Analysis for an SP_1-N_ν-band model in radiative heat transfer

Debora Clever
Graduate School CE and Department of Mathematics, Technische Universität Darmstadt,
Dolivostr. 15, D-64293 Darmstadt, Germany

TECHNISCHE UNIVERSITÄT DARMSTADT

Numerical Analysis

1 Introduction

Optimal control of radiative heat transfer (RHT) is a challenging task due to the high complexity of the underlying partial differential equations. For solving this problem more efficiently there exist various approximation concepts, which reduce the numerical effort significantly while representing the main physical behavior [5, 6, 9, 12, 19–21].

In the publications [3, 4] we consider an SP_1-approximation with N_ν frequency bands to describe the cooling process of glass during manufacturing. The objective functional takes into account not only the tracking of the glass temperature and a regularization of the control, but also a minimization of mean internal stresses, special treatment at final time and a regularization of the control slope.

This work is devoted to the analysis of this model and the resulting optimal control problem. A detailed examination of the so called gray scale model with $N_\nu = 1$ is published by R. Pinnau in [15]. In [18] by A. Schulze the analytical background of a basis-transformed N_ν-band model is discussed. We build the following work on results achieved in Reference [15], Reference [18] and some recent work of R. Pinnau and O. Tse for a simplified natural convection-radiation model [16]. We augment the analytical studies to the needs of the considered SP_1-N_ν-band model and an objective which includes:

- tracking of the glass temperature to a desired spatially constant profile
- minimization of the L^2-norm of the glass temperature gradients
- special treatment of these two terms at final time
- Tikhonov regularization of the control and its time derivative.

The optimal control problem, including the SP_1 N_ν-band model and the considered objective is defined in Section 2. The analytical studies including an index analysis, existence and uniqueness of the state, existence of an optimal control and existence of the reduced derivatives are presented in Section 3. In Section 4 we finally formulate the optimality systems.

2 The Optimal Control Problem

One important step in glass manufacturing is the cooling of the hot and already formed glass down to room temperature. Because the quality of the final product depends highly on the temperature evolution within the glass during the cooling process, there is the need to control the behavior of the glass temperature. To this end, the hot glass is cooled within a furnace, which is preheated in the beginning. Choosing an optimal course for the temperature reduction within the oven, the temperature evolution within the glass can be influenced in such a way that the resulting product is of high quality. Within this setting it is reasonable to consider the furnace temperature and hence the control as spatially constant.

2.1 The Objective

To determine an appropriate optimal control, it is essential to formulate a sound objective. In the context of glass cooling one important aim is to force the glass temperature function T as close as possible to a desired temperature profile T_d. Such a profile, for which good performance of the involved chemical processes is known, is generally given by engineers. A common approach is to choose the tracking function for the glass temperature spatially constant in order to enforce a homogeneous cooling with small temperature gradients. This is necessary to reduce internal stresses and avoid cracks within the glass. Note that, because the cooling is controlled at the boundary only, such a guiding function can only be approached but generally not reached within the entire domain. To be able to reduce internal stresses independently of the temperature tracking, we also consider objective functionals that include an integral mean of the glass temperature gradient ∇T. Furthermore, it is desirable to pay certain attention to the glass temperature and its gradient at the final time. Especially in the context of the continuous adjoint calculus, such a term is of great importance, since it affects the initial values of the adjoint systems.

Finally, the objective has to include a regularization of the control. To this end, we consider a tracking of the control itself and a minimization of the time derivative of u. The first term can either be used to search an optimal control close to a preferable profile or to minimize the manufacturing costs. The second term can be introduced to avoid unphysical fast changes within the furnace temperature. An objective functional that meets all the requirements stated above can be defined by

$$\begin{aligned} J(y,u) :=& \frac{1}{2}\int_0^{t_e} \left\|T - T_d\right\|^2_{L^2(\Omega)} \mathrm{d}t + \frac{1}{2}\int_0^{t_e} \delta_g(t)\|\nabla T\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{\delta_e}{2}\left\|(T-T_d)(t_e)\right\|^2_{L^2(\Omega)} \\ &+ \frac{\delta_{ge}}{2}\|\nabla T(t_e)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^{t_e} \delta_u(t)(u-u_d)^2\mathrm{d}t + \frac{1}{2}\int_0^{t_e} \delta_d(t)(\partial_t u)^2\mathrm{d}t, \end{aligned} \tag{1}$$

with spatial domain Ω, process interval $(0,t_e)$, glass temperature distribution $T(x,t)$, desired glass temperature distribution $T_d(x,t)$, furnace temperature (control) $u(t)$, guideline for the control $u_d(t)$ and the positive weights $\delta_g(t)$, δ_e, δ_{ge}, $\delta_u(t)$, and $\delta_d(t)$, that are either constants or functions in $L^2(0,t_e)$.

To account for the operation interval of the furnace, it is important to restrict the control u to the feasible set

$$U_{ad} := \{u \in L^2(0,t_e;\mathbb{R}) : u_{\text{low}}(t) \le u(t) \le u_{\text{up}}(t), \forall t \in [0,t_e]\}, \tag{2}$$

with lower bound $u_{\text{low}}(t)$ and upper bound $u_{\text{up}}(t)$.

2.2 The Glass Cooling Model

To model the cooling process itself, it is an important observation, that because of the high temperatures that occur especially at the beginning of the cooling process, the direction- and frequency-dependent thermal radiation field and the spectral radiative properties of semi-transparent glass play a dominant role. In the following we describe radiation by the mean radiative intensities $\phi_i(x,t)$, $(x,t) \in \Omega \times [0,t_e]$, $i = 1,\dots,N_\nu$, where we discretize the semi-transparent region of the continuous frequency spectrum into N_ν bands $[\nu_{i-1},\nu_i]$, $i = 0,\dots,N_\nu$ and formally set $\nu_{N_\nu} := \infty$ and $\nu_{-1} := 0$. On each of the bands we interpret frequency dependent quantities as constants and define the frequency-independent mean

$$B^{(i)}(v) := \int_{\nu_{i-1}}^{\nu_i} B(v,\nu)\mathrm{d}\nu, \; i = 0,\dots,N_\nu, \tag{3}$$

of the Planck function $B(v,\nu)$. Then, the SP$_1$-N$_\nu$-band approximation of the full RHT equation is given by the following system of space-time dependent partial differential algebraic equations of mixed parabolic-elliptic type in $N_\nu + 1$ components $y := (T,\phi_1,\dots,\phi_{N_\nu})^T$,

$$\partial_t T - k_c \Delta T - \sum_{i=1}^{N_\nu} \frac{1}{3\left(\sigma_i + \kappa_i\right)} \Delta\phi_i = 0, \tag{4}$$

$$-\frac{\epsilon^2}{3\left(\sigma_i + \kappa_i\right)} \Delta\phi_i = -\kappa_i\phi_i + 4\pi\kappa_i B^{(i)}(T), \;\; i = 1,\dots,N_\nu, \tag{5}$$

with boundary and initial conditions

$$\begin{aligned} k_c n \cdot \nabla T + \sum_{i=1}^{N_\nu} \frac{1}{3\left(\sigma_i+\kappa_i\right)} n \cdot \nabla\phi_i =& \frac{h_c}{\epsilon}(u-T) + \\ &+ \frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2 \left(B^{(0)}(u) - B^{(0)}(T)\right) + \frac{a_1}{\epsilon}\sum_{i=1}^{N_\nu}\left(4\pi B^{(i)}(u) - \phi_i\right), \end{aligned} \tag{6}$$

$$\frac{\epsilon^2}{3\left(\sigma_i+\kappa_i\right)} n \cdot \nabla\phi_i = a_1\epsilon\left(4\pi B^{(i)}(u) - \phi_i\right), \;\; i = 1,\dots,N_\nu, \tag{7}$$

$$T(x,0) = T_0(x). \tag{8}$$

and with glass temperature $T(x,t)$, $(x,t) \in \Omega \times [0,t_e]$, furnace temperature $u(t) \in U_{ad}$, boundary condition coefficient $a_1 = 1.149e-1$ and piecewise constant scattering and absorption coefficients σ_i and κ_i, $i = 1,\dots,N_\nu$.

