# Homogenization for dislocation based gradient visco-plasticity

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**Abstract**

In this work we study the homogenization for infinitesimal dislocation based gradient viscoplasticity with linear kinematic hardening and general non-associative monotone plastic flows. The constitutive equations in the models we study are assumed to be only of monotone type. Based on the generalized version of Korn's inequality for incompatible tensor fields (the non-symmetric plastic distortion) due to Neff et al., we derive uniform estimates for the solutions of quasistatic initial-boundary value problems under consideration and then using an unfolding operator technique and a monotone operator method we obtain the homogenized system of equations.

**Key words:** plasticity, gradient plasticity, viscoplasticity, dislocations, plastic spin, homogenization, periodic unfolding, Korn's inequality, Rothe's time-discretization method, rate-dependent models.

**AMS 2000 subject classification:** 35B65, 35D10, 74C10, 74D10, 35J25, 34G20, 34G25, 47H04, 47H05

## 1 Introduction

We study the homogenization of quasistatic initial-boundary value problems arising in gradient viscoplasticity. The models we study use rate-dependent constitutive equations with internal variables to describe the deformation behaviour of metals at infinitesimally small strain.

Our focus is on a phenomenological model on the macroscale not including the case of single crystal plasticity. From a mathematical point of view, the maze of equations, slip systems and physical mechanisms in single crystal plasticity is only obscuring the mathematical structure of the problem.

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Our model has been first presented in [37]. It is inspired by the early work of Menzel and Steinmann [33]. Contrary to more classical strain gradient approaches, the model features from the outset a non-symmetric plastic distortion field $p \in \mathcal{M}^3$ [9], a dislocation based energy storage based solely on $|\operatorname{Curl} p|$ (and not $\nabla p$) and therefore second gradients of the plastic distortion in the form of $\operatorname{Curl}\operatorname{Curl} p$ acting as dislocation based kinematical backstresses. We only consider energetic length scale effects and not higher gradients in the dissipation.

Uniqueness of classical solutions in the subdifferential case (associated plasticity) for rate-independent and rate-dependent formulations is shown in [36]. The existence question for the rate-independent model in terms of a weak reformulation is addressed in [37]. The rate-independent model with isotropic hardening is treated in [19]. First numerical results for a simplified rate-independent irrotational formulation (no plastic spin, symmetric plastic distortion $p$) are presented in [41]. In [22, 50] well-posedness for a rate-independent model of Gurtin and Anand [24] is shown under the decisive assumption that the plastic distortion is symmetric (the irrotational case), in which case one may really speak of a strain gradient plasticity model, since the full gradient acts on the symmetric plastic strain.

In order to appreciate the simplicity and elegance of our model we sketch some of its ingredients. First, as is usual in infinitesimal plasticity theory, we split the total displacement gradient into non-symmetric elastic and non-symmetric plastic distortions

$$\nabla u = e + p\,.$$

For invariance reasons, the elastic energy contribution may only depend on the elastic strains $\operatorname{sym} e = \operatorname{sym}(\nabla u - p)$. While $p$ is non-symmetric, a distinguishing feature of our model is that, similar to classical approaches, only the symmetric part $\varepsilon_p := \operatorname{sym} p$ of the plastic distortion appears in the local Cauchy stress $\sigma$, while only the higher order stresses are non-symmetric. The reason for this is that we assume that $p$ has to obey the same transformation behavior as $\nabla u$ does, and thus the energy storage due to kinematical hardening should depend only on the plastic strains $\operatorname{sym} p$. For more on the basic invariance questions related to this issue dictating this type of behaviour, see [54, 35]. We assume as well plastic incompressibility $\operatorname{tr} p = 0$, as is usual.

The thermodynamic potential of our model can therefore be written as

$$\int_\Omega \Big( \underbrace{\mathbb{C}[x](\operatorname{sym}(\nabla u - p))(\operatorname{sym}(\nabla u - p))}_{\text{elastic energy}} \tag{1}$$

$$+ \underbrace{\frac{C_1[x]}{2}\,|\operatorname{dev}\operatorname{sym} p|^2}_{\text{kinematical hardening}} + \underbrace{\frac{C_2}{2}|\operatorname{Curl} p|^2}_{\text{dislocation storage}} + \underbrace{u \cdot b}_{\text{external volume forces}} \Big)\, dx$$

The positive definite elasticity tensor $\mathbb{C}$ is able to represent the elastic anisotropy of the material. The evolution equations for the plastic distortion $p$ are taken such that the stored energy is non-increasing along trajectories of $p$ at frozen displacement $u$, see [37]. This ensures the validity of the second law of thermodynamics in the form of the reduced dissipation inequality.

For the reduced dissipation inequality we fix $u$ in time and consider the time derivative of the free energy (and taking into account that Curl is a self-adjoint operator provided that the appropriate boundary conditions are specified), we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega W(\nabla u(t_0)-p(t),p(t),\operatorname{Curl}p(t))\,dx\\
&\quad=\int_\Omega D_1W\cdot(-\partial_t p)+D_2W\cdot\partial_t p+D_3W\cdot\operatorname{Curl}\partial_t p\,dx\\
&\quad=-\int_\Omega (D_1W-D_2W-\operatorname{Curl}D_3W)\cdot\partial_t p\,dx\,.
\end{aligned}$$

Choosing $\partial_t p\in g(D_1W-D_2W-\operatorname{Curl}D_3W)$ with a monotone function $g$ we obtain the reduced dissipation inequality

$$\frac{d}{dt}\int_\Omega W(\nabla u(t_0)-p(t),p(t),\operatorname{Curl}p(t))\,dx\le 0\,.$$

Adapted to our situation, the plastic flow has the form

$$\partial_t p\in g(\sigma-C_1[x]\operatorname{dev}\operatorname{sym}p-C_2\operatorname{Curl}\operatorname{Curl}p)\,, \tag{2}$$

where $\sigma=\mathbb{C}[x]\operatorname{sym}(\nabla u-p)$ is the elastic symmetric Cauchy stress of the material and $g$ is a multivalued monotone flow function which is not necessary the subdifferential of a convex plastic potential (associative plasticity). In this generality, our formulation comprises certain non-associative plastic flows in which the yield condition and the flow direction are independent and governed by distinct functions. Moreover, the flow function $g$ is supposed to induce a rate-dependent response as all materials are, in reality, rate-dependent.

Clearly, in the absence of energetic length scale effects ($C_2=0$), the $\operatorname{Curl}\operatorname{Curl}p$-term is absent. In general we assume that $g$ maps symmetric tensors to symmetric tensors. Thus, for $C_2=0$ the plastic distortion remains always symmetric and the model reduces to a classical plasticity model. Therefore, the energetic length scale is solely responsible for the plastic spin in the model. The appearance of the $\operatorname{Curl}\operatorname{Curl}p$-term in the argument of $g$ is clear: the argument of $g$ consists of the Eshelby-stress tensor $\Sigma$ driving the plastic evolution, see [37].

Regarding the boundary conditions necessary for the formulation of the higher order theory we assume that the boundary is a perfect conductor, this means that the tangential component of $p$ vanishes on $\partial\Omega$. In the context of dislocation dynamics these conditions express the requirement that there is no flux of the Burgers vector across a hard boundary. Gurtin [25] introduces the following different types of boundary conditions for the plastic distortion[1]

$$\begin{aligned}
\partial_t p\times n|_{\Gamma_{\text{hard}}}&=0 \quad \text{"micro-hard" (perfect conductor)}\\
\partial_t p|_{\Gamma_{\text{hard}}}&=0 \quad \text{"hard-slip"}\\
\operatorname{Curl}p\times n|_{\Gamma_{\text{hard}}}&=0 \quad \text{"micro-free"}\,.
\end{aligned} \tag{3}$$

We specify a sufficient condition for the micro-hard boundary condition, namely

$$p\times n|_{\Gamma_{\text{hard}}}=0$$

and assume for simplicity $\Gamma_{\text{hard}}=\partial\Omega$. This is the correct boundary condition for tensor fields in $H(\operatorname{Curl})$ which admits tangential traces.

[1] Here, $v\times n$ with $v\in\mathcal{M}^3$ and $n\in\mathbb{R}^3$ denotes a row by column operation.

We combine this with a new inequality extending Korn's inequality to incompatible tensor fields, namely

$$\exists\, C = C(\Omega) > 0\ \forall p \in H(\mathrm{Curl}): \quad p \times n|_{\Gamma_{\mathrm{hard}}} = 0: \tag{4}$$
$$\underbrace{\|p\|_{L^2(\Omega)}}_{\text{plastic distortion}} \le C(\Omega)\left(\underbrace{\|\operatorname{sym} p\|_{L^2(\Omega)}}_{\text{plastic strain}} + \underbrace{\|\operatorname{Curl} p\|_{L^2(\Omega)}}_{\text{dislocation density}}\right).$$

Here, $\Gamma_{\mathrm{hard}} \subset \partial\Omega$ with full two-dimensional surface measure and the domain $\Omega$ needs to be **sliceable**, i.e. cuttable into finitely many simply connected subdomains with Lipschitz boundaries. This inequality has been derived in [38, 39, 40] and is precisely motivated by the well-posedness question for our model [37]. The inequality (4) expresses the fact that controlling the plastic strain $\operatorname{sym} p$ and the dislocation density $\operatorname{Curl} p$ in $L^2(\Omega)$ gives a control of the plastic distortion $p$ in $L^2(\Omega)$ provided the correct boundary conditions are specified: namely the micro-hard boundary condition. Since we assume that $\operatorname{tr}(p) = 0$ (plastic incompressibility) the quadratic terms in the thermodynamic potential provide a control of the right hand side in (4).

It is worthy to note that with $g$ only monotone and not necessarily a subdifferential the powerful energetic solution concept [32, 22, 31] cannot be applied. In our model we face the combined challenge of a gradient plasticity model based on the dislocation density tensor $\operatorname{Curl} p$ involving the plastic spin, a general non-associative monotone flow-rule and a rate-dependent response.

**Setting of the homogenization problem.** Let $\Omega \subset \mathbb{R}^3$ be an open bounded set, the set of material points of the solid body, with a $C^1$-boundary and $Y \subset \mathbb{R}^3$ be a set having the paving property with respect to a basis $(b_1, b_2, b_3)$ defining the periods, a reference cell. By $T_e$ we denote a positive number (time of existence), which can be chosen arbitrarily large, and for $0 < t \le T_e$

$$\Omega_t = \Omega \times (0, t).$$

The sets, $\mathcal{M}^3$ and $\mathcal{S}^3$ denote the sets of all $3\times 3$–matrices and of all symmetric $3\times 3$–matrices, respectively. Let $\mathfrak{sl}(3)$ be the set of all traceless $3\times 3$–matrices, i.e.

$$\mathfrak{sl}(3) = \{v \in \mathcal{M}^3 \mid \operatorname{tr} v = 0\}.$$

Unknown in our small strain formulation are the displacement $u_\eta(x,t) \in \mathbb{R}^3$ of the material point $x$ at time $t$ and the non-symmetric infinitesimal plastic distortion $p_\eta(x,t) \in \mathfrak{sl}(3)$.

The model equations of the problem are

$$-\operatorname{div}_x \sigma_\eta(x,t) = b(x,t), \tag{5}$$
$$\sigma_\eta(x,t) = \mathbb{C}[x/\eta](\operatorname{sym}(\nabla_x u_\eta(x,t) - p_\eta(x,t))), \tag{6}$$
$$\partial_t p_\eta(x,t) \in g\big(x/\eta, \Sigma_\eta^{\mathrm{lin}}(x,t)\big), \quad \Sigma_\eta^{\mathrm{lin}} = \Sigma_{e,\eta}^{\mathrm{lin}} + \Sigma_{\mathrm{sh},\eta}^{\mathrm{lin}} + \Sigma_{\mathrm{curl},\eta}^{\mathrm{lin}}, \tag{7}$$
$$\Sigma_{\mathrm{e},\eta}^{\mathrm{lin}} = \sigma_\eta,\ \Sigma_{\mathrm{sh},\eta}^{\mathrm{lin}} = -C_1[x/\eta] \operatorname{dev}\operatorname{sym} p_\eta,\ \Sigma_{\mathrm{curl},\eta}^{\mathrm{lin}} = -C_2 \operatorname{Curl}\operatorname{Curl} p_\eta\,,$$

which must be satisfied in $\Omega \times [0, T_e)$. Here, $C_2 \ge 0$ is a given material constant independent of $\eta$ and $\Sigma_\eta^{\mathrm{lin}}$ is the infinitesimal Eshelby stress tensor driving the evolution of the plastic distortion $p_\eta$ and $\eta$ is a scaling parameter of the microstructure. The initial condition and Dirichlet boundary condition are

$$\begin{aligned} p_\eta(x,0) &= p^{(0)}(x), \quad x \in \Omega, & (8)\\ p_\eta(x,t) \times n(x) &= 0, \qquad (x,t) \in \partial\Omega \times [0,T_e), & (9)\\ u_\eta(x,t) &= 0, \qquad (x,t) \in \partial\Omega \times [0,T_e)\,, & (10)\end{aligned}$$

where $n$ is a normal vector on the boundary $\partial\Omega$[2]. For simplicity we consider only homogeneous boundary condition and we assume that the cell of periodicity is given by $Y = [0,1)^3$. Then, we assume that $C_1 : Y \to \mathbb{R}$, a given material function, is measurable, periodic with the periodicity cell $Y$ and satisfies the inequality

$$C_1[y] \geq \alpha_1 > 0$$

for all $y \in Y$ and some positive constant $\alpha_1$. For every $y \in Y$ the elasticity tensor $\mathbb{C}[y] : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, positive definite mapping and the mapping $y \mapsto \mathbb{C}[y] : \mathbb{R}^3 \to \mathcal{S}^3$ is measurable and periodic with the same periodicity cell $Y$. Due to the above assumption ($C_1 > 0$), the classical linear kinematic hardening is included in the model. Here, the nonlocal backstress contribution is given by the dislocation density motivated term $\Sigma^{\text{lin}}_{\text{curl},\eta} = -C_2 \operatorname{Curl}\operatorname{Curl} p_\eta$ together with corresponding Neumann conditions.

