Asymmetric hydrodynamics of suspensions. Nonlinear case.

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A viscous incompressible fluid with a large number of small axially symmetric solid particles is considered. It is assumed that the particles are identically oriented and under the influence of the fluid they move translationally or rotate around symmetry axis but the direction of their symmetry axes does not change. The asymptotic behavior of oscillations of the system is studied, when the diameters of particles and distances between the nearest particles are decreased. The equations, describing the homogenized model of the system, are derived. It is shown that the homogenized equations correspond to a non-standard hydrodynamics. Namely, the homogenized stress tensor linearly depends not only on the strain tensor but also on the rotation tensor.

Keywords: Microstructure; suspension; anisotropic material; inhomogeneous material; viscous incompressible fluid; asymmetric hydrodynamics; asymptotic analysis.

1 Introduction

Mechanics of suspensions is a part of a general physical-chemical sphere of knowledge about dispersions. Dispersion is a mixture of 2 phases one of which forms a continuum medium (we will call it a dispersive phase) and the other one is dispersed and distributed in the form of separate volume elements inside the first one (we will call it a disperse phase). In this work it is supposed that the dispersive phase is a viscous incompressible fluid and the disperse phase consists of a great number of small solid ferromagnetic particles suspended in the fluid. The sizes of particles are assumed to be of the same order as the distances between the nearest particles.

When neglecting all physical-chemical processses, the study of the suspension motion can be considered as a problem of pure classical mechanics. In this case the motion of the dispersive phase is governed by the Navier-Stockes equations, and the motion of solid particles forming a dispersed phase is described by the equations of continuum mechanics. However, the study of the properties of the fluids in the framework of such a model by using both analytical and numerical methods appears to be an unsurmountable problem because of a great number of the small particles. Therefore it is necessary to develop adequate macroscopic models that can help in studying such fluids. It is known that under the absence of external forces the motion of the compound is governed by the following homogenized equations:

$$\rho\frac{\partial \underline{v}}{\partial t} + (\underline{v}, \nabla)\underline{v} - \operatorname{div}\sigma[\underline{\mathrm{v}}] = \rho\underline{\mathrm{f}}, \quad \operatorname{div}\underline{\mathrm{v}} = 0,$$

where $\rho = \rho(\underline{x})$ is the homogenized specific mass density of the mixture, $\underline{v}(\underline{x}, t)$ is the homogenized velocity of the suspension, $\sigma[\underline{v}] = \{\sigma_{ij}[\underline{v}]\}_{i,j=1}^{3}$ is the homogenized stress tensor, and $\underline{f}$ is the external force acting on the suspension. Moreover, the stress tensor linearly depends on the strain tensor $e[\underline{v}] = \{e_{ij}[\underline{v}] = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})\}_{i,j=1}^{3}$:

$$\sigma[\underline{v}] = Ae[\underline{v}] - Ep,$$

where $A = \{a_{npqr}(\underline{x}, t)\}_{n,p,q,r=1}^{3}$ is the effective viscousity tensor (it is symmetrical with respect to permutation of pairs of subscripts and of subscripts in pairs themselves), $E = \{\delta_{ij}\}_{i,j=1}^{3}$ is the unity matrix, and $p(\underline{x}, t)$ is the pressure. The result is qualitatively the same in the case of weak electric or magnetic forces affecting the suspension.

If the suspension is subjected to the influence of a very strong electric or magnetic field then its behavior appears to be different. The study of such a behavior leads to a development of the so-called asymmetric hydrodynamics in which case the stress tensor appears to be non-symmetric (see, for example, the pioneer works [1] and [15] where this fact was established from physical considerations, and our previous work [4] where the same linear problem was considered):

$$\sigma[\underline{v}] = A^D e[\underline{v}] + A^R \omega[\underline{v}] - Ep. \tag{1}$$

Here A^D and A^R are the deformative and rotational parts of the effective viscousity tensor, and $\omega[\underline{v}] = \{\omega_{ij}[\underline{v}] = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i})\}_{i,j=1}^{3}$.

In this paper we suggest a non-linear mathematical model of a suspension which is a mixture of a viscous incompressible fluid with a large number of small perfectly rigid inclusions which are the prolate particles oriented along the fixed direction $\underline{l}$. Under the influence of the surrounding fluid the particles can move translationally or rotate around symmetry axis but the direction of their symmetry axes does not change. Such a motion of the composite can be realized, for example, if the particles are strongly magnetizable and subjected to the influence of the strong magnetic field, so that they are oriented along the field direction B (see Figure1).

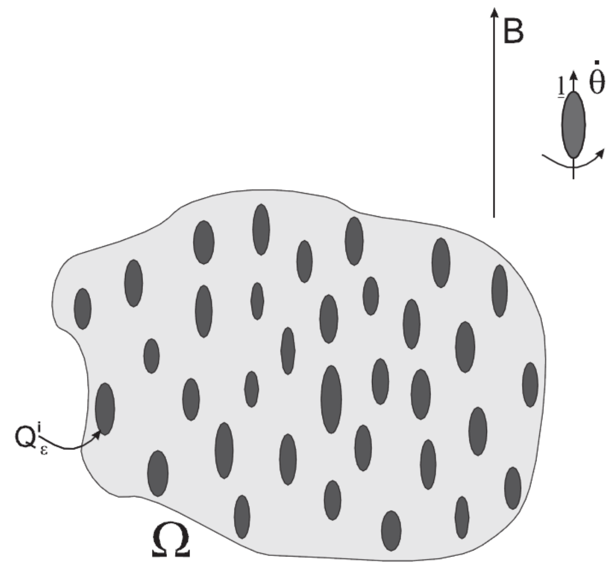

Figure 1: The suspension with oriented particles

We study the asymptotic behavior of such a mixture when the diameters of inclusions tend to zero and the inclusions are distributed in the whole volume. As a result, we obtain the homogenized equations corresponding to asymmetric hydrodynamics.