The Optimal Control Problem

The optimal control problem can now be formulated as follows:

$$\min_{u \in U_{ad}} J(y,u) \text{ such that (4)-(8) hold.} \tag{9}$$

3 Analysis of the Optimal Control Problem

Given the space-time cylinder $Q := \Omega \times (0,t_e)$ and its spatial boundary $\Sigma := \partial\Omega \times (0,t_e)$, with convex, bounded domain $\Omega \in \mathbb{R}^d$, $d = 1,2,3$ and sufficiently smooth Lipschitz boundary $\partial\Omega$, see [15, Remark 1.4], we define the following spaces

$$U := L^2(0,t_e;\mathbb{R}), \tag{10}$$
$$W := L^2(0,t_e;H^1(\Omega)), \tag{11}$$
$$X := \{w \in W : \partial_t w \in W^*\}, \tag{12}$$
$$Y := [X \times W^{N_\nu}] \cap [L^\infty(Q)]^{N_\nu+1}, \tag{13}$$
$$V := Y \times U, \tag{14}$$
$$Z := W^{N_\nu+1} \times L^2(\Omega). \tag{15}$$

In the following we refer to U as *space of controls* or *control space*, to Y as *space of states* or *state space*, and to Z as *space of adjoint states* or *adjoint space*. Furthermore, we identify the dual space $W^* = L^2(0,t_e;H^1(\Omega))^*$ with $L^2(0,t_e;(H^1)^*(\Omega))$ and the dual adjoint space $Z^* = (W^{N_\nu+1} \times L^2(\Omega))^*$ with $[W^*]^{N_\nu+1} \times L^2(\Omega)$. Note, that especially in three spatial dimensions it is not sufficient to consider the more general space $X \times W^{N_\nu}$ instead of $Y := [X \times W^{N_\nu}] \cap [L^\infty(Q)]^{N_\nu+1}$ as the space of states. This is due to the fact that the non-linear terms on the boundary are not necessarily integrable, see [10], and therefore it is not guaranteed that the state operator $e(y,u)$ is well defined. For the N_ν-band model the non-linear state operator $e := (e_0,\dots,e_{N_\nu+1}) : V \to Z^*$, for all $\xi \in W$, is given by

$$\begin{aligned}
\langle e_0(y,u),\xi\rangle_{W^*,W} :=\ & \langle \partial_t T,\xi\rangle_{W^*,W} + k_c(\nabla T,\nabla\xi)_{L^2(Q)} + \sum_{i=1}^{N_\nu} \frac{1}{3(\sigma_i+\kappa_i)}(\nabla\phi_i,\nabla\xi)_{L^2(Q)} \\
&+ \frac{h_c}{\epsilon}(T-u,\xi)_{L^2(\Sigma)} + \frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2 (B^{(0)}(T)-B^{(0)}(u),\xi)_{L^2(\Sigma)} \\
&+ \frac{a_1}{\epsilon}\sum_{i=1}^{N_\nu}(\phi_i - 4\pi B^{(i)}(u),\xi)_{L^2(\Sigma)}
\end{aligned} \tag{16}$$

$$\begin{aligned}
\langle e_i(y,u),\xi\rangle_{W^*,W} :=\ & \frac{\epsilon^2}{3(\sigma_i+\kappa_i)}(\nabla\phi_i,\nabla\xi)_{L^2(Q)} + \kappa_i(\phi_i - 4\pi B^{(i)}(T),\xi)_{L^2(Q)} \\
&+ a_1\epsilon(\phi_i - 4\pi B^{(i)}(u),\xi)_{L^2(\Sigma)}, \quad i = 1,\dots,N_\nu
\end{aligned} \tag{17}$$

and

$$e_{N_\nu+1} := T(0) - T_0. \tag{18}$$

In this subsection we state the following two definitions and determine the corresponding differential indices for the presented SP_1-N_ν-band glass cooling model.

Definition 3.1 *Let a generalized class of PDAEs of the form*

$$H(x,t,v,\nabla v)\partial_t v(x,t) = \nabla \cdot (D(x,t,v,\nabla v)\nabla v(x,t)) + F(x,t,v,\nabla v), \tag{19}$$

be given. If the matrix H is regular, the ***differential time index*** d^t_{idx} *of the PDAE* (19) *is defined to be zero. If H is singular* d^t_{idx} *is the minimum number of times, that all or part of the PDAE* (19) *must be differentiated with respect to* t *in order to obtain* $\partial_t v$ *as a continuous function of* x*,* t *and* v*. This function may include partial differential operators in* x*, see [11, 13, 14].*

To be able to define the differential spatial index we consider the transformed PDAE

$$\bar{H}\partial_t \bar{v} - \nabla \cdot (\bar{D}\nabla \bar{v}) = \bar{F}, \tag{20}$$

which is quasi-linear with respect to the second spatial derivatives, by defining

$$\bar{D} = S_0 D S_1^{-1} = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}, \tag{21}$$

$$\bar{H} = S_0 H S_1^{-1},\ \bar{F} = S_0 F S_1^{-1},\ \text{and}\ \bar{v} = S_1 v \tag{22}$$

with constant and regular matrices $S_0, S_1 \in \mathbb{R}^{n\times n}$. For more details we refer to [13].

Definition 3.2 *Let a generalized class of PDAEs of form* (19) *be given. If the matrix D is regular, the* ***differential spatial index*** d^x_{idx} *of the PDAE* (19) *is defined to be zero. If D is singular,* d^x_{idx} *is the minimum number of times, that the quasi-linear PDAE* (20) *must be differentiated with respect to* x *in order to obtain*

$$\bar{V} := \left(\Delta \bar{u}_1, \dots, \Delta \bar{u}_m, \nabla \bar{u}_{n-m}, \dots, \nabla \bar{u}_n\right) \tag{23}$$

as a continuous function of x*,* t*,* $\bar{v}$*,* $\partial_t \bar{v}$ *and* $\nabla \bar{v}_1, \dots, \nabla \bar{v}_m$*, see [13].*

To apply these definitions we exploit the semi-explicit structure of the underlying PDAEs. Alternatively, one can also follow the proof of Lemma 2 in [13] and augment the examination to the case where F in (19) may depend non-linearly on the solution y, to be able to handle the Planck function B and the non-linearity T^4, see also [2].

To show that the SP_1-N_ν-band system has a differential time index of one $d^t_{\text{idx}} = 1$, we consider the abstract formulation

$$\partial_t T = \tilde{f}(T, \Phi), \tag{24}$$

$$0 = g(T, \Phi) \tag{25}$$

with $\Phi := (\phi_1, \dots, \phi_{N_\nu})^T$ and show

$$\begin{aligned} & 0 = \partial_t g(T,\Phi) \\ \Leftrightarrow\quad & 0 = \partial_T g \cdot \partial_t T + \partial_\Phi g \cdot \partial_t \Phi \\ \Leftrightarrow\quad & 0 = \partial_T g \cdot \tilde{f} + \partial_\Phi g \cdot \partial_t \Phi, \end{aligned} \tag{26}$$

is uniquely solvable for $\partial_t \Phi$.