For the model we require that the nonlinear constitutive mapping $v \mapsto g(y,v) : \mathcal{M}^3 \to 2^{\mathfrak{sl}(3)}$ is monotone for all $y \in Y$, i.e. it satisfies

$$0 \leq (v_1 - v_2) \cdot (v_1^* - v_2^*), \tag{11}$$

for all $v_i \in \mathcal{M}^3,\ v_i^* \in g(y,v_i),\ i = 1,2$ and all $y \in Y$. We also require that

$$0 \in g(y,0), \qquad \text{a.e. } y \in Y. \tag{12}$$

The mapping $y \mapsto g(y,\cdot) : \mathbb{R}^3 \to 2^{\mathfrak{sl}(3)}$ is periodic with the same periodicity cell $Y$. Given are the volume force $b(x,t) \in \mathbb{R}^3$ and the initial datum $p^{(0)}(x) \in \mathfrak{sl}(3)$.

*Remark* 1.1. *It is well known that classical viscoplasticity (without gradient effects) gives rise to a well-posed problem. We extend this result to our formulation of rate-dependent gradient plasticity. The presence of the classical linear kinematic hardening in our model is related to $C_1 > 0$ whereas the presence of the nonlocal gradient term is always related to $C_2 > 0$.*

The development of the homogenization theory for the quasi-static initial boundary value problem of monotone type in the classical elasto/visco-plasticity introduced by Alber in [2] has started with the work [3], where the homogenized system of equations has been derived using the formal asymptotic ansatz. In the following work [4] Alber justified the formal asymptotic ansatz for the case of positive definite free energy[3], employing the energy method of Murat-Tartar, yet only for local smooth solutions of the homogenized problem. It is shown there that the solutions of elasto/visco-plasticity problems can be approximated in the $L^2(\Omega)-$norm by the smooth functions constructed from the solutions of the homogenized problem. Later in [42], under the assumption that the free energy is positive definite, it is proved that the difference of the solutions of

[2]Here, $v \times n$ with $v \in \mathcal{M}^3$ and $n \in \mathbb{R}^3$ denotes a row by column operation.

[3]Positive definite energy corresponds to linear kinematic hardening behavior of materials.

the microscopic problem and the solutions constructed from the homogenized problem, which both need not be smooth, tends to zero in the $L^2(\Omega\times Y)-$norm, where $Y$ is the periodicity cell. Based on the results obtained in [42], in [5] the convergence in $L^2(\Omega\times Y)$ is replaced by convergence in $L^2(\Omega)$. In the meantime, for the rate-independent problems in plasticity similar results are obtained in [34] using the unfolding operator method (see Section 3) and methods of energetic solutions due to Mielke. For special rate-dependent models of monotone type, namely for rate-dependent generalized standard materials, the two-scale convergence of the solutions of the microscopic problem to the solutions of the homogenized problem has been shown in [56, 57]. The homogenization of the Prandtl-Reuss model is performed in [52, 57]. In [43] the author considered the rate-dependent problems of monotone type with constitutive functions $g$, which need not be subdifferentials, but which belong to the class of functions $\mathbb{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$ introduced in Section 5. Using the unfolding operator method and in particular the homogenization methods developed in [16], for this class of functions the homogenized equations for the viscoplactic problems of monotone type are obtained in [43].

In the present work the construction of the homogenization theory for the initial boundary value problem (5) - (10) is based on the existence result derived in [45] (see Theorem 5.6) and on the homogenization techniques developed in [43] for classical viscoplasticity of monotone type. The existence result in [45] extends the well-posedness for infinitesimal dislocation based gradient viscoplasticity with linear kinematic hardening from the subdifferential case (see [44]) to general non-associative monotone plastic flows for sliceable domains. In this work we also assume that the domain $\Omega$ is sliceable and that the monotone function $g : \mathbb{R}^3 \times \mathcal{M}^3 \to 2^{\mathfrak{sl}(3)}$ belongs to the class $\mathbb{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$. For sliceable domains $\Omega$, based on the inequality (4), we are able to derive then uniform estimates for the solutions of (5) - (10) in Lemma 5.8. Using the uniform estimates for the solutions of (5) - (10), the unfolding operator method and the homogenization techniques developed in [16, 43], for the class of functions $\mathbb{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$ we obtain easily the homogenized equations for the original problem under consideration (see Theorem 5.7). To the best our knowledge this is the first homogenization result obtained for the problem (5) - (10). We note that similar homogenization results for the strain-gradient model of Fleck and Willis [20] are derived in [21, 23, 27] using the unfolding method together with the $\Gamma$-convergence method in the rate-independent setting. In [21] the authors, based on the assumption that the model under consideration is of rate-independent type, are able to treat the case when $C_2$ is a $Y$-periodic function as well. In the rate-independent setting this is possible due to the fact that the whole system (5) - (10) can be rewritten as a standard variational inequality (see [26]) and then the subsequant usage of the techniques of the convex analysis enable the passage to the limit in the model equations. Contrary to this, in the rate-independent case this reduction to a single variational inequality is not possible and one is forced to use the monotonicity argument to study the asymptotic behavior of the third term $\Sigma^{\text{lin}}_{\text{curl},\eta}$ in (7).

**Notation.** Suppose that $\Omega$ is a bounded domain with $C^\infty$-boundary $\partial\Omega$. Throughout the whole work we choose the numbers $q, q^*$ satisfying the following conditions

$$1 < q, q^* < \infty \ \text{ and } \ 1/q + 1/q^* = 1,$$

and $|\cdot|$ denotes a norm in $\mathbb{R}^k$. Moreover, the following notations are used in this work. The space $W^{m,q}(\Omega, \mathbb{R}^k)$ with $q \in [1, \infty]$ consists of all functions in $L^q(\Omega, \mathbb{R}^k)$ with weak derivatives in $L^q(\Omega, \mathbb{R}^k)$ up to order $m$. If $m$ is not integer, then $W^{m,q}(\Omega, \mathbb{R}^k)$ denotes the corresponding Sobolev-Slobodecki space. We set $H^m(\Omega, \mathbb{R}^k) = W^{m,2}(\Omega, \mathbb{R}^k)$. The norm in $W^{m,q}(\Omega, \mathbb{R}^k)$ is denoted by $\|\cdot\|_{m,q,\Omega}$ ($\|\cdot\|_q := \|\cdot\|_{0,q,\Omega}$). The operator $\Gamma_0$ defined by

$$\Gamma_0 : v \in W^{1,q}(\Omega, \mathbb{R}^k) \mapsto W^{1-1/q,q}(\partial\Omega, \mathbb{R}^k)$$

denotes the usual trace operator. The space $W_0^{m,q}(\Omega, \mathbb{R}^k)$ with $q \in [1, \infty]$ consists of all functions $v$ in $W^{m,q}(\Omega, \mathbb{R}^k)$ with $\Gamma_0 v = 0$. One can define the bilinear form on the product space $L^q(\Omega, \mathcal{M}^3) \times L^{q^*}(\Omega, \mathcal{M}^3)$ by

$$(\xi, \zeta)_\Omega = \int_\Omega \xi(x) \cdot \zeta(x) dx.$$

The space

$$L^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3) = \{v \in L^q(\Omega, \mathcal{M}^3) \mid \operatorname{Curl} v \in L^q(\Omega, \mathcal{M}^3)\}$$

is a Banach space with respect to the norm

$$\|v\|_{q,\mathrm{Curl}} = \|v\|_q + \|\operatorname{Curl} v\|_q.$$

By $H(\mathrm{Curl})$ we denote the space of measurable functions in $L^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$, i.e. $H(\mathrm{Curl}) = L^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$. The well known result on the generalized trace operator can be easily adopted to the functions with values in $\mathcal{M}^3$ (see [53, Section II.1.2]). Then, according to this result, there is a bounded operator $\Gamma_n$ on $L^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$

$$\Gamma_n : v \in L^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3) \mapsto \left(W^{1-1/q^*,q^*}(\partial\Omega, \mathcal{M}^3)\right)^*$$

with

$$\Gamma_n v = v \times n\big|_{\partial\Omega} \text{ if } v \in C^1(\bar{\Omega}, \mathcal{M}^3),$$

where $X^*$ denotes the dual of a Banach space $X$. Next,

$$L^q_{\mathrm{Curl},0}(\Omega, \mathcal{M}^3) = \{w \in L^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3) \mid \Gamma_n(w) = 0\}.$$

Let us define spaces $V^q(\Omega, \mathcal{M}^3)$ and $X^q(\Omega, \mathcal{M}^3)$ by

$$V^q(\Omega, \mathcal{M}^3) = \{v \in L^q(\Omega, \mathcal{M}^3) \mid \operatorname{div} v, \operatorname{Curl} v \in L^q(\Omega, \mathcal{M}^3), \Gamma_n v = 0\},$$

$$X^q(\Omega, \mathcal{M}^3) = \{v \in L^q(\Omega, \mathcal{M}^3) \mid \operatorname{div} v, \operatorname{Curl} v \in L^q(\Omega, \mathcal{M}^3), \Gamma_0 v = 0\},$$

which are Banach spaces with respect to the norm

$$\|v\|_{V^q} (\|v\|_{X^q}) = \|v\|_q + \|\operatorname{Curl} v\|_q + \|\operatorname{div} v\|_q.$$

According to [30, Theorem 2][4] the spaces $V^q(\Omega, \mathcal{M}^3)$ and $X^q(\Omega, \mathcal{M}^3)$ are continuously imbedded into $W^{1,q}(\Omega, \mathcal{M}^3)$. We define $V^q_\sigma(\Omega, \mathcal{M}^3)$ and $X^q_\sigma(\Omega, \mathcal{M}^3)$ by

$$V^q_\sigma(\Omega, \mathbb{R}^3) := \{v \in V^q(\Omega, \mathbb{R}^3) \mid \operatorname{div} v = 0\},$$

[4] This theorem has to be applied to each row of a function with values in $\mathcal{M}^3$ to obtain the desired result.

$$X^q_\sigma(\Omega,\mathbb{R}^3) := \{v \in X^q(\Omega,\mathbb{R}^3) \mid \operatorname{div} v = 0\},$$

and denote by $V^q_{har}(\Omega,\mathbb{R}^3)$ and $X^q_{har}(\Omega,\mathbb{R}^3)$ the $L^q$-spaces of harmonic functions on $\Omega$ as

$$V^q_{har}(\Omega,\mathbb{R}^3) := \{v \in V^q_\sigma(\Omega,\mathbb{R}^3) \mid \operatorname{Curl} v = 0\},$$

$$X^q_{har}(\Omega,\mathbb{R}^3) := \{v \in X^q_\sigma(\Omega,\mathbb{R}^3) \mid \operatorname{Curl} v = 0\},$$

Then the spaces $V^q_{har}(\Omega,\mathbb{R}^3)$ and $X^q_{har}(\Omega,\mathbb{R}^3)$ for every fixed $q$, $1 < q < \infty$, coincides with the spaces $V_{har}(\Omega,\mathbb{R}^3)$ and $X_{har}(\Omega,\mathbb{R}^3)$ given by

$$V_{har}(\Omega,\mathbb{R}^3) = \{v \in C^\infty(\bar{\Omega},\mathbb{R}^3) \mid \operatorname{div} v = 0, \operatorname{Curl} v = 0 \text{ with } v \cdot n = 0 \text{ on } \partial\Omega\},$$

$$X_{har}(\Omega,\mathbb{R}^3) = \{v \in C^\infty(\bar{\Omega},\mathbb{R}^3) \mid \operatorname{div} v = 0, \operatorname{Curl} v = 0 \text{ with } v \times n = 0 \text{ on } \partial\Omega\},$$

respectively (see [30, Theorem 2.1(1)]). The spaces $V_{har}(\Omega,\mathbb{R}^3)$ and $X_{har}(\Omega,\mathbb{R}^3)$ are finite dimensional vector spaces ([30, Theorem 1]).

We also define the space $Z^q_{\text{Curl}}(\Omega,\mathcal{M}^3)$ by

$$Z^q_{\text{Curl}}(\Omega,\mathcal{M}^3) = \{v \in L^q_{\text{Curl},0}(\Omega,\mathcal{M}^3) \mid \operatorname{Curl}\operatorname{Curl} v \in L^q(\Omega,\mathcal{M}^3)\},$$

which is a Banach space with respect to the norm

$$\|v\|_{Z^q_{\text{Curl}}} = \|v\|_{q,\text{Curl}} + \|\operatorname{Curl}\operatorname{Curl} v\|_q.$$

The space $W^{m,q}_{per}(Y,\mathbb{R}^k)$ denotes the Banach space of $Y$-periodic functions in $W^{m,q}_{loc}(\mathbb{R}^k,\mathbb{R}^k)$ equipped with the $W^{m,q}(Y,\mathbb{R}^k)$-norm.

