The Fundamental Solution of Linearized Nonstationary Navier-Stokes Equations of Motion around a Rotating and Translating Body

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Abstract. *We derive the fundamental solution of the linearized problem of the motion of a viscous fluid around a rotating body when the axis of rotation of the body is not parallel to the velocity of the fluid at infinity.*

1 Introduction

We consider a rigid body $\mathcal{B}$ moving in a viscous, incompressible liquid that fills the whole space $\mathbb{R}^3$; here $\mathcal{B}$ is assumed to be an open, bounded set with smooth boundary. Let $V = V(y,t)$ be the velocity field associated with the motion of the body $\mathcal{B}$ with respect to an inertial frame $\mathcal{I}$ with origin O. Denoting by $y_C = y_C(t)$ the path of the center of mass of $\mathcal{B}$ and by $\tilde{\omega} = \tilde{\omega}(t) \in \mathbb{R}^3$ the angular velocity of $\mathcal{B}$ around its center of mass, we have

$$V(y,t) = \dot{y}_C(t) + \tilde{\omega}(t) \times (y - y_C(t)), \tag{1.1}$$

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where $\dot{y}_C = dy_C/dt$ is the translational velocity of $\mathcal{B}$ and, for simplicity, $y_C(0) = 0$. Let the Eulerian velocity field and pressure associated with the motion of the liquid in $\mathcal{I}$ be denoted by $v = v(y,t)$, and $q = q(y,t)$, respectively. The equations of conservation of linear momentum and mass of the fluid are then modeled by the Navier-Stokes equations. Given a kinematic viscosity $\nu > 0$ and an external force $\tilde{f} = \tilde{f}(y,t)$, the unknowns v, q solve the nonlinear system

$$\begin{array}{rcll} \partial_t v - \nu\Delta v + (v\cdot\nabla)\, v + \nabla q & = & \tilde{f} & \text{in} \quad \mathcal{D}(t),\ t\in(0,\infty) \\ \operatorname{div} v & = & 0 & \text{in} \quad \mathcal{D}(t),\ t\in(0,\infty) \\ v(y,t) & = & V(y,t) & \text{on} \quad \partial\mathcal{D}(t),\ t\in(0,\infty) \\ v(y,t) & \to & 0 & \text{as} \quad |y|\to\infty \end{array} \tag{1.2}$$

in a time-dependent exterior domain $\mathcal{D}(t) \subset \mathbb{R}^3$.

In this paper we discuss the case of a time-independent angular velocity $\tilde{\omega} = ke_3$ and constant translational velocity $0 \neq \dot{y}_C = u_\infty \in \mathbb{R}^3$ so that $y_C(t) = u_\infty t$. For this reason we introduce the change of variables

$$x = O(t)^T\big(y - y_C(t)\big) \tag{1.3}$$

and the new functions

$$u(x,t) = O(t)^T v(y,t),\ \ p(x,t) = q(y,t),\ \ f(x,t) = O(t)^T \tilde{f}(y,t)\,, \tag{1.4}$$

where the matrix of rotation is defined by

$$O(t) = \begin{pmatrix} \cos kt & -\sin kt & 0 \\ \sin kt & \cos kt & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.5}$$

Then (u,p) satisfies - after a linearization around $u = 0$ - the system

$$\begin{array}{rcll} \partial_t u - \nu\Delta u + \nabla p - & & & \\ -\big[\big(\omega\wedge x + O(t)^T u_\infty\big)\cdot\nabla\big]u + \omega\wedge u & = & f & \text{in} \quad \mathcal{D}\times(0,\infty) \\ \operatorname{div} u & = & 0 & \text{in} \quad \mathcal{D}\times(0,\infty) \\ u & = & u_{\partial\mathcal{D}} & \text{on} \quad \partial\mathcal{D}\times(0,\infty) \\ u(x,t) & \to & 0 & \text{as} \quad |x|\to\infty \end{array} \tag{1.6}$$

in a time-independent exterior domain $\mathcal{D} \subset \mathbb{R}^3$, where $\omega = \tilde{\omega} = ke_3$ and $u_{\partial\mathcal{D}}(x,t) = \omega\wedge x + O(t)^T u_\infty$. For details of this coordinate transform in an even more general setting leading from (1.2) to (1.6) see Section 2 and also [21, Ch. 1]. Note that if

u_∞ is transversal or even orthogonal to e_3, then (1.6) contains the time-dependent term $(O(t)^T u_\infty) \cdot \nabla u$ which appears in a natural way for an observer sitting on the rotating and translating obstacle and seeing the fluid flowing past him from the time-dependent direction $O(t)^T u_\infty$.

Our aim is to find an explicit formula of the fundamental solution of (1.6) and to discuss the asymptotic behavior as $|x| \to \infty$. In particular, we are interested in the existence of a wake for any angular velocity ω and translational velocity $u_\infty \neq 0$, see Remark 4.3 below.

To describe the main results we assume, for simplicity, that $\nu = 1$ and introduce the y- and t-dependent Oseen-type operator

$$\mathcal{L}v = \mathcal{L}_{y,t}v = -\Delta v - (O(t)^T u_\infty + \omega \wedge y) \cdot \nabla v + \omega \wedge v\,.$$

Then the fundamental tensor of (1.6) comprises a 3×3–matrix of distributions $\Gamma(y,z,t,s)$ and a three–dimensional vector of distributions $Q(y,z,t,s)$ such that for any vector $a \in \mathbb{R}^3$ the distributions

$$\begin{aligned} v_{z,s}(y,t) &= \Gamma(y,z,t,s)a,\ t \geq s, \quad v_{z,s}(y,t) = 0,\ t < s, \\ \pi_{z,s}(y,t) &= Q(y,z,t,s)a,\ t \geq s, \quad \pi_{z,s}(y,t) = 0,\ t < s \end{aligned}$$

solve the system

$$\begin{aligned} \frac{\partial v_{z,s}}{\partial t} + \mathcal{L}v_{z,s} + \nabla \pi_{z,s} &= \delta_s(t)\delta_z(y)a \\ \operatorname{div} v_{z,s} &= 0 \end{aligned} \tag{1.7}$$

in the sense of distributions. I.e., for all test functions $\varphi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})^3$

$$\langle \partial_t v_{z,s} + \mathcal{L}v_{z,s} + \nabla \pi_{z,s}, \varphi \rangle = \varphi(z,s) \cdot a = \langle \delta_z \otimes \delta_s, \varphi \cdot a \rangle$$

and div $v_{z,s} = 0$ for all $t > s$. Here $\delta_s(t), \delta_z(y)$ denote the point masses concentrated at $t = s, y = z$, respectively.

Moreover, we introduce the fundamental solution of the heat equation in $\mathbb{R}^3$,

$$K(x,t) = \frac{1}{(4\pi t)^{3/2}} \exp\big(- \frac{|x|^2}{4t} \big).$$

Let ${}_1F_1(a,c,\cdot)$, a, $c > 0$, denote the Kummer function

$${}_1F_1(a,c,\lambda) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{\lambda^n}{n!} \tag{1.8}$$

where $(b)_n = \Gamma(b+n)/(\Gamma(b))$ is the Pochhammer symbol; for classical results on Kummer functions needed in this paper see Lemma 2.1, and [35] for a more comprehensive treatment. Furthermore, let $Y(t,s)$ be the solution of the ordinary differential equation

$$\frac{\partial Y}{\partial t} + \omega \wedge Y = O(t)^T u_\infty, \quad t > s, \quad Y(s,s) = 0, \tag{1.9}$$

i.e., $Y(t,s) = (t-s)O(t)^T u_\infty$, see also (3.6), (3.11). Finally, for $z \in \mathbb{R}^3$, let

$$\tilde{z}(t,s,z) = O(t-s)^T z - Y(t,s) = O(s-t)\big(z - (t-s)O(s)^T u_\infty\big). \tag{1.10}$$

Theorem 1.1 *The fundamental tensor $\Gamma(y,z,t,s)$, $Q(y,z,t,s)$ of the linearized problem (1.7) can be written in the form*

$$\Gamma(y,z,t,s) = \Gamma_0(y - \tilde{z}(t,s,z), t-s), \quad Q(y,z,t,s) = Q_0(y - \tilde{z}(t,s,z), t-s)$$

where $\tilde{z}(t,s,z)$ was defined in (1.10) above and

$$\Gamma_0(y,\tau) = K(y,\tau)\Big\{ \Big[I - \frac{y\otimes y}{|y|^2}\Big] - {}_1F_1\Big(1, \frac{5}{2}, \frac{|y|^2}{4\tau}\Big)\Big[\frac{1}{3}I - \frac{y \otimes y}{|y|^2}\Big]\Big\} O(\tau)^T$$

$$Q_0(y,\tau) = Q^*(y)\delta_0(\tau), \quad Q^*(y) = -\frac{1}{4\pi}\nabla_y \frac{1}{|y|}\,.$$

In particular, for every initial value $u_0 \in \mathcal{S}(\mathbb{R}^3)^3$ and $s \in \mathbb{R}$

$$\lim_{t\to s+} \int_{\mathbb{R}^3} \Gamma_0(y - \tilde{z}(t,s,z), t-s) u_0(z)\, dz = P u_0(y), \quad y \in \mathbb{R}^3,$$

where P denotes the Helmholtz decomposition on $\mathbb{R}^3$.

