabla\gamma_j, \end{aligned} \tag{44}$$

where

$$\begin{aligned} v_j &= -(\nabla\varphi_j)(\psi - c_j) - 2(\nabla\varphi_j)\nabla w - (\Delta\varphi_j) w \\ \nabla\gamma_j &= (\nabla\varphi_j)(\nabla w) + (\nabla^2\varphi_j)\cdot w. \end{aligned} \tag{45}$$

Since all terms on the right-hand side of (45) have compact support, $v_j \in L^s(H_j)$ and $\gamma_j \in \hat{W}^{1,s}(H_j)$ for each $s \in (1, q]$. Hence, by the regularity results in Remark 3.2.2 and the compactness of the supports of $\nabla\varphi_j$, every $(\varphi_j w, \varphi_j\psi) \in \hat{W}^{2,s}(H_j) \times \hat{W}^{1,s}(H_j)$ and summation over j yields $(w, \psi) \in \dot{W}^{2,s}(\Omega) \times \hat{W}^{1,s}(\Omega)$, $1 < s \le q$. Moreover, by Sobolev's embedding theorem ∇w, $\psi \in L^r(\Omega)$ for all sufficiently small $r > \frac{n}{n-1}$ and $w \in L^\rho(\Omega)$ for all sufficiently small $\rho > \frac{n}{n-2}$.

We need to extend the interval of admissible exponents s from $(1, s_0]$, $s_0 = q$, to $(1, n-1)$. Therefore, define $s_1 > q$ by $\frac{1}{n} + \frac{1}{s_1} = \frac{1}{q}$, which is possible, since $q < \frac{n}{2} < n$. Then Sobolev's imbedding theorem yields for a bounded $C^{1,1}$-domain G_j containing $\Omega \cap \operatorname{supp} \nabla\varphi_j$ (and $G_0 \supset (G \setminus B) \cap \mathbb{R}^n_+$, see §2.2)

$$\begin{aligned} \|(\nabla\varphi_j)\nabla w\|_{s_1,H_j} &\le c\|\nabla w\|_{s_1,G_j} \le c\|\nabla w\|_{1,q,G_j} < \infty, \\ \|(\Delta\varphi_j)w\|_{s_1,H_j} &\le c\|\nabla w\|_{q,G_j} \le c\|\nabla w\|_{1,q,G_j} < \infty, \\ \|(\nabla\varphi_j)(\psi_j - c_j)\|_{s_1,H_j} &\le c\|\psi_j - c_j\|_{s_1,G_j} \le c\|\nabla\psi\|_{1,q,G_j} < \infty, \end{aligned} \tag{46}$$

where $c_j = \frac{1}{|G_j|}\int_{G_j} \psi\,dx$. Hence $v_j \in L^{s_1}(H_j)$, and a similar argument shows that $\gamma_j \in \hat{W}^{1,s_1}(H_j)$. Now, if $s_1 < n-1$, Remark 3.2.2 together with (42) yields $(\varphi_j w, \varphi_j \psi) \in \hat{W}^{2,s_1}(H_j) \times \hat{W}^{1,s_1}(H_j)$, and summation over j gives us $(w,\psi) \in \hat{W}^{2,s_1}(\Omega) \times \hat{W}^{1,s_1}(\Omega)$. If however $s_1 \geq n-1$, we may replace s_1 by any $s_1 \in (s_0, n-1)$ and apply Remark 3.2.2 and (42) to obtain $(\varphi_j w, \varphi_j \psi) \in \hat{W}^{2,s_1}(H_j) \times \hat{W}^{1,s_1}(H_j)$ and thus $(w,\psi) \in \hat{W}^{2,s_1}(\Omega) \times \hat{W}^{1,s_1}(\Omega)$. In any case, repeating this procedure a finite number of times we receive exponents $q < s_1 < \ldots < s_k < n-1$ such that s_k is arbitrarily close to $n-1$. Summarizing, we get $(w,\psi) \in \hat{W}^{2,s}(\Omega) \times \hat{W}^{1,s}(\Omega)$ for every $1 < s < n-1$ and by Sobolev's imbedding theorem

$$\nabla w,\, \psi \in L^r(\Omega) \text{ for all } \frac{n}{n-1} < r < n(n-1), \quad w \in L^\rho(\Omega) \text{ for all } \frac{n}{n-2} < \rho < \infty.$$