In the following Section we show that the state system is uniquely solvable. Under the assumption of suitable differentiation properties, this solution is also a solution of (26). Then, uniqueness of this solution follows from continuous invertibility of $\partial_\phi g$, with

$$\partial_\Phi g = \begin{pmatrix} \kappa_1 - \frac{\epsilon^2}{3(\sigma_1+\kappa_1)}\Delta & 0 & \dots & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & & \dots & 0 & \kappa_{N_\nu} - \frac{\epsilon^2}{3(\sigma_{N_\nu}+\kappa_{N_\nu})}\Delta \end{pmatrix}. \tag{27}$$

Furthermore, because of the regular diffusion matrix D it has the differential spatial index $d^x_{\text{idx}} = 0$.

3.2 Existence and Uniqueness of the State

To be able to reduce the SP_1-N_ν-band based glass cooling problem to its control component, we first have to show that there exists a unique solution y of the state system (4)-(8).

As presented in [15] for the gray scale model, for the N_ν-band model it is also possible to employ the fixed point theorem of Leray-Schauder (see e.g. [7, Theorem 10.6]) and the Stampacchia truncation method to show the existence of uniformly bounded states. Uniqueness is then shown in a second step by contradiction.

Existence.

Let $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, be a bounded domain with Lipschitz boundary and let $u \in U_{ad} \subset U$ and $T_0 \in L^\infty(\Omega)$ be given. To apply the theorem of Leray-Schauder, for the case of the N_ν-band model a suitable fixed point mapping is given by

$$G : L^2(Q) \times [0,1] \to L^2(Q), \quad (w,\sigma) \mapsto G(w,\sigma) = T, \tag{28}$$

with $w \in L^2(Q)$ and $\sigma \in [0,1]$ fulfilling the auxiliary problem

$$\partial_t T - \nabla \cdot (k_c \nabla T) = \sigma \sum_{i=1}^{N_\nu} \nabla \cdot \left(\frac{1}{3(\sigma_i + \kappa_i)} \nabla \phi_i \right) \tag{29}$$

$$-\epsilon^2 \nabla \cdot \left(\frac{1}{3(\sigma_i+\kappa_i)} \nabla \phi_i \right) + \kappa_i \phi_i = 4\pi\kappa_i B^{(i)}([w]_{\underline{T},\overline{T}}), \quad i = 1\dots,N_\nu \tag{30}$$

with boundary conditions

$$\frac{h_c}{\epsilon} T + k_c n \cdot \nabla T + \frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2 B^{(0)}(T) = \sigma \left(\frac{h_c}{\epsilon} u + \frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2 B^{(0)}(u) \right) \tag{31}$$

$$a_1 \epsilon \phi_i + \frac{\epsilon^2}{3(\sigma_i+\kappa_i)} n \cdot \nabla \phi_i = a_1 \epsilon 4\pi B^{(i)}(u), \quad i = 1\dots,N_\nu \tag{32}$$

$$T(0,x) = \sigma T_0, \text{ in } L^2(\Omega), \tag{33}$$

and the cut-off operator

$$[w]_{\underline{T},\overline{T}} = \begin{cases} \overline{T}, & w \geq \overline{T} \\ w, & \overline{T} > w \geq \underline{T} \\ \underline{T}, & w < \underline{T}. \end{cases} \tag{34}$$

Since the $N_v + 1$ equations of the auxiliary problem decouple, we can show existence and uniqueness of T and ϕ_i, $i = 1, \dots, N_v$ separately and conclude that the fix point mapping (28) is well defined. To show, that each equation (30) has a unique solution ϕ_i we employ Lax-Milgram's lemma, see e.g. [8, Lemma 1.8]. Therefore, we define $a_1(y, v) = F_1(v)$, with

$$a_1(y, v) = \frac{\epsilon^2}{3(\sigma_i + \kappa_i)}(\nabla y, \nabla v)_{L^2(Q)} + \kappa_i(y, v)_{L^2(Q)}, \quad F_1(v) = (4\pi\kappa_i B^{(i)}([w]_{\underline{T},\overline{T}}), v)_{L^2(Q)}. \tag{35}$$

Obviously, the mapping a_1 is a bilinear form. Boundedness of a_1 follows by the trace theorem, see [8, Theorem 1.2.1] and V-coercivity is shown by Poincaré's inequality. Then, Lax-Milgram's lemma can be applied.

Remark 3.3 *Since a_1 is symmetric the existence of a unique solution also follows directly from Riesz representation theorem.*

Identifying the space $L^2(0, t_e; (H^1)^*(\Omega))$ with $W^* = L^2((0, t_e); (H^1)(\Omega))^*$ and observing that $\sum_{i=1}^{N_v} \nabla \cdot \left(\frac{1}{3(\sigma_i+\kappa_i)}\nabla\phi_i\right) \in W^*$, the uniqueness of $T \in X$ follows from [17, Theorem 10.3]. To this end, we define $\partial_t y = A(t)y + f_2(t)$, with $y(0) = y_0$ and

$$-(A(t)y, v) = a_2(t, y, v) = k_c(\nabla y, \nabla v)_{L^2(\Omega)}, \quad f_2(t) = \sigma \sum_{i=1}^{N_v} \nabla \cdot \left(\frac{1}{3(\sigma_i + \kappa_i)}\nabla\phi_i\right). \tag{36}$$

We consider the Gelfand triple $H^1(\Omega) \subset L^2(\Omega) \subset (H^1)^*(\Omega)$. Because $A(t) \in \mathcal{L}(H^1(\Omega), (H^1)^*(\Omega))$ depends continuously on $t \in [0, t_e]$, because the parameterized quadratic form $a_2(t, y, v)$ satisfies the coercivity condition and because $\sum_{i=1}^{N_v} \nabla \cdot \left(\frac{1}{3(\sigma_i+\kappa_i)}\nabla\phi_i\right) \in W^*$ [15], uniqueness of $T \in X$ follows from [17, Theorem 10.3].

With G (28) compact (shown analogously to [15]), and w bounded in $L^2(Q)$ for all (w, σ), which satisfy $w = G(w, \sigma)$, the Leray-Schauder fixed point theorem proofs, that the mapping $G(\cdot, 1)$ has a fixed point. Because the solution $(T, \phi_1, \dots, \phi_{N_v})$ is uniformly bounded with

$$\underline{T} = \min\left(\inf_{t\in(0,t_e)} u(t), \inf_{x\in\Omega} T_0(x)\right), \tag{37}$$

$$\overline{T} = \min\left(\sup_{t\in(0,t_e)} u(t), \sup_{x\in\Omega} T_0(x)\right), \tag{38}$$

$$\underline{\phi}_i = 4\pi B^{(i)}(\underline{T}),\ i = 1, \dots, N_v, \tag{39}$$

$$\overline{\phi}_i = 4\pi B^{(i)}(\overline{T}),\ i = 1, \dots, N_v, \tag{40}$$

Gronwall's lemma shows that every fixed point of $G(\cdot, 1)$ is a solution of the state equation (16)-(18). Hence, there exists at least one solution $(T, \phi_1, \dots, \phi_N) \in Y$.