In the following we will also use cylindrical coordinates $r, \theta, x_3 \in [0,\infty) \times [0, 2\pi) \times \mathbb{R}$ for x such that $(\omega \wedge x)\cdot \nabla u = \partial_\theta u$ where ∂_θ denotes the angular derivative with respect to θ. Obviously $-\Delta$ commutes with ∂_θ. Let $\nabla' = (\partial_1, \partial_2)$.

Recall the function space $\mathcal{J}_T^{q,s}$, $1 < q, s < \infty$, of initial values with norm

$$\|u_0\|_{\mathcal{J}_T^{q,s}} = \Big(\int_0^T \big(\|e^{-tA_q}P_q u_0\|_q^s + \|A_q e^{-tA_q} P_q u_0\|_q^s\big)\, dt\Big)^{1/s},$$

where P_q is the Helmholtz projection on $L^q(\mathbb{R}^3)^3$ and $A_q = -P_q\Delta$ is the Stokes operator. The following theorem states that the equation under consideration is well posed in this space.

Theorem 1.2 *Let $0 < T < \infty$ and assume that for some $1 < q, s < \infty$ the data $u_0 \in L^q_\sigma(\mathbb{R}^3)^3$ and $f \in L^s\big(0,T; L^q(\mathbb{R}^3)^3\big)$ satisfy*

$$f,\, \partial_\theta f,\, t\nabla' f \in L^s\big(0,T; L^q(\mathbb{R}^3)^3\big)$$

and $u_0,\, \partial_\theta u_0 \in \mathcal{J}_T^{q,s}$. Then the unique solution $(v, \nabla p) \in L^s(0,T; (L^q(\mathbb{R}^3))^6$ of

$$\begin{aligned} \frac{\partial v}{\partial t} + \mathcal{L}v + \nabla p &= f \qquad in \quad \mathbb{R}^3 \times (0,\infty) \\ \nabla \cdot v &= 0 \qquad in \quad \mathbb{R}^3 \times (0,\infty) \end{aligned} \tag{1.11}$$

with initial data $v(0,y) = u_0(y)$ is given by

$$v(y,t) = \int_0^t \int_{\mathbb{R}^3} \Gamma_0(y - \tilde{z}(t,s,z), t-s) f(z,s)\, dz\, ds + \int_{\mathbb{R}^3} \Gamma_0(y - \tilde{z}(t,0,z), t) u_0(z)\, dz, \tag{1.12}$$

$$\begin{aligned} p(y,t) &= \int_0^t \int_{\mathbb{R}^3} Q_0(y - \tilde{z}(t,s,z), t-s) \cdot f(z,s)\, dz\, ds \\ &= \int_{\mathbb{R}^3} Q^*(y-z) \cdot f(z,t)\, dz. \end{aligned} \tag{1.13}$$

Moreover, v, p satisfies the a priori estimate

$$\begin{aligned} &\|v;\, \nabla v;\, \nabla^2 v;\, v_t;\, \partial_\theta v;\, \nabla p\|_{L^s(0,T;L^q)} \;\le\; C(1+T)\|u_0;\, \partial_\theta u_0\|_{\mathcal{J}_T^{q,s}} + \\ &+ C(1+T)\big[\|f;\, \partial_\theta f\|_{L^s(0,T;L^q)} + |\omega \wedge u_\infty|\big((1+T)\|f\|_{L^s(0,T;L^q)} + \|t\nabla' f\||_{L^s(0,T;L^q)}\big)\big] \end{aligned} \tag{1.14}$$

where the constant C depends on q, s and ω, u_∞, but not on T.

Remark 1.3 We note that in the simpler case when u_∞ is parallel to ω and consequently $\omega \wedge u_\infty = 0$ the terms $|\omega \wedge u_\infty|(1+T)\big((1+T)\|f\|_{L^s(0,T;L^q)} + \|t\nabla' f\|_{L^s(0,T;L^q)}\big)$ are not needed in (1.14). Other terms which are already present in an estimate of v_t and when $\omega || u_\infty$ are due to the fact that the operator $\mathcal{L}$ does not generate an analytic semigroup and will not satisfy the standard maximal regularity estimate, see [16], [17], [27], [29].

Corollary 1.4 *(i) The fundamental solution Γ from Theorem 1.1 is unique.*

(ii) For any $y, z \in \mathbb{R}^3$ and $s < \tau < t$ one has the semigroup property

$$\int_{\mathbb{R}^3} \Gamma(y, z', t, \tau)\, \Gamma(z', z, \tau, s)\, dz' = \Gamma(y, z, t, s). \tag{1.15}$$

(iii) For $u \in \mathcal{S}(\mathbb{R}^3)^3$

$$\lim_{(y,t)\to(y^0,0^+)} \int_{\mathbb{R}^3} \Gamma_0(y - \tilde{z}(t,0,z),t)u(z)\,dz = Pu(y^0)\,.$$

(iv) The (backward in time) adjoint problem

$$(-\partial_s + \mathcal{L}^*)w + \nabla\pi = g,\ \nabla\cdot w = 0 \quad \text{on } \mathbb{R}^3\times(0,T),\ w(T) = 0$$

with the operator $\mathcal{L}^*w = -\Delta w + (O(t)^T\cdot u_\infty + \omega\wedge y)\cdot\nabla w - \omega\wedge w$ *has the fundamental solution*

$$\Gamma'(z,y,s,t) = \Gamma_0(z - \tilde{y}(s,t,y), s-t)$$

where $\tilde{y}(s,t,y) = O(t-s)\big(y + (t-s)O(t)^T u_\infty\big)$.

Almost all results known to the authors so far concern the case when the velocity at infinity, u_∞, vanishes or is parallel to the angular velocity ω. Concerning the linear steady case we mention the work of Farwig, Hishida, Müller [13, 6, 7] in L^q for the whole space and Hishida [29, 30] for an exterior domain. A generalization to weighted spaces was performed by Farwig, Krbec, Nečasová [14, 15] and by Kračmar, Nečasová, Penel [33]. The nonlinear steady situation was e.g. investigated in L^2 by Galdi [22] proving pointwise estimates for Navier-Stokes equations with rotating terms; in particular, he obtained for a steady solution u_s, p_s that

$$|u_s(x)| \le \frac{c}{|x|},\ |\nabla u_s(x)| + |p_s(x)| \le \frac{c}{|x|^2}\,.$$

An extension of this result was obtained by Deuring, Kračmar, Nečasová, see [1]-[4]. Moreover, Galdi, Kyed [23] prove that every Leray solution (finite Dirichlet integral of the velocity) satisfying an energy inequality is physically reasonable. Another outlook on estimates in $L^{q,\infty}$, the weak L^q-spaces, has been considered by Farwig, Hishida [9]. Further, Galdi and Silvestre [26] have proved a stability result for steady solutions u_s. A generalization of this result to the $L^{3,\infty}$ setting was obtained by Hishida and Shibata [32]. Concerning the nonsteady Navier-Stokes case with rotating terms we mention the work of Hishida [31] and of Geissert, Heck, Hieber [27] in the L^2-framework and L^q-framework, respectively. The fundamental solution $\Gamma(x,z,t)$ in the nonsteady linear case was investigated by Guenther and Thomann [38]. For a recent result in the case of a time-dependent ω we refer to Hansel [28].

In the steady case the fundamental solution is obtained via an integration in time $t \in (0,\infty)$ of $\Gamma(x,z,t)$. When $u_\infty = 0$, the asymptotic profile of steady solutions is analyzed by Farwig, Hishida in [10] for the linear problem and in [11], [12] for the nonlinear problem using Landau solutions; we also refer to Farwig, Galdi, Kyed [8] for the case of Leray solutions. In [24] Galdi, Kyed discuss properties of Leray solutions of a special model when the constant vectors u_∞ and ω are arbitrarily oriented.