Unfortunately, this argument needs the smallness assumption (42) for s arbitrarily close to $n-1$. But actually, we do need this argument only for $s < 2$ defined by $\frac{1}{n} + \frac{1}{2} = \frac{1}{s}$ and for s close to $\frac{n}{3}$ where $\nabla^2 w \in L^s(\Omega)$ implies $w \in L^{n+\varepsilon}(\Omega)$ for $\varepsilon > 0$ sufficiently small. In the first case when $\frac{1}{n} + \frac{1}{2} = \frac{1}{s}$ we have $\Delta w \in L^s(\Omega)$, $\nabla w \in L^2(\Omega)$ and $w \in L^{s'}(\Omega)$, the latter because of $\frac{2}{n} + \frac{1}{s'} = \frac{1}{s}$. Therefore, we may test (23) with w and write

$$\begin{aligned} 0 &= -\int_{\Omega_R} \Delta w \cdot w\,dx - \int_{\Omega_R} \nabla\psi \cdot w\,dx \\ &= \int_{\Omega_R} |\nabla w|^2\,dx - \int_{\partial\Omega'_R} \big(w \cdot \frac{\partial w}{\partial \mathbf{n}} + \psi w \cdot \mathbf{n}\big)\,d\sigma, \end{aligned} \tag{47}$$

where $\Omega_R = \Omega \cap B_R$ and $\partial\Omega'_R = \partial\Omega_R \backslash \partial\Omega$, because $w|_{\partial\Omega} = 0$. Moreover, the integral over $\psi\mathrm{div}\, w$ vanishes, since $\nabla\mathrm{div}\, w = 0$ and the constant div w lies in $L^2(\Omega)$.

Next we want to show that the boundary integrals in (47) vanish, if a suitable sequence of radii $(R_i)_{i\in\mathbb{N}}$ tends to infinity. First observe that for any function $f \in L^1(\Omega)$ there exists a sequence of radii $(R_i)_{i\in\mathbb{N}}$ with $R_i \to \infty$ for $i \to \infty$ such that

$$\int_{\partial\Omega'_{R_i}} |f|\,d\sigma \leq cR_i^{-1} \to 0. \tag{48}$$

Due to the regularity of w and ψ already shown, we know that $w \in L^{n+\varepsilon}(\Omega)$ and $\nabla w, \psi \in L^{\frac{n}{n-1}+\varepsilon}(\Omega)$ for any small $\varepsilon > 0$. Hence for sufficiently small $\varepsilon > 0$ we find $\theta_\varepsilon > n$ such that $\frac{1}{n+\varepsilon} + \frac{1}{\frac{n}{n-1}+\varepsilon} + \frac{1}{\theta_\varepsilon} = 1$. By Hölder's inequality we thus get

$$\int_{\partial\Omega'_R} \left| w \cdot \frac{\partial w}{\partial \mathbf{n}} \right| d\sigma \leq cR^{\frac{n-1}{\theta_\varepsilon}} \|w\|_{n+\varepsilon,\partial\Omega'_R} \|\nabla w\|_{\frac{n}{n-1}+\varepsilon,\partial\Omega'_R}, \tag{49}$$

where $c^{1/\theta_\varepsilon} = \frac{1}{2}|\partial B_1(0)|$; an analogous estimate holds for the integral $\int_{\partial\Omega'_R} |\psi w \cdot \mathbf{n}| d\sigma$. But since $|w|^{n+\varepsilon} + |\nabla w|^{\frac{n}{n-1}+\varepsilon} + |\psi|^{\frac{n}{n-1}+\varepsilon} \in L^1(\Omega)$, we find by (48) a sequence of radii $(R_i)_{i\in\mathbb{N}}$ with $R_i \to \infty$ for $i \to \infty$ such that

$$\|w\|_{n+\varepsilon,\partial\Omega'_{R_i}} \leq cR_i^{-\frac{1}{n+\varepsilon}}, \quad \|\nabla w\|_{\frac{n}{n-1}+\varepsilon,\partial\Omega'_{R_i}} + \|\psi\|_{\frac{n}{n-1}+\varepsilon,\partial\Omega'_{R_i}} \leq cR_i^{-\frac{1}{\frac{n}{n-1}+\varepsilon}}.$$

Now it follows that due to $\theta_\varepsilon > n$ the right-hand side of (49) tends to zero as $R_i \to \infty$. The analogous result holds for $\int_{\partial\Omega'_R} |\psi w \cdot \mathbf{n}| d\sigma$.