Uniqueness.

In the following paragraph uniqueness of the state is shown by contradiction. Therefore, we assume, that there exist two solutions $y_1 := (T_1, \phi_{1,1}, \ldots, \phi_{N_v,1})$, $y_2 := (T_2, \phi_{1,2}, \ldots, \phi_{N_v,2}) \in Y$. Then, the difference $\hat{y} := (\hat{T}, \hat{\phi}_1, \ldots, \hat{\phi}_{N_v}) := y_1 - y_2$ is a solution of

$$\partial_t \hat{T} - \nabla \cdot (k_c \nabla \hat{T}) - \sum_{i=1}^{N_v} \nabla \cdot \left(\frac{1}{3(\sigma_i + \kappa_i)} \nabla \hat{\phi}_i \right) = 0, \tag{41}$$

$$-\epsilon^2 \nabla \cdot \left(\frac{1}{3(\sigma_i + \kappa_i)} \nabla \hat{\phi}_i \right) = -\kappa_i \hat{\phi}_i + 4\pi\kappa_i \left(B^{(i)}(T_1) - B^{(i)}(T_2) \right), \; i = 1, \ldots, N_v, \tag{42}$$

$$k_c n \cdot \nabla \hat{T} + \sum_{i=1}^{N_v} \frac{1}{3(\sigma_i + \kappa_i)} n \cdot \nabla \hat{\phi}_i = -\frac{h_c}{\epsilon} \hat{T} - \frac{\alpha\pi}{\epsilon} \left(\frac{n_a}{n_g} \right)^2 \left(B^{(0)}(T_1) - B^{(0)}(T_2) \right) - \frac{a_1}{\epsilon} \sum_{i=1}^{N_v} \hat{\phi}_i, \tag{43}$$

$$\frac{\epsilon^2}{3(\sigma_i + \kappa_i)} n \cdot \nabla \hat{\phi}_i = -a_1 \epsilon \hat{\phi}_i, \; i = 1, \ldots, N_v, \tag{44}$$

$$\hat{T}(x, 0) = 0. \tag{45}$$

To show $\hat{T} = 0$ and $\hat{\phi}_i = 0$, $i = 1, \ldots, N_v$, we first test each of the N_v equations of (42) with the corresponding difference $\hat{\phi}_i$. After integration by parts we get

$$\frac{\epsilon^2}{3(\sigma_i + \kappa_i)} \|\nabla \hat{\phi}_i\|^2_{L^2(Q)} + \kappa_i \|\hat{\phi}_i\|^2_{L^2(Q)} \leq 4\pi\kappa_i \left(B^{(i)}(T_1) - B^{(i)}(T_2) \right), \hat{\phi}_i)_{L^2(Q)}, \; i = 1, \ldots, N_v. \tag{46}$$

Using the Cauchy-Schwarz inequality and the fact that $\|\hat{\phi}_i\|_{L^2(Q)} \leq \|\hat{\phi}_i\|_W$ we get

$$\|\hat{\phi}_i\|_W \leq c_{h1} \|B^{(i)}(T_1) - B^{(i)}(T_2)\|_{L^2(Q)}, \tag{47}$$

$c_{h1} > 0$. Substituting $B^{(i)}(T_1)$ by its Taylor expansion at T_2 we can modify the difference of the right hand side as follows

$$B^{(i)}(T_1) - B^{(i)}(T_2) = B^{(i)}(T_2 + \hat{T}) - B^{(i)}(T_2) \tag{48}$$

$$= B^{(i)}(T_2) + \sum_{j=1}^{\infty} \frac{1}{j!} (\partial_T)^j B^{(i)}(T_2) \hat{T}^j - B^{(i)}(T_2) \tag{49}$$

$$= \frac{1}{2} \hat{T} \left(\partial_T B^{(i)}(T_2) + \sum_{j=0}^{\infty} \frac{1}{j!} (\partial_T)^{(j+1)} B^{(i)}(T_2) \hat{T}^j \right) \tag{50}$$

$$= \frac{1}{2} \hat{T} \left(\partial_T B^{(i)}(T_2) + \partial_T B^{(i)}(T_1) \right) = c_{h2} \hat{T}, \tag{51}$$

with $0 \leq c_{h2} < \infty$, because of the continuity and the monotonicity of the Planck function $B^{(i)}$. This gives

$$\|\hat{\phi}_i\|_W \leq c_{h3} \|\hat{T}\|_{L^2(Q)}, \; c_{h3} > 0. \tag{52}$$

Second, we test (41) with $\hat{T}(t)$, where we substitute

$$\nabla \cdot \left(\frac{1}{3(\sigma_i + \kappa_i)} \nabla \hat{\phi}_i \right) = \frac{\kappa_i}{\epsilon^2} \hat{\phi}_i - \frac{4\pi\kappa_i}{\epsilon^2} \left(B^{(i)}(T_1) - B^{(i)}(T_2) \right),$$

and account for the monotonicity of the Planck function which implies that

$$\left(\frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2\left(B^{(0)}(T_1)-B^{(0)}(T_2)\right),\hat{T}(t)\right)_{L^2(\Omega)} > 0.$$

After some minor transformations we get

$$\frac{1}{2}\partial_t\|\hat{T}(t)\|^2_{L^2(\Omega)} + k_c\|\nabla\hat{T}(t)\|^2_{L^2(\Omega)} \leq \sum_{i=1}^{N_\nu}\frac{\kappa_i}{\epsilon}(\hat{\phi}_i(t),\hat{T}(t))_{L^2(\Omega)} \leq \sum_{i=1}^{N_\nu}\frac{\kappa_i}{\epsilon}|(\hat{\phi}_i(t),\hat{T}(t))_{L^2(\Omega)}|. \tag{53}$$

Using the Cauchy-Schwarz inequality and (52) we finally have

$$\frac{1}{2}\partial_t\|\hat{T}(t)\|^2_{L^2(\Omega)} + k_c\|\nabla\hat{T}(t)\|^2_{L^2(\Omega)} \leq \sum_{i=1}^{N_\nu}\frac{\kappa_i}{\epsilon}\|\hat{\phi}_i(t)\|_{L^2(\Omega)}\|\hat{T}(t)\|_{L^2(\Omega)} \leq c\|\hat{T}(t)\|^2_{L^2(\Omega)}, \tag{54}$$

for all $t \in (0, t_e)$. Analogously to [15] we can now use Gronwall's Lemma to deduce

$$\|\hat{T}(t)\|_{L^2(\Omega)} = 0 \text{ for all } t \in (0, t_e). \tag{55}$$

Hence, we have $\hat{T} = 0$ a.e. in Q and consequently $\hat{\phi}_i = 0$, $i = 1, \ldots, N_\nu$, a.e. in Q, which finally shows uniqueness of the state.