2 Preliminaries

To describe the general procedure leading from (1.2) to (1.6) we introduce the skew-symmetric matrix $\widetilde{\Omega}(t) \in \mathbb{R}^{3,3}$, $t \geq 0$, defined by the property $\widetilde{\Omega}a = \tilde{\omega} \wedge a$ for all $a \in \mathbb{R}^3$ and the orthogonal matrix of rotation $O(t) \in \mathbb{R}^{3,3}$ defined by the linear system of ordinary differential equations

$$\dot{O} = \widetilde{\Omega}\, O, \quad O(0) = I.$$

Note that $\dot{O}\, O^T = \widetilde{\Omega} = -O\, \dot{O}^T$. Then the domain $\mathcal{D}(t)$ occupied by the fluid at time $t \geq 0$ is given by

$$\mathcal{D}(t) = O(t)\mathcal{D} + y_C(t)$$

where $\mathcal{D} = \mathbb{R}^3 \setminus \mathcal{B}$ is the given exterior domain at time $t = 0$.

Now we introduce the change of variables

$$x = O(t)^T\big(y - y_C(t)\big) \tag{2.1}$$

and the new functions, cf. (1.4),

$$u(x,t) = O(t)^T v(y,t), \;\; p(x,t) = q(y,t), \;\; f(x,t) = O(t)^T \tilde{f}(y,t)\,. \tag{2.2}$$

Then $v(y,t) = O(t)\, u(x,t) = O(t)\, u(O^T(y - y_C(t)), t)$ has the time derivative

$$\begin{aligned} v_t &= \dot{O}\, u + O\, u_t + O\big[\big((\dot{O}^T(y - y_C(t)) - O^T \dot{y}_C\big) \cdot \nabla_x\big]u \\ &= O\big(u_t + O^T\dot{O}u + \big[\big(\dot{O}^T O x - O^T \dot{y}_C\big) \cdot \nabla_x\big]u\big). \end{aligned}$$

Moreover, a simple calculation implies that $v \cdot \nabla_y v = O\big(u \cdot \nabla_x u\big)$, $\Delta_y v = O\Delta_x u$ and $\nabla_y q = O\, \nabla_x p$. Hence from (1.2) we get that in $\mathcal{D}$

$$u_t - \nu\Delta_x u + u \cdot \nabla_x u + \big[\big(\dot{O}^T O x - O^T \dot{y}_C\big) \cdot \nabla_x\big]u + O^T\dot{O}u + \nabla_x p = f\,. \tag{2.3}$$

To simplify (2.3) we define in addition to $\widetilde{\Omega}$, $\tilde{\omega}$ the skew-symmetric matrix and angular velocity, $\Omega = \Omega(t), \omega = \omega(t)$, $t \geq 0$, respectively, by

$$\Omega = O^T\dot{O} \quad \text{and} \quad \Omega a = \omega \wedge a \text{ for all } a \in \mathbb{R}^3\,,$$

so that $\dot{O}^T O = -\Omega$, and the "new" path of the center of mass, $x_C(t)$, defined by

$$\dot{x}_C = O^T \dot{y}_C,\ x_C(0) = y_C(0) = 0. \tag{2.4}$$

Then (2.3) reads

$$u_t - \nu\Delta_x u + u\cdot\nabla_x u - \big[\big(\omega\wedge x + \dot{x}_C\big)\cdot\nabla_x\big]u + \omega\wedge u + \nabla_x p = f \tag{2.5}$$

where $\omega\wedge u$ except for the factor 2 is the Coriolis force, $\dot{x}_C(t)$ denotes the velocity of the center of mass in the new coordinate system attached to the rotating obstacle and $(\omega\wedge x)\cdot\nabla u$ is a new term not subordinate to the Laplacian in the exterior domain $\mathcal{D}$. Note that $\mathrm{div}_x u = \mathrm{div}_y v = 0$.

Summarizing the previous results we get that $(u,\, p)$ is a solution of the nonlinear system

$$\begin{array}{rcll} \partial_t u - \nu\Delta u + (u\cdot\nabla)\,u + \nabla p - & & & \\ -\big[\big(\omega\wedge x + \dot{x}_C\big)\cdot\nabla\big]u + \omega\wedge u & = & f & \text{in}\quad \mathcal{D}\times(0,\infty) \\ \mathrm{div}\, u & = & 0 & \text{in}\quad \mathcal{D}\times(0,\infty) \\ u & = & u_{\partial\mathcal{D}} & \text{on}\quad \partial\mathcal{D}\times(0,\infty) \\ u(x,t) & \to & 0 & \text{as}\quad |x|\to\infty \end{array} \tag{2.6}$$

where $u_{\partial\mathcal{D}}(x,t) = \omega\wedge x + \dot{x}_C$. For more details on this change of coordinates including stress and inertia tensors we refer to [21, Ch. 1].

In this paper $\tilde{\omega} \equiv ke_3$, $k\neq 0$, i.e., in the inertial frame $\mathcal{I}$ the angular velocity is a constant multiple of the third unit vector e_3, and the path of the center of mass of the obstacle is given by its translational velocity $\dot{y}_C(t)\equiv u_\infty$, where $0\neq u_\infty\in\mathbb{R}^3$ is a vector transversal or even orthogonal to $\tilde{\omega}$. Then

$$O(t) = \begin{pmatrix} \cos kt & -\sin kt & 0 \\ \sin kt & \cos kt & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.7}$$

$$\widetilde{\Omega}(t) = \Omega(t) = k\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \tilde{\omega}(t) = \omega(t) = ke_3\,, \tag{2.8}$$

and

$$\dot{x}_C(t) = U(t) := O(t)^T u_\infty = \begin{pmatrix} \cos kt\, u_{\infty,1} + \sin kt\, u_{\infty,2} \\ -\sin kt\, u_{\infty,1} + \cos kt\, u_{\infty,2} \\ u_{\infty,3} \end{pmatrix} \tag{2.9}$$

is *time-dependent* since u_∞ is *not parallel* to e_3. Linearizing (2.6) around $u = 0$, we are left with the system (1.6) the fundamental solution of which we are looking for.

To this aim, recall that the Riesz transforms R_j, $j = 1, 2, 3$, can be defined by their symbol $-i\frac{\xi_j}{|\xi|}$ in Fourier space defining continuous linear operators on $L^p(\mathbb{R}^3)$, $1 < p < \infty$. Here we use the Fourier transform $\mathcal{F}$ in the form

$$\mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{3/2}} \int e^{-ix\cdot\xi} f(x)\, dx\,,$$

e.g., for f in $\mathcal{S}(\mathbb{R}^3)$, the Schwartz class of rapidly decreasing test functions. Let P denote the Helmholtz projection of vector fields on $\mathbb{R}^3$ onto divergence free vector fields. Then,

$$P = I + R = I + \nabla \mathrm{div}(-\Delta)^{-1},$$

where R is the 3×3-matrix operator with entries $(R_i R_j)_{i,j}$.

As basic results for Kummer functions we mention the following facts:

Lemma 2.1 *For a, $c > 0$ the following results hold:*

(1)

$${}_1F_1(1, c, \lambda) = \sum_{n=0}^{\infty} \frac{1}{(c)_n} \lambda^n.$$

(2)

$$\frac{d}{d\lambda}\, {}_1F_1(a, c, \lambda) = \frac{a}{c}\, {}_1F_1(a+1, c+1, \lambda).$$

(3) There exists a constant $C > 0$ such that for all $\lambda > 0$

$$\left| e^{-\lambda}\big({}_1F_1(1, c, \lambda) - 1\big) \right| \le C \frac{\lambda}{(1+\lambda)^c}\,.$$

(4)

$$\frac{d}{d\lambda}\Big(e^{-\lambda}\big({}_1F_1(1,c,\lambda) - 1\big)\Big) = \frac{1}{c} e^{-\lambda}\, {}_1F_1(1, c+1, \lambda) - \frac{\lambda}{c+1} e^{-\lambda}\, {}_1F_1(1, c+2, \lambda),$$

$$\frac{d^2}{d\lambda^2}\Big(e^{-\lambda}\big({}_1F_1(1,c,\lambda) - 1\big)\Big) = \frac{-2}{c+1} e^{-\lambda}\, {}_1F_1(1, c+2, \lambda) + \frac{\lambda}{c+2} e^{-\lambda}\, {}_1F_1(1, c+3, \lambda).$$

In particular, there exists a constant $C > 0$ such that for all $\lambda > 0$ and for $j = 1, 2$

$$\left| \frac{d^j}{d\lambda^j}\Big(e^{-\lambda}\big({}_1F_1(1,c,\lambda) - 1\big)\Big) \right| \le C \frac{1}{(1+\lambda)^{c+j-1}}\,.$$