2 Statement of the problem

Consider a bounded domain Ω in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Suppose that this domain is filled with a mixture consisting of a viscous incompressible fluid with a large number $N_\varepsilon = O(\varepsilon^{-3})$ of small solids $Q^i_\varepsilon(t)$ bounded by smooth surfaces $\partial Q^i_\varepsilon(t)$ and suspended in the fluid. Further we will call them "the particles".

Let $\Omega_\varepsilon(t) = \Omega \setminus \bigcup\limits_{i=1}^{N} Q^i_\varepsilon(t)$ be a domain filled with the fluid, ρ_f and ρ_s be the specific mass density of the fluid and of solid particles respectively, μ be the dynamic viscosity of the fluid, $\underline{v}_\varepsilon = \underline{v}_\varepsilon(x,t)$ be the velocity of the fluid, $p_\varepsilon = p_\varepsilon(x,t)$ be the pressure, $\underline{f}_\varepsilon = \underline{f}_\varepsilon(\underline{x},t)$ be the external force acting on the suspension, $e[\underline{v}_\varepsilon] = \Big\{ e_{np}[\underline{v}] = \frac{1}{2}\big(\frac{\partial v_n}{\partial x_p} + \frac{\partial v_p}{\partial x_n}\big)\Big\}_{n,p=1}^3$ be the strain tensor in the fluid, $\sigma[\underline{v}_\varepsilon] = \big\{\sigma_{np}[\underline{v}] = 2\mu e_{np}[\underline{v}] - p_\varepsilon \delta_{np}\big\}_{n,p=1}^3$ be the stress tensor in the fluid, $\underline{x}^i_\varepsilon(t)$ be the position of the center of mass of $Q^i_\varepsilon(t)$, $\underline{u}^i_\varepsilon(t)$ be the displacement of the center of mass of $Q^i_\varepsilon(t)$, $\underline{\theta}^i_\varepsilon(t)$ be the rotation vector of $Q^i_\varepsilon(t)$, m^i_ε be the mass of $Q^i_\varepsilon(t)$, $I^i_\varepsilon(t)$ be the inertia tensor of $Q^i_\varepsilon(t)$.

Consider the following system of equations:

$$\rho_f \frac{\partial \underline{v}_\varepsilon}{\partial t} + \rho_f(\underline{v}_\varepsilon, \nabla)\underline{v}_\varepsilon - \mu \triangle \underline{v}_\varepsilon = \nabla p_\varepsilon + \rho_f \underline{f}_\varepsilon, \quad \operatorname{div} \underline{v}_\varepsilon = 0, \quad \underline{x} \in \Omega_\varepsilon; \tag{2}$$

$$\underline{v}_\varepsilon = \underline{\dot{u}}^i_\varepsilon + \underline{\dot{\theta}}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon), \quad \underline{\dot{\theta}}^i_\varepsilon = P^d \underline{\dot{\theta}}^i_\varepsilon, \quad \underline{x} \in \partial Q^i_\varepsilon; \tag{3}$$

$$m^i_\varepsilon \underline{\ddot{u}}^i_\varepsilon + \int\limits_{S^i_\varepsilon} \sigma[\underline{v}_\varepsilon]\nu\, ds = \int\limits_{Q^i_\varepsilon} \rho_s \underline{f}_\varepsilon\, d\underline{x}; \tag{4}$$

$$P^d \frac{d}{dt}\left[I^i_\varepsilon \underline{\dot{\theta}}^i_\varepsilon\right] + P^d \int\limits_{\partial Q^i_\varepsilon} (\underline{x} - \underline{x}^i_\varepsilon) \times \sigma[\underline{v}_\varepsilon]\underline{\nu}\, ds = P^d \int\limits_{Q^i_\varepsilon} (\underline{x} - \underline{x}^i_\varepsilon) \times \rho_s \underline{f}_\varepsilon\, d\underline{x}, \tag{5}$$

where $\underline{f}_\varepsilon = \underline{f}_\varepsilon(\underline{x}, t)$ is the external force acting on the mixture, ν is the unit inner normal vector to the surface $\partial Q^i_\varepsilon(t)$, and P^d is a projection operator onto some fixed d-dimensional subspace $S^d \subset \mathbb{R}^3$.

Depending on d, such a system describes non-stationary motions of the mixture under various regimes of particles rotations. Namely, if $d = 3$ then the particles can rotate without any constraints. Such a situation was considered in [11] (for the case of an elastic medium filled with the particles) and in [2],[12],[17],[26] (for the case of a viscous incompressible fluid filled with the particles). If $d = 0$ then the particles move translationally without any rotations. In this paper, we focuss on the non-standard cases where $d = 1$ or $d = 2$ (a similar linear problems for the case of elastic and fluid media were considered in our previous works [9] and [4]; see also [10]).

The case $d = 1$ can be realized, for example, if we consider strongly magnetizable prolate ellipsoidal particles in the strong magnetic field directed along a constant vector $\underline{B}$. Then all the particles are aligned along $\underline{B}$ ([20]), and under the influence of elastic forces they can move translationally or rotate only around their symmetry axis $\underline{l} = \underline{B}$, but the direction of their symmetry axis does not change (see Figure1). In this case, the subspace S^1 is a linear subspace spanned by the vector $\underline{l}$.

The case $d = 2$ can be realized, for example, if we consider strongly magnetizable oblate ellipsoidal particles in the strong magnetic field. Moreover, it is assumed that the particles are aligned in such a way that their symmetry axes are identically oriented along the direction $\underline{l}$ perpendicular to the field direction $\underline{B}$ and they can rotate both around their symmetry axis and around the field direction. In this case, subspace S^2 is a linear subspace spanned by vectors $\underline{l}$ and $\underline{B}$. The result both in case $d = 1$ and in case $d = 2$ is qualitatively the same: the stress tensor in the homogenized model is expressed via the strain tensor and the rotation tensor in accordance with (1).