The Reduced Problem

Based on existence and uniqueness of uniformly bounded states, we can post the reduced optimal control problem as follows:

$$\begin{aligned}\min_{u\in U_{ad}} \hat{J}(u) := J(y(u),u) :=& \frac{1}{2}\int_0^{t_e}\left\|T-T_d\right\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{1}{2}\int_0^{t_e}\delta_g(t)\|\nabla T\|^2_{L^2(\Omega)}\mathrm{d}t\\ &+\frac{1}{2}\delta_e\left\|(T-T_d)(t_e)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\delta_{ge}\|\nabla T(t_e)\|^2_{L^2(\Omega)}\\ &+\frac{1}{2}\int_0^{t_e}\delta_u(t)(u-u_d)^2\mathrm{d}t + \frac{1}{2}\int_0^{t_e}\delta_d(t)(\partial_t u)^2\mathrm{d}t,\\ \text{where } y(u)\in Y \text{ satisfies } e(y(u),u)=0.&\end{aligned} \tag{56}$$

3.3 Existence of an optimal control

Showing that the considered objective functional (1) is

A1. of separated type,

A2. twice continuously Fréchet differentiable with locally Lipschitz continuous second derivative,

A3. radially unbounded with respect to u for every $y \in Y$,

A4. bounded from below,

A5. weakly lower semi-continuous,

the existence of a minimizer $(y^*,u^*) \in Y \times U$ and finally that of an optimal control $u \in U$ can be shown analogously to [15]. Therefore, in this subsection we only sketch the proof, and show that the considered objective fulfills the assumptions A1-A5.

Given a minimizing sequence $(y_k,u_k)_{k\in\mathbb{N}} \in Y \times U$, radially unboundedness of J with respect to u, boundedness of $(\|y_k\|_Y)_{k\in\mathbb{N}}$ (see (37)) and Sobolev's embedding theorem [1] imply that there exist subsequences, such that

$$\begin{aligned} u_k &\rightharpoonup u^* && \text{weakly in } U, && (57)\\ T_k &\rightharpoonup T^* && \text{weakly in } W, && (58)\\ \partial_t T_k &\rightharpoonup \partial_t T^* && \text{weakly in } W^*, && (59)\\ (\phi_i)_k &\rightharpoonup \phi_i^* && \text{weakly in } W,\ i=1,\dots,N_v, && (60)\end{aligned}$$

for $k \to \infty$. Weak lower semi-continuity of J implies

$$J(y^*,u^*) = \inf_{Y\times U} J(y,u) > 0. \tag{61}$$

Uniform boundedness of the control u and the state y, Sobolev's embedding theorem and Aubin's lemma finally show that for the minimizer $(y^*,u^*) = (T^*,\phi_1^*,\dots,\phi_{N_v}^*)$, the state systems

$$e(y^*,u^*) = 0 \in Z^* \tag{62}$$

holds, which implies the existence of at least one optimal control u. Note, that generally such an optimal control is not unique, because the set of states given by the constraint e is not convex.

In the following, we show that the objective functional (1) fulfills the five assumptions made above.

A1. Separation.

Obviously, the considered objective functional (1) can be separated into two parts, such that

$$J(y,u) = J_1(y) + J_2(u) \tag{63}$$

with

$$\begin{aligned} J_1(y) = \frac{1}{2}\int_0^{t_e} \left\|T - T_d\right\|^2_{L^2(\Omega)} \mathrm{d}t + \frac{1}{2}\int_0^{t_e} \delta_g(t)\|\nabla T\|^2_{L^2(\Omega)}\mathrm{d}t \\ + \frac{\delta_e}{2}\left\|(T-T_d)(t_e)\right\|^2_{L^2(\Omega)} + \frac{\delta_{ge}}{2}\|\nabla T(t_e)\|^2_{L^2(\Omega)}, \qquad (64)\\ J_2(u) = \frac{1}{2}\int_0^{t_e} \delta_u(t)(u-u_d)^2\mathrm{d}t + \frac{1}{2}\int_0^{t_e} \delta_d(t)(\partial_t u)^2\mathrm{d}t. \qquad (65)\end{aligned}$$

A2. Fréchet differentiability.

Being of quadratic type the considered objective is two times Fréchet differentiable. The first derivative is given by

$$\langle \partial_{(y,u)} J(y,u),(s,d)\rangle_{V^*,V} = \langle \partial_y J_1(y),s\rangle_{Y^*,Y} + \langle \partial_u J_2(u),d\rangle_{U^*,U}, \tag{66}$$

with

$$\begin{aligned} \langle \partial_y J_1(y),s\rangle_{Y^*,Y} = \int_Q (T-T_d)s\mathrm{d}x\mathrm{d}t + \int_Q \delta_g(t)\nabla T\nabla s\mathrm{d}x\mathrm{d}t \\ +\delta_e\int_\Omega (T-T_d)(t_e)\boldsymbol{\delta}(t_e)s\mathrm{d}x + \delta_{ge}\int_\Omega \nabla T(t_e)\boldsymbol{\delta}(t_e)\nabla s\mathrm{d}x, \qquad (67)\\ \langle \partial_u J_2(u),d\rangle_{U^*,U} = \int_0^{t_e} \delta_u(t)(u-u_d)d\mathrm{d}t + \int_0^{t_e} \delta_d(t)\partial_t u\partial_t d\mathrm{d}t, \qquad (68)\end{aligned}$$

where $\boldsymbol{\delta}$ represents the Dirac delta function. The second derivative is given by

$$\langle \partial_{(y,u)(y,u)} J(y,u)(s_1,d_1),(s_2,d_2)\rangle_{V^*,V} = \langle \partial_{yy} J_1(y)s_1,s_2\rangle_{Y^*,Y} + \langle \partial_{uu} J_2(u)d_1,d_2\rangle_{U^*,U}, \tag{69}$$

with

$$\begin{aligned}\langle \partial_{yy} J_1(y),s\rangle_{Y^*,Y} &= \int_Q s_1 s_2 \mathrm{d}x\mathrm{d}t + \int_Q \delta_g(t)\nabla s_1 \nabla s_2 \mathrm{d}x\mathrm{d}t \\ &\quad + \delta_e \int_\Omega \boldsymbol{\delta}(t_e)s_1 \boldsymbol{\delta}(t_e)s_2 \mathrm{d}x + \delta_{ge}\int_\Omega \boldsymbol{\delta}(t_e)\nabla s_1 \boldsymbol{\delta}(t_e)\nabla s_2 \mathrm{d}x, \end{aligned} \tag{70}$$

$$\langle \partial_{uu} J_2(u),d\rangle_{U^*,U} = \int_0^{t_e} \delta_u(t) d_1 d_2 \mathrm{d}t + \int_0^{t_e} \delta_d(t)\partial_t d_1 \partial_t d_2 \mathrm{d}t. \tag{71}$$

A3. Radially unboundedness.

Since $\|u\|_U \to \infty$ implies $J_2(u) \to \infty$ and $J_1 \geq 0$ for every y by definition, we can show that $\|u\|_U \to \infty$ also implies $J(y,u) \to \infty$ for every y. Hence, the considered objective functional is radially unbounded with respect to u.

A4. Boundedness from below.

By definition the objective functional is always greater or equal to zero and hence bounded from below.

A5. Weak lower semi-continuity.

By definition, for a Banach space X, any continuous, convex functional $F : X \to \mathbb{R}$ is weakly lower semi-continuous, e.g. [8]. Continuity is obvious and also convexity

$$J(\omega y_1 + (1-\omega)y_2, \omega u_1 + (1-\omega)u_2) \leq \omega J(y_1,u_1) + (1-\omega)J(y_2,u_2). \tag{72}$$

y_1, $y_2 \in Y$, u_1, $u_2 \in U$ and $\omega \in [0,1]$ can easily be shown by using the quadratic structure of J with respect to y and u.