Proof: (1)-(2) can be found in [38, pp. 82f]. For the proof of (3) we use the Gamma function Γ, the asymptotic result

$$e^{-\lambda}\,{}_1F_1(1,c,\lambda) \sim \Gamma(c)\frac{1}{\lambda^{c-1}} \quad \text{as} \quad \lambda \to \infty, \tag{2.10}$$

see [38, p. 82], and that ${}_1F_1(1,c,0)=1$. $(4)_1$ follows from the formula

$$\frac{d}{d\lambda}\big(e^{-\lambda}\,{}_1F_1(1,c,\lambda)\big) = \frac{1-c}{c}e^{-\lambda}\,{}_1F_1(1,c+1,\lambda)\,,$$

see [38, Lemma 2.1], and the identity

$${}_1F_1(1,c,\lambda)-1 = \frac{1}{c}\lambda\,{}_1F_1(1,c+1,\lambda),$$

see [38, (4.9)]. The second equation in (4) is proved analogously. The estimates are a consequence of (2.10). ∎

3 Proof of the main theorem

First, ignoring the pressure term and the solenoidality condition in (1.7), we consider, with $U(t) = O(t)^T u_\infty$, the linear operator

$$\widetilde{\mathcal{L}}w = \widetilde{\mathcal{L}}_{y,t}w = -\Delta w - (U(t) + \omega \wedge y)\cdot \nabla w + \omega \wedge w. \tag{3.1}$$

Proposition 3.1 *Assume $w_0 \in \mathcal{S}(\mathbb{R}^3)^3$. Then the solution of the initial value problem*

$$\frac{\partial w}{\partial t} + \widetilde{\mathcal{L}}w = 0 \quad \text{in } (s,\infty), \quad w(\cdot,s) = w_0, \tag{3.2}$$

is given by

$$w(y,t) = \int_{\mathbb{R}^3} \widetilde{\Gamma}(y - \tilde{z}(t,s,z), t-s) w_0(z)\,dz \tag{3.3}$$

where

$$\widetilde{\Gamma}(y,\tau) = K(y,\tau)O(\tau)^T\,, \tag{3.4}$$

$$\tilde{z}(t,s,z) = O(s-t)z - Y(t,s)\,, \tag{3.5}$$

cf. (1.10), and

$$Y(t,s) = (t-s)O(t)^T u_\infty = (t-s)\begin{pmatrix} \cos kt\, u_{\infty,1} + \sin kt\, u_{\infty,2} \\ -\sin kt\, u_{\infty,1} + \cos kt\, u_{\infty,2} \\ u_{\infty,3} \end{pmatrix}. \tag{3.6}$$

Proof: First we consider the case when $s = 0$. By two elementary transformations we will reduce problem (3.2) with $s = 0$ to the simpler problem

$$\frac{\partial v}{\partial t} - (\omega \wedge y) \cdot \nabla v - \Delta v = 0 \quad \text{in } (0, \infty),\ v(0) = w_0\,. \tag{3.7}$$

First let $w^*(t) = O(t)w(t)$. Then

$$\frac{\partial w^*}{\partial t} - (U + \omega \wedge y) \cdot \nabla w^* - \Delta w^* = 0 \quad \text{in } (0, \infty),\ w^*(0) = w_0. \tag{3.8}$$

Next, we are looking for a matrix field $Y(t)$ with $Y(0) = 0$, and let

$$v(y, t) = w^*(y - Y(t), t). \tag{3.9}$$

Taking into account that v is evaluated at (y, t), but w^* at $(y - Y(t), t)$, we get in view of (2.7)-(2.9) that

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial w^*}{\partial t} - \frac{\partial Y}{\partial t} \cdot \nabla w^* \\ &= (U + \omega \wedge (y - Y(t))) \cdot \nabla w^* + \Delta w^* - \frac{\partial Y}{\partial t} \cdot \nabla w^* \\ &= (U + \omega \wedge (y - Y(t))) \cdot \nabla v + \Delta v - \frac{\partial Y}{\partial t} \cdot \nabla v \\ &= (\omega \wedge y) \cdot \nabla v + \Delta v + \Big(U - \omega \wedge Y - \frac{\partial Y}{\partial t}\Big) \cdot \nabla v\,. \end{aligned} \tag{3.10}$$

In order to let vanish the last term in brackets on the right-hand side of (3.10) Y must satisfy the linear ordinary differential equation (1.9), i.e.,

$$\partial_t Y + \omega \wedge Y = U \quad \text{in } (0, \infty),\ Y(0) = 0. \tag{3.11}$$

Note that the system (3.11) is in the state of resonance, and its solution with vanishing initial value at $t = 0$ is given by $Y(t) = tU(t) = tO(t)^T u_\infty$.

Now the solution v to (3.7) can be written in the form

$$\begin{aligned} v(y, t) &= \int_{\mathbb{R}^3} K(z, t) w_0(O(t)y - z)\, dz \\ &= \frac{1}{(4\pi t)^{3/2}} \int_{\mathbb{R}^3} \exp\left(-\frac{|O(t)y - z|^2}{4t}\right) w_0(z)\, dz\,, \end{aligned}$$

see e.g. DaPrato and Lunardi [36]. Hence we get

$$
\begin{aligned}
w(y,t) &= O(t)^T w^*(y,t) = O(t)^T v(y+Y(t),t) \\
&= \frac{1}{(4\pi t)^{3/2}} O(t)^T \int_{\mathbb{R}^3} \exp\left(-\frac{|O(t)(y+Y(t))-z|^2}{4t}\right) w_0(z)\,dz\,.
\end{aligned}
$$

Finally we note that $|O(t)(y+Y(t))-z| = |y-\tilde{z}(t,0,z)|$.

In the more general case $s>0$ we easily see that problem (3.2) can be reduced to the previous case $s=0$ by replacing t by $t-s$ and u_∞ by $O(s)^T u_\infty$. This argument immediately yields the assertion when $s>0$.

Now it is straightforward to see that $\widetilde{\Gamma}(y-\tilde{z}(t,s,z),t-s)$ extended by 0 for $t \le s$ solves the equation $\big(\partial_t + \widetilde{\mathcal{L}}\big)\widetilde{\Gamma} = \delta_y(z)\delta_s(t)$ in the sense of distributions on $\mathbb{R}^3 \times \mathbb{R}$. ■

To obtain the fundamental solution of the linearized problem (1.7) taking into account the incompressibility condition, we have to adapt Proposition 3.1, cf. [38]. Using the Helmholtz projection P it is easy to see that for every fixed $a \in \mathbb{R}^3$

$$
\begin{aligned}
\Gamma(y,z,t,s)a &= \Gamma_0(y-\tilde{z}(t,s,z),t-s)a &&= P(\widetilde{\Gamma}(y-\tilde{z}(t,s,z),t-s)a), \\
Q(y,z,t,s)a &= Q_0(y-\tilde{z}(t,s,z),t-s)a &&= -\frac{1}{4\pi}\, a\cdot\nabla\frac{1}{|y-z|}\delta_s(t)
\end{aligned}
$$

is the fundamental tensor for the linear equation (1.7); here, P acts on the variable y. In particular, for $t>s$

$$
\big(\frac{\partial}{\partial t} + \mathcal{L}\big)(\Gamma a) + \nabla Q a = 0,\ \nabla\cdot(\Gamma a) = 0. \tag{3.12}
$$

Since

$$
\Gamma_0(y,\tau)a = P\widetilde{\Gamma}(y,\tau)a = (I+R)\widetilde{\Gamma}(y,\tau)a = [(I+R)K(y,\tau]O(\tau)^T a
$$

and $R_iR_jf = \frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}(-\Delta)^{-1}$ we get

$$
\Gamma_0(y,\tau)a = [K(y,\tau)I + \text{Hess}\ \psi(y,\tau)]O(\tau)^T a\,; \tag{3.13}
$$

here $\psi(y,\tau)$ is a solution of the equation $-\Delta_y\psi(y,\tau) = K(y,\tau)$, i.e.,

$$
\psi(y,\tau) = \frac{1}{4\pi}\frac{1}{(4\pi\tau)^{3/2}} \int_{R^3} \frac{1}{|y-x|}\exp\left(-\frac{|x|^2}{4\tau}\right) dx\,, \tag{3.14}
$$

and Hess $\psi(y,\tau) = \big(\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\big)\psi(y,\tau)$ denotes the Hessian of ψ.