For functions $v$ defined on $\Omega \times [0,\infty)$ we denote by $v(t)$ the mapping $x \mapsto v(x,t)$, which is defined on $\Omega$. The space $L^q(0,T_e;X)$ denotes the Banach space of all Bochner-measurable functions $u : [0,T_e) \to X$ such that $t \mapsto \|u(t)\|^q_X$ is integrable on $[0,T_e)$. Finally, we frequently use the spaces $W^{m,q}(0,T_e;X)$, which consist of Bochner measurable functions having $q$-integrable weak derivatives up to order $m$.

# 2 Maximal monotone operators

In this section we recall some basics about monotone and maximal monotone operators. For more details see [8, 28, 48], for example.

Let $V$ be a reflexive Banach space with the norm $\|\cdot\|$, $V^*$ be its dual space with the norm $\|\cdot\|_*$. The brackets $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $V$ and $V^*$. Under $V$ we shall always mean a reflexive Banach space throughout this section. For a multivalued mapping $A : V \to 2^{V^*}$ the sets

$$D(A) = \{v \in V \mid Av \neq \emptyset\}$$

and

$$GrA = \{[v,v^*] \in V \times V^* \mid v \in D(A),\ v^* \in Av\}$$

are called the *effective domain* and the *graph* of $A$, respectively.

**Definition 2.1.** *A mapping $A : V \to 2^{V^*}$ is called* monotone *if and only if the inequality holds*

$$\langle v^* - u^*, v - u\rangle \geq 0 \qquad \forall\ [v,v^*],[u,u^*] \in GrA.$$

*A monotone mapping* $A : V \to 2^{V^*}$ *is called* maximal monotone *iff the inequality*

$$\langle v^* - u^*, v - u\rangle \geq 0 \quad \forall\ [u, u^*] \in GrA$$

*implies* $[v, v^*] \in GrA$.

*A mapping* $A : V \to 2^{V^*}$ *is called* generalized pseudomonotone *iff the set* $Av$ *is closed, convex and bounded for all* $v \in D(A)$ *and for every pair of sequences* $\{v_n\}$ *and* $\{v_n^*\}$ *such that* $v_n^* \in Av_n$, $v_n \rightharpoonup v_0$, $v_n^* \rightharpoonup v_0^* \in V^*$ *and*

$$\limsup_{n\to\infty} \langle v_n^*, v_n - v_0\rangle \leq 0,$$

*we have that* $[v_0, v_0^*] \in GrA$ *and* $\langle v_n^*, v_n\rangle \to \langle v_0^*, v_0\rangle$.

*A mapping* $A : V \to 2^{V^*}$ *is called* strongly coercive *iff either* $D(A)$ *is bounded or* $D(A)$ *is unbounded and the condition*

$$\frac{\langle v^*, v - w\rangle}{\|v\|} \to +\infty \quad \text{as } \|v\| \to \infty, \quad [v, v^*] \in GrA,$$

*is satisfied for each* $w \in D(A)$.

It is well known ([48, p. 105]) that if $A$ is a maximal monotone operator, then for any $v \in D(A)$ the image $Av$ is a closed convex subset of $V^*$ and the graph $GrA$ is demi-closed.[5] A maximal monotone operator is also generalized pseudomonotone (see [8, 28, 48]).

*Remark* 2.2. We recall that the subdifferential of a lower semi-continuous and convex function is maximal monotone (see [49, Theorem 2.25]).

**Definition 2.3.** *The* duality mapping $J : V \to 2^{V^*}$ *is defined by*

$$J(v) = \{v^* \in V^* \mid \langle v^*, v\rangle = \|v\|^2 = \|v^*\|_*^2\ \}$$

*for all* $v \in V$.

Without loss of generality (due to Asplund's theorem) we can assume that both $V$ and $V^*$ are strictly convex, i.e. that the unit ball in the corresponding space is strictly convex. In virtue of [8, Theorem II.1.2], the equation

$$J(v_\lambda - v) + \lambda A v_\lambda \ni 0$$

has a solution $v_\lambda \in D(A)$ for every $v \in V$ and $\lambda > 0$ if $A$ is maximal monotone. The solution is unique (see [8, p. 41]).

**Definition 2.4.** *Setting*

$$v_\lambda = j_\lambda^A v \quad \text{and} \quad A_\lambda v = -\lambda^{-1} J(v_\lambda - v)$$

*we define two single valued operators: the* Yosida approximation $A_\lambda : V \to V^*$ *and the* resolvent $j_\lambda^A : V \to D(A)$ *with* $D(A_\lambda) = D(j_\lambda^A) = V$.

By the definition, one immediately sees that $A_\lambda v \in A\big(j_\lambda^A v\big)$. For the main properties of the Yosida approximation we refer to [8, 28, 48] and mention only that both are continuous operators and that $A_\lambda$ is bounded and maximal monotone.

[5]A set $A \in V \times V^*$ is demi-closed if $v_n$ converges strongly to $v_0$ in $V$ and $v_n^*$ converges weakly to $v_0^*$ in $V^*$ (or $v_n$ converges weakly to $v_0$ in $V$ and $v_n^*$ converges strongly to $v_0^*$ in $V^*$) and $[v_n, v_n^*] \in GrA$, then $[v, v^*] \in GrA$

**Convergence of maximal monotone graphs** In the presentation of the next subsections we follow the work [16], where the reader can also find the proofs of the results mentioned here.

The derivation of the homogenized equations for the initial boundary value problem (5) - (10) is based on the notion of the convergence of the graphs of maximal monotone operators.

According to Brezis [10] and Attouch [7], the convergence of the graphs of maximal monotone operators is defined as follows.

**Definition 2.5.** *Let $A^n$, $A : V \to 2^{V^*}$ be maximal monotone operators. The sequence $A^n$ converges to $A$ as $n \to \infty$, ($A^n \rightarrowtail A$), if for every $[v, v^*] \in GrA$ there exists a sequence $[v_n, v_n^*] \in GrA^n$ such that $[v_n, v_n^*] \to [v, v^*]$ strongly in $V \times V^*$ as $n \to \infty$.*

Obviously, if $A^n$ and $A$ are everywhere defined, continuous and monotone, then the pointwise convergence, i.e. if for every $v \in V$, $A^n(v) \to A(v)$, implies the convergence of the graphs. The converse is true in finite-dimensional spaces.

The next theorem is the main mathematical tool in the derivation of the homogenized equations for the problem (5) - (10).

**Theorem 2.6.** *Let $A^n$, $A : V \to 2^{V^*}$ be maximal monotone operators, and let $[v_n, v_n^*] \in GrA^n$ and $[v, v^*] \in V \times V^*$. If, as $n \to \infty$, $A^n \rightarrowtail A$, $v_n \rightharpoonup v_0$, $v_n^* \rightharpoonup v_0^* \in V^*$ and*

$$\limsup_{n\to\infty} \langle v_n^*, v_n \rangle \le \langle v_0^*, v_0 \rangle \,, \tag{13}$$

*then $[v_0, v_0^*] \in GrA$ and*

$$\liminf_{n\to\infty} \langle v_n^*, v_n \rangle = \langle v_0^*, v_0 \rangle \,.$$

*Proof.* See [16, Theorem 2.8]. □

The convergence of the graphs of multi-valued maximal monotone operators can be equivalently stated in term of the pointwise convergence of the corresponding single-valued Yosida approximations and resolvents as the following result shows.

**Theorem 2.7.** *Let $A^n$, $A : V \to 2^{V^*}$ be maximal monotone operators and $\lambda > 0$. The following statements are equivalent:*

*(a) $A^n \rightarrowtail A$ as $n \to \infty$;*

*(b) for every $v \in V$, $j_\lambda^{A^n} v \to j_\lambda^A v$ as $n \to \infty$;*

*(c) for every $v \in V$, $A^n{}_\lambda v \to A_\lambda v$ as $n \to \infty$;*

*(d) $A_\lambda^n \rightarrowtail A_\lambda$ as $n \to \infty$.*

*Moreover, the convergences $j_\lambda^{A^n} v \to j_\lambda^A v$ and $A^n{}_\lambda v \to A_\lambda v$ are uniform on strongly compact subsets of $V$.*

*Proof.* See [16, Theorem 2.9]. □

**Measurability of multi-valued mappings.** In this subsection we present briefly some facts about measurable multi-valued mappings. We assume that $V$, and hence $V^*$, is separable and denote the set of maximal monotone operators from $V$ to $V^*$ by $\mathfrak{M}(V \times V^*)$. Further, let $(S, \Sigma(S), \mu)$ be a $\sigma-$finite $\mu-$complete measurable space.

**Definition 2.8.** *A function $A : S \to \mathfrak{M}(V \times V^*)$ is measurable iff for every open set $U \in V \times V^*$ (respectively closed set, Borel set, open ball, closed ball),*

$$\{x \in S \mid A(x) \cap U \neq \emptyset\}$$

*is measurable in $S$.*

The next result states that the notion of measurability for maximal monotone mappins can be equivalently defined in terms of the measurability for appropriate single-valued mappings.

**Proposition 2.9.** *Let $A : S \to \mathfrak{M}(V \times V^*)$, let $\lambda > 0$ and let $E$ be dense in $V$. The following are equivalent:*

*(a) $A$ is measurable;*

*(b) for every $v \in E$, $x \mapsto j_\lambda^{A(x)} v$ is measurable;*

*(c) $v \in E$, $x \mapsto A_\lambda(x) v$ is measurable.*

*Proof.* See [16, Proposition 2.11]. □

For further reading on measurable multi-valued mappings we refer the reader to [11, 28, 47].

**Canonical extensions of maximal monotone operators.** Given a mapping $A : S \to \mathfrak{M}(V \times V^*)$, one can define a monotone graph from $L^p(S, V)$ to $L^q(S, V^*)$, where $1/p + 1/q = 1$, as follows:

**Definition 2.10.** *Let $A : S \to \mathfrak{M}(V \times V^*)$, the canonical extension of $A$ from $L^p(S, V)$ to $L^q(S, V^*)$, where $1/p + 1/q = 1$, is defined by:*

$$Gr\mathcal{A} = \{[v, v^*] \in L^p(S, V) \times L^q(S, V^*) \mid [v(x), v^*(x)] \in GrA(x) \ for \ a.e. \ x \in S\}.$$

Monotonicity of $\mathcal{A}$ defined in Definition 2.10 is obvious, while its maximality follows from the next proposition.

**Proposition 2.11.** *Let $A : S \to \mathfrak{M}(V \times V^*)$ be measurable. If $Gr\mathcal{A} \neq \emptyset$, then $\mathcal{A}$ is maximal monotone.*

*Proof.* See [16, Proposition 2.13]. □

We have to point out here that the maximality of $A(x)$ for almost every $x \in S$ does not imply the maximality of $\mathcal{A}$ as the latter can be empty ([16]): $S = (0, 1)$, and $GrA(x) = \{[v, v^*] \in \mathbb{R} \times \mathbb{R} \mid v^* = x^{-1/q}\}$.

For given mappings $A, A^n : S \to \mathfrak{M}(V \times V^*)$ and their canonical extensions $\mathcal{A}, \mathcal{A}^n$, one can ask whether the pointwise convergence $A^n(x) \rightarrowtail A(x)$ implies the convergence of the graphs of the corresponding canonical extensions $\mathcal{A}^n \rightarrowtail \mathcal{A}$. The answer is given by the next theorem.

**Theorem 2.12.** *Let $A, A^n : S \to \mathfrak{M}(V \times V^*)$ be measurable. Assume*

*(a) for almost every $x \in S$, $A^n(x) \rightarrowtail A(x)$ as $n \to \infty$,*

*(b) $\mathcal{A}$ and $\mathcal{A}^n$ are maximal monotone,*

*(c) there exists $[\alpha_n, \beta_n] \in Gr\mathcal{A}^n$ and $[\alpha, \beta] \in L^p(S, V) \times L^q(S, V^*)$ such that $[\alpha, \beta] \to [\alpha, \beta]$ strongly in $L^p(S, V) \times L^q(S, V^*)$ as $n \to \infty$,*

*then $\mathcal{A}^n \rightarrowtail \mathcal{A}$.*

*Proof.* See [16, Proposition 2.16]. □

We note that assumption (c) in Theorem 2.12 can not be dropped in virtue of Remark 2.16 in [16].

# 3 The periodic unfolding

The derivation of the homogenized problem for (5) - (10) is based on the periodic unfolding operator method introduced by Cioranescu, Damlamian and Griso [12]. For the reader unfamiliar with this method we recall here some properties of this operator. The proofs of all results mentioned here as well as examples of applications of the method can be found in [12, 13, 15] and in the literature cited there.

Let $\Omega \subset \mathbb{R}^3$ be an open set and $Y = [0,1)^3$. Let $(e_1, e_2, e_3)$ denote the standard basis in $\mathbb{R}^3$. For $z \in \mathbb{R}^3$, $[z]_Y$ denotes a linear combination $\sum_{j=1}^3 d_j e_j$ with $\{d_1, d_2, d_3\} \in \mathbb{Z}$ such that $z - [z]_Y$ belongs to $Y$, and set

$$\{z\}_Y := z - [z]_Y \in Y \quad v \in \mathbb{R}^3.$$

Then, for each $x \in \mathbb{R}^3$, one has

$$x = \eta \left( \left[ \frac{x}{\eta} \right]_Y + y \right).$$

We use the following notations:

$$\Xi_\eta = \{\xi \in \mathbb{Z}^k \mid \eta(\xi + Y) \subset \Omega\}, \quad \hat{\Omega}_\eta = \operatorname{int} \left\{ \bigcup_{\xi \in \Xi_\eta} \left( \eta\xi + \eta\overline{Y} \right) \right\}, \quad \Lambda_\eta = \Omega \setminus \hat{\Omega}_\eta.$$

The set $\hat{\Omega}_\eta$ is the largest union of $\eta(\xi + \overline{Y})$ cells ($\xi \in \mathbb{Z}^3$) included in $\Omega$, while $\Lambda_\eta$ is the subset of $\Omega$ containing the parts from $\eta(\xi + \overline{Y})$ cells intersecting the boundary $\partial\Omega$.