Summing up, we get in virtue of (47) by Lebesgue's Theorem that $\int_\Omega |\nabla w|^2\,dx = 0$ for all $n \geq 3$ and thus by the boundary condition $w = 0$. This leads immediately to $\nabla\psi = 0$. The uniqueness part is proven.

The existence of the solution and the estimate follow by the unique solvability of the corresponding resolvent problem [10, Theorem 1.2] via an approximation procedure for the resolvent parameter $\lambda \to 0+$. Therefore, let $v \in L^q(\Omega)$ and for the moment $\gamma \in W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega)$. Let

$(\lambda_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ be a sequence with $\lambda_i \to 0$ as $i \to \infty$. By virtue of the unique solvability of the resolvent problem, we receive corresponding solutions $(w_i, \psi_i) \in (W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)) \times \hat{W}^{1,q}(\Omega)$ with

$$\sup_{i\in\mathbb{N}} \|(\lambda_i w_i, \nabla^2 w_i, \nabla\psi_i)\|_q \le c(\|v\|_q + \|\nabla\gamma\|_q + \sup_{i\in\mathbb{N}} |\lambda_i|\, \|\gamma\|_{\hat{W}^{-1,q}(\Omega)}) < \infty.$$

Therefore there exists a subsequence (which we will denote with index i again), such that we have the weak convergences

$$\begin{aligned} \lambda_i w_i &\rightharpoonup \Phi && \text{in } L^q(\Omega),\\ \nabla^2 w_i &\rightharpoonup \tilde{w} && \text{in } L^q(\Omega),\\ \nabla\psi_i &\rightharpoonup \tilde{\psi} && \text{in } L^q(\Omega). \end{aligned} \tag{50}$$

Moreover, by the compact embedding $\hat{W}^{1,q}(\Omega') \subset L_0^q(\Omega')$ for any compact $\Omega' \subset \overline{\Omega}$ of class $C^{0,1}$ we find constants c_i and linear polynomials $a_i + A_i x$, such that $w_i - (a_i + A_i x)$ converges in $W^{1,q}_{\text{loc}}(\overline{\Omega})$ to some $w \in W^{1,q}_{\text{loc}}(\overline{\Omega})$ with $\nabla^2 w = \tilde{w}$ and $\psi_i - c_i$ converges locally in $L^q(\Omega)$ to some $\psi \in L^q_{\text{loc}}(\overline{\Omega})$ with $\nabla\psi = \tilde{\psi}$. Then for any smooth $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega (\lambda_i w_i)\cdot \nabla^2\varphi\, dx = \lambda_i \int_\Omega (\nabla^2 w_i)\varphi\, dx \to 0 \cdot \int_\Omega \tilde{w}\varphi\, dx = 0, \qquad i \to \infty$$

by (50). This ensures that $\nabla^2\Phi = 0$ and even $\Phi = 0$, since $\Phi \in L^q(\Omega)$. But then, the weak convergences (50) assure that $-\Delta w - \nabla\psi = v$ and $\nabla\text{div } w = \nabla\gamma$. Moreover, we get (43) by

$$\begin{aligned} \|\nabla^2 w\|_q + \|\nabla\psi\|_q &\le \liminf_{i\to\infty} \big(\|\lambda_i w_i\|_q + \|\nabla^2 w_i\|_q + \|\nabla\psi_i\|_q\big)\\ &\le \lim_{i\to\infty} c\big(\|v\|_q + \|\nabla\gamma\|_q + |\lambda_i|\|\gamma\|_{-1,q}\big)\\ &\le c\big(\|v\|_q + \|\nabla\gamma\|_q\big). \end{aligned} \tag{51}$$