To compute ψ and its Hessian we follow [38] and introduce the *error function*

$$\operatorname{Erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-u^2} du = \frac{2s}{\sqrt{\pi}} e^{-s^2} {}_1F_1(1, 3/2, s^2).$$

Lemma 3.2 *For all $\tau > 0$*

$$\psi(y,\tau) = \frac{1}{4\pi|y|} \operatorname{Erf}\left(\frac{|y|}{\sqrt{4\tau}}\right) \tag{3.15}$$

$$= \frac{1}{2\pi\sqrt{4\pi\tau}} \exp\left(-\frac{|y|^2}{4\tau}\right) {}_1F_1\left(1, \frac{3}{2}, \frac{|y|^2}{4\tau}\right)$$

$$= 2\tau K(y,\tau)\, {}_1F_1\left(1, \frac{3}{2}, \frac{|y|^2}{4\tau}\right) \tag{3.16}$$

and

$$\frac{\partial^2}{\partial y_i \partial y_j} \psi(y,\tau) = K(y,\tau)\left(-\frac{1}{3}{}_1F_1\left(1, \frac{5}{2}, \frac{|y|^2}{4\tau}\right)\delta_{ij} + \frac{y_i y_j}{|y|^2}\left[{}_1F_1\left(1, \frac{5}{2}, \frac{|y|^2}{4\tau}\right) - 1\right]\right).$$

Proof: See [38, Lemma 3.1, Prop. 3.2]. ∎

Proof of Theorem 1.1. From Proposition 3.1 and Lemma 3.2 it follows for all $a \in \mathbb{R}^3$ that $(\partial_t + \mathcal{L})(\Gamma a) + \nabla(Qa) = 0$ for $t > s$ and $\operatorname{div}(\Gamma a) = 0$. It remains to show for every initial value $u_0 \in \mathcal{S}(\mathbb{R}^3)^3$ with Helmholtz decomposition $u_0 = h + \nabla q$ that

$$\lim_{t \to s+} \int_{\mathbb{R}^3} \Gamma_0(y - \tilde{z}(t,s,z), t-s) u_0(z)\, dz + \nabla_y \int_{\mathbb{R}^3} Q^*(y-z) u_0(z)\, dz = u_0(y)\,. \tag{3.17}$$

We note that h, $q \in W^{2,2}(\mathbb{R}^3)$ and $\nabla \cdot h = 0$. Hence

$$\int_{\mathbb{R}^3} Q^*(y-z) u_0(z)\, dz = \int_{\mathbb{R}^3} -\frac{1}{4\pi} \nabla_y \frac{1}{|y-z|} \nabla q(z)\, dz$$
$$= -\int_{\mathbb{R}^3} \frac{1}{4\pi} \frac{1}{|y-z|} \Delta q(z)\, dz = q(y)$$

and consequently

$$\nabla_y \int_{\mathbb{R}^3} Q^*(y-z) u_0(z)\, dz = \nabla_y q(y). \tag{3.18}$$

By Lemma 3.1 and the transformation $\tilde{z}(t,s,z) = O(s-t)z - Y(t,s)$ it is easy to see for $\psi = \psi(y - \tilde{z}(t,s,z), t-s)$ that

$$\mathrm{Hess}_y\psi = O(s-t)\mathrm{Hess}_z\psi \; O(s-t)^T.$$

Then, using (3.13)

$$\int_{\mathbb{R}^3} \Gamma(y - \tilde{z}(t,s,z), t-s)\, u_0(z)\, dz = O(s-t) \int_{\mathbb{R}^3} K(y - \tilde{z}(t,s,z), t-s)\, u_0(z)\, dz$$
$$+ O(s-t) \int_{\mathbb{R}^3} \mathrm{Hess}_z\psi(y - \tilde{z}(t,s,z), t-s)\, u_0(z)\, dz\,.$$

Standard properties of the heat kernel give

$$\lim_{t\to s+} O(s-t) \int_{\mathbb{R}^3} K(y - \tilde{z}(t,s,z), t-s) u_0(z)\, dz = u_0(y). \tag{3.19}$$

Finally, similarly as in [38, p. 88] and using the Helmholtz decomposition of u_0, we get for $i = 1,2,3$ that

$$\lim_{t\to s+} \int_{\mathbb{R}^3} \big(\mathrm{Hess}_y\psi(y - \tilde{z}(t,s,z), t-s)\, u_0(z)\big)_i\, dz$$
$$= \lim_{t\to 0+} -O(s-t) \int_{\mathbb{R}^3} K(y - \tilde{z}(t,s,z), t-s)\, \frac{\partial q}{\partial z_i}\, dz = -\frac{\partial q}{\partial y_i}(y). \tag{3.20}$$

Then (3.17) follows from (3.18), (3.19) and (3.20).

Now Theorem 1.1 is proved. ∎

4 Basic properties of the fundamental solution

We will use the following notation:

$$w = y - \tilde{z}(t,s,z), \quad \hat{w} = \frac{\hat{w}}{\hat{w}}$$
$$\Lambda(\hat{w}) = \hat{w} \otimes \hat{w}$$
$$\lambda = \frac{|y - \tilde{z}(t,s,z)|^2}{4(t-s)} = \frac{|w|^2}{4(t-s)}$$
$$\mathcal{F}(\lambda) = {}_1F_1(1, 5/2, \lambda)$$
$$M(y,z,t,s) = \frac{1}{3}\frac{1}{(4\pi(t-s))^{3/2}} e^{-\lambda}\, \mathcal{F}(\lambda)[I - 3\Lambda(\hat{w})]$$

so that

$$\Gamma(y,z,t,s) = \big[K(y-\tilde z(t,s,z), t-s)\{I-\Lambda(\hat w)\} - M(y,z,t,s)\big]O(s-t).$$

Proposition 4.1 *The fundamental solution* Γ *has (in each component of the* 3×3-*matrix) the following asymptotic properties:*

(i) For any vectors $y,\, z\in\mathbb{R}^3$, $y\neq z$,

$$\Gamma(y,z,t,s) \sim -\frac{1}{4\pi}\frac{1}{|y-z|^3}\left[I - 3\frac{(y-z)\otimes(y-z)}{|y-z|^2}\right] \quad \textit{as}\ \ t\to s+\ .$$

(ii) For any vectors $y,\, z\in\mathbb{R}^3$ *and* $t>s$

$$\Gamma(y,z,t,s) \sim \frac{2}{3}\frac{1}{(4\pi(t-s))^{3/2}}O(s-t) \quad \textit{as} \quad \frac{|y-\tilde z(t,s,z)|^2}{4(t-s)} \to 0\,.$$

(iii) Let $y^0, z, \eta\in\mathbb{R}^3$, $|\eta|=1$, *be fixed and let* $y=y^0+\rho\eta$, $\rho>0$. *Then for* $t>s$

$$\Gamma(y,z,t,s) \sim -\frac{1}{4\pi}\frac{1}{|y-\tilde z(t,s,z)|^3}[I-3\eta\otimes\eta]O(s-t) \quad \textit{as}\ \ \rho\to\infty\,.$$

(iv) For any vectors $y, z\in\mathbb{R}^3$, *as* $t\to\infty$,

$$O(t-s)\Gamma(y,z,t,s) \sim -\frac{1}{4\pi}\frac{1}{|tu_\infty|^3}\left[I - 3\frac{O(t)^T u_\infty\otimes O(t)^T u_\infty}{|u_\infty|^2}\right]. \tag{4.1}$$

Proof: (i) Since $y\neq z$, the term $\lambda\to\infty$ as $t\to s+$. Hence the leading term in Γ is determined by M where by Lemma 2.1 (3) $e^{-\lambda}\mathcal{F}(\lambda)\sim\Gamma(5/2)\lambda^{-3/2} = \frac{3}{4}\sqrt{\pi}\lambda^{-3/2}$. This proves (i).

(ii) By assumption $\lambda\to 0$. Since $\mathcal{F}(\lambda)\to 1$, $e^{-\lambda}\to 1$ as $\lambda\to 0$, the term $\Lambda(\theta)$ in Γ will be canceled in the limit, and the asymptotic behavior is determined by the remaining terms leading to (ii), see also (4.2) below.

(iii) In this case $\lambda\to\infty$ and the leading term in Γ is determined by M, cf. (i). Since $\Lambda(\hat w)\sim\eta\otimes\eta$ as $\rho\to\infty$ for $t>s$ fixed, we get (iii).