3.4 Existence of reduced gradient and reduced Hessian

To show that the reduced derivatives

$$\nabla \hat{J}(u) = \nabla_u J(y,u) + \nabla_u e^*(y,u)\xi \in U, \tag{73}$$

and

$$\hat{J}''(u)s_u = \partial_{uu} J(y,u)s_u + \partial_{uu} e^*(y,u)\xi s_u + \partial_u e^*(y,u)w + \partial_{uy} J(y,u)s_y + \partial_{uy} e^*(y,u)\xi s_y, \tag{74}$$

are well defined, it has to be shown, that the mapping $y : U \to Y$, $u \mapsto y(u)$ is continuously Fréchet differentiable with derivative

$$y'(u) = -\partial_y e^{-1}(y,u)\partial_u e(y,u). \tag{75}$$

Usually, this is done my means of the implicit function theorem by showing that the PDAE-constraint $e(y,u) = 0$ is twice continuously Fréchet differentiable and that $\partial_y e(y,u)$ has a bounded inverse. However, non-linearities in form of Nemyzki operators are only differentiable from L^∞ to L^∞ or with norm gap. To this end we define an operator $R : Y \times U \to Y$, as suggested in [15], that separates linear and non-linear parts, given by

$$R(y,u) = y + D^{-1}N(y) + D^{-1}B(u),\ D : Y \to Z^*,\ N : Y \to L^\infty(Q)^{N_v+2},\ B : U \to L^\infty(\Sigma)^{N_v+2}, \tag{76}$$

with

$$\langle D_0(y), \lambda \rangle_{W^*,W} := \langle \partial_t T, \lambda \rangle_{W^*,W} + k_c (\nabla T, \nabla \lambda)_{L^2(Q)} - \sum_{i=1}^{N_v} \frac{\kappa_i}{\epsilon^2} (\phi_i, \lambda)_{L^2(Q)} + \frac{h_c}{\epsilon} (T, \lambda)_{L^2(\Sigma)}, \tag{77}$$

$$\langle D_i(y), \lambda \rangle_{W^*,W} := \frac{\epsilon^2}{3(\sigma_i + \kappa_i)} (\nabla \phi_i, \lambda)_{L^2(Q)} + \kappa_i (\phi_i, \lambda)_{L^2(Q)} + a_1 \epsilon (\phi_i, \lambda)_{L^2(\Sigma)}, \; i = 1, \ldots, N_v, \tag{78}$$

$$\langle N_0(y), \lambda \rangle_{W^*,W} := \sum_{i=1}^{N_v} \frac{\kappa_i}{\epsilon^2} (4\pi \kappa_i B^{(i)}(T), \lambda)_{L^2(Q)} + \frac{\alpha \pi}{\epsilon} \left(\frac{n_a}{n_g} \right)^2 (B^{(0)}(T), \lambda)_{L^2(\Sigma)}, \tag{79}$$

$$\langle N_i(y), \lambda \rangle_{W^*,W} := -\kappa_i (4\pi \kappa_i B^{(i)}(T), \lambda)_{L^2(Q)}, \; i = 1, \ldots, N_v, \tag{80}$$

$$\langle B_0(u), \lambda \rangle_{W^*,W} := -\frac{h_c}{\epsilon} (u, \lambda)_{L^2(\Sigma)} - \frac{\alpha \pi}{\epsilon} \left(\frac{n_a}{n_g} \right)^2 (B^{(0)}(u), \lambda)_{L^2(\Sigma)}, \tag{81}$$

$$\langle B_i(u), \lambda \rangle_{W^*,W} := -a_1 \epsilon (4\pi B^{(i)}(u), \lambda)_{L^2(\Sigma)}, \; i = 1, \ldots, N_v, \tag{82}$$

$$\tag{83}$$

$D_{N_v+1}(y) := T(0) - T_0$, and $N_{N_v+1}(y) = B_{N_v+2}(y) = 0$, such that

$$e(y(u), u) = 0 \Leftrightarrow R(y(u), u) = 0. \tag{84}$$

Because the linear operator D is bounded invertible with $D^{-1} \in \mathcal{L}(L^\infty(Q)^{N_v+2}, Y)$ and $D^{-1} \in \mathcal{L}(L^\infty(\Sigma)^{N_v+2}, Y)$ [17], and because $N : Y \to L^\infty(Q)^{N_v+2}$ and $B : U \to L^\infty(\Sigma)^{N_v+2}$ are continuously Fréchet differentiable, the operator R is continuously Fréchet differentiable [15]. From [18] we can deduce that the linearized state system

$$Dv + \partial_y N(y) v = -\partial_y N(y) g \tag{85}$$

has a unique solution $v \in Y$. The substitution $v := w - g$ and the fact that the linear operator D is bounded invertible implies that the system

$$w + D^{-1} \partial_y N(y) w = g \tag{86}$$

has a unique solution $w \in Y$. Hence, there exists a unique solution $w \in Y$ with

$$\partial_y R(y, u) w = g, \tag{87}$$

which shows bounded invertibility of $\partial_y R$ and hence the applicability of the implicit function theorem to $R(y(u), u)$. Hence, the Fréchet derivative of $u \mapsto y(u)$ exists and is given by

$$y'(u) = -\partial_y e^{-1}(y, u) \partial_u e(y, u). \tag{88}$$

The acting of the first derivative of $e(y,u)$ at $(y,u) \in V$ in the direction $(s,d) \in V$, $s := (s_T, s_{\phi_1}, \ldots, s_{\phi_{N_\nu}})$ is given by

$$\langle \partial_y e_0(y,u)s, \xi_T \rangle_{W^*,W} = \langle \partial_t s_T, \xi_T \rangle_{W^*,W} + k_c (\nabla s_T, \nabla \xi_T)_{L^2(Q)} + \sum_{i=1}^{N_\nu} \frac{1}{3(\sigma_i + \kappa_i)} (\nabla s_{\phi_i}, \nabla \xi_T)_{L^2(Q)}$$
$$+ \frac{h_c}{\epsilon}(s_T, \xi_T)_{L^2(\Sigma)} + \frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2 (\partial_T B^{(0)}(T) s_T, \xi_T)_{L^2(\Sigma)}$$
$$+ \frac{a_1}{\epsilon} \sum_{i=1}^{N_\nu} (s_{\phi_i}, \xi_T)_{L^2(\Sigma)}, \tag{89}$$
$$\langle \partial_y e_i(y,u)s, \xi_{\phi_i} \rangle_{W^*,W} = \frac{\epsilon^2}{3(\sigma_i + \kappa_i)} (\nabla s_{\phi_i}, \nabla \xi_{\phi_i})_{L^2(Q)} + \kappa_i (s_{\phi_i} - 4\pi \partial_T B^{(i)}(T) s_T, \xi_{\phi_i})_{L^2(Q)}$$
$$+ a_1 \epsilon (s_{\phi_i}, \xi_{\phi_i})_{L^2(\Sigma)}, \quad i = 1, \ldots, N_\nu, \tag{90}$$
$$\langle \partial_u e_0(y,u)d, \xi_T \rangle_{W^*,W} = -\frac{h_c}{\epsilon}(d, \xi_T)_{L^2(\Sigma)} - \frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2 (\partial_T B^{(0)}(u) d, \xi_T)_{L^2(\Sigma)}$$
$$- \frac{4\pi a_1}{\epsilon} \sum_{i=1}^{N_\nu} (\partial_T B^{(i)}(u) d, \xi_T)_{L^2(\Sigma)}, \tag{91}$$
$$\langle \partial_u e_i(y,u)d, \xi_{\phi_i} \rangle_{W^*,W} := -4\pi a_1 \epsilon (\partial_T B^{(i)}(u) d, \xi_{\phi_i})_{L^2(\Sigma)}, \quad i = 1, \ldots, N_\nu, \tag{92}$$