**Definition 3.1.** *Let $Y$ be a reference cell, $\eta$ be a positive number and a map $v : \Omega \to \mathbb{R}^k$. The unfolding operator $\mathcal{T}_\eta(v) : \Omega \times Y \to \mathbb{R}^k$ is defined by*

$$\mathcal{T}_\eta(v) := \begin{cases} v\left( \eta \left[ \frac{x}{\eta} \right]_Y + \eta y \right), & \text{a.e. } (x, y) \in \hat{\Omega}_\eta \times Y, \\ 0, & \text{a.e. } (x, y) \in \Lambda_\eta \times Y. \end{cases}$$

The next results are straightforward from Definition 3.1.

**Proposition 3.2.** *For $q \in [1, \infty[$, the operator $\mathcal{T}_\eta$ is linear and continuous from $L^q(\Omega, \mathbb{R}^k)$ to $L^q(\Omega \times Y, \mathbb{R}^k)$. For every $\phi$ in $L^1(\Omega, \mathbb{R}^k)$ one has*

*(a)* $\frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\eta(\phi)(x, y) dx dy = \int_{\hat{\Omega}_\eta} \phi(x) dx,$

*(b)* $\frac{1}{|Y|} \int_{\Omega \times Y} |\mathcal{T}_\eta(\phi)(x, y)| dx dy \leq \int_\Omega |\phi(x)| dx,$

*(c)* $\left| \int_{\hat{\Omega}_\eta} \phi(x) dx - \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\eta(\phi)(x, y) dx dy \right| \leq \int_{\Lambda_\eta} |\phi(x)| dx,$

*(d)* $\|\mathcal{T}_\eta(\phi)\|_{p, \Omega \times Y} = |Y|^{1/p} \|\phi I_{\hat{\Omega}_\eta}\|_q \leq |Y|^{1/p} \|\phi\|_q.$

*Proof.* See [13, Proposition 2.5]. □

Obviously, if $\phi_\eta \in L^1(\Omega, \mathbb{R}^k)$ satisfies

$$\int_{\Lambda_\eta} |\phi_\eta(x)| dx \to 0, \tag{14}$$

then

$$\int_\Omega \phi_\eta(x) dx - \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\eta(\phi_\eta)(x, y) dx dy \to 0.$$

If a sequence $\phi_\eta$ satisfies (14), we shall write

$$\int_\Omega \phi_\eta(x) dx \overset{\mathcal{T}_\eta}{\simeq} \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\eta(\phi_\eta)(x, y) dx dy.$$

**Proposition 3.3.** *Let $q$ belong to $[1, \infty[$.*

*(a) For any $v \in L^q(\Omega, \mathbb{R}^k)$, $\mathcal{T}_\eta(v) \to v$ strongly in $L^q(\Omega \times Y, \mathbb{R}^k)$,*

*(b) Let $v_\eta$ be a bounded sequence in $L^q(\Omega, \mathbb{R}^k)$ such that $v_\eta \to v$ strongly in $L^q(\Omega, \mathbb{R}^k)$, then*

$$\mathcal{T}_\eta(v_\eta) \to v, \quad \textit{strongly in } L^q(\Omega \times Y, \mathbb{R}^k).$$

*(c) For every relatively weakly compact sequence $v_\eta$ in $L^q(\Omega, \mathbb{R}^k)$, the corresponding $\mathcal{T}_\eta(v_\eta)$ is relatively weakly compact in $L^q(\Omega \times Y, \mathbb{R}^k)$. Furthermore, if*

$$\mathcal{T}_\eta(v_\eta) \rightharpoonup \hat{v} \quad \textit{in } L^q(\Omega \times Y, \mathbb{R}^k),$$

*then*

$$v_\eta \rightharpoonup \frac{1}{|Y|} \int_Y \hat{v} dy \quad \textit{in } L^q(\Omega, \mathbb{R}^k).$$

*Proof.* See [13, Proposition 2.9]. □

We note that the strong/weak convergence of $\mathcal{T}_\eta(v_\eta)$ does not imply, unless $\Lambda_\eta$ has the measure 0 for every $\eta$, the strong/weak convergence of $v_\eta$, since the information concerning the behavior of $v_\eta$ on $\Lambda_\eta$ is missing.

Next results present some properties of the restriction of the unfolding operator to the space $W^{1,q}(\Omega, \mathbb{R}^k)$.

**Proposition 3.4.** *Let $q$ belong to $]1,\infty[$.*

*(a) Suppose that $v_\eta \in W^{1,q}(\Omega,\mathbb{R}^k)$ is bounded in $L^q(\Omega,\mathbb{R}^k)$ and satisfies*

$$\eta\|\nabla v_\eta\|_q \le C.$$

*Then, there exists a subsequence and $\hat{v} \in L^p(\Omega, W^{1,q}_{per}(Y,\mathbb{R}^k))$ such that*

$$\begin{aligned} \mathcal{T}_\eta(v_\eta) &\rightharpoonup \hat{v} \quad \text{in } L^q(\Omega, W^{1,q}_{per}(Y,\mathbb{R}^k)), \\ \mathcal{T}_\eta(\nabla v_\eta) &\rightharpoonup \nabla_y \hat{v} \quad \text{in } L^q(\Omega\times Y,\mathbb{R}^k). \end{aligned}$$

*(b) Let $v_\eta$ converge weakly in $W^{1,q}(\Omega,\mathbb{R}^k)$ to $v$. Then*

$$\mathcal{T}_\eta(v_\eta) \rightharpoonup v \quad \text{in } L^q(\Omega, W^{1,q}_{per}(Y,\mathbb{R}^k)).$$

*Proof.* See [13, Corollary 3.2, Corollary 3.3]. □

**Proposition 3.5.** *Let $q$ belong to $]1,\infty[$. Let $v_\eta$ converge weakly in $W^{1,q}(\Omega,\mathbb{R}^k)$ to some $v$. Then, up to a subsequence, there exists some $\hat{v} \in L^q(\Omega, W^{1,q}_{per}(Y,\mathbb{R}^k))$ such that*

$$\mathcal{T}_\eta(\nabla v_\eta) \rightharpoonup \nabla v + \nabla_y \hat{v} \quad \text{in } L^q(\Omega\times Y,\mathbb{R}^k).$$

*Proof.* See [13, Theorem 3.5, (i)]. □

The last proposition can be generalized to $W^{m,q}(\Omega,\mathbb{R}^k)$-spaces with $m \ge 1$.

**Proposition 3.6.** *Let $q$ belong to $]1,\infty[$ and $m \ge 1$. Let $v_\eta$ converge weakly in $W^{m,q}(\Omega,\mathbb{R}^k)$ to some $v$. Then, up to a subsequence, there exists some $\hat{v} \in L^q(\Omega, W^{m,q}_{per}(Y,\mathbb{R}^k))$ such that*

$$\begin{aligned} \mathcal{T}_\eta(D^l v_\eta) &\rightharpoonup D^l v \quad \text{in } L^q(\Omega, W^{m-l,q}(Y,\mathbb{R}^k)) \text{ for } |l| \le m-1, \\ \mathcal{T}_\eta(D^l v_\eta) &\rightharpoonup D^l v + D^l_y \hat{v} \quad \text{in } L^q(\Omega\times Y,\mathbb{R}^k) \text{ for } |l| = m \end{aligned}$$

*Proof.* See [13, Theorem 3.6]. □

We note that the periodic unfolding method described above is an alternative to the two-scale convergence method introduced in [46] and further developed in [6]. More precisely, the two-scale convergence of a bounded sequence $v_\eta$ in $L^p(\Omega,\mathbb{R}^k)$ is equivalent to the weak convergence of the corresponding unfolded sequence $\mathcal{T}_\eta(v_\eta)$ in $L^p(\Omega\times Y,\mathbb{R}^k)$ (see [13, Proposition 2.14]).

For a multi-valued function $h \in \mathbb{M}(\Omega,\mathbb{R}^k,\alpha,m)$[6] we define the unfolding operator as follows.

**Definition 3.7.** *Let $Y$ be a periodicity cell, $\eta$ be a positive number and a map $h \in \mathbb{M}(\Omega,\mathbb{R}^k,p,\alpha,m)$. The unfolding operator $\mathcal{T}_\eta(h) : \Omega\times Y\times\mathbb{R}^k \to 2^{\mathbb{R}^k}$ is defined by*

$$\mathcal{T}_\eta(h)(x,y,v) := \begin{cases} h\left(\eta\left[\frac{x}{\eta}\right]_Y + \eta y, v\right), & \text{a.e. } (x,y)\in\hat{\Omega}_\eta\times Y,\ v\in\mathbb{R}^k, \\ |v|^{p-2}v, & \text{a.e. } (x,y)\in\Lambda_\eta\times Y,\ v\in\mathbb{R}^k. \end{cases}$$

[6] The class of functions $h \in \mathbb{M}(\Omega,\mathbb{R}^k,\alpha,m)$ is defined in Definition 5.1.

Obviously, by its definition the unfolding operator of a multi-valued function from $\mathbb{M}(\Omega, \mathbb{R}^k, \alpha, m)$ belongs to the set $\mathbb{M}(\Omega \times Y, \mathbb{R}^k, \alpha, m)$.

We note that the periodic unfolding method described above is an alternative to the two-scale convergence method introduced in [46] and further developed in [6]. More precisely, the two-scale convergence of a bounded sequence $v_\eta$ in $L^p(\Omega, \mathbb{R}^k)$ is equivalent to the weak convergence of the corresponding unfolded sequence $\mathcal{T}_\eta(v_\eta)$ in $L^p(\Omega \times Y, \mathbb{R}^k)$ (see [13, Proposition 2.14]).

**Homogenization of the linear elasticity problem.** In this section we apply the periodic unfolding method to the homogenization of linear elasticity systems[7] with periodically highly oscillating coefficients (see [14] for properties of periodically oscillating functions). We show the strong convergence of the unfolded sequence of the gradients of the solutions of linear elasticity problem (see Theorem 3.8 below). The proof of the mentioned result applied to an elliptic partial differential equation is performed in [13] and can be carried over to linear elasticity systems without significant modifications. Therefore, we sketch here only the proof in [13] adopted to our needs.

In linear elasticity theory it is well known (see [55, Theorem 4.2]) that a Dirichlet boundary value problem formed by the equations

$$-\operatorname{div}_x \sigma_\eta(x) = \hat{b}(x), \qquad x \in \Omega, \tag{15}$$

$$\sigma_\eta(x) = \mathbb{C}[x/\eta](\operatorname{sym}(\nabla_x u_\eta(x)) - \hat{\varepsilon}_\eta(x)), \qquad x \in \Omega, \tag{16}$$

$$u_\eta(x) = 0, \qquad x \in \partial\Omega, \tag{17}$$

to given $\hat{b} \in H^{-1}(\Omega, \mathbb{R}^3)$ and $\hat{\varepsilon}_\eta \in L^2(\Omega, \mathcal{S}^3)$ has a unique weak solution $(u_\eta, \sigma_\eta) \in H_0^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3)$. We require that $\hat{\varepsilon}_\eta$ converges to $\hat{\varepsilon}_0$ strongly in $L^2(\Omega, \mathcal{S}^3)$ as $\eta \to 0$. The following result holds.

**Theorem 3.8.** *There exist $u_0 \in H_0^1(\Omega, \mathbb{R}^3)$, $\sigma_0 \in L^2(\Omega \times Y, \mathcal{S}^3)$ and $u_1 \in L^2(\Omega, H^1_{per}(Y, \mathbb{R}^3))$ such that*

$$u_\eta \rightharpoonup u_0 \quad in\ H_0^1(\Omega, \mathbb{R}^3), \tag{18}$$

$$\mathcal{T}_\eta(u_\eta) \rightharpoonup u_0 \quad in\ L^2(\Omega, H^1_{per}(Y, \mathbb{R}^3)), \tag{19}$$

$$\mathcal{T}_\eta(\nabla u_\eta) \rightharpoonup \nabla u_0 + \nabla_y u_1 \quad in\ L^2(\Omega \times Y, \mathbb{R}^3), \tag{20}$$

$$\mathcal{T}_\eta(\sigma_\eta) \rightharpoonup \sigma_0 \quad in\ L^2(\Omega \times Y, \mathcal{S}^3), \tag{21}$$

*and $(u_0, \sigma_0, u_1)$ is the unique solution of the homogenized system:*

$$-\operatorname{div}_y \sigma_0(x, y) = 0, \tag{22}$$

$$\sigma_0(x, y) = \mathbb{C}[y]\big(\operatorname{sym}(\nabla u_0(x) + \nabla_y u_1(x, y)) - \hat{\varepsilon}_0(x)\big), \tag{23}$$

$$y \mapsto u_1(x, y), \quad Y-periodic, \tag{24}$$

$$-\operatorname{div}_x \int_Y \sigma_0(x, y)dy = \hat{b}(x), \tag{25}$$

$$u_0(x) = 0, \qquad x \in \partial\Omega. \tag{26}$$

[7] A survey on other applications of the method can be found in [13].