(iv) We use $\tilde z(t,s,z) = O(s-t)z-(t-s)O(t)^T u_\infty)$ and get for large t that

$$\lambda = \frac{|y-\tilde z(t,s,z)|^2}{4t} \sim \frac{t|u_\infty|^2}{4}\,,$$

$$\hat w = \frac{y-\tilde z(t,s,z)}{|y-\tilde z(t,s,t)|} \sim \frac{tO(t)^T u_\infty}{|tu_\infty|} = O(t)^T\hat u_\infty\,, \quad \hat u_\infty = \frac{u_\infty}{|u_\infty|}\,;$$

moreover,

$$\Lambda(\hat{w}) = \hat{w} \otimes \hat{w} \sim O(t)^T \hat{u}_\infty \otimes O(t)^T \hat{u}_\infty\,.$$

Since by Lemma 2.1 (3) the leading term in Γ is determined by M,

$$\begin{aligned} O(t-s)\Gamma(y,z,t,s) \ &\sim\ -\frac{1}{3}\frac{\Gamma(5/2)}{(4\pi t)^{3/2}}\Big(\frac{t|u_\infty|^2}{4}\Big)^{-3/2}\big[I - 3\Lambda(\hat{w})\big] \\ &= -\frac{1}{4\pi}\frac{1}{|tu_\infty|^3}\big[I - 3\Lambda(\hat{w})\big] \end{aligned}$$

as $t \to \infty$. ■

Global space-time estimates of Γ and of its derivatives can be obtained in terms of t and the spatial variable $w = y - \tilde{z}(t,s,z)$. For simplicity we let $s = 0$, use the notation $\lambda = \frac{|w|^2}{4t}$ and rewrite Γ_0 in the form

$$\Gamma_0(w,t) = \Big\{\frac{2}{3}\frac{e^{-\lambda}}{(4\pi t)^{3/2}}I - \frac{e^{-\lambda}}{(4\pi t)^{3/2}}\big(\mathcal{F}(\lambda) - 1\big)\Big(\frac{1}{3}I - \frac{w \otimes w}{|w|^2}\Big)\Big\}\, O(t)^T\,. \qquad (4.2)$$

Proposition 4.2 *There exist a constant $C > 0$ independent of $w \in \mathbb{R}^3$, $t > 0$ such that*

$$\begin{aligned} |\Gamma_0(w,t)| \ &\le\ \frac{C}{(t+|w|^2)^{3/2}}\,, \\ |\nabla_w \Gamma_0(w,t)| \ &\le\ \frac{C|w|}{(t+|w|^2)^{5/2}} \\ |\nabla_w^2 \Gamma_0(w,t)| \ &\le\ \frac{C}{(t+|w|^2)^{5/2}}\,. \end{aligned}$$

In particular Γ_0, $(1+|w|)\nabla_w\Gamma_0$, $\nabla_w^2\Gamma_0(\cdot,t) \in L^p(\mathbb{R}^3)$ for all $p \in (1,\infty)$ and all $t > 0$. Moreover, $\nabla_y\Gamma(\cdot,z,t,s)$, $\nabla_z\Gamma(y,\cdot,t,s) \in L^p(\mathbb{R}^3)$ for all $p \in (1,\infty)$, all $t > s$, and all $z \in \mathbb{R}^3$ or $y \in \mathbb{R}^3$, respectively.

Proof: By Lemma 2.1 $|e^{-\lambda}\mathcal{F}(\lambda)| \le C\big(1 + \lambda^{3/2}\big)^{-1}$ as $\lambda \to \infty$ and also as $\lambda \to 0$. Hence

$$|\Gamma_0(w,t)| \le \frac{ce^{-|w|^2/(4t)}}{t^{3/2}} + \frac{c}{t^{3/2}(1+|w|^2/t)^{3/2}} \le \frac{C}{(t+|w|^2)^{3/2}}\,.$$

To discuss estimates of derivatives we consider Γ_0 as in (4.2) and use that $\frac{d\lambda}{dw_j} = \frac{w_j}{2t}$, $j = 1, 2, 3$. Then Lemma 2.1 (4) yields the first order derivative

$$\begin{aligned} \frac{\partial}{\partial w_j}\Gamma_0(w,t) = \frac{1}{(4\pi t)^{3/2}}\frac{w_j}{2t}\Big\{ -\frac{2}{3}e^{-\lambda} I \\ -\Big(\frac{2}{5}e^{-\lambda}\,{}_1F_1(1,7/2,\lambda) - \frac{2\lambda}{7}e^{-\lambda}\,{}_1F_1(1,9/2,\lambda)\Big)\Big(\frac{1}{3}I - \frac{w\otimes w}{|w|^2}\Big)\Big\} \\ + \frac{1}{(4\pi t)^{3/2}}e^{-\lambda}\big(\mathcal{F}(\lambda)-1\big)\frac{\partial}{\partial w_j}\Big(\frac{w\otimes w}{|w|^2}\Big)O(t)^T\Big|_{\lambda=|w|^2/(4t)} \end{aligned} \tag{4.3}$$

and together with (2.10) the assertion for $|\partial\Gamma_0(w,t)/\partial w_j|$. Differentiating (4.3) with respect to w_k, taking into account $\frac{d\lambda}{dw_k} = \frac{w_k}{2t}$, $k = 1, 2, 3$, and Lemma 2.1 we finally get the estimate for $|\nabla^2\Gamma_0(w,t)|$.

Since $\nabla_y w = I$, $\nabla_z v = O(t)^T$ and $|w| \sim |y|$ or $|w| \sim |z|$ as $|y| \to \infty$ or $|z| \to \infty$, respectively, the assertions on $\nabla_y\Gamma(\cdot, z, t, s)$, $\nabla_z\Gamma(y, \cdot, t, s)$ are immediate. ■

Remark 4.3 Fixing the initial time $s = 0$ we would like to explain the meaning of the term

$$|y - \tilde{z}(t,0,z)| = |y - (O(t)^T z - tO(t)^T u_\infty)| = |O(t)y - z + tu_\infty|$$

occurring in Proposition 4.1 (iii) in the denominator of the asymptotic expansion of the fundamental solution Γ and in Proposition 4.2. For simplicity let us fix also $z = 0$, i.e., we consider an initial value and an external force concentrated near $z = 0$. Then we will work in an inertial frame with spatial variable $x = O(t)y$ so that the obstacle is rotating with angular velocity ω and its center of mass is not moving. Hence the fluid is moving past the obstacle with constant velocity $-u_\infty$, and by Proposition 4.2 the term

$$\frac{1}{(t + |x + tu_\infty|^2)^{3/2}}$$

plays a decisive role in the asymptotic expansion of $\Gamma(y, 0, t, 0)$.

First we consider points x, t with x either in the upstream direction $x = +u_\infty$ or orthogonal to u_∞ or even in the downstream direction, but not parallel to $-u_\infty$, i.e., $0 < \sphericalangle(x, -u_\infty) < 2\pi$. In that case, if $|x + tu_\infty|^2 > t$, then $\Gamma(y, 0, t, 0)$ decays as fast as $|x + tu_\infty|^{-3}$. Next let x move in the downstream direction $-u_\infty$ and assume $|x + tu_\infty|^2 \le t$, i.e., x lies in the closed ball $\overline{B}_{\sqrt{t}}(-tu_\infty)$. Then $\Gamma(y, 0, t, 0)$ is bounded by a constant $C(t) = t^{-3/2}$. The set of balls $B_{\sqrt{t}}(-tu_\infty)$, $t > 0$, defines a paraboloid oriented in the direction $-u_\infty$. Actually, let us assume for simplicity that $-u_\infty = e_1$. Then the condition $|x + tu_\infty|^2 \le t$ is equivalent to

$$|x - te_1|^2 \le t \Leftrightarrow |x'|^2 - 2tx_1 + x_1^2 + t^2 \le t \Leftrightarrow |x'|^2 \le \frac{1}{4} + x_1 - \big(t - x_1 - \frac{1}{2}\big)^2 .$$

Choosing $t = x_1 + 1/2$ we get the condition $|x'|^2 \le \frac{1}{4} + x_1$ which is equivalent to the well-known characterization $s(x) := |x| - x_1 \le 1/4$ of the wake in the stationary Navier-Oseen problem of fluid flow past an obstacle with velocity e_1 at infinity, see [5], [18]. A simple rotation and scaling argument yields a similar result when $u_\infty \neq 0$ is arbitrary. This proves the existence of a wake of paraboloidal shape in the downstream direction for any angular velocity ω and translational velocity $u_\infty \neq 0$.