The system of equations (2)-(5) is supplemented by the initial conditions

$$\underline{v}_\varepsilon(\underline{x}, 0) = \underline{v}_{\varepsilon 0}(\underline{x}), \quad \underline{x} \in \Omega_\varepsilon(0); \tag{6}$$

$$\underline{u}^i_\varepsilon(0) = 0,\ \underline{\dot{u}}^i_\varepsilon(0) = \underline{v}^i_\varepsilon,\ \theta^i_\varepsilon(0) = 0,\ \dot{\theta}^i_\varepsilon(0) = \omega^i_\varepsilon \tag{7}$$

(div $\underline{v}_{\varepsilon 0} = 0$ at $\underline{x} \in \Omega_\varepsilon(0)$ and $\underline{v}_{\varepsilon 0}(\underline{x}) = \underline{v}^i_\varepsilon + \underline{\omega}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon(0))$ at $\underline{x} \in \partial Q^i_\varepsilon(0)$) and the boundary condition on $\partial\Omega$

$$\underline{v}_\varepsilon(\underline{x}, t) = \underline{0}, \quad \underline{x} \in \partial\Omega. \tag{8}$$

Theorem 1. *There exists a unique solution of the problem* $(2) - (8)$ *for* $t \in [0, T]$ *(local in time* $(0 < T < \infty)$ *or even global* $(T = \infty)$ *if the data are small enough and the particles do not collide with each other and with the boundary* $\partial\Omega$*; for more details see, for example, [28] and references therein).*

The main goal of the paper is to study the asymptotic behavior of the solution of problem $(2) - (8)$ as $\varepsilon \to 0$.

At first, we get uniform (with respect to ε) bounds for the derivatives of that solution extended onto the particles $Q^i_\varepsilon(t)$ by equality (3).

3 A priori estimations of the solution of the problem $(2) - (8)$

Starting from the solution $\{\underline{v}_\varepsilon(\underline{x}, t),\ \underline{u}^i_\varepsilon(t),\ \underline{\theta}^i_\varepsilon(t) = P^d\underline{\theta}^i_\varepsilon(t),\ i = \overline{1, N_\varepsilon}\}$ of the problem $(2) - (8)$ we construct the vector function

$$\underline{\tilde{v}}_\varepsilon(\underline{x}, t) = \chi_\varepsilon(\underline{x}, t)\underline{v}_\varepsilon(\underline{x}, t) + \sum_{i=1}^{N_\varepsilon} \chi^i_\varepsilon(\underline{x}, t)[\underline{\dot{u}}^i_\varepsilon(t) + \underline{\dot{\theta}}^i_\varepsilon(t) \times (\underline{x} - \underline{x}^i_\varepsilon(t))], \tag{9}$$

where $\chi_\varepsilon(\underline{x}, t)$ is the characteristic function of the domain $\Omega_\varepsilon(t)$, filled with the fluid, and $\chi^i_\varepsilon(\underline{x}, t)$ is the characteristic function of a particle $Q^i_\varepsilon(t)$. We also denote by

$$\rho_\varepsilon(\underline{x}, t) = \rho_f \chi_\varepsilon(\underline{x}, t) + \rho_s \sum_{i=1}^{N_\varepsilon} \chi^i_\varepsilon(\underline{x}, t)$$

the density of the suspension "the fluid-the particles".

1. At the first step we estimate $\|\nabla\underline{\tilde{v}}_\varepsilon(\underline{x}, t)\|_{\mathbf{L_2}(\Omega_T)}$, where $\Omega_T = \Omega \times [0, T]$. To do so, we multiply equation (2) by $\underline{v}_\varepsilon(\underline{x}, t)$ and integrate over domain $\Omega^T_\varepsilon = [0, T] \times \Omega_\varepsilon(t)$. Using Green's formula, we get

$$\int_0^T \int_{\Omega_\varepsilon(t)} \rho_f \Big(\underline{v}_\varepsilon, \frac{\partial \underline{v}_\varepsilon}{\partial t}\Big)\, d\underline{x}dt + \int_0^T \int_{\Omega_\varepsilon(t)} \rho_f (\underline{v}_\varepsilon, (\underline{v}_\varepsilon \cdot \nabla)\underline{v}_\varepsilon)\, d\underline{x}dt + 2\mu \int_0^T \int_{\Omega_\varepsilon(t)} \sum_{k,l=1}^{3} e^2_{kl}[\underline{v}_\varepsilon]\, d\underline{x}dt -$$

$$- \int_0^T \int_{\partial\Omega_\varepsilon(t)} (\sigma[\underline{v}_\varepsilon], \underline{v}_\varepsilon)\, dSdt = \rho_f \int_0^T \int_{\Omega_\varepsilon(t)} (\underline{v}_\varepsilon, \underline{f}_\varepsilon)\, d\underline{x}dt. \tag{10}$$

With the aid of the boundary conditions (3), equations (4)-(5) and Reynolds transport theorem (see, for example, [3]), the surface integral in (10) can be transformed as follows:

$$-\int_0^T \int_{\partial\Omega_\varepsilon(t)} (\sigma[\underline{v}_\varepsilon], \underline{v}_\varepsilon)\, dSdt = \int_0^T \frac{\rho_s}{2} \sum_{i=1}^{N_\varepsilon} \frac{d}{dt} \int_{Q_\varepsilon^i(t)} |\underline{\dot{u}}_\varepsilon^i(t) + \underline{\dot{\theta}}_\varepsilon^i(t) \times (\underline{x} - \underline{x}_\varepsilon^i(t))|^2 \, d\underline{x}dt -$$

$$-\rho_s \int_0^T \sum_{i=1}^{N_\varepsilon} \int_{Q_\varepsilon^i(t)} \Big(\underline{\dot{u}}_\varepsilon^i(t) + \underline{\dot{\theta}}_\varepsilon^i(t) \times (\underline{x} - \underline{x}_\varepsilon^i(t)), \underline{f}_\varepsilon(\underline{x}, t) \Big)\, d\underline{x}dt. \tag{11}$$