*Moreover, the following convergences hold*

$$\lim_{\eta\to 0}\int_{\Omega}\mathbb{C}\,[\cdot/\eta]\,\mathrm{sym}(\nabla_x u_\eta)\,\mathrm{sym}(\nabla_x u_\eta)dx$$
$$=\int_{\Omega\times Y}\mathbb{C}[y]\,\mathrm{sym}\left(\nabla u_0+\nabla_y u_1\right)\mathrm{sym}\left(\nabla u_0+\nabla_y u_1\right)dxdy, \quad (27)$$

$$\lim_{\eta\to 0}\int_{\Lambda_\eta}|\,\mathrm{sym}(\nabla_x u_\eta)|^2dx=0, \quad (28)$$

*and*

$$\mathcal{T}_\eta(\nabla u_\eta)\to\nabla u_0+\nabla_y u_1 \quad in\ L^2(\Omega\times Y,\mathbb{R}^3), \quad (29)$$
$$\mathcal{T}_\eta(\sigma_\eta)\to\sigma_0 \quad in\ L^2(\Omega\times Y,\mathcal{S}^3). \quad (30)$$

*Proof.* See [43, Theorem 4.1]. □

# 4 Unfolding the Curl Curl-operator

Our method is based on the Helmholtz-Weyl decomposition for vector fields in general $L^q$-spaces over a domain $\Omega$. It turns out (see [30, Theorem 2.1(2)]) that the following theorem holds.

**Theorem 4.1.** *Let $1<q<\infty$. Every $v\in L^q(\Omega,\mathbb{R}^3)$ can be uniquely decompose as*

$$v=h+\mathrm{Curl}\,w+\nabla z, \quad (31)$$

*where $h\in X^q_{har}(\Omega,\mathbb{R}^3)$, $w\in V^q_\sigma(\Omega,\mathbb{R}^3)$ and $z\in W^{1,q}(\Omega,\mathbb{R}^3)$, and the triple $(h,w,z)$ satisfies the inequality*

$$\|h\|_q+\|w\|_{1,q,\Omega}+\|z\|_{1,q,\Omega}\le C\|v\|_q, \quad (32)$$

*where $C$ is a constant depending on $\Omega$ and $q$. If there is another triple of functions $(\tilde h,\tilde w,\tilde z)$ such that $v$ can be written in the form*

$$v=\tilde h+\mathrm{Curl}\,\tilde w+\nabla\tilde z,$$

*with $\tilde h\in X^q_{har}(\Omega,\mathbb{R}^3)$, $\tilde w\in V^q_\sigma(\Omega,\mathbb{R}^3)$ and $\tilde z\in W^{1,q}(\Omega,\mathbb{R}^3)$, then it holds*

$$h=\tilde h,\quad \mathrm{Curl}\,w=\mathrm{Curl}\,\tilde w,\quad \nabla z=\nabla\tilde z.$$

*Remark* 4.2. If $L$ denotes the dimension of $V_{har}(\Omega,\mathbb{R}^3)$, i.e. $\dim V_{har}(\Omega,\mathbb{R}^3)=L$, and $\{\phi_1,...,\phi_2\}$ is a basis of $V_{har}(\Omega,\mathbb{R}^3)$, then it holds $V^q(\Omega,\mathbb{R}^3)\subset W^{1,q}(\Omega,\mathbb{R}^3)$ with the estimate

$$\|v\|_q+\|\nabla v\|_q\le C(\|\,\mathrm{Curl}\,v\|_q+\|\,\mathrm{div}\,v\|_q+\sum_{i=1}^L|(v,\phi_i)|)$$

for all $v\in V^q(\Omega,\mathbb{R}^3)$, where $C=C(\Omega,q)$ ([30, Theorem 2.4(2)]). The proof of the inequality (33) with $\sum_{i=1}^L|(v,\phi_i)|$ replaced by $\|v\|_q$ is performed in [30, Lemma 4.5] (for $q = 2$ it can be found in [18, Theorem VII.6.1]). If we assume that the boundary $\partial\Omega$ has $L+1$ smooth connected components $\Gamma_0, \Gamma_1, ..., \Gamma_L$ such that $\Gamma_1, ..., \Gamma_L$ lie inside $\Gamma_0$ with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$ and

$$\partial\Omega = \cup_{i=0}^{L} \Gamma_i,$$

then it holds ([30, Appendix A])

$$\dim V_{har}(\Omega, \mathbb{R}^3) = L.$$

If the function $v$ in (31) is more regular, then the function $w$ can be chosen from a better space as the next theorem shows.

**Theorem 4.3.** *Let $1 < q < \infty$. Assume that decomposition (31) holds. If, additionally $v \in Z^q_{\mathrm{Curl}}(\Omega, \mathbb{R}^3)$, then $w$ in (31) can be chosen from $W^{3,q}(\Omega, \mathbb{R}^3) \cap V^q_\sigma(\Omega, \mathbb{R}^3)$ satisfying the estimate*

$$\|w\|_{3,q,\Omega} \leq C(\|\operatorname{Curl} v\|_{1,q,\Omega} + \|v\|_q), \tag{33}$$

*where $C$ is a constant depending on $\Omega$ and $q$.*

*Proof.* For $v \in L^q_{\mathrm{Curl}}(\Omega, \mathbb{R}^3)$ this result is proved in [29]. For $v \in Z^q_{\mathrm{Curl}}(\Omega, \mathbb{R}^3)$ the proof runs the same lines. We repeat them.

As it is shown in [30, Lemma 4.2(2)], we can choose the function $w \in V^q_\sigma(\Omega, \mathbb{R}^3)$ satisfying the equation

$$(\operatorname{Curl} w, \operatorname{Curl} \psi)_\Omega = (v, \operatorname{Curl} \psi)_\Omega, \quad \text{for all } \psi \in V^{q^*}_\sigma(\Omega, \mathbb{R}^3) \tag{34}$$

with the estimate

$$\|w\|_{1,q,\Omega} \leq C\|v\|_q, \tag{35}$$

where $C$ depends only on $\Omega$ and $q$. Since $\operatorname{div} w = 0$ in $\Omega$ and $v \in Z^q_{\mathrm{Curl}}(\Omega, \mathbb{R}^3)$, it follows from (34) that $-\Delta w = \operatorname{Curl} v$ in the sense of distributions, and we may regard $w$ as a weak solution of the following boundary value problem

$$\begin{aligned} -\Delta w &= \operatorname{Curl} v, && \text{in } \Omega, && (36)\\ \operatorname{div} w &= 0, && \text{on } \partial\Omega, && (37)\\ w \cdot n &= 0, && \text{on } \partial\Omega. && (38) \end{aligned}$$

Since $\operatorname{Curl} v \in W^{1,q}(\Omega, \mathbb{R}^3)$, it follows from [30, Lemma 4.3(1)] and the classical theory of Agmon, Douglas and Nirenberg [1] that the solution $w$ of the homogeneous boundary value problem (36) belongs to $W^{3,q}(\Omega, \mathbb{R}^3)$ and the estimate

$$\|w\|_{3,q,\Omega} \leq C(\|\operatorname{Curl} v\|_{1,q,\Omega} + \|w\|_q), \tag{39}$$

is valid with the constant $C$ dependent of $\Omega$ and $q$. Due to (35), the estimate (39) implies (33). This completes the proof. □

Now we can state the main result of this section.

**Theorem 4.4.** *Let $1 < q < \infty$. Suppose that sequence $v_\eta$ is weakly compact in $Z^q_{\mathrm{Curl}}(\Omega, \mathbb{R}^3)$. Then there exist*

$$v \in Z^q_{\mathrm{Curl}}(\Omega, \mathbb{R}^3), \quad v_0 \in L^q(\Omega \times Y, \mathbb{R}^3) \ \ \textit{and}$$
$$v_1 \in L^q(\Omega, W^{2,q}_{per}(Y, \mathbb{R}^3)) \ \textit{with} \ \operatorname{div}_y v_1 = 0,$$

*such that*

$$v_\eta \rightharpoonup v \ \ \textit{in} \ Z^q_{\mathrm{Curl}}(\Omega, \mathbb{R}^3), \tag{40}$$

$$\mathcal{T}_\eta(v_\eta) \rightharpoonup v_0 \ \ \textit{in} \ L^q(\Omega \times Y, \mathbb{R}^3), \tag{41}$$

$$\mathcal{T}_\eta(\mathrm{Curl}\, v_\eta) \rightharpoonup \mathrm{Curl}\, v \ \ \textit{in} \ L^q(\Omega, W^{1,q}_{per}(Y, \mathbb{R}^3)), \tag{42}$$

$$\mathcal{T}_\eta(\mathrm{Curl}\,\mathrm{Curl}\, v_\eta) \rightharpoonup \mathrm{Curl}\,\mathrm{Curl}\, v + \mathrm{Curl}_y\,\mathrm{Curl}_y\, v_1, \ \textit{in} \ L^q(\Omega \times Y, \mathbb{R}^3). \tag{43}$$

*Moreover, $v(x) = \int_Y v_0(x, y) dy$.*

*Proof.* Convergence (41) and the last statement of the theorem follow from Proposition 3.3(c). Convergence (40) is obvious. Next, we prove convergences (42) and (43). According to Theorem 4.1, there exist $h_\eta \in X^q_{har}(\Omega, \mathbb{R}^3)$, $w_\eta \in V^q_\sigma(\Omega, \mathbb{R}^3)$ and $z_\eta \in W^{1,q}(\Omega, \mathbb{R}^3)$ satisfying the inequality

$$\|h_\eta\|_q + \|w_\eta\|_{1,q,\Omega} + \|z_\eta\|_{1,q,\Omega} \le C\|v_\eta\|_q \tag{44}$$

with the constant $C$ independent of $\eta$, and such that

$$v_\eta = h_\eta + \mathrm{Curl}\, w_\eta + \nabla z_\eta. \tag{45}$$

Moreover, due to Theorem 4.3, $w_\eta$ in (45) enjoys the inequality

$$\|w_\eta\|_{3,q,\Omega} \le C(\|\,\mathrm{Curl}\, v_\eta\|_{1,q,\Omega} + \|v_\eta\|_q) \tag{46}$$

with the constant $C$ independent of $\eta$. Therefore, the weak compactness of $v_\eta$ in $Z^q_{\mathrm{Curl}}(\Omega, \mathbb{R}^3)$ and (46) imply that $w_\eta$ is weakly compact in $W^{3,q}(\Omega, \mathbb{R}^3)$. Thus, in virtue of Proposition 3.6 we conclude that there exist

$$w \in W^{3,q}(\Omega, \mathbb{R}^3) \ \ \text{and} \ \ w_1 \in L^q(\Omega, W^{3,q}_{per}(Y, \mathbb{R}^3))$$

such that

$$\mathcal{T}_\eta(D^l w_\eta) \rightharpoonup D^l w \ \ \textit{in} \ L^q(\Omega, W^{3-l,q}(Y, \mathbb{R}^k)) \ \textit{for} \ |l| \le 2,$$
$$\mathcal{T}_\eta(D^l w_\eta) \rightharpoonup D^l w + D^l_y w_1 \ \ \textit{in} \ L^q(\Omega \times Y, \mathbb{R}^k) \ \textit{for} \ |l| = 3.$$

Since $\mathrm{Curl}\, v_\eta = \mathrm{Curl}\, w_\eta$ and $\mathrm{Curl}\, v = \mathrm{Curl}\, w$, we get that

$$\mathcal{T}_\eta(\mathrm{Curl}\, v_\eta) \rightharpoonup \mathrm{Curl}\, v \ \ \textit{in} \ L^q(\Omega, W^{1,q}_{per}(Y, \mathbb{R}^3)),$$

$$\mathcal{T}_\eta(\mathrm{Curl}\,\mathrm{Curl}\, v_\eta) \rightharpoonup \mathrm{Curl}\,\mathrm{Curl}\, v + \mathrm{Curl}_y\,\mathrm{Curl}_y\, v_1, \ \textit{in} \ L^q(\Omega \times Y, \mathbb{R}^3).$$

The proof of Theorem 4.4 is complete. □

# 5 Homogenized system of equations

**Main result.** First, we define a class of maximal monotone functions we deal with in this work.

**Definition 5.1.** *For* $m \in L^1(\Omega, \mathbb{R})$, $\alpha \in \mathbb{R}_+$ *and* $q > 1$, $\mathbb{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$ *is the set of multi-valued functions* $h : \Omega \times \mathbb{R}^k \to 2^{\mathbb{R}^k}$ *with the following properties*

- $v \mapsto h(x, v)$ *is maximal monotone for almost all* $x \in \Omega$,
- *the mapping* $x \mapsto j_\lambda(x, v) : \Omega \to \mathbb{R}^k$ *is measurable for all* $\lambda > 0$, *where* $j_\lambda(x, v)$ *is the inverse of* $v \mapsto v + \lambda h(x, v)$,
- *for a.e.* $x \in \Omega$ *and every* $v^* \in h(x, v)$

$$\alpha \left( \frac{|v|^q}{q} + \frac{|v^*|^{q^*}}{q^*} \right) \leq (v, v^*) + m(x), \tag{47}$$

*where* $1/q + 1/q^* = 1$.

*Remark* 5.2. We note that the condition (47) is equivalent to the following two inequalities

$$|v^*|^{q^*} \leq m_1(x) + \alpha_1 |v|^q, \tag{48}$$

$$(v, v^*) \geq m_2(x) + \alpha_2 |v|^q, \tag{49}$$

for a.e. $x \in \Omega$ and every $v^* \in h(x, v)$ and with suitable functions $m_1, m_2 \in L^1(\Omega, \mathbb{R})$ and numbers $\alpha_1, \alpha_2 \in \mathbb{R}_+$.