Concerning the trace $w|_{\partial\Omega}$, we know that $w_i|_{\partial\Omega} = 0$ for all $i \in \mathbb{N}$ and thus $a_i + A_i x|_{\partial\Omega}$ converges in $W^{1-1/q,q}_{\text{loc}}(\partial\Omega)$ – and therefore componentwise – to some $a + Ax|_{\partial\Omega}$. Now consider $\tilde{w} := w - (a + Ax) \in W^{2,q}_{\text{loc}}(\Omega)$. Then $\|\nabla^2\tilde{w}\|_q = \|\nabla^2 w\|_q$ and $(\tilde{w}, \psi) \in W^{2,q}_{\text{loc}}(\Omega) \times \hat{W}^{1,q}(\Omega)$ is a solution of (23) with the desired properties for $\gamma \in W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega)$.

Now let $v \in L^q(\Omega)$ and $\gamma \in \hat{W}^{1,q}(\Omega)$. In view of Lemma 2.1.1 there exists a sequence $(\gamma_i) \subset W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega)$, such that $\|\nabla(\gamma - \gamma_i)\|_q \to 0$ as $i \to \infty$. To each (v, γ_i) corresponds a solution $(w_i, \psi_i) \in \dot{W}^{2,q}(\Omega) \times \hat{W}^{1,q}(\Omega)$ satisfying the estimate (43). This ensures, that both (w_i) and (ψ_i) are Cauchy sequences in their respective spaces and hence converge to some $w \in \dot{W}^{2,q}(\Omega)$ and $\psi \in \hat{W}^{1,q}(\Omega)$. The pair (w, ψ) actually solves the system (23) and satisfies the *a priori* estimate (43). The proof is complete. □

Remark 3.3.2. *If in the situation of Theorem 3.3.1 we have additionally $v \in L^s(\Omega)$ and $\gamma \in \hat{W}^{1,s}(\Omega)$ for some $1 < s < \frac{n}{2}$, then $\|\nabla^2 w\|_s < \infty$ and $\|\nabla\psi\|_s < \infty$. The construction of a solution $(w_s, \psi_s) \in (\hat{W}^{2,q}(\Omega) \cap \hat{W}^{2,s}(\Omega)) \times (\hat{W}^{1,q}(\Omega) \cap \hat{W}^{1,s}(\Omega))$ follows analogously to the construction in the proof. Then the uniqueness assertion of Theorem 3.3.1 yields $(w_s, \psi_s) = (w, \psi)$.*

From Theorem 3.3.1 we deduce analogously to the proof of Theorem 3.1.4 the result on very weak solutions in the perturbed half space in the abstract setting.

Theorem 3.3.3. *Let $n \ge 3$, $\frac{n}{n-2} < q < \infty$ and $\mathcal{F} \in \hat{Y}_\sigma^{-2,q}(\Omega)$, $\mathcal{K} \in \hat{W}^{-1,q}(\Omega)$, where Ω is a perturbed half space. Then the problem (28) has a unique very weak solution $u \in L^q(\Omega)$ satisfying*

$$\|u\|_q \le c(\|\mathcal{F}\|_{\hat{Y}_\sigma^{-2,q}(\Omega)} + \|\mathcal{K}\|_{\hat{W}^{-1,q}(\Omega)}), \tag{52}$$

where $c = c(\Omega, q) > 0$ is a constant.

This directly implies the assertion of Theorem 1.2 in case of a perturbed half space.

3.4 Very Weak Solutions in the Aperture Domain

In the case of an aperture domain, the two function spaces $\dot{W}^{1,q}(\Omega)$ and $\hat{W}^{1,q}(\Omega)$ do not necessarily coincide. In fact, we have the following characterization [11, Lemma 3.1].