Consider now the second term in the LHS of (10). Taking into account the boundary condition (8) and the divergence-free condition for the velocity $\underline{v}_\varepsilon(\underline{x}, t)$, with integrating by parts we get

$$\int_0^T \int_{\Omega(t)} \rho_f(\underline{v}_\varepsilon, (\underline{v}_\varepsilon \cdot \nabla)\underline{v}_\varepsilon)\, d\underline{x}dt = 0,$$

whence it follows that

$$\int_0^T \int_{\Omega_\varepsilon(t)} \rho_f(\underline{v}_\varepsilon, (\underline{v}_\varepsilon \cdot \nabla)\underline{v}_\varepsilon)\, d\underline{x}dt = -\rho_f \int_0^T \sum_{i=1}^{N_\varepsilon} \int_{Q_\varepsilon^i(t)} (\underline{v}_\varepsilon, (\underline{v}_\varepsilon \cdot \nabla)\underline{v}_\varepsilon)\, d\underline{x}dt. \tag{12}$$

With the help of equality (3) one can easily check that

$$\int_{Q_\varepsilon^i(t)} (\underline{v}_\varepsilon, (\underline{v}_\varepsilon \cdot \nabla)\underline{v}_\varepsilon)\, d\underline{x} = 0, \quad i = \overline{1, N_\varepsilon}. \tag{13}$$

Combining now (12) and (13), we conclude that the second term in the LHS of (10) is equal to 0.

Consider now the first term in the LHS of (10). It is easy to see that the following identity holds:

$$\int_0^T \int_{\Omega_\varepsilon(t)} \rho_f\Big(\underline{v}_\varepsilon, \frac{\partial \underline{v}_\varepsilon}{\partial t}\Big)\, d\underline{x}dt = \int_0^T \frac{\rho_f}{2} \frac{d}{dt} \int_{\Omega_\varepsilon(t)} |\underline{v}_\varepsilon|^2 \, d\underline{x}dt +$$

$$+ \int_0^T \frac{\rho_f}{2} \frac{d}{dt} \sum_{i=1}^{N_\varepsilon} \int_{Q_\varepsilon^i(t)} |\underline{v}_\varepsilon|^2 \, d\underline{x}dt - \int_0^T \frac{\rho_f}{2} \sum_{i=1}^{N_\varepsilon} \int_{Q_\varepsilon^i(t)} \frac{d|\underline{v}_\varepsilon|^2}{dt}\, d\underline{x}dt.$$

Using Reynolds transport theorem and recalling that $\underline{v}_\varepsilon = \underline{\dot{u}}_\varepsilon^i + \underline{\dot{\theta}}_\varepsilon^i \times (\underline{x} - \underline{x}_\varepsilon^i)$ for $\underline{x} \in Q_\varepsilon^i(t)$ $(i = \overline{1, N_\varepsilon})$, one can be convinced that the second and the third term in the RHS of that equality coincide with each other. Hence,

$$\int_0^T \int_{\Omega_\varepsilon(t)} \rho_f \Big(\underline{v}_\varepsilon, \frac{\partial \underline{v}_\varepsilon}{\partial t}\Big)\, d\underline{x} dt = \int_0^T \frac{\rho_f}{2} \frac{d}{dt} \int_{\Omega_\varepsilon(t)} |\underline{v}_\varepsilon|^2 \, d\underline{x} dt. \tag{14}$$

Combining now equalities (10)-(14), we obtain

$$\int_0^T \frac{\rho_f}{2} \frac{d}{dt} \int_{\Omega_\varepsilon(t)} |\underline{v}_\varepsilon|^2 \, d\underline{x} dt + \int_0^T \frac{\rho_s}{2} \sum_{i=1}^{N_\varepsilon} \frac{d}{dt} \int_{Q^i_\varepsilon(t)} |\underline{\dot{u}}^i_\varepsilon(t) + \underline{\dot{\theta}}^i_\varepsilon(t) \times (\underline{x} - \underline{x}^i_\varepsilon(t))|^2 \, d\underline{x} dt -$$

$$-\rho_s \int_0^T \sum_{i=1}^{N_\varepsilon} \int_{Q^i_\varepsilon(t)} \Big(\underline{\dot{u}}^i_\varepsilon(t) + \underline{\dot{\theta}}^i_\varepsilon(t) \times (\underline{x} - \underline{x}^i_\varepsilon(t)), \underline{f}_\varepsilon(\underline{x}, t)\Big)\, d\underline{x} dt + 2\mu \int_0^T \int_{\Omega_\varepsilon(t)} \sum_{k,l=1}^{3} e^2_{kl}[\underline{v}_\varepsilon] \, d\underline{x} dt =$$

$$= \rho_f \int_0^T \int_{\Omega_\varepsilon(t)} (\underline{v}_\varepsilon, \underline{f}_\varepsilon) \, d\underline{x} dt.$$

It is easy to see that we can rewrite this equality as follows:

$$\frac{1}{2} \int_0^T \frac{d}{dt} \Big\{ \int_\Omega \rho_\varepsilon(\underline{x}, t) |\underline{\tilde{v}}_\varepsilon(\underline{x}, t)|^2 \, d\underline{x} \Big\} dt + 2\mu \int_0^T \int_\Omega \sum_{k,l=1}^{3} e^2_{kl}[\underline{\tilde{v}}_\varepsilon] \, d\underline{x} dt =$$

$$= \int_0^T \int_\Omega \rho_\varepsilon(\underline{x}, t) (\underline{\tilde{v}}_\varepsilon, \underline{f}_\varepsilon) \, d\underline{x} dt.$$

Due to the first Korn's inequality (see [25])

$$\|\underline{\tilde{v}}_\varepsilon\|^2_{\overset{\circ}{H}{}^1(\Omega)} \le 2 \int_\Omega \sum_{k,l=1}^{3} e^2_{kl}[\underline{\tilde{v}}_\varepsilon] \, d\underline{x}, \tag{15}$$

the last identity gives us the required bound

$$\|\nabla \underline{\tilde{v}}_\varepsilon\|^2_{\mathbf{L_2}(\Omega_T)} \le C \tag{16}$$

provided the external force $\underline{f}_\varepsilon$ is bounded.