*Remark* 5.3. Visco-plasticity is typically included in the former conditions by choosing the function $g$ to be in Norton-Hoff form, i.e.

$$g(\Sigma) = [|\Sigma| - \sigma_{\mathrm{y}}]_+^r \, \frac{\Sigma}{|\Sigma|} , \quad \Sigma \in \mathcal{M}^3 ,$$

where $\sigma_{\mathrm{y}}$ is the flow stress and $r$ is some parameter together with $[x]_+ := \max(x, 0)$. If $g : \mathcal{M}^3 \mapsto \mathcal{S}^3$ then the flow is called irrotational (no plastic spin).

The main properties of the class $\mathbb{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$ are collected in the following proposition.

**Proposition 5.4.** *Let* $\mathcal{H}$ *be a canonical extension of a function* $h : \mathbb{R}^k \to 2^{\mathbb{R}^k}$, *which belongs to* $\mathbb{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$. *Then* $\mathcal{H}$ *is maximal monotone, surjective and* $D(\mathcal{H}) = L^p(\Omega, \mathbb{R}^k)$.

*Proof.* See Corollary 2.15 in [16]. □

Next, we define the notion of strong solutions for the initial boundary value problem (5) - (10).

**Definition 5.5. (Strong solutions)** *A function* $(u_\eta, \sigma_\eta, p_\eta)$ *such that*

$$(u_\eta, \sigma_\eta) \in H^1(0, T_e; H_0^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3)), \quad \Sigma_\eta^{\mathrm{lin}} \in L^q(\Omega_{T_e}, \mathcal{M}^3),$$

$$p_\eta \in H^1(0, T_e; L^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)) \cap L^2(0, T_e; Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3))$$

*is called a* strong solution *of the initial boundary value problem (5) - (10), if for every* $t \in [0, T_e]$ *the function* $(u_\eta(t), \sigma_\eta(t))$ *is a weak solution of the boundary value problem (15) - (17) with* $\hat{\varepsilon}_p = \operatorname{sym} p_\eta(t)$ *and* $\hat{b} = b(t)$, *the evolution inclusion (7) and the initial condition (8) are satisfied pointwise.*

Next, we state the existence result (see [45]).

**Theorem 5.6.** *Suppose that $1 < q^* \leq 2 \leq q < \infty$. Assume that $\Omega \subset \mathbb{R}^3$ is a sliceable domain with a $C^1$-boundary, $\mathbb{C} \in L^\infty(\Omega, \mathcal{S}^3)$ and $C_1 \in L^\infty(\Omega, \mathbb{R})$. Let the functions $b \in W^{1,q}(0, T_e; L^q(\Omega, \mathbb{R}^3))$ be given and $g \in \mathbb{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$. Suppose that for a.e. $x \in \Omega$ the relations*

$$p^{(0)}(x) = 0 \quad \text{and} \quad 0 \in g(x/\eta, \sigma^{(0)}(x)) \tag{50}$$

*hold, where the function $\sigma^{(0)} \in L^2(\Omega, \mathcal{S}^3)$ is determined by equations (15) - (17) for $\hat{\varepsilon}_p = 0$ and $\hat{b} = b(0)$. Then there exists a solution $(u_\eta, \sigma_\eta, p_\eta)$ of the initial boundary value problem (5) - (10).*

Now we can formulate the main result of this work.

**Theorem 5.7.** *Suppose that all assumptions of Theorem 5.6 are fulfilled. Then there exists*

$$\begin{aligned}
&u_0 \in H^1(0, T_e; H_0^1(\Omega, \mathbb{R}^3)), \quad u_1 \in H^1(0, T_e; L^2(\Omega, H^1_{per}(Y, \mathbb{R}^3))),\\
&\sigma_0 \in L^\infty(0, T_e; L^2(\Omega \times Y, \mathcal{S}^3)), \quad p_0 \in H^1(0, T_e; L^2(\Omega \times Y, \mathcal{M}^3)),\\
&p \in H^1(0, T_e; L^2(\Omega, \mathcal{M}^3)) \cap L^2(0, T_e; Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)),\\
&p_1 \in L^2(0, T_e; L^2(\Omega, W^{2,q^*}_{per}(Y, \mathcal{M}^3))) \text{ with } \operatorname{div}_y p_1 = 0,
\end{aligned}$$

*and*

$$\sigma \in L^\infty(0, T_e; L^2(\Omega, \mathcal{S}^3)),$$

*such that*

$$u_\eta \rightharpoonup u_0 \quad \text{in } H^1(0, T_e; H_0^1(\Omega, \mathbb{R}^3)), \tag{51}$$

$$p_\eta \rightharpoonup p \quad \text{in } H^1(0, T_e; L^2(\Omega, \mathcal{M}^3)) \cap L^2(0, T_e; Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)), \tag{52}$$

$$\mathcal{T}_\eta(\nabla u_\eta) \rightharpoonup \nabla u_0 + \nabla_y u_1 \quad \text{in } H^1(0, T_e; L^2(\Omega \times Y, \mathbb{R}^3)), \tag{53}$$

$$\sigma_\eta \overset{*}{\rightharpoonup} \sigma_0 \quad \text{in } L^\infty(0, T_e; L^2(\Omega, \mathcal{S}^3)), \tag{54}$$

$$\mathcal{T}_\eta(\sigma_\eta) \overset{*}{\rightharpoonup} \sigma_0 \quad \text{in } L^\infty(0, T_e; L^2(\Omega \times Y, \mathcal{S}^3)), \tag{55}$$

$$\mathcal{T}_\eta(p_\eta) \rightharpoonup p_0 \quad \text{in } L^2(0, T_e; L^2(\Omega \times Y, \mathcal{M}^3)), \tag{56}$$

$$\mathcal{T}_\eta(\partial_t p_\eta) \rightharpoonup \partial_t p_0 \quad \text{in } L^2(0, T_e; L^2(\Omega \times Y, \mathcal{M}^3)), \tag{57}$$

*and*

$$\mathcal{T}_\eta(\operatorname{Curl} p_\eta) \rightharpoonup \operatorname{Curl} p \quad \text{in } L^2(0, T_e; L^2(\Omega, H^1_{per}(Y, \mathcal{M}^3))), \tag{58}$$

$$\mathcal{T}_\eta(\operatorname{dev}\operatorname{sym} p_\eta) \rightharpoonup \operatorname{dev}\operatorname{sym} p_0 \quad \text{in } L^2(\Omega_{T_e} \times Y, \mathcal{M}^3), \tag{59}$$

$$\mathcal{T}_\eta(\operatorname{Curl}\operatorname{Curl} p_\eta) \rightharpoonup \tilde{p} \quad \text{in } L^2(\Omega_{T_e} \times Y, \mathcal{M}^3), \tag{60}$$

$$\mathcal{T}_\eta(\Sigma^{\mathrm{lin}}_\eta) \rightharpoonup \Sigma^{\mathrm{lin}}_0 \quad \text{in } L^q(\Omega_{T_e} \times Y, \mathcal{M}^3), \tag{61}$$

*where*

$$\begin{aligned}
\tilde{p} &:= \operatorname{Curl}\operatorname{Curl} p + \operatorname{Curl}_y \operatorname{Curl}_y p_1,\\
\Sigma^{\mathrm{lin}}_0 &:= \sigma_0 - C_1[y] \operatorname{dev}\operatorname{sym} p_0 - C_2 \tilde{p},
\end{aligned}$$

*and* $(u_0, u_1, \sigma, \sigma_0, p, p_0, p_1)$ *is a solution of the following system of equations:*

$$
\begin{aligned}
-\operatorname{div}_x \sigma(x,t) &= b(x,t), & (62)\\
-\operatorname{div}_y \sigma_0(x,y,t) &= 0, & (63)\\
\sigma_0(x,y,t) &= \mathbb{C}[y](\operatorname{sym}(\nabla_x u_0(x,t) + \nabla_y u_1(x,y,t) - p_0(x,y,t))), & (64)\\
\partial_t p_0(x,y,t) &\in g(y, \Sigma_0^{\text{lin}}(x,y,t)), & (65)
\end{aligned}
$$

*which holds for* $(x,y,t) \in \Omega \times \mathbb{R}^3 \times [0,T_e]$*, and the initial condition and boundary condition*

$$
\begin{aligned}
p(x,0) &= p^{(0)}(x), \quad x \in \Omega, & (66)\\
p(x,t) \times n(x) &= 0, \qquad (x,t) \in \partial\Omega \times [0,T_e), & (67)\\
u_0(x,t) &= 0, \qquad (x,t) \in \partial\Omega \times [0,T_e).\,, & (68)
\end{aligned}
$$

*The functions* $\sigma$ *and* $p$ *are related to* $\sigma_0$ *and* $p_0$ *in the following ways*

$$\sigma(x,t) = \int_Y \sigma_0(x,y,t)dy, \quad p(x,t) = \int_Y p_0(x,y,t)dy.$$

The proof of Theorem 5.7 is divided into two parts. In the next lemma we derive the uniform estimates for $(u_\eta, \sigma_\eta, p_\eta)$ and then, based on these estimates, we show the convergence result.

## 5.1 Uniform estimates

First, we show that the sequence of solutions $(u_\eta, \sigma_\eta, p_\eta)$ is weakly compact.

**Lemma 5.8.** *Let all assumptions of Theorem 5.7 be satisfied. Then the sequence of solutions* $(u_\eta, \sigma_\eta)$ *is weakly compact in* $H^1(0,T_e; H_0^1(\Omega,\mathbb{R}^3) \times L^2(\Omega,\mathcal{S}^3))$ *and* $p_\eta$ *is weakly compact in* $H^1(0,T_e; L^2(\Omega,\mathcal{M}^3)) \cap L^2(0,T_e, Z^2_{\text{Curl}}(\Omega,\mathcal{M}^3))$*.*

*Proof.* To prove the lemma we recall the basic steps in the proof of the existence result (Theorem 5.6). For more details the reader is referred to [45]. The time-discretized problem for (5) - (10) is introduced as follows:
Let us fix any $m \in \mathbb{N}$ and set

$$h := \frac{T_e}{2^m},\ p^0_{\eta,m} := 0\ b^n_m := \frac{1}{h}\int_{(n-1)h}^{nh} b(s)ds \in L^q(\Omega,\mathbb{R}^3), \quad n = 1, ..., 2^m.$$

Then we are looking for functions $u^n_{\eta,m} \in H^1(\Omega,\mathbb{R}^3)$, $\sigma^n_{\eta,m} \in L^2(\Omega,\mathcal{S}^3)$ and $p^n_{\eta,m} \in Z^2_{\text{Curl}}(\Omega,\mathcal{M}^3)$ with $p^n_{\eta,m}(x) \in \mathfrak{sl}(3)$ for a.e. $x \in \Omega$ and

$$\Sigma^{\text{lin}}_{n,m} := \sigma^n_{\eta,m} - C_1[x/\eta] \operatorname{dev}\operatorname{sym} p^n_{\eta,m} - \frac{1}{m} p^n_{\eta,m} - C_2 \operatorname{Curl}\operatorname{Curl} p^n_{\eta,m} \in L^q(\Omega,\mathcal{M}^3)$$

solving the following problem

$$
\begin{aligned}
-\operatorname{div}_x \sigma^n_{\eta,m}(x) &= b^n_m(x), & (69)\\
\sigma^n_{\eta,m}(x) &= \mathbb{C}[x/\eta](\operatorname{sym}(\nabla_x u^n_{\eta,m}(x) - p^n_{\eta,m}(x))) & (70)\\
\frac{p^n_{\eta,m}(x) - p^{n-1}_{\eta,m}(x)}{h} &\in g\big(x/\eta, \Sigma^{\text{lin}}_{n,m}(x)\big), & (71)
\end{aligned}
$$

together with the boundary conditions

$$p^n_{\eta,m}(x) \times n(x) = 0, \qquad x \in \partial\Omega, \tag{72}$$
$$u^n_{\eta,m}(x) = 0, \qquad x \in \partial\Omega\,. \tag{73}$$

Such functions $(u^n_{\eta,m}, \sigma^n_{\eta,m}, p^n_{\eta,m})$ exist and satisfy the following estimate

$$\begin{aligned}
&\frac{1}{2}\Big(\|\mathbb{B}^{1/2}\sigma^l_{\eta,m}\|_2^2 + \alpha_1\|\operatorname{dev}\operatorname{sym} p^l_{\eta,m}\|_2^2 + \frac{1}{m}\|p^l_{\eta,m}\|_2^2 + C_2\|\operatorname{Curl} p^l_{\eta,m}\|_2^2\Big) \\
&+h\hat{C}\sum_{n=1}^{l}\left(\big\|\Sigma^{\mathrm{lin}}_{n,m}\big\|_q^q + \left\|\frac{p^n_{\eta,m}-p^{n-1}_{\eta,m}}{h}\right\|_{q^*}^{q^*}\right) \le C^{(0)} + \int_\Omega m(x)dx \\
&\quad +h\tilde{C}\sum_{n=1}^{l}\Big(\|b^n_m\|_q^q + \|(b^n_m - b^{n-1}_m)/h\|_2^2\Big)
\end{aligned} \tag{74}$$

for any fixed $l \in [1, 2^m]$, where (here $\mathbb{B} := \mathbb{C}^{-1}$)

$$2C^{(0)} := \|\mathbb{B}^{1/2}\sigma^{(0)}\|_2^2$$

and $\tilde{C}$, $\hat{C}$ are some positive constants independent of $\eta$ (see [45] for details). To proceed further we introduce the Rothe approximation functions.