Lemma 3.4.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an aperture domain.*

(i) Suppose $1 < q < n$ and let $r \in (\frac{n}{n-1}, \infty)$ be defined via $\frac{1}{n} + \frac{1}{r} = \frac{1}{q}$. Then for every $\psi \in \dot{W}^{1,q}(\Omega)$ there are constants $\psi_\pm \in \mathbb{C}$ such that $\psi - \psi_\pm \in L^r(\Omega_\pm)$ and

$$\|\psi - \psi_+\|_{L^r(\Omega_+)} + \|\psi - \psi_-\|_{L^r(\Omega_-)} + |\psi_+ - \psi_-| \leq c\|\nabla\psi\|_q.$$

Thus, the map $[\cdot] : \dot{W}^{1,q}(\Omega) \to \mathbb{C}, [\psi] = \psi_+ - \psi_-$ is a continuous linear functional and

$$\hat{W}^{1,q}(\Omega) = \{\psi \in \dot{W}^{1,q}(\Omega) : [\psi] = 0\}.$$

Suppose φ^0 is a smooth funtion with $\varphi^0 = 1$ on $\Omega_+ \backslash B$ and $\varphi^0 = 0$ on $\Omega_- \backslash B$. Then each $\psi \in \dot{W}^{1,q}(\Omega)$ has the unique decomposition $\psi = \psi_0 + [\psi]\varphi^0$ with $\psi_0 \in \hat{W}^{1,q}(\Omega)$, $\|\nabla\psi_0\|_q \leq c\|\nabla\psi\|_q$ and

$$\dot{W}^{1,q}(\Omega) = \hat{W}^{1,q}(\Omega) \oplus \{K\varphi^0 : K \in \mathbb{C}\}$$

is a direct sum.

(ii) Suppose $q \geq n$. Then $\dot{W}^{1,q}(\Omega) = \hat{W}^{1,q}(\Omega)$.

It is convenient to think of $\varphi^0 \in C^\infty(\overline{\Omega})$ as a function satisfying

$$\varphi^0(x) = \begin{cases} 1 & \text{for} \quad x \in \Omega_+ \\ 0 & \text{for} \quad x \in \Omega_- \backslash B \end{cases} \qquad \text{and} \qquad \int\limits_{B\cap\Omega_-} \varphi^0 \, dx = 0. \tag{53}$$

Then $\varphi^0 \in \dot{W}^{1,q'}(\Omega)$ for all $1 < q' < \infty$. Moreover, for all $u \in L^q(\Omega)$ with $u \cdot N|_{\partial\Omega} = 0$ and $\operatorname{div} u = 0$, we have for the flux $\phi(u)$ through the aperture of Ω

$$\phi(u) := \int_S u \cdot N \, d\sigma = -\int_\Omega u \cdot \nabla\varphi^0 \, dx. \tag{54}$$

Here, $u \cdot N|_{\partial\Omega}$ can only be defined locally as an element of $W^{-\frac{1}{q},q}(\Sigma)$ with $\Sigma \subset \Omega$ bounded. The flux integral $\int_S u \cdot N \, d\sigma$ thus has to be understood in the sense of evaluation of the functional $u \cdot N|_S$ at $1 \in W^{\frac{1}{q},q'}(S)$, see [11] for details. Identity (54) motivates the definition of the *generalized flux*

$$\hat{\phi}(u) := -\int\limits_\Omega u \cdot \nabla\varphi^0 \, dx, \qquad u \in L^q(\Omega). \tag{55}$$

We have the following result on the strong solutions, which is mainly due to Farwig and Sohr [4, 11].

Theorem 3.4.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an aperture domain and let $v \in L^q(\Omega)$, $\gamma \in W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega)$, $1 < q < \frac{n}{2}$. Furthermore let r, ρ be defined via $\frac{1}{n} + \frac{1}{r} = \frac{1}{q}$ and $\frac{2}{n} + \frac{1}{\rho} = \frac{1}{q}$, respectively.*

(i) For every $\alpha \in \mathbb{C}$ there is a unique solution $(w, \psi) \in L^q_{\text{loc}}(\overline{\Omega}) \times \dot{W}^{1,q}(\Omega)$ with $\|\nabla^2 w\|_q + \|\nabla w\|_r + \|w\|_\rho < \infty$ of (23) and $\hat{\phi}(w) = \alpha$. Moreover,

$$\|w\|_\rho + \|\nabla w\|_r + \|\nabla^2 w\|_q + \|\nabla\psi\|_q \leq c(\|v\|_q + \|\nabla\gamma\|_q + |\alpha|) \tag{56}$$

for some $c = c(n, q, \Omega)$, where the term $|\langle\gamma, \varphi^0\rangle|$ must be added to the right-hand side of (56) if $1 < q \leq \frac{n}{n-1}$. Moreover $[\psi]$ is a linear functional of v, γ and α.

