Multilevel optimization for PDAE-constrained optimal control problems with pointwise constraints on control and state

Debora Clever and Jens Lang

Graduate School CE and Department of Mathematics, Technische Universität Darmstadt, Dolivostr. 15, D-64293 Darmstadt, Germany
Center of Smart Interfaces, Technische Universität Darmstadt, Petersenstr. 30, D-64287 Darmstadt, Germany

Preprint Department of Mathematics, Technische Universität Darmstadt, 2012

Abstract

We have developed a fully adaptive optimization environment suitable to solve complex optimal control problems restricted by partial differential algebraic equations (PDAEs) and pointwise constraints on the control [1, 2]. This contribution is devoted to the inclusion of pointwise constraints on the state within the optimization environment. To this end we first give a brief introduction into the architecture of the environment and the inclusion of pointwise constraints on the state by Moreau-Yosida regularization. Then, we test the new tool by applying it to an optimal boundary control problem for the cooling of hot glass down to room temperature, modeled by radiative heat transfer and semi-transparent boundary conditions.

1 The KARDOS-based optimization environment

In the following we consider PDAE-constrained optimal control problems of the form

$$\min_{(y,u)\in U\times Y} J(y,u) \;\text{ s.t. }\; e(y,u)=0, \tag{1}$$

$$b_{\text{y,low}} \le y \le b_{\text{y,up}}, \tag{2}$$

$$b_{\text{u,low}} \le u \le b_{\text{u,up}}, \tag{3}$$

with state $y \in Y$ and state bounds $b_{\text{y,low}}, b_{\text{y,up}} \in Y$ defining the feasible set of states Y_{ad}, control $u \in U$ and control bounds $b_{\text{u,low}}, b_{\text{u,up}} \in U$ defining the feasible control set U_{ad}, objective functional $J(y,u)$, and PDAE-constraint $e(y,u)=0$. We will assume that the control space U and the state space Y are Hilbert spaces. Furthermore, we assume $J(y,u)$ and $e(y,u)$ to be twice continuously Fréchet differentiable and that the PDAE admits a unique and Fréchet differentiable solution operator $y : u \in U \mapsto y(u) \in Y$. Then, we can reduce the optimal control problem (1)-(3) to the control component u

$$\min_{u\in U_{ad}} \hat{J}(u) := J(y(u),u), \quad \text{where } y = y(u) \text{ satisfies } e(y,u)=0 \text{ and } y \in Y_{ad}. \tag{4}$$

The optimization environment is built up in a modular way, such that the user can choose between different optimization techniques, different time integration methods and different error estimation approaches. Furthermore, the spatial domain may be one, two or three dimensional, by considering the software packages KARDOS1D,KARDOS2D or KARDOS3D, respectively [3].

In this section we focus on the implementation of the generalized multilevel SQP method developed in [5], linearly implicit one-step methods of Rosenbrock type for the time integration and multilevel linear finite elements on a triangular mesh for the space discretization of a two-dimensional spatial domain. Local discretization errors in time are estimated by embedded Rosenbrock schemes of inferior order and local errors in space by hierarchical bases. The space-time grids are locally adapted with respect to these error estimates such that the overall discretization error serves a desired accuracy. The level of accuracy is controlled autonomously with respect to convergence criteria and optimization progress.

Pointwise constraints on the control are realized by a projected Newton method with a modified version of BICGSTAB, computing an SQP-step on an ϵ-inactive subset of the control space and a gradient step on the corresponding ϵ-active subset. We use Armijo-line search to determine a proper scaling, such that the considered direction is a descent direction. Global convergence is ensured by including a trust region strategy. For more details we refer to [1] and [2].

To include pointwise state constraints of the form (2) we consider a Moreau-Yosida regularization within the objective. Given the original objective $J(y,u)$ from (1) and the space-time cylinder $Q := \Omega \times [0, t_e]$, with spatial domain Ω and final time t_e, we consider its regularized counterpart

$$J_{\text{reg}}(y,u) = J(y,u) + \frac{\gamma_k}{3}\|\max(0, y - b_{\text{y,up}})\|^3_{L^3(Q)} + \frac{\gamma_k}{3}\|\max(0, b_{\text{y,low}} - y)\|^3_{L^3(Q)} \tag{5}$$

$$+ \frac{\gamma_{e,k}}{3}\|\max(0, y(t_e) - b_{\text{y,up}}(t_e))\|^3_{L^3(\Omega)} + \frac{\gamma_{e,k}}{3}\|\max(0, b_{\text{y,low}}(t_e) - y(t_e))\|^3_{L^3(\Omega)} + \frac{\delta_{ut}}{2}\|\partial_t u\|^2_{L^2(0,t_e)}. \tag{6}$$

Note, that due to differentiability arguments it is not sufficient to consider quadratic penalty terms. Furthermore, following continuous adjoint calculus a consideration of the state constraints also at final time can only be ensured by the inclusion of final value terms, weighted with penalty parameter $\gamma_{e,k}$.