the acting of the second derivative at $(y,u) \in V$ in the direction $[(s_1,d_1),(s_2,d_2)] \in V \times V$, with $s_j := (s_{T,j}, s_{\phi_1,j}, \ldots, s_{\phi_{N_\nu},j})$ by

$$\langle \partial_{yy} e_0(y,u)[s_1,s_2], \xi_T \rangle_{W^*,W} = \frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2 (\partial_{TT} B^{(0)}(T) s_{T,1} s_{T,2}, \xi_T)_{L^2(\Sigma)}, \tag{93}$$
$$\langle \partial_{yy} e_i(y,u)[s_1,s_2], \xi_{\phi_i} \rangle_{W^*,W} = -4\pi\kappa_i (\partial_{TT} B^{(i)}(T) s_{T,1} s_{T,2}, \xi_{\phi_i})_{L^2(Q)}, \quad i = 1, \ldots, N_\nu, \tag{94}$$
$$\langle \partial_{uu} e_0(y,u)[d_1,d_2], \xi_T \rangle_{W^*,W} = -\frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2 (\partial_{TT} B^{(0)}(u) d_1 d_2, \xi_T)_{L^2(\Sigma)}$$
$$- \frac{4\pi a_1}{\epsilon} \sum_{i=1}^{N_\nu} (\partial_{TT} B^{(i)}(u) d_1 d_2, \xi_T)_{L^2(\Sigma)}, \tag{95}$$
$$\langle \partial_{uu} e_i(y,u)[d_1,d_2], \xi_{\phi_i} \rangle_{W^*,W} := -4\pi a_1 \epsilon (\partial_{TT} B^{(i)}(u) d_1 d_2, \xi_{\phi_i})_{L^2(\Sigma)}, \quad i = 1, \ldots, N_\nu, \tag{96}$$

and the mixed terms by

$$\partial_{yu} e(y,u) = 0, \qquad \partial_{uy} e(y,u) = 0. \tag{97}$$

4 Optimality Systems

Assuming uniqueness of the adjoint system and hence the existence of Lagrange multipliers we have all ingredients at hand to state the first-order optimality condition. A promising idea to show solvability of the adjoint system is published in a recent work by R. Pinnau and O. Tse for a simplified natural convection-radiation model [16].

4.1 First-Order Optimality System, N-band

For the N_ν-band model the reduced gradient is given by

$$\nabla \hat{J}(u)d_u = \Big(\delta_u(t)(u-u_d), d_u\Big)_{U,U} + \Big(\delta_d(t)\partial_t u, \partial_t d_u\Big)_{U,U} - \int_0^{t_e}\int_{\partial\Omega}\Big(\frac{h_c}{\epsilon}\xi_T + \frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2 \partial_T B^{(0)}(u)\xi_T + \frac{4\pi a_1}{\epsilon}\sum_{i=1}^{N_\nu}\partial_T B^{(i)}(u)\xi_T + 4\pi a_1 \epsilon \sum_{i=1}^{N_\nu}\partial_T B^{(i)}(u)\xi_{\phi_i}\Big) d_u \mathrm{d}x\mathrm{d}t, \tag{98}$$

with adjoint state $\xi := (\xi_T, \xi_{\phi_1}, \dots, \xi_{\phi_{N_\nu}})^T$, fulfilling the adjoint system

$$-\partial_t \xi_T - k_c \Delta \xi_T - 4\pi \sum_{i=1}^{N_\nu} \kappa_i \partial_T B^{(i)}(T)\xi_{\phi_i} = -(T-T_d) + \delta_g(t)\Delta T, \tag{99}$$

$$-\frac{\epsilon^2}{3(\sigma_i+\kappa_i)}\Delta\xi_{\phi_i} - \frac{1}{3(\sigma_i+\kappa_i)}\Delta\xi_T = -\kappa_i \xi_{\phi_i}, \;\; i=1,\dots,N_\nu, \tag{100}$$

with boundary and terminal conditions

$$k_c n\cdot\nabla\xi_T = -\left(\frac{h_c}{\epsilon} + \frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2 \partial_T B^{(0)}(T)\right)\xi_T - \delta_g(t) n\cdot\nabla T, \tag{101}$$

$$\frac{\epsilon^2}{3(\sigma_i+\kappa_i)} n\cdot\nabla\xi_{\phi_i} + \frac{1}{3(\sigma_i+\kappa_i)} n\cdot\nabla\xi_T = -a_1\epsilon\xi_{\phi_i} - \frac{a_1}{\epsilon}\xi_T, \;\; i=1,\dots,N_\nu, \tag{102}$$

$$\xi_T(t_e) = -\delta_e(T-T_d)(t_e) + \delta_{ge}\Delta T(t_e) - \delta_{ge} n\cdot\nabla T(t_e), \tag{103}$$

where we identify the component ξ_{T_0} by $\xi_T(0)$ and formally set the outer normal n of an inner point $x\in\Omega$ equal to zero.

Finally, state system (4)-(8), adjoint system (99)-(103) and the optimality condition

$$\int_\Sigma \Big(-\frac{h_c}{\epsilon}\xi_T - \frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2 \partial_T B^{(0)}(u)\xi_T - 4\pi a_1 \sum_{i=1}^{N}\partial_T B^{(i)}(u)\left(\frac{1}{\epsilon}\xi_T + \epsilon\xi_{\phi_i}\right) + \frac{\delta_u}{|\partial\Omega|}(u-u_d)\Big)(d_u - u) + \frac{\delta_d}{|\partial\Omega|}\partial_t u \partial_t(d_u-u)\mathrm{d}x\mathrm{d}t \geq 0, \;\; \forall d_u \in U_{ad}. \tag{104}$$

accumulate to first-order optimality system.

4.2 Second-Order Optimality System, N-band

For the N_ν-band model the acting of the reduced Hessian in a direction $[s_u, d_u]$ is given by

$$\begin{aligned}\langle \hat{J}''(u)s_u, d_u\rangle_{U^*,U} = &\Big(\delta_u(t)s_u, d_u\Big)_{U,U} + \Big(\delta_d(t)\partial_t s_u, \partial_t d_u\Big)_{U,U} \\ &- \int_0^{t_e}\int_{\partial\Omega}\Big(\frac{h_c}{\epsilon}w_T + \frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2\Big(\partial_T B^{(0)}(u)w_T + \partial_{TT}B^{(0)}(u)s_u\xi_T\Big) \\ &+ \frac{4\pi a_1}{\epsilon}\sum_{i=1}^{N_\nu}\Big(\partial_T B^{(i)}(u)w_T + \partial_{TT}B^{(i)}(u)s_u\xi_T\Big) \\ &+ 4\pi a_1\epsilon\sum_{i=1}^{N_\nu}\Big(\partial_T B^{(i)}(u)w_{\phi_i} + \partial_{TT}B^{(i)}(u)s_u\xi_{\phi_i}\Big)\Big) d_u\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{105}$$

such that the linearized state system

$$\partial_t s_{y,T} - k_c \Delta s_{y,T} - \sum_{i=1}^{N_\nu} \frac{1}{3\left(\sigma_i + \kappa_i\right)} \Delta s_{y,\phi_i} = 0, \tag{106}$$