Before coming to the proof of Theorem 1.2 we need a lemma on the nonstationary Stokes system

$$u_t - \Delta u + \nabla p = f, \ \ \operatorname{div} u = 0 \quad \text{in} \quad \mathbb{R}^3, \ u(0) = u_0 \quad \text{at} \quad t = 0 \tag{4.4}$$

on finite time intervals $(0,T)$, see Lemma 4.4 below. Recall $A_q = -P_q\Delta$ denote the Stokes operator on $\mathbb{R}^3$. It is well known that A_q generates a bounded analytic semigroup e^{-tA_q} by which the unique solution u of the Stokes problem can explicitly be written in the form

$$u(t) = e^{-tA_q} P_q u_0 + \int_0^t e^{-(t-\tau)A_q} P_q f(\tau)\, d\tau\,, \quad 0 < t < T\,. \tag{4.5}$$

Here we assume that $f \in L^s(0,T; L^q(\mathbb{R}^3))$ and u_0 lies in the space of initial values, $\mathcal{J}_T^{q,s}$, defined before Theorem 1.2. By the maximal regularity estimate, see [37], we know that

$$\|u_t;\, \nabla^2 u;\, \nabla p\|_{L^s(0,T;L^q)} \le c\big(\|u_0\|_{\mathcal{J}_T^{q,s}} + \|f\|_{L^s(0,T;L^q)}\big) \tag{4.6}$$

with a constant $c = c(q,s) > 0$ independent of T. Actually, only the $L^s(L^q)$-norm of $A_q e^{-tA_q} P_q u_0$ is needed in the term $\|u_0\|_{\mathcal{J}_T^{q,s}}$ in (4.6). Concerning the terms u, ∇u we note that by (4.5) $\|u(t)\|_q \le \|e^{-tA_q} P_q u_0\|_q + ct^{1/s'}\big(\int_0^t \|f\|_q^s\, d\tau\big)^{1/s}$ so that with the help of interpolation

$$\|u;\, \nabla u\|_{L^s(0,T;L^q)} \le c\big(\|u_0\|_{\mathcal{J}_T^{q,s}} + (1+T)\|f\|_{L^s(0,T;L^q)}\big) \tag{4.7}$$

where $c = c(q,s) > 0$ is independent of T.

Moreover, we note that the Stokes fundamental solution coincides with $\Gamma_0(x,\tau)$, cf. Theorem 1.1, up to the last factor $O(\tau)^T$ which has to be omitted, i.e.,

$$\Gamma_{\mathrm{St}}(x,\tau) = \Gamma_0(x,\tau) O(\tau)\,,$$

and the solution can explicitly be written in the form

$$u(x,t) = \int_0^t \int_{\mathbb{R}^3} \Gamma_{\rm St}(x-z,t-s) f(z,s)\,dz\,ds + \int_{\mathbb{R}^3} \Gamma_{\rm St}(x-z,t) u_0(z)\,dz. \qquad (4.8)$$

Lemma 4.4 *Let $1 < s, q < \infty$, $0 < T < \infty$, let the initial value u_0 satisfy u_0, $\partial_\theta u_0 \in \mathcal{J}_T^{q,s}$ and let $f \in L^s(0,T;L^q(\mathbb{R}^3))$ be given with $\partial_\theta f \in L^s(0,T;L^q(\mathbb{R}^3))$. Then the solution u of the Stokes system (4.4) satisfies, in addition to (4.6),*

$$\partial_\theta u_t, \partial_\theta u, \nabla\partial_\theta u, \nabla^2\partial_\theta u \in L^s(0,T;L^q(\mathbb{R}^3))$$

and the estimate

$$\|\partial_\theta u_t;\ \partial_\theta u;\ \nabla\partial_\theta u;\ \nabla^2\partial_\theta u\|_{L^s(0,T;L^q)} \le c(1+T)\big(\|u_0;\ \partial_\theta u_0\|_{\mathcal{J}_T^{q,s}} + \|f;\ \partial_\theta f\|_{L^s(0,T;L^q)}\big)$$

with a constant $c = c(q,s) > 0$ independent of T.

Proof: Given the solution u of the Stokes system (4.4) satisfying the estimate (4.6) we apply the differential operator $\partial_\theta = (\omega \wedge x)\cdot\nabla$ to (4.4). We easily get that

$$\begin{aligned} \partial_\theta \nabla p &= \nabla(\partial_\theta p) + \nabla^\perp p, \quad \nabla^\perp p = (-\partial_2 p, \partial_1 p, 0),\\ \partial_\theta\, {\rm div} u &= {\rm div}(\partial_\theta u) - {\rm div}(\omega\wedge u) = {\rm div}(\partial_\theta u) + ({\rm rot}\ u)_3\,. \end{aligned}$$

Hence $\partial_\theta u$, $\partial_\theta p$ is a solution of the generalized Stokes system

$$v_t - \Delta v + \nabla\pi = \partial_\theta f + \nabla^\perp p,\ \ {\rm div}\, v = -({\rm rot}\ u)_3 \ \text{ in } \mathbb{R}^3, \quad v(0) = \partial_\theta u_0 \ \text{ at } \ t=0\,.$$

To reduce this system to the Stokes system with solenoidal solutions we solve for almost all $t \in [0,T)$ the Poisson problem

$$\Delta\psi = -({\rm rot}\ u)_3$$

(with $\Delta\psi(0) = ({\rm rot}\ u_0)_3$) and get a solution $\psi = (-\Delta)^{-1}({\rm rot}\ u)_3$ satisfying the estimate $\|\Delta\nabla\psi;\, \partial_t\nabla\psi\|_q \le c\|\nabla^2 u;\, u_t\|_q$ for a.a. $t \in (0,T)$ and consequently

$$\|\nabla^2\nabla\psi;\, \partial_t\nabla\psi\|_{L^s(0,T;L^q)} \le c\big(\|u_0\|_{\mathcal{J}_T^{q,s}} + \|f\|_{L^s(0,T;L^q)}\big)\,.$$

Then $w = \partial_\theta u - \nabla\psi$ solves the Stokes system

$$\begin{aligned} w_t - \Delta w + \nabla\pi = \partial_\theta f + \nabla^\perp p - \partial_t\nabla\psi + \Delta\nabla\psi,\ \ {\rm div}\, w = 0 \ \text{ in } \ \mathbb{R}^3,\\ w(0) = \partial_\theta u_0 - \nabla\psi(0) \ \text{ at } \ t = 0 \end{aligned}$$

where $\operatorname{div} w(0) = 0$ and $P_q w(0) = P_q \partial_\theta u_0$. By the previous estimates we conclude with the maximal regularity estimate for w that

$$\|\partial_t \partial_\theta u;\, \nabla^2 \partial_\theta u\|_{L^s(0,T;L^q)} \le c\big(\|u_0;\, \partial_\theta u_0\|_{\mathcal{J}_T^{q,s}} + \|f;\, \partial_\theta f\|_{L^s(0,T;L^q)}\big)$$

with a constant $c = c(q,s) > 0$ independent of T. Moreover, as for the proof of (4.7) and with the estimates $\|\nabla\psi;\, \nabla^2\psi\|_{L^s(0,T;L^q)} \le c\|u;\, \nabla u\|_{L^s(0,T;L^q)}$, we get that

$$\|\partial_\theta u;\, \nabla \partial_\theta u\|_{L^s(0,T;L^q)} \le c(1+T)\big(\|u_0;\, \partial_\theta u_0\|_{\mathcal{J}_T^{q,s}} + \|f;\, \partial_\theta f\|_{L^s(0,T;L^q)}\big)\,.$$

Now the proof of the lemma is complete. ■

Proof of Theorem 1.2 Looking for a solution $(v,p)(y,t)$ of (1.11) for data f, v_0 we can solve the usual Stokes system

$$u_t - \nu\Delta u + \nabla\tilde{p} = \tilde{f}, \;\; \operatorname{div} u = 0 \;\text{ in }\; \mathbb{R}^3 \times (0,T), \;\; u(0) = u_0 := v_0$$

for a solution $(u,\, \tilde{p})(x,t)$ where

$$v(y,t) = O(t)^T u(x,t),\; f(y,t) = O(t)^T \tilde{f}(x,t),\; \tilde{p}(x,t) = p(y,t)$$

using the coordinate transform $y = O(t)^T(x - u_\infty t)$, cf. (1.3), (1.4) together with a change of notation. By the change of coordinates formula on $\mathbb{R}^3$ we obviously get for a.a. $t \in (0,T)$ that $\|\tilde{f}\|_q = \|f\|_q$ and consequently

$$\|\tilde{f}\|_{L^s(0,T;L^q)} = \|f\|_{L^s(0,T;L^q)}\,.$$

Concerning the angular derivative of $\tilde{f}(x,t) = O(t)f(O(t)^T(x - u_\infty t), t)$ we compute that

$$\begin{aligned}
\partial_\theta \tilde{f}(x,t) = (\omega\wedge x)\cdot\nabla_x\tilde{f}(x,t) &= O(t)\big[(\omega\wedge x)\cdot\nabla_x\big(f(O(t)^T(x-u_\infty t),t)\big)\big] \\
&= O(t)\big[O(t)^T(\omega\wedge x)\cdot(\nabla f)(O(t)^T(x-u_\infty t),t)\big] \\
&= O(t)\big[(\omega\wedge O(t)^T x)\cdot(\nabla f)(O(t)^T(x-u_\infty t),t)\big]
\end{aligned}$$

where we used the simple identity $O^T(\omega\wedge x) = (O^T\omega)\wedge(O^T x) = \omega\wedge O^T x$. Then by the change of variables formula we see that

$$\begin{aligned}
\|\partial_\theta \tilde{f}\|_q &= \|(\omega\wedge x')\cdot\nabla f(x' - O(t)^T u_\infty t, t)\|_q \\
&\le c\big(\|\partial_\theta f\|_q + |\omega\wedge O(t)^T u_\infty t|\,\|\nabla' f\|_q\big)\,.
\end{aligned}$$