2. At the second step we prove the following estimation:

$$\rho_f \|\frac{\partial \underline{v}_\varepsilon}{\partial t}\|^2_{\mathbf{L_2}(\Omega^T_\varepsilon)} + 2\mu \int_{\Omega_\varepsilon(T)} \sum_{k,l=1}^{3} e_{kl}[\underline{v}_\varepsilon(\underline{x}, T)] \, d\underline{x} \le C. \tag{17}$$

To prove this bound we assume that the following conditions hold.

3.1) Let d^i_ε be the diameter of the ellipsoidal particle $Q^i_\varepsilon(t)$, $B(Q^i_\varepsilon(t))$ be a minimal ball containing $Q^i_\varepsilon(t)$, and $R^i_\varepsilon(t)$ be a distance from the ball $B(Q^i_\varepsilon(t))$ to other minimal balls and to the boundary $\partial\Omega$. We suppose that both d^i_ε and $R^i_\varepsilon(t)$ (for any fixed $t \in [0,T]$) satisfy the following inequalities:

$$C_1\varepsilon \le d^i_\varepsilon, R^i_\varepsilon(t) \le C_2\varepsilon, \tag{18}$$

where constants C_1 and C_2 do not depend on ε ($0 < C_1 < C_2 < \infty$).

3.2) The velocities $\dot{\underline{u}}^i_\varepsilon(t)$ of the centers of mass of $Q^i_\varepsilon(t)$ for any fixed $t \in [0,T]$ change in a smooth way when passing from one particle to another, i.e. there exist smooth vector functions $\underline{V}_\varepsilon(\underline{x},t)$ such that $\dot{\underline{u}}^i_\varepsilon(t) = \underline{V}_\varepsilon(\underline{x}^i_\varepsilon, t)$, $\underline{V}_\varepsilon(\underline{x},t) = 0$ for $(\underline{x},t) \in \partial\Omega \times [0,T]$, $\left|\dfrac{\partial \underline{V}_\varepsilon}{\partial t}\right| < C$ and $|\nabla_{\underline{x}}\underline{V}_\varepsilon| < C$.

3.3) The instant angular velocities and accelerations of the particles $Q^i_\varepsilon(t)$ are bounded, so that the following estimates hold: $|\dot{\underline{\theta}}^i_\varepsilon(t)| < C$, $\sum_{i=1}^{N_\varepsilon} |\ddot{\underline{\theta}}^i_\varepsilon|^2 (d^i_\varepsilon)^3 < C$.

3.4) $\max\limits_{(\underline{x},t)\in\Omega^T_\varepsilon} |\underline{v}_\varepsilon(\underline{x},t)| < C$.

3.5) The external force $\underline{f}_\varepsilon(\underline{x},t)$ and the initial velocity $\underline{v}_{\varepsilon 0}(\underline{x})$ are bounded: $\|\underline{f}_\varepsilon\|_{L_2(\Omega_T)} \le C$, $\|\underline{v}_{\varepsilon 0}\|_{H^1(\Omega)} \le C$.

Here, all constants C do not depend on ε,

Let $\varphi^i_\varepsilon(\underline{x},t) \in C^\infty(\Omega \times [0,T])$ be the functions satisfying the following conditions for any $t \in [0,T]$: $\varphi^i_\varepsilon = 1$ for $\underline{x} \in B(Q^i_\varepsilon(t))$, $\varphi^i_\varepsilon = 0$ for $\underline{x} \notin B_{(1+\alpha)d^i_\varepsilon}(t)$, $0 \le \varphi^i_\varepsilon \le 1$ for $\underline{x} \in B_{(1+\alpha)d^i_\varepsilon}(t)$ and $|\nabla_{\underline{x}}\varphi^i_\varepsilon(\underline{x},t)| \le \dfrac{C}{d^i_\varepsilon}$, where $B_{(1+\alpha)d^i_\varepsilon}(t)$ denotes a ball of diameter $(1+\alpha)d^i_\varepsilon$ concentric with the minimal ball $B(Q^i_\varepsilon(t))$, and constants $\alpha > 0$, $C > 0$ do not depend on ε.

Introduce the following vector field:

$$\underline{l}_\varepsilon(\underline{x},t) = \underline{V}_\varepsilon(\underline{x},t) + \sum_{i=1}^{N_\varepsilon} [\dot{\underline{u}}^i_\varepsilon(t) + \dot{\underline{\theta}}^i_\varepsilon(t) \times (\underline{x} - \underline{x}^i_\varepsilon(t)) - \underline{V}_\varepsilon(\underline{x},t)]\varphi^i_\varepsilon(\underline{x},t), \quad (\underline{x},t) \in \Omega^T. \tag{19}$$

From the properties of $\underline{V}^i_\varepsilon(\underline{x},t)$ and $\varphi^i_\varepsilon(\underline{x},t)$ follows that

$$\begin{gathered} \underline{l}_\varepsilon(\underline{x},t) = \dot{\underline{u}}^i_\varepsilon(t) + \dot{\underline{\theta}}^i_\varepsilon(t) \times (\underline{x} - \underline{x}^i_\varepsilon(t)) \quad \text{for } \underline{x} \in Q^i_\varepsilon(t) \quad (i = \overline{1, N_\varepsilon}); \\ \left|\frac{\partial \underline{l}_\varepsilon(\underline{x},t)}{\partial t}\right| < C; \quad |\nabla_{\underline{x}}\underline{l}_\varepsilon(\underline{x},t)| < C; \quad \underline{l}_\varepsilon(\underline{x},t) = 0 \quad \text{for } (\underline{x},t) \in \partial\Omega \times [0,T], \end{gathered} \tag{20}$$

where $C > 0$ does not depend on ε.