**Rothe approximation functions:** For any family $\{\xi^n_m\}_{n=0,\dots,2m}$ of functions in a reflexive Banach space $X$, we define *the piecewise affine interpolant* $\xi_m \in C([0,T_e],X)$ by

$$\xi_m(t) := \left(\frac{t}{h} - (n-1)\right)\xi^n_m + \left(n - \frac{t}{h}\right)\xi^{n-1}_m \;\text{ for } (n-1)h \le t \le nh \tag{75}$$

and *the piecewise constant interpolant* $\bar{\xi}_m \in L^\infty(0,T_e;X)$ by

$$\bar{\xi}_m(t) := \xi^n_m \text{ for } (n-1)h < t \le nh,\; n = 1,...,2^m, \text{ and } \bar{\xi}_m(0) := \xi^0_m. \tag{76}$$

For the further analysis we recall the following property of $\bar{\xi}_m$ and $\xi_m$:

$$\|\xi_m\|_{L^q(0,T_e;X)} \le \|\bar{\xi}_m\|_{L^q(-h,T_e;X)} \le \Big(h\|\xi^0_m\|_X^q + \|\bar{\xi}_m\|^q_{L^q(0,T_e;X)}\Big)^{1/q}, \tag{77}$$

where $\bar{\xi}_m$ is formally extended to $t \le 0$ by $\xi^0_m$ and $1 \le q \le \infty$ (see [51]).

Now, from (74) we get immediately that

$$\begin{aligned}
&\bar{C}\|\bar{\sigma}_{\eta,m}(t)\|_\Omega^2 + \alpha_1\|\operatorname{dev}\operatorname{sym}\bar{p}_{\eta,m}(t)\|_2^2 + \frac{1}{m}\|\bar{p}_{\eta,m}(t)\|_2^2 + C_2\|\operatorname{Curl}\bar{p}_{\eta,m}(t)\|_2^2 \\
&\quad +2\hat{C}\Big(\|\partial_t p_{\eta,m}\|^{q^*}_{q^*,\Omega\times(0,T_e)} + \|\bar{\Sigma}^{\mathrm{lin}}_m\|^q_{q,\Omega\times(0,T_e)}\Big) \\
&\quad \le 2C^{(0)} + 2\|m\|_{1,\Omega} + 2\tilde{C}\|b\|^q_{W^{1,q}(0,T_e;L^q(\Omega,\mathcal{S}^3))},
\end{aligned} \tag{78}$$

where $\bar{C}$ is some other constant independent of $\eta$. In [45] it is shown that the Rothe approximation functions $(u_{\eta,m}, \sigma_{\eta,m}, p_{\eta,m})$ and $(\bar{u}_{\eta,m}, \bar{\sigma}_{\eta,m}, \bar{p}_{\eta,m})$ converge to the same limit $(u_\eta, \sigma_\eta, p_\eta)$. Due to the lower semi-continuity of the norm and (78) this convergence is uniform with respect to $\eta$. Therefore, estimate (78) provides that

$$\{\sigma_\eta\}_\eta \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{S}^3)), \tag{79}$$

$$\{\operatorname{dev}\operatorname{sym} p_\eta\}_\eta \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{M}^3)), \tag{80}$$

$$\{\operatorname{Curl} p_\eta\}_\eta \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{M}^3)), \tag{81}$$

$$\{p_\eta\}_\eta \text{ is uniformly bounded in } W^{1,q^*}(0,T_e;L^{q^*}(\Omega,\mathcal{M}^3)), \tag{82}$$

$$\{\Sigma_\eta^{\text{lin}}\}_\eta \text{ is uniformly bounded in } L^q(\Omega_{T_e},\mathcal{M}^3). \tag{83}$$

Furthermore, from estimates (4), (79) - (83) we obtain easily that

$$\{u_\eta\}_\eta \text{ is uniformly bounded in } L^2(0,T_e;H_0^1(\Omega,\mathbb{R}^3)), \tag{84}$$

$$\{p_\eta\}_\eta \text{ is uniformly bounded in } L^2(0,T_e;Z^2_{\text{Curl}}(\Omega,\mathcal{M}^3)). \tag{85}$$

**Additional regularity of discrete solutions.** In order to get the additional a'priori estimates, we extend the function $b$ to $t<0$ by setting $b(t)=b(0)$. The extended function $b$ is in the space $W^{1,p}(-2h,T_e;W^{-1,p}(\Omega,\mathbb{R}^3))$. Then, we set $b_m^0 = b_m^{-1} := b(0)$. Let us further set

$$p_{\eta,m}^{-1} := p_{\eta,m}^0 - h\mathcal{G}_\eta(\Sigma_{0,m}^{\text{lin}}),$$

where $\mathcal{G}_\eta : L^p(\Omega,\mathcal{M}^3) \to 2^{L^q(\Omega,\mathfrak{sl}(3))}$ denotes the canonical extensions of $g(x/\eta,\cdot) : \mathcal{M}^3 \to 2^{\mathfrak{sl}(3)}$. The assumption (50) implies that $p_{\eta,m}^{-1}=0$. Next, we define functions $(u_{\eta,m}^{-1},\sigma_{\eta,m}^{-1})$ and $(u_{\eta,m}^0,\sigma_{\eta,m}^0)$ as solutions of the linear elasticity problem (15) - (17) to the data $\hat{b}=b_m^{-1}$, $\hat{\gamma}=0$, $\hat{\varepsilon}_p=0$ and $\hat{b}=b_m^0$, $\hat{\gamma}=0$, $\hat{\varepsilon}_p=0$, respectively. Obviously, the following estimate holds

$$\left\{\left\|\frac{u_{\eta,m}^0 - u_{\eta,m}^{-1}}{h}\right\|_2, \left\|\frac{\sigma_{\eta,m}^0-\sigma_{\eta,m}^{-1}}{h}\right\|_2\right\} \le C, \tag{86}$$

where $C$ is some positive constant independent of $m$ and $\eta$. Taking now the incremental ratio of (71) for $n=1,...,2^m$, we obtain[8]

$$\operatorname{rt} p_{\eta,m}^n - \operatorname{rt} p_{\eta,m}^{n-1} = \mathcal{G}_\eta(\Sigma_{n,m}^{\text{lin}}) - \mathcal{G}_\eta(\Sigma_{(n-1),m}^{\text{lin}}).$$

Let us now multiply the last identity by $-(\Sigma_{n,m}^{\text{lin}} - \Sigma_{(n-1),m}^{\text{lin}})/h$. Then using the monotonicity of $\mathcal{G}_\eta$ we obtain that

$$\frac{1}{m}\Big(\operatorname{rt} p_{\eta,m}^n - \operatorname{rt} p_{\eta,m}^{n-1}, \operatorname{rt} p_{\eta,m}^n\Big)_\Omega + \big(\operatorname{rt} p_{\eta,m}^n - \operatorname{rt} p_{\eta,m}^{n-1}, C_1 \operatorname{dev}\operatorname{sym}(\operatorname{rt} p_{\eta,m}^n)\big)_\Omega$$
$$+\big(\operatorname{rt} p_{\eta,m}^n - \operatorname{rt} p_{\eta,m}^{n-1}, C_2 \operatorname{Curl}\operatorname{Curl}(\operatorname{rt} p_{\eta,m}^n)\big)_\Omega \le \big(\operatorname{rt} p_{\eta,m}^n - \operatorname{rt} p_{\eta,m}^{n-1}, \operatorname{rt}\sigma_{\eta,m}^n\big)_\Omega.$$

With (69) and (70) the previus inequality can be rewritten as follows

$$\frac{1}{m}\Big(\operatorname{rt} p_{\eta,m}^n - \operatorname{rt} p_{\eta,m}^{n-1}, \operatorname{rt} p_{\eta,m}^n\Big)_\Omega + \big(\operatorname{rt} p_{\eta,m}^n - \operatorname{rt} p_{\eta,m}^{n-1}, C_1 \operatorname{dev}\operatorname{sym}(\operatorname{rt} p_{\eta,m}^n)\big)_\Omega$$

[8] For sake of simplicity we use the following notation $\operatorname{rt}\phi_m^n := (\phi_m^n - \phi_m^{n-1})/h$, where $\phi_m^0, \phi_m^1, ..., \phi_m^{2m}$ is any family of functions.

$$+\big(\operatorname{rt} p^n_{\eta,m} - \operatorname{rt} p^{n-1}_{\eta,m}, C_2 \operatorname{Curl}\operatorname{Curl}(\operatorname{rt} p^n_{\eta,m})\big)_\Omega + \big(\operatorname{rt}\sigma^n_{\eta,m} - \operatorname{rt}\sigma^{n-1}_{\eta,m}, \mathbb{C}^{-1}\operatorname{rt}\sigma^n_{\eta,m}\big)_\Omega$$
$$\le \big(\operatorname{rt} u^n_{\eta,m} - \operatorname{rt} u^{n-1}_{\eta,m}, \operatorname{rt} b^n_m\big)_\Omega.$$

As in the proof of (74), multiplying the last inequality by $h$ and summing with respect to $n$ from 1 to $l$ for any fixed $l \in [1, 2^m]$ we get the estimate

$$\frac{h}{m}\|\operatorname{rt} p^l_{\eta,m}\|_2^2 + h\alpha_1\|\operatorname{dev}\operatorname{sym}\operatorname{rt} p^l_{\eta,m}\|_2^2 + h\|\mathbb{B}^{1/2}\operatorname{rt}\sigma^l_{\eta,m}\|_2^2 + hC_2\|\operatorname{Curl}\operatorname{rt} p^l_{\eta,m}\|_2^2$$
$$\le 2hC^{(0)} + 2h\sum_{n=1}^{l}\big(\operatorname{rt} u^n_{\eta,m} - \operatorname{rt} u^{n-1}_{\eta,m}, \operatorname{rt} b^n_m\big)_\Omega, \tag{87}$$

where now $C^{(0)}$ denotes

$$2C^{(0)} := \|\mathbb{B}^{1/2}\operatorname{rt}\sigma^0_{\eta,m}\|_2^2.$$

We note that (86) yields the uniform boundness of $C^{(0)}$ with respect to $m$. Now, using Young's inequality with $\epsilon > 0$ in (87) and then summing the resulting inequality for $l = 1, ..., 2^m$ we derive the inequality

$$\frac{1}{m}\|\partial_t p_m\|^2_{2,\Omega_{T_e}} + \alpha_1\|\operatorname{dev}\operatorname{sym}(\partial_t p_m)\|^2_{2,\Omega_{T_e}} + C_2\|\operatorname{Curl}(\partial_t p_m)\|^2_{2,\Omega_{T_e}}$$
$$+C\|\partial_t\sigma_m\|^2_{2,\Omega_{T_e}} \le C_\varepsilon\|\partial_t b_m\|^2_{2,\Omega_{T_e}} + 2\varepsilon\|\partial_t u_m\|^2_{2,\Omega_{T_e}}, \tag{88}$$

where $C_\varepsilon$ is some positive constant independent of $m$ and $\eta$. Using now inequality (4), the condition $\partial_t p_m(x,t) \in \mathfrak{sl}(3)$ for a.e. $(x,t) \in \Omega_{T_e}$, and the ellipticity theory of linear systems we obtain that

$$\frac{1}{m}\|\partial_t p_m\|^2_{2,\Omega_{T_e}} + C_\epsilon(\Omega)\|\partial_t p_m\|^2_{2,\Omega_{T_e}} + C\|\partial_t\sigma_m\|^2_{2,\Omega_{T_e}} \le C_\varepsilon\|\partial_t b_m\|^2_{2,\Omega_{T_e}}, \tag{89}$$

where $C_\varepsilon(\Omega)$ is some further positive constant independent of $m$ and $\eta$. Since $b_m$ is uniformly bounded in $W^{1,q}(\Omega_{T_e}, \mathcal{S}^3)$, estimates (88) and (89) imply

$$\{\operatorname{dev}\operatorname{sym}\partial_t p_\eta\}_\eta \text{ is uniformly bounded in } L^2(0,T_e;L^2(\Omega,\mathcal{M}^3)), \tag{90}$$
$$\{\partial_t\sigma_\eta\}_\eta \text{ is uniformly bounded in } L^2(0,T_e;L^2(\Omega,\mathcal{M}^3)), \tag{91}$$
$$\{\operatorname{Curl}\partial_t p_\eta\}_\eta \text{ is uniformly bounded in } L^2(0,T_e;L^2(\Omega,\mathcal{M}^3)), \tag{92}$$
$$\{p_\eta\}_\eta \text{ is uniformly bounded in } H^1(0,T_e;L^2_{\operatorname{Curl}}(\Omega,\mathcal{M}^3)). \tag{93}$$

The proof of the lemma is complete. □

## 5.2 Proof of Theorem 5.7

Now, we can prove Theorem 5.7.