Since $|\omega\wedge O(t)^T u_\infty| = |\omega\wedge u_\infty|$,

$$\|\partial_\theta \tilde{f}\|_{L^s(0,T;L^q)} \le c\big(\|\partial_\theta f\|_{L^s(0,T;L^q)} + |\omega\wedge u_\infty|\,\|t\,\nabla' f\|_{L^s(0,T;L^q)}\big)\,.$$

(i) To prove that the integral representation (1.12) of v is well-defined and defines a strong solution to (1.11) we again exploit the classical Stokes system and its fundamental solution Γ_{St}. By Proposition 4.2 for any fixed $t > 0$ and all $r > 1$ we have $\Gamma_{\mathrm{St}}(t) \in L^r(\mathbb{R}^3)$ with $\|\Gamma_{\mathrm{St}}(t)\|_r \le ct^{-3/(2r')}$ with a positive constant c independent of t. Hence the convolution integral $|\Gamma_{\mathrm{St}}(\cdot,t)| * |\tilde f(\cdot,t)|$ is well-defined for $t > 0$ and, choosing r sufficiently close to 1, Young's inequality shows that

$$u^{(0)}(y,t) = \int_0^t \int_{\mathbb{R}^3} \Gamma_{\mathrm{St}}(y-z,t-s)\tilde f(z,s)\,dz\,ds$$

is well-defined in $L^{\tilde q}(\mathbb{R}^3)$ where $\frac{1}{\tilde q} = \frac{1}{r} + \frac{1}{q} - 1$ and hence for a.a. $y \in \mathbb{R}^3$. A similar result holds for the integral

$$\int_0^t \int_{\mathbb{R}^3} \Gamma_0(y - \tilde z(t,s,z), t-s) f(z,s)\,dz\,ds\,.$$

Now we may apply Lemma 4.4 to get a solution u, p satisfying $u, \nabla u,\ \nabla^2 u,\ \nabla\tilde p \in L^s(0,T;L^q(\mathbb{R}^3))$ as well as $\partial_\theta u_t,\ \partial_\theta u, \nabla\partial_\theta u,\ \nabla^2\partial_\theta u \in L^s(0,T;L^q(\mathbb{R}^3))$ with corresponding estimates. By this means, (4.6), (4.7), and the above coordinate transform we also find a solution v, p of (1.11) satisfying

$$\begin{aligned} \|v;\,\nabla v;\,\nabla^2 v\|_{L^s(0,T;L^q)} &\le c\|u;\,\nabla u;\,\nabla^2 u\|_{L^s(0,T;L^q)} \\ &\le c\big(\|u_0\|_{\mathcal{J}_T^{q,s}} + (1+T)\|\tilde f\|_{L^s(0,T;L^q)}\big) \le c\big(\|v_0\|_{\mathcal{J}_T^{q,s}} + (1+T)\|f\|_{L^s(0,T;L^q)}\big)\,. \end{aligned}$$

For the time derivative v_t we use $v(y,t) = O(t)^T u(O(t)y + u_\infty t, t)$ and get that

$$\begin{aligned} v_t &= \dot O^T u + O^T u_t + O^T\big((((\dot O y + u_\infty)\cdot\nabla)u)(Oy + u_\infty t, t)\big) \\ &= O^T\big(O\dot O^T u + u_t + (((\dot O y + u_\infty)\cdot\nabla)u)(Oy + u_\infty t, t)\big) \\ &= O^T\big(-\omega\wedge u + u_t + ((\omega\wedge Oy + u_\infty)\cdot\nabla u)(Oy + u_\infty t, t)\big)\,. \end{aligned}$$

Consequently, by the change of variables formula we are led to the estimate

$$\begin{aligned} \|v_t\|_q &\le \|u;\,u_t\|_q + \|((\omega\wedge y' + u_\infty)\cdot\nabla u)(y' + u_\infty t, t)\|_q \\ &\le c\big(\|u;\,u_t;\,\partial_\theta u\|_q + \|\nabla u\|_q + |\omega\wedge u_\infty|\|t\nabla' u\|_q\big) \end{aligned}$$

and by integration over time to the corresponding estimate in $L^s(0,T;L^q)$. Finally we consider $\partial_\theta v$ and compute more or less as in the preceding steps that

$$(\omega\wedge y)\cdot\nabla v = O^T\big((\omega\wedge Oy)\cdot(\nabla_y u)(Oy + u_\infty t, t)\big);$$

thus

$$\begin{aligned}\|\partial_\theta v\|_q &\le \|(\omega\wedge y')\cdot\nabla u(y'+u_\infty t,t)\|_q\\ &\le c\big(\|\partial_\theta u\|_q + |\omega\wedge u_\infty|\,\|t\nabla' u\|_q\big)\end{aligned}$$

and consequently

$$\|\partial_\theta v\|_{L^s(0,T;L^q)} \le \|\partial_\theta u\|_{L^s(0,T;L^q)} + |\omega\wedge u_\infty|\,\|t\,\nabla' u\|_{L^s(0,T;L^q)}\,.$$

Concerning $t\,\nabla' u$ note that tu solves a nonstationary Stokes system with right-hand side $t\tilde f + u$ and vanishing initial value. Hence by (4.7)

$$\|t\nabla' u\|_{L^s(0,T;L^q)} \le c(1+T)\big(\|u_0\|_{\mathcal{J}_T^{q,s}} + (1+T)\|f\|_{L^s(0,T;L^q)}\big)\,.$$

Summarizing the previous estimates of v, ∇v, $\nabla^2 v$ and of v_t, $\partial_\theta v$ we get the estimate (1.14) with a constant C depending on q, s and ω, u_∞, but not on T. ■

Proof of Corollary 1.4 (i) For $(x,t)\in\mathbb{R}^3\times\mathbb{R}$ put

$$\begin{aligned}&y - y_C(t) = O(t)x\\ &v(y,t) = u(x,t)O(t),\ q(y,t) = p(x,t),\ \tilde f(y,t) = O(t)f(x,t)\,.\end{aligned}\tag{4.9}$$

Then the uniqueness property for (v,q) in the nonstationary Stokes problem

$$\begin{aligned}\partial_t v - \Delta v + \nabla q &= \tilde f \quad \text{in } \mathbb{R}^3\times(0,\infty)\\ \operatorname{div} v &= 0 \quad \text{in } \mathbb{R}^3\times(0,\infty)\\ v(y,t) &\to 0 \quad \text{as } |y|\to\infty\end{aligned}\tag{4.10}$$

and hence for (u,p) follows from classical results, see [34, Ch. 4, Sect. 6, Thm. 10].
, beland

(ii) Denote the left-hand side of (1.15) by $\gamma(y,z,t,s)$. By Theorem 1.2 γ as a function of y, t is a solution of the system $(\partial_t+\mathcal{L})\gamma = 0$ for $t>\tau$ and initial value $\Gamma(y,z,\tau,s)$ at $t=\tau$. Since $\Gamma(y,z,t,s)$, the right-hand side of (1.15), has the same properties, the uniqueness assertion of Theorem 1.2 completes the proof of this semigroup property.

(iii) This assertion is proved as the analogous result in Theorem 1.1.

(iv) It is easy to see that $\mathcal{L}$ yields the adjoint operator $\mathcal{L}^*$ modeling flow past a rotating obstacle with angular velocity $-\omega$ and translational velocity $-u_\infty$. To be more precise, on the interval $(0,T)$, $T>0$, the Oseen term $O(t)^T u_\infty$ should be written as $-O_-(T-t)^T(-O(T)^T u_\infty)$ where O_- is the matrix of rotation defined by $-\omega$ instead of ω; i.e., the initial velocity of the center of mass at time T is $-O(T)^T u_\infty = O(T)^T(-u_\infty)$ and will be rotated by $O_-(T-t)^T$ for $T > t > 0$.

Concerning the fundamental solution we note that $y - \tilde{z}(t,s,z) = -O(s-t)\big(z - \tilde{y}(s,t,y)\big)$ with $\tilde{y} = O(t-s)\big(y + (t-s)O(t)^T u_\infty\big)$, that $|y-\tilde{z}| = |z-\tilde{y}|$ and $O(t-s)\,(y-\tilde{z}) \otimes (y-\tilde{z}) = (z-\tilde{y}) \otimes (z-\tilde{y})\,O(s-t)^T$. ∎

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