Consider in Ω_T the following vector function:

$$\underline{q}_\varepsilon(\underline{x},t) = \frac{\partial \tilde{\underline{v}}_\varepsilon}{\partial t} + \sum_{i=1}^{3} l_{\varepsilon i}\frac{\partial \tilde{\underline{v}}_\varepsilon}{\partial x_i} = \left(1 + |\underline{l}_\varepsilon|^2\right)^{\frac{1}{2}} \frac{\partial \tilde{\underline{v}}_\varepsilon}{\partial \hat{\underline{l}}_\varepsilon}, \tag{21}$$

where $\dfrac{\partial \tilde{\underline{v}}_\varepsilon}{\partial \hat{\underline{l}}_\varepsilon}$ denotes the directional derivative of $\tilde{\underline{u}}_\varepsilon$ along the vector field

$$\hat{\underline{l}}_\varepsilon = \left\{ \frac{1}{\left(1+|\underline{l}_\varepsilon|^2\right)^{\frac{1}{2}}}, \frac{l_{\varepsilon 1}}{\left(1+|\underline{l}_\varepsilon|^2\right)^{\frac{1}{2}}}, \frac{l_{\varepsilon 2}}{\left(1+|\underline{l}_\varepsilon|^2\right)^{\frac{1}{2}}}, \frac{l_{\varepsilon 3}}{\left(1+|\underline{l}_\varepsilon|^2\right)^{\frac{1}{2}}} \right\}.$$

Since $\tilde{\underline{v}}_\varepsilon(\underline{x},t)$ is continuous everywhere in Ω_T, $\nabla_{\underline{x}} \tilde{\underline{v}}_\varepsilon(\underline{x},t)$ and $\dfrac{\partial \tilde{\underline{v}}_\varepsilon(\underline{x},t)}{\partial t}$ are continuous both in Ω_ε^T and in $Q_\varepsilon^T = \Omega_T \setminus \Omega_\varepsilon^T$, $\tilde{\underline{v}}_\varepsilon(\underline{x},t) = 0$ for $(\underline{x},t) \in \partial\Omega \times [0,T]$, and the vector field $\hat{\underline{l}}_\varepsilon$ is tangent to the lateral surface of $\partial\Omega_\varepsilon^T$, then $\underline{q}_\varepsilon(\underline{x},t)$ is continuous everywhere in Ω^T and $\underline{q}_\varepsilon(\underline{x},t) = 0$ for $(\underline{x},t) \in \partial\Omega \times [0,T]$. Therefore, from (19) and (21), it follows that for any $t \in [0,T]$ $\underline{q}_\varepsilon(\underline{x},t) \in \overset{\circ}{H}{}^1(\Omega)$. Moreover, since div $\tilde{\underline{v}}_\varepsilon = 0$, we get

$$\operatorname{div} \underline{q}_\varepsilon = \sum_{i,j=1}^{3} \frac{\partial l_{\varepsilon i}}{\partial x_j} \frac{\partial \tilde{v}_{\varepsilon j}}{\partial x_i}. \tag{22}$$

For further considerations we need a technical lemma which can be proved analogously to [12].

Lemma 1. *For any vector function $\underline{q}_\varepsilon(\underline{x}) \in \overset{\circ}{H}{}^1(\Omega)$ there exists a vector function $\underline{z}_\varepsilon(\underline{x}) \in \overset{\circ}{H}{}^1(\Omega)$ such that div $\underline{z}_\varepsilon(\underline{x}) =$ div $\underline{q}_\varepsilon(\underline{x})$ for $\underline{x} \in \Omega$, $\underline{z}_\varepsilon(\underline{x}) = \underline{q}_\varepsilon(\underline{x}) + \underline{a}_\varepsilon^i$ for $\underline{x} \in Q_\varepsilon^i$ $(i = \overline{1, N_\varepsilon})$ and*

$$\|\underline{z}_\varepsilon\|^2_{H^1(\Omega)} \le C\Big(\|div\, \underline{q}_\varepsilon\|^2_{L_2(\Omega)} + \sum_{i=1}^{N_\varepsilon} \|\nabla \underline{q}_\varepsilon\|^2_{L_2(Q_\varepsilon^i)} \Big), \tag{23}$$

where $\underline{a}_\varepsilon^i$ are constant vectors and C does not depend on ε.

Applying this Lemma to the function $\underline{q}_\varepsilon(\underline{x})$ defined by equality (21) for every $t \in [0,T]$, we construct a vector function $\underline{z}_\varepsilon(\underline{x},t)$. From estimate (23), using equalities (21) and (22) and taking into account the properties of the vector field $\underline{l}_\varepsilon(\underline{x},t)$, we get

$$\|\underline{z}_\varepsilon(t)\|^2_{H^1(\Omega)} \le C\Big\{ \|\nabla \tilde{\underline{v}}_\varepsilon(t)\|^2_{L_2(\Omega)} + \sum_{i=1}^{N_\varepsilon} \big(|\ddot{\underline{\theta}}_\varepsilon^i(t)|^2 + |\dot{\underline{\theta}}_\varepsilon^i(t)|^4 \big) (d_\varepsilon^i)^3 \Big\}.$$

Hence, due to the estimate (16) and conditions 3.1) and 3.3), the following inequality follows

$$\int_0^T \|\underline{z}_\varepsilon(t)\|^2_{H^1(\Omega)}\, dt \le C, \tag{24}$$

where C does not depend on ε.