(ii) For every $\beta \in \mathbb{C}$ there is a unique strong solution (w, ψ) of (23) with $[\psi] = \beta$. Moreover

$$\|w\|_\rho + \|\nabla w\|_r + \|\nabla^2 w\|_q + \|\nabla \psi\|_q \leq c(\|v\|_q + \|\nabla \gamma\|_q + \|\gamma\|_{\hat{W}^{-1,q}(\Omega)} + |\beta|), \tag{57}$$

where $c = c(n, q, \Omega) > 0$, and $\hat{\phi}(w)$ is a linear functional of v, γ and β.

Remark 3.4.3. *(i) Since $w \in L^\rho(\Omega)$, $\nabla w \in L^r(\Omega)$, we get by Lemma 3.4.1 $\|w\|_\rho \leq c\|\nabla^2 w\|_q$ and thus $w \in \hat{Y}^{2,q}(\Omega)$.*

(ii) The estimate (57) may be improved. As shown by Franzke [14], the term $\|\gamma\|_{\hat{W}^{-1,q}(\Omega)}$ on the right-hand side is not needed.

(iii) Denote by w_0 the solution corresponding to the data $v = 0$, $\gamma = 0$ and $\alpha = 1$. Then $w_0 \in L^q(\Omega)$ for all $\frac{n}{n-1} < q < \infty$, see [4, Lemma 3.3.]. This lower bound is sharp: Assume $w_0 \in L^q(\Omega)$ for some $1 < q \leq \frac{n}{n-1}$ and choose $\varphi_k \in C_0^\infty(\overline{\Omega})$ with $\|\nabla \varphi_k\|_{q'} \to \|\nabla \varphi^0\|_{q'}$ as $k \to \infty$, which is possible in virtue of Lemma 3.4.1 (ii). Then we get the contradiction

$$0 = \langle \gamma, \varphi^0 \rangle = \lim_{k\to\infty} \langle \gamma, \varphi_k \rangle = \lim_{k\to\infty} -\int_\Omega w_0 \cdot \nabla \varphi_k \, dx = \hat{\phi}(w_0) = 1.$$

Proof of Theorem 3.4.2. If one neglects the statement about the regularity of $w \in L^\rho(\Omega)$ itself, the interval of admissible exponents can be even extended to $1 < q < n$. In this formulation, the theorem has been proven in [11, Corollary 2.4] for the case $\frac{n}{n-1} < q < n$ and in [4, Theorem 1.4] for the case $1 < q \leq \frac{n}{n-1}$. The proofs rely on the unique solvability of the corresponding resolvent problem [4, Theorem 1.2] via an approximation procedure of the resolvent parameter $\lambda \to 0+$. However, if we restrict ourselves to $1 < q < \frac{n}{2}$, we get by Sobolev's embedding theorem $w_\lambda \in L^\rho(\Omega)$ for each $\lambda > 0$, where w_λ is the corresponding solution to the resolvent problem with parameter λ. This regularity then carries over to the solution w, being the weak limit of a subsequence of the w_λ. □

Theorem 3.4.4. *Let $\Omega \subset \mathbb{R}^n_+$ be an aperture domain, $q > \frac{n}{n-2}$ and $\mathcal{F} \in \hat{Y}_\sigma^{-2,q}(\Omega)$, $\mathcal{K} \in \hat{W}^{-1,q}(\Omega)$, $\alpha \in \mathbb{C}$. Then the problem (28) has a unique very weak solution $u \in L^q(\Omega)$ satisfying $\hat{\phi}(u) = \alpha$. This solution satisfies the estimate*

$$\|u\|_q \leq c(\|\mathcal{F}\|_{\hat{Y}_\sigma^{-2,q}(\Omega)} + \|\mathcal{K}\|_{\hat{W}^{-1,q}(\Omega)} + |\alpha|), \tag{58}$$

where $c = c(n, q) > 0$ is a constant.