*Proof.* Due to Lemma 5.8, we have that the sequence of solutions $(u_\eta, \sigma_\eta)$ is weakly compact in $H^1(0,T_e;H^1_0(\Omega,\mathbb{R}^3)\times L^2(\Omega,\mathcal{S}^3))$ and the sequence $p_\eta$ is weakly compact in $H^1(0,T_e;L^2(\Omega,\mathcal{M}^3)) \cap L^2(0,T_e;Z^2_{\operatorname{Curl}}(\Omega,\mathcal{M}^3))$. Thus, by Proposition 3.3, Proposition 3.5 and Theorem 4.4, the uniform estimates (79) - (93) yield that there exist functions $u_0$, $u_1$, $\sigma$, $\sigma_0$, $p$, $p_0$ and $p_1$ with the prescribed regularities in Theorem 5.6 such that the convergences in (51) - (61) hold. Note that (53) - (56) give the equation (64), i.e

$$\sigma_0(x,y,t) = \mathbb{C}[y]\big(\operatorname{sym}(\nabla_x u_0(x,t) + \nabla_y u_1(x,y,t) - p_0(x,y,t))\big), \quad \text{a.e.} \tag{94}$$

By Proposition 3.3, the weak-star limit $\sigma$ of $\sigma_\eta$ in $L^\infty(0,T_e;L^2(\Omega,\mathcal{S}^3))$ and the weak limit $p$ of $p_\eta$ in $L^2(0,T_e;L^2(\Omega,\mathcal{M}^3))$ are related to $\sigma_0$ and $p_0$ in the following ways

$$\sigma(x,t) = \int_Y \sigma_0(x,y,t)dy, \quad p(x,t) = \int_Y p_0(x,y,t)dy.$$

Now, as in [17], we consider any $\phi \in C_0^\infty(\Omega,\mathbb{R}^3)$. Then, by the weak convergence of $\sigma_\eta$, the passage to the weak limit in (5) yields

$$\int_\Omega (\sigma(x,t), \nabla\phi(x))dx = \int_\Omega (b(x,t),\phi(x))dx, \tag{95}$$

i.e $\operatorname{div}_x \sigma = b$ in the sense of distributions. Next, define $\phi_\eta(x) = \eta\phi(x)\psi(x/\eta)$, where $\phi \in C_0^\infty(\Omega,\mathbb{R}^3)$ and $\psi \in C_{per}^\infty(Y,\mathbb{R}^3)$. Then, one obtains that

$$\phi_\eta \rightharpoonup 0, \quad \text{in } H_0^1(\Omega,\mathbb{R}^3), \quad \text{and} \quad \mathcal{T}_\eta(\nabla\phi_\eta) \to \phi\nabla_y\psi, \quad \text{in } L^2(\Omega, H^1_{per}(Y,\mathbb{R}^3)).$$

Therefore, since $\phi_\eta$ has a compact support,

$$\int_{\Omega\times Y} (\mathcal{T}_\eta(\sigma_\eta(t)), \mathcal{T}_\eta(\nabla\phi_\eta))dxdy \overset{\mathcal{T}_\eta}{\simeq} \int_\Omega (b(t),\phi_\eta)dx. \tag{96}$$

The passage to the limit in (96) leads to

$$\int_{\Omega\times Y} (\sigma_0(x,y,t), \phi(x)\nabla_y\psi(y))dxdy = 0.$$

Thus, in virtue of the arbitrariness of $\phi$, one can conclude that

$$\int_{\Omega\times Y} (\sigma_0(x,y,t), \nabla_y\psi(y))dxdy = 0. \tag{97}$$

i.e $\operatorname{div}_y \sigma_0(x,\cdot,t) = 0$ in the sense of distributions.

Next, let $\mathcal{T}_\eta(\mathcal{G}_\eta) : L^p(\Omega\times Y,\mathbb{R}^N) \to 2^{L^q(\Omega\times Y,\mathbb{R}^N)}$ and $\mathcal{G} : L^p(\Omega,\mathbb{R}^N) \to 2^{L^q(\Omega,\mathbb{R}^N)}$ denote the canonical extensions of $\mathcal{T}_\eta(g_\eta)(x,y) : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ and $g(y) : \mathbb{R}^N \to 2^{\mathbb{R}^N}$, respectively. Here, $g(y)$ is the pointwise limit graph of the convergent sequence of graphs $\mathcal{T}_\eta(g_\eta)(x,y)$. The existence of the limit graph for $\mathcal{T}_\eta(g_\eta)(x,y)$ guaranteed by Theorem 2.7. Indeed, the resolvent $j_\lambda^{\mathcal{T}_\eta(g_\eta)}$ converges pointwise to the resolvent $j_\lambda^g$, what follows from the periodicity of the mapping $y \to g(y,z) : Y \to 2^{\mathbb{R}^N}$ and the simple computations:

$$j_\lambda^{\mathcal{T}_\eta(g_\eta)}(x,y,z) = \mathcal{T}_\eta(j_\lambda^{g_\eta})(x,y,z) = j_\lambda^g(y,z),$$

for a.e. $(x,y) \in \Omega\times Y$ and every $z \in \mathbb{R}^N$. Thus, by Theorem 2.7 we get that

$$\mathcal{T}_\eta(g_\eta)(x,y) \rightarrowtail g(y) \tag{98}$$

holds for a.e. $(x,y) \in \Omega \times Y$. Since $g_\eta \in \mathcal{M}(\Omega, \mathbb{R}^N, p, \alpha, m)$, by Definition 3.7 of the unfolding operator for a multi-valued function it follows that $\mathcal{T}_\eta(g_\eta) \in \mathcal{M}(\Omega \times Y, \mathbb{R}^N, p, \alpha, m)$. Therefore, due to this and convergence (98), by Propositon 5.4(b) we obtain that

$$\mathcal{T}_\eta(\mathcal{G}_\eta) \rightarrowtail \mathcal{G}. \tag{99}$$

To prove that the limit functions $(\sigma_0, p_0)$ satisfy (65), we apply Theorem 2.6. Since the graph convergence is already established, we show that condition (13) is fulfilled. Using equations (5) and (6), we successfully compute that

$$\begin{aligned}
&\frac{1}{|Y|}\int_{\Omega\times Y}(\mathcal{T}_\eta(\partial_t p_\eta(t)), \mathcal{T}_\eta(\Sigma_\eta^{\text{lin}}(t)))dxdy \\
&= \frac{1}{|Y|}\int_{\Omega\times Y}\big(\mathcal{T}_\eta\big(\partial_t(\varepsilon(\nabla u_\eta(t)) - \mathbb{C}^{-1}\sigma_\eta(t))\big), \mathcal{T}_\eta(\sigma_\eta(t))\big)dxdy \\
&+ \frac{1}{|Y|}\int_{\Omega\times Y}\big(\mathcal{T}_\eta(\partial_t p_\eta(t)), \mathcal{T}_\eta(\Sigma_{\text{sh},\eta}^{\text{lin}}(t) + \Sigma_{\text{curl},\eta}^{\text{lin}}(t))\big)dxdy \\
&= \int_\Omega (b(t), \partial_t u_\eta(t))dx - \int_{\Lambda_\eta}(b(t), \partial_t u_\eta(t))dx \\
&- \frac{1}{|Y|}\int_{\Omega\times Y}\big(\mathcal{T}_\eta(\partial_t \mathbb{C}^{-1}\sigma_\eta(t))), \mathcal{T}_\eta(\sigma_\eta(t))\big)dxdy \\
&- \frac{1}{|Y|}\int_{\Omega\times Y}\big(C_1\mathcal{T}_\eta(\partial_t \operatorname{dev}\operatorname{sym} p_\eta(t)), \mathcal{T}_\eta(\operatorname{dev}\operatorname{sym} p_\eta(t))\big)dxdy \\
&- \frac{1}{|Y|}\int_{\Omega\times Y}\big(C_2\mathcal{T}_\eta(\partial_t \operatorname{Curl} p_\eta(t)), \mathcal{T}_\eta(\operatorname{Curl} p_\eta(t))\big)dxdy.
\end{aligned}$$

Integrating the last identity over $(0,t)$ and using the integration-by-parts formula we get that

$$\begin{aligned}
&\frac{1}{|Y|}\int_0^t (\mathcal{T}_\eta(\partial_t p_\eta(t)), \mathcal{T}_\eta(\Sigma_\eta^{\text{lin}}(t)))_{\Omega\times Y}dt \\
&\qquad = \int_0^t (b(t), \partial_t u_\eta(t))_\Omega dt - \int_0^t (b(t), \partial_t u_\eta(t))_{\Lambda_\eta}dt \\
&\qquad - \frac{1}{2}\|\mathcal{T}_\eta(\mathcal{B}^{1/2}\sigma_\eta(t))\|_{2,\Omega\times Y}^2 + \frac{1}{2}\|\mathcal{T}_\eta(\mathcal{B}^{1/2}\sigma_\eta(0))\|_{2,\Omega\times Y}^2 \\
&\qquad - \frac{1}{2}\|C_1^{1/2}\mathcal{T}_\eta(\operatorname{dev}\operatorname{sym} p_\eta(t))\|_{2,\Omega\times Y}^2 - \frac{1}{2}\|C_2^{1/2}\mathcal{T}_\eta(\operatorname{Curl} p_\eta(t))\|_{2,\Omega\times Y}^2,
\end{aligned} \tag{100}$$

where $\mathcal{B} = \mathbb{C}^{-1}$. Due to the uniform boundness of $\{\partial_t u_\eta\}$ in $L^2(0, T_e; H_0^1(\Omega, \mathbb{R}^3))$, we easily obtain that

$$\lim_{\eta\to 0}\int_0^t (b(t), \partial_t u_\eta(t))_{\Lambda_\eta}dt = 0.$$

Moreover, since $\sigma_\eta(0)$ solves the linear elasticity problem (15) - (17) with $\hat{\varepsilon}_\eta = 0$ and $\hat{b} = b(t)$, by Theorem 3.8, we can conclude that $\mathcal{T}_\eta(\mathcal{B}^{1/2}\sigma_\eta(0))$ converges to $\mathcal{B}^{1/2}\sigma_0(0)$ strongly in $L^2(\Omega \times Y, \mathcal{S}^3)$. Thus, by the lower semi-continuity of the norm the passing to the limit in (100) yields

$$\limsup_{n\to\infty} \frac{1}{|Y|}\int_0^t (\mathcal{T}_\eta(\partial_t p_\eta(t)), \mathcal{T}_\eta(\Sigma_\eta^{\rm lin}(t)))_{\Omega\times Y} dt$$
$$\le \int_0^t (b(t), \partial_t u_0(t))_\Omega dt - \frac{1}{2}\|\mathcal{B}^{1/2}\sigma_0(t)\|^2_{2,\Omega\times Y} + \frac{1}{2}\|\mathcal{B}^{1/2}\sigma_0(0)\|^2_{2,\Omega\times Y}$$
$$-\frac{1}{2}\|C_1 \operatorname{dev}\operatorname{sym} p_0(t)\|^2_{2,\Omega\times Y} - \frac{1}{2}\|C_2 \operatorname{Curl} p_0(t)\|^2_{2,\Omega\times Y},$$

or

$$\limsup_{n\to\infty} \frac{1}{|Y|}\int_0^t (\mathcal{T}_\eta(\partial_t p_\eta(t)), \mathcal{T}_\eta(\Sigma_\eta^{\rm lin}(t)))_{\Omega\times Y} dt$$
$$\le \int_0^t (b(t), \partial_t u_0(t))_\Omega dt - \frac{1}{|Y|}\int_0^t \big(\partial_t \mathbb{C}^{-1}\sigma_0(t), \sigma_0(t)\big)_{\Omega\times Y} dt \qquad (101)$$
$$-\frac{1}{|Y|}\int_0^t \big(\partial_t \operatorname{dev}\operatorname{sym} p_0(t), C_1 \operatorname{dev}\operatorname{sym} p_0(t)\big)_{\Omega\times Y} dt$$
$$-\frac{1}{|Y|}\int_0^t \big(\partial_t \operatorname{Curl} p_0(t), C_2 \operatorname{Curl} p_0(t)\big)_{\Omega\times Y} dt$$

We note that (95) and (97) imply

$$\int_\Omega (b(t), \partial_t u_0(t)) dx = \frac{1}{|Y|}\int_{\Omega\times Y} \big(\sigma_0(t), \partial_t\varepsilon(\nabla u_0(t) + \nabla_y u_1(t))\big) dxdy. \qquad (102)$$

And, since for almost all $(x,y,t)\in\Omega\times Y\times(0,T_e)$ one has

$$\big(\partial_t \operatorname{dev}\operatorname{sym} p_0(x,y,t), C_1[y] \operatorname{dev}\operatorname{sym} p_0(x,y,t)\big) = \big(\partial_t p_0(x,y,t), C_1[y] \operatorname{dev}\operatorname{sym} p_0(x,y,t)\big),$$

and that for almost all $t\in(0,T_e)$

$$\big(\partial_t \operatorname{Curl} p_0(t), C_2 \operatorname{Curl} p_0(t)\big)_{\Omega\times Y} = \big(\partial_t p_0(t), C_2 \operatorname{Curl}\operatorname{Curl} p_0(t)\big)_{\Omega\times Y},$$

the relations (101) and (102) together with (94) yield

$$\limsup_{n\to\infty} \frac{1}{|Y|}\int_0^t (\mathcal{T}_\eta(\partial_t p_\eta(t)), \mathcal{T}_\eta(\Sigma_\eta^{\rm lin}(t)))_{\Omega\times Y} dt$$
$$\le \frac{1}{|Y|}\int_0^t \big(\partial_t p_0(t), \Sigma_0^{\rm lin}(t)\big)_{\Omega\times Y} dt. \qquad (103)$$

In virtue of convergence (99) and inequality (103), Theorem 2.6 yields that

$$[\Sigma_0^{\rm lin}(x,y,t), \partial_t p_0(x,y,t)] \in Grg(y)$$

or, equivalently, that

$$\partial_t p_0(x,y,t) \in g(y, \Sigma_0^{\rm lin}(x,y,t)).$$

The initial and boundary conditions (66) - (68) for the limit functions $u_0$ and $p$ are easily obtained from the weak compactness of $u_\eta$ and $p_\eta$ in the spaces $H^1(0,T_e;H^1_0(\Omega,\mathbb{R}^3))$ and $H^1(0,T_e;L^2(\Omega,\mathcal{M}^3))\cap L^2(0,T_e;Z^2_{\rm Curl}(\Omega,\mathcal{M}^3))$, respectively. Therefore, summarizing everything done above, we conclude that the functions $(u_0,u_1,\sigma,\sigma_0,p,p_0,p_1)$ satisfy the homogenized initial-boundary value problem formed by the equations/inequalities (62) - (68). □

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