Set now $\underline{w}_\varepsilon = \underline{q}_\varepsilon - \underline{z}_\varepsilon$. It is clear that $\underline{w}_\varepsilon(\underline{x},t) \in L_2(0,T; \overset{\circ}{H}{}^1(\Omega))$, div $\underline{w}_\varepsilon = 0$ and $\underline{w}_\varepsilon(\underline{x},t) = \underline{a}_\varepsilon^i(t)$ for $\underline{x} \in Q_\varepsilon^i(t)$ $(i = \overline{1, N_\varepsilon})$. Multiply equation (2) by $\underline{w}_\varepsilon$ and integrate over domain Ω_ε^T. Using Green's formula, we get

$$\int_0^T \int_{\Omega_\varepsilon(t)} \rho_f\Big(\underline{w}_\varepsilon, \frac{\partial \underline{v}_\varepsilon}{\partial t}\Big)\, dxdt + \int_0^T \int_{\Omega_\varepsilon(t)} \rho_f(\underline{w}_\varepsilon, (\underline{v}_\varepsilon \cdot \nabla)\underline{v}_\varepsilon)\, dxdt + 2\mu \int_0^T \int_{\Omega_\varepsilon(t)} \sum_{k,l=1}^3 e_{kl}[\underline{w}_\varepsilon] e_{kl}[\underline{v}_\varepsilon]\, dxdt -$$

$$- \int_0^T \sum_{i=1}^{N_\varepsilon} \Big(\int_{\partial Q_\varepsilon^i(t)} \sigma[\underline{v}_\varepsilon]\, dS, \underline{a}_\varepsilon^i(t)\Big)\, dt = \rho_f \int_0^T \int_{\Omega_\varepsilon(t)} (\underline{w}_\varepsilon, \underline{f}_\varepsilon)\, dxdt. \tag{25}$$

Estimate now each of the terms in (25). Taking into account the form of the vector function $\underline{q}_\varepsilon(\underline{x}, t)$ and using inequality (20), Cauchy-Schwarz and Young's inequalities (with arbitrary $\delta > 0$), estimate from above the RHS of equality (25):

$$\Big|\rho_f \int_0^T \int_{\Omega_\varepsilon(t)} (\underline{w}_\varepsilon, \underline{f}_\varepsilon)\, dxdt\Big| = \rho_f \Big|\int_0^T \int_{\Omega_\varepsilon(t)} (\underline{q}_\varepsilon - \underline{z}_\varepsilon, \underline{f}_\varepsilon)\, dxdt\Big| \leq$$

$$\leq \rho_f \|\underline{f}_\varepsilon\|_{L_2(\Omega_\varepsilon^T)} \Big(\|\frac{\partial \underline{v}_\varepsilon}{\partial t}\|_{L_2(\Omega_\varepsilon^T)} + C\|\nabla \underline{v}_\varepsilon\|_{L_2(\Omega_\varepsilon^T)} + \|\underline{z}_\varepsilon\|_{L_2(\Omega_\varepsilon^T)}\Big) \leq \tag{26}$$

$$\leq \frac{\rho_f}{4\delta} \|\underline{f}_\varepsilon\|^2_{L_2(\Omega_T)} + 3\rho_f \delta \Big(\|\frac{\partial \underline{v}_\varepsilon}{\partial t}\|^2_{L_2(\Omega_\varepsilon^T)} + C^2 \|\nabla \underline{v}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)} + \|\underline{z}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)}\Big).$$

Consider now the first term on the LHS of equality (25). With the help of the same arguments as before, we obtain the following lower bound:

$$\int_0^T \int_{\Omega_\varepsilon(t)} \rho_f\Big(\underline{w}_\varepsilon, \frac{\partial \underline{v}_\varepsilon}{\partial t}\Big)\, dxdt = \int_0^T \int_{\Omega_\varepsilon(t)} \rho_f\Big(\underline{q}_\varepsilon - \underline{z}_\varepsilon, \frac{\partial \underline{v}_\varepsilon}{\partial t}\Big)\, dxdt =$$

$$= \int_0^T \int_{\Omega_\varepsilon(t)} \rho_f \Big|\frac{\partial \underline{v}_\varepsilon}{\partial t}\Big|^2 dxdt + \sum_{i=1}^3 \int_0^T \int_{\Omega_\varepsilon(t)} \rho_f l_{\varepsilon i} \Big(\frac{\partial \underline{v}_\varepsilon}{\partial x_i}, \frac{\partial \underline{v}_\varepsilon}{\partial t}\Big)\, dxdt - \int_0^T \int_{\Omega_\varepsilon(t)} \rho_f \Big(z_\varepsilon, \frac{\partial \underline{v}_\varepsilon}{\partial t}\Big)\, dxdt \geq \tag{27}$$

$$\geq \rho_f \Big\|\frac{\partial \underline{v}_\varepsilon}{\partial t}\Big\|^2_{L_2(\Omega_\varepsilon^T)} - 3C\rho_f \Big(\frac{1}{4\delta}\|\nabla \underline{v}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)} + \delta \Big\|\frac{\partial \underline{v}_\varepsilon}{\partial t}\Big\|^2_{L_2(\Omega_\varepsilon^T)}\Big) - \frac{\rho_f}{4\delta}\|\underline{z}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)} - \rho_f \delta \Big\|\frac{\partial \underline{v}_\varepsilon}{\partial t}\Big\|^2_{L_2(\Omega_\varepsilon^T)}.$$

Next, analogously to (12)-(13), it can be shown that

$$\int_0^T \int_{\Omega_\varepsilon(t)} \rho_f(\underline{w}_\varepsilon, (\underline{v}_\varepsilon \cdot \nabla)\underline{v}_\varepsilon)\, dxdt = 0. \tag{28}$$