Starting with moderate values for γ_0 and $\gamma_{e,0}$ the penalty parameters γ_k and $\gamma_{e,k}$ are increased by a factor c_{increase}, if

$$m_k < c_1 \gamma_k^{-0.5}, \tag{7}$$

with criticality measure $m_k = P_{U_{ad}-u_k}(-\nabla \hat{J}_{\text{reg}}(u_k))$, and some constant $c_1 > 0$. We want to point out that the criticality measure m_k, which is a projection of the reduced gradient to the shifted feasible control set $U_{ad} - u_k$ is also used to control the grid refining multilevel strategy. The counter k describes the number of the current optimization iteration.

Finally, we include a regularization of $\partial_t u$ to avoid undesired fast changes within the control.

2 Application to the glass cooling problem: focus on state constraints

We use the augmented environment to solve an optimal boundary control problem which occurs in the context of glass manufacturing. Usually, the hot glass is cooled within a furnace to allow the control of the evolution of the glass temperature $T(x,t)$ by choosing an appropriate furnace temperature profile $u(t)$. Due to the high temperatures the process has to be modeled by radiative heat transfer. Here, we use the so called gray scale problem, see e.g. [4], which is a semi-linear PDAE in two components. Whereas the glass temperature distribution is described by the differential component the mean radiative intensities are given by the algebraic component. Furthermore there is a high non-linear coupling between temperature field and radiative field resulting in non-linearities in u and T of the power of four. Note, that the furnace temperature u only occurs in the boundary conditions.

We consider the computational domain $\Omega = [0,1] \times [0,1]$, $t_e = 0.1$ and the objective

$$J_{\text{reg}}(y,u) = \frac{1}{2}\int_0^{t_e} \left\|T - T_d\right\|^2_{L^2(\Omega)} \mathrm{d}t + \frac{0.1}{2}\left\|(T - T_d)(t_e)\right\|^2_{L^2(\Omega)} + \frac{0.1}{2}\int_0^{t_e}(u-u_d)^2 \mathrm{d}t + \frac{\gamma_k}{3}\|\max(0, T - b_{\text{T,up}})\|^3_{L^3(Q)}$$

$$+\frac{\gamma_k}{3}\|\max(0, b_{\text{T,low}} - T)\|^3_{L^3(Q)} + \frac{0.1\gamma_k}{3}\|\max(0,(T - b_{\text{T,up}})(t_e))\|^3_{L^3(\Omega)} + \frac{0.1\gamma_k}{3}\|\max(0,(b_{\text{T,low}} - T)(t_e))\|^3_{L^3(\Omega)} + \frac{\delta_{ut}}{2}\|\partial_t u\|^2_{L^2(0,t_e)},$$

with $b_{\text{T,up}} = T_d = 900\exp\left(\frac{-\log(3)t}{t_e}\right)$, $b_{\text{T,low}} = 0.0$, $\gamma_0 = 1.0e+1$, $\gamma_{\max} = 1.0e+4$, $\delta_{ut} = 1.0e-5$.

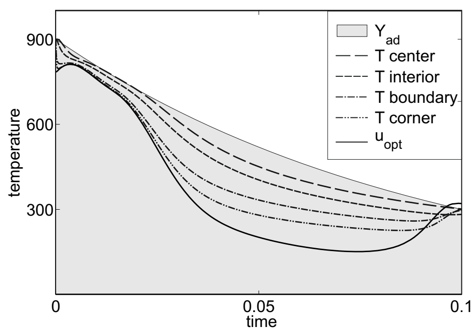

Figure 1: Optimal control and resulting state in four significant points

Figure 1 shows the optimal control and the resulting glass temperature in four significant points, namely $p_1 = (0,0)$ in a corner, $p_2 = (0.5,0)$ on an edge, $p_3 = (0.25,0.25)$ in the interior and $p_4 = (0.5,0.5)$ in the center. As desired, the glass temperature stays within the feasible set of states, marked in gray.

Acknowledgement

The authors gratefully acknowledge the support of the DFG Priority Program 1253 entitled Optimization with Partial Differential Equations and thank the project partners S. Ulbrich and J. C. Ziems for the fruitful cooperation.

References

[1] D. Clever, J. Lang, S. Ulbrich, and J. C. Ziems. Combination of an adaptive multilevel SQP method and a space-time adaptive PDAE solver for optimal control problems. *Procedia Computer Science*, pages 1429–1437, 2010.

[2] D. Clever, J. Lang, S. Ulbrich, and J. C. Ziems. *Constrained Optimization and Optimal Control for Partial Differential Equations*, volume 160 of *International Series of Numerical Mathematics*, chapter Generalized Multilevel SQP-methods for PDAE-constrained Optimization Based on Space-Time Adaptive PDAE Solvers, pages 37–60. Springer Verlag, 2012.

[3] B. Erdmann, J. Lang, and R. Roitzsch. KARDOS-User's Guide. Manual, Konrad-Zuse-Zentrum Berlin, 2002.

[4] R. Pinnau. Analysis of optimal boundary control for radiative heat transfer modeled by the SP_n-system. *Communications in Mathematical Sciences*, 5(4):951–969, 2007.

[5] J. C. Ziems. *Adaptive multilevel SQP-methods for PDE-constrained optimization*. PhD thesis, Technische Universität Darmstadt, 2010.