Proof. The arguments follow the proof of Theorem 3.1.4. Define $u \in L^q(\Omega)$ via

$$(u, v) = \langle \mathcal{F}, w \rangle + \langle \mathcal{K}, \psi_0 \rangle + \alpha[\psi] \text{ for } v \in L^{q'}(\Omega),$$

where $w \in \hat{Y}_\sigma^{2,q'}(\Omega)$, $\psi \in \dot{W}^{1,q'}(\Omega)$ is the unique solution to (32) with $\hat{\phi}(w) = 0$, and $\psi_0 = \psi - [\psi]\varphi^0$. Then Lemma 3.4.1 and (56) yield the estimate

$$\begin{aligned}|(u, v)| &\leq \|\mathcal{F}\|_{\hat{Y}_\sigma^{-2,q}(\Omega)} \left\|\nabla^2 w\right\|_{q'} + \|\mathcal{K}\|_{\hat{W}^{-1,q}(\Omega)} \|\nabla \psi_0\|_{q'} + |\alpha| \|\nabla \psi\|_{q'} \\ &\leq c(\|\mathcal{F}\|_{\hat{Y}_\sigma^{-2,q}(\Omega)} + \|\mathcal{K}\|_{\hat{W}^{-1,q}(\Omega)} + |\alpha|) \, \|v\|_{q'},\end{aligned}$$

for some constant $c = c(\Omega, q) > 0$. Since $\nabla \varphi^0 \in L^{q'}(\Omega)$ decomposes in the above sense with $w = 0$, $\psi_0 = 0$ and $\psi = -\varphi^0$, the flux condition $\hat{\phi}(u) = -(u, \nabla \varphi^0) = \alpha$ is automatically fulfilled. Furthermore, u actually solves the problem (28), since for test functions $w \in \hat{Y}_\sigma^{2,q'}(\Omega)$, $\psi \in \hat{W}^{1,q'}(\Omega)$, we have $[\psi] = 0$ and thus

$$-(u, \Delta w) - (u, \nabla \psi) = \langle \mathcal{F}, w \rangle + \langle \mathcal{K}, \psi \rangle.$$

Uniqueness may be seen by considering a very weak solution $u \in L^q(\Omega)$ to the data $\mathcal{F} = 0$, $\mathcal{K} = 0$ and $\alpha = 0$. Then $(u, v) = -(u, \Delta w) - (u, \nabla \psi_0) - (u, [\psi]\nabla \varphi^0) = \langle \mathcal{F}, w \rangle + \langle \mathcal{K}, \psi_0 \rangle + \alpha[\psi] = 0$ for all $v \in L^{q'}(\Omega)$ and thus $u = 0$. The proof is complete. □

Now, we may prove Theorem 1.3.

Proof of Theorem 1.3. It still remains to prove the assertion about the pressure. Consider test functions $w \in C_{0,\sigma}^{\infty}(\Omega)$. In the sense of distributions we thus have $\langle \operatorname{div} F + \Delta u, w \rangle = 0$. Then de Rham's argument [23, Chapter I, Proposition 1.1], yields a distribution $p \in C_0^{\infty}(\Omega)'$ with $\operatorname{div} F + \Delta u = \nabla p$. □

Remark 3.4.5. *(i) Note that a similar construction as in the proof of Theorem 1.2 for the functional p fails here, as we cannot derive from Theorem 3.4.2, that each $\gamma \in \hat{W}^{1,q'}(\Omega)$ can be written in the form*

$$\gamma = \operatorname{div} w, \qquad w \in \hat{Y}^{2,q'}(\Omega) \text{ with } \hat{\phi}(w) = 0. \tag{59}$$

The no-flux condition is crucial here, as otherwise the pressure p will not be well-defined. However, this problem may be resolved for $n \geq 4$, $\frac{n}{n-1} < q' < \frac{n}{2}$. In that case, estimate (56) may be used without the term $|\langle \gamma, \varphi^0 \rangle|$ on the right-hand side, and thus (59) is ensured due to Lemma 2.1.1. So for $\frac{n}{n-2} < q < n$, it still holds true that $p \in \hat{W}^{-1,q}(\Omega)$.

(ii) Defining a concept similar to the pressure drop in the situation of strong solutions seems out of reach, as p itself is not contained in any function space anymore. Thus our setting is too coarse to reflect local phenomena.

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