$$-\frac{\epsilon^2}{3\left(\sigma_i + \kappa_i\right)} \Delta s_{y,\phi_i} = 4\pi\kappa_i \partial_T B^{(i)}(T) s_{y,T} - \kappa_i s_{y,\phi_i}, \;\; i = 1,\dots,N_\nu, \tag{107}$$

with boundary and initial conditions

$$k_c n \cdot \nabla s_{y,T} + \sum_{i=1}^{N_\nu} \frac{1}{3\left(\sigma_i + \kappa_i\right)} n \cdot \nabla s_{y,\phi_i} = \frac{h_c}{\epsilon}(s_u - s_{y,T}) \tag{108}$$

$$+ \frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2 \left(\partial_T B^{(0)}(u) s_u - \partial_T B^{(0)}(T) s_{y,T}\right) + \frac{a_1}{\epsilon} \sum_{i=1}^{N_\nu} \left(4\pi \partial_T B^{(i)}(u) s_u - s_{y,\phi_i}\right),$$

$$\frac{\epsilon^2}{3(\sigma_i + \kappa_i)} n \cdot \nabla s_{y,\phi_i} = a_1 \epsilon \left(4\pi \partial_T B^{(i)}(u) s_u - s_{y,\phi_i}\right), \;\; i = 1,\dots,N_\nu, \tag{109}$$

$$s_{y,T}(x,0) = 0, \tag{110}$$

and the second adjoint system

$$-\partial_t w_T - k_c \Delta w_T = -s_{y,T} + 4\pi \sum_{i=1}^{N_\nu} \kappa_i \left(\partial_T B^{(i)}(T) w_{\phi_i} + \partial_{TT} B^{(i)}(T) s_{y,T} \xi_{\phi_i}\right) + \delta_g(t) \Delta s_{y,T}, \tag{111}$$

$$-\frac{\epsilon^2}{3(\sigma_i + \kappa_i)} \Delta w_{\phi_i} - \frac{1}{3(\sigma_i + \kappa_i)} \Delta w_T = -\kappa_i w_{\phi_i}, \;\; i = 1,\dots,N_\nu, \tag{112}$$

with boundary and terminal conditions

$$k_c n \cdot \nabla w_T = -\frac{h_c}{\epsilon} w_T - \frac{\alpha\pi}{\epsilon}\left(\frac{n_a}{n_g}\right)^2 \left(\partial_T B^{(0)}(T) w_T + \partial_{TT} B^{(0)}(T) s_{y,T} \xi_T\right) - \delta_g(t) n \cdot \nabla s_{y,T}, \tag{113}$$

$$\frac{\epsilon^2}{3(\sigma_i + \kappa_i)} n \cdot \nabla w_{\phi_i} + \frac{1}{3(\sigma_i + \kappa_i)} n \cdot \nabla w_T = -a_1 \epsilon w_{\phi_i} - \frac{a_1}{\epsilon} w_T, \;\; i = 1,\dots,N_\nu, \tag{114}$$

$$w_T(t_e) = -\delta_e s_{y,T}(t_e) + \delta_{ge} \Delta s_{y,T}(t_e) - \delta_{ge} n \cdot \nabla s_{y,T}(t_e), \tag{115}$$

are fulfilled for the given direction $s_u \in U$. Analogously to the adjoint system, we identify the component w_{T_0} by $w_T(0)$ and formally set the outer normal n of an inner point $x \in \Omega$ equal to zero.

Thinking in terms of reduced quantities, the second-order optimality condition is equivalent with positive definiteness of the reduced Hessian in a minimizer $(\bar{y}, \bar{u}, \bar{\xi})$, that fulfills the necessary first-order optimality system.

5 Conclusion

In this preprint we have presented first promising ideas to show existence and uniqueness of the state for and SP_1-N_ν-band approximation of radiative heat transfer. For a founded formulation of first and second optimality condition it is necessary to examine the adjoint system and the second adjoint system (adjoint for Hessian) in detail. From the current point of view, success seems most likely, however some function spaces might need to be adapted.

References

[1] K. Atkinson and W. Han. *Theoretical numerical analysis: a functional analysis framework*. Springer, 2 edition, 2000.

[2] K. Chudej, P. Heidebrecht, V. Petzet, S. Scherdel, K. Schittkowski, H. J. Pesch, and K. Sundmacher. Index analysis and numerical solution of a large scale nonlinear PDAE system describing the dynamical behaviour of molten carbonate fuel cells. *ZAMM - Journal of Applied Mathematics and Mechanics*, 85(2):132–140, 2005.

[3] D. Clever. *Adaptive Multilevel Methods for PDAE-Constrained Optimal Control Problems*. PhD thesis, TU Darmstadt, 2012.

[4] D. Clever and J. Lang. Optimal control of radiative heat transfer in glass cooling with restrictions on the temperature gradient. *Optimal Control Applications and Methods*, 33(2):157–175, 2012.

[5] M. Frank. Approximate models for radiative transfer. *Bulletin of the Institute of Mathematics Academia Sinica (New Series)*, 2(2):409–432, 2007.

[6] M. Frank and A. Klar. *Mathematical Models in the Manufacturing of Glass*, chapter Radiative Heat Transfer and Applications for Glass Production Processes, pages 57–134. Lecture Notes in Mathematics, 2011.

[7] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Springer, 1977.

[8] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich. *Optimization with PDE constraints*. Springer, 2009.

[9] A. Klar, J. Lang, and M. Seaïd. Adaptive solution of SP_N-approximations to radiative heat transfer in glass. *International Journal of Thermal Science*, 44:1013–1023, 2005.

[10] M. T. Laitinen and T. Tiihonen. Integro-differential equation modelling heat transfer in conducting, radiating and semitransparent materials. *Mathematical Methods in the Applied Sciences*, 21:375–392, 1998.

[11] J. Lang. *Adaptive multilevel solution of nonlinear parabolic PDE systems*. Springer, 2001.

[12] E. W. Larsen, G. Thömmes, A. Klar, M. Seaïd, and T. Götz. Simplified P_N approximations to the equations of radiative heat transfer and applications. *Journal of Computational Physics*, 183:652–675, 2002.

[13] W. Lucht and K. Debrabant. On quasi-linear PDAEs with convection: application, indices, numerical solution. *Applied Numerical Mathematics*, 42:297–314, 2002.

[14] W. S. Martinson and P. I. Barton. Index and characteristic analysis of linear PDAE systems. *SIAM Journal on Scientific Computing*, 24(3):905–923, 2002.

[15] R. Pinnau. Analysis of optimal boundary control for radiative heat transfer modeled by the SP_n-system. *Communications in Mathematical Sciences*, 5(4):951–969, 2007.

[16] R. Pinnau and O. Tse. Optimal control of a simplified natural convection-radiation model.

[17] M. Renardy and R. C. Rogers. *An introduction to partial differential equations*. Springer, 1992.

[18] C. A. T. Schulze. *Minimizing thermal stress in glass production processes: model reduction and optimal control*. Shaker, 2006.

[19] G. Thömmes. *Radiative heat transfer equations for glass cooling problems: Analysis and Numerics*. PhD thesis, Technische Universität Darmstadt, 2002.

[20] R. Turpault. A consistent multigroup model for radiative transfer and its underlying mean opacities. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 94(3-4):357–371, 2005.

[21] R. Turpault, M. Frank, B. Dubroca, and A. Klar. Multigroup half space moment approximations to the radiative heat transfer equations. *Journal of Computational Physics*, 198:364–371, 2004.