Estimate now the fourth term in the LHS of (25):

$$- \int_0^T \sum_{i=1}^{N_\varepsilon} \Big(\int_{\partial Q_\varepsilon^i(t)} \sigma[\underline{v}_\varepsilon]\, dS, \underline{a}_\varepsilon^i(t)\Big)\, dt = \int_0^T \sum_{i=1}^{N_\varepsilon} \big(m_\varepsilon^i \ddot{\underline{u}}_\varepsilon^i(t), \underline{a}_\varepsilon^i(t)\big)\, dt -$$

$$-\rho_s \int_0^T \sum_{i=1}^{N_\varepsilon} \Big(\int_{Q_\varepsilon^i(t)} \underline{f}_\varepsilon(\underline{x},t)\,dx, \underline{a}_\varepsilon^i(t) \Big)\,dt = J_\varepsilon^1 - J_\varepsilon^2.$$

Using Cauchy-Schwarz inequality and identity $\underline{a}_\varepsilon^i(t) = \underline{q}_\varepsilon(\underline{x},t) - \underline{z}_\varepsilon(\underline{x},t)$ for $\underline{x} \in Q_\varepsilon^i(t)$ $(i = \overline{1, N_\varepsilon})$, we get:

$$|J_\varepsilon^1| \le \sum_{i=1}^{N_\varepsilon} \Big\{ \int_0^T m_\varepsilon^i |\underline{\ddot{u}}_\varepsilon^i|^2\,dt \Big\}^{\frac{1}{2}} \Big\{ \int_0^T m_\varepsilon^i |\underline{a}_\varepsilon^i|^2\,dt \Big\}^{\frac{1}{2}} \le$$

$$\le \frac{1}{2} \int_0^T \sum_{i=1}^{N_\varepsilon} m_\varepsilon^i |\underline{\ddot{u}}_\varepsilon^i|^2\,dt + \frac{\rho_s}{2} \int_0^T \sum_{i=1}^{N_\varepsilon} \int_{Q_\varepsilon^i(t)} \big(|\underline{q}_\varepsilon(\underline{x},t)|^2 + |\underline{z}_\varepsilon(\underline{x},t)|^2 \big)\,dxdt,$$

whence, taking into account the form of the vector function $\underline{q}_\varepsilon(\underline{x},t)$ on the particles $Q_\varepsilon^i(t)$ and conditions 3.2)-3.3), the following upper bound follows:

$$|J_\varepsilon^1| \le \frac{\rho_s}{2} \|\underline{z}_\varepsilon\|^2_{L_2(Q_\varepsilon^T)} + C \max_{0\le t\le T} \Big\{ \sum_{i=1}^{N_\varepsilon} |\underline{\ddot{\theta}}_\varepsilon^i|^2 (d_\varepsilon^i)^5 + \sum_{i=1}^{N_\varepsilon} |\underline{\ddot{u}}_\varepsilon^i|^2 (d_\varepsilon^i)^3 + \sum_{i=1}^{N_\varepsilon} |\underline{\dot{\theta}}_\varepsilon^i(t)|^2 (d_\varepsilon^i)^3 \Big\} \le$$

$$\le \frac{\rho_s}{2} \|\underline{z}_\varepsilon\|^2_{L_2(Q_\varepsilon^T)} + C.$$

The term J_ε^2 can be estimated analogously:

$$|J_\varepsilon^2| \le \frac{\rho_s}{2} \|\underline{z}_\varepsilon\|^2_{L_2(Q_\varepsilon^T)} + \rho_s \|\underline{f}_\varepsilon\|^2_{L_2(Q_\varepsilon^T)} + C.$$

Thus,

$$\Big| \int_0^T \sum_{i=1}^{N_\varepsilon} \Big(\int_{\partial Q_\varepsilon^i(t)} \sigma[\underline{v}_\varepsilon]\,dS, \underline{a}_\varepsilon^i(t) \Big)\,dt \Big| \le \rho_s \|\underline{z}_\varepsilon\|^2_{L_2(Q_\varepsilon^T)} + \rho_s \|\underline{f}_\varepsilon\|^2_{L_2(Q_\varepsilon^T)} + 2C. \tag{29}$$

It remains to consider only the third term in the LHS of (25). Taking into account that $\underline{w}_\varepsilon = \underline{q}_\varepsilon - \underline{z}_\varepsilon$, we have:

$$2\mu \int_0^T \int_{\Omega_\varepsilon(t)} \sum_{k,l=1}^3 e_{kl}[\underline{w}_\varepsilon] e_{kl}[\underline{v}_\varepsilon]\,dxdt =$$

$$= 2\mu \int_0^T \int_{\Omega_\varepsilon(t)} \sum_{k,l=1}^3 e_{kl}[\underline{q}_\varepsilon] e_{kl}[\underline{v}_\varepsilon]\,dxdt - 2\mu \int_0^T \int_{\Omega_\varepsilon(t)} \sum_{k,l=1}^3 e_{kl}[\underline{z}_\varepsilon] e_{kl}[\underline{v}_\varepsilon]\,dxdt. \tag{30}$$

The second term in the RHS of (30) we can estimate as follows:

$$2\mu\Big|\int\limits_0^T\int\limits_{\Omega_\varepsilon(t)}\sum_{k,l=1}^3 e_{kl}[\underline{z}_\varepsilon]e_{kl}[\underline{v}_\varepsilon]\,dxdt\Big| \le \big(\|\nabla\underline{z}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)}+\|\nabla\underline{v}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)}\big). \tag{31}$$

Recalling the definition of $\underline{q}_\varepsilon$ and $\hat{\underline{l}}_\varepsilon$, the first term in the RHS of (30) can be written as follows $\Big(\dfrac{\partial}{\partial x_0}\equiv\dfrac{\partial}{\partial t}\Big)$:

$$2\mu\int\limits_0^T\int\limits_{\Omega_\varepsilon(t)}\sum_{k,l=1}^3 e_{kl}[\underline{q}_\varepsilon]e_{kl}[\underline{v}_\varepsilon]\,dxdt = \mu\int\limits_0^T\int\limits_{\Omega_\varepsilon(t)}\sum_{i=0}^3\frac{\partial}{\partial x_i}\Big(\hat{l}_{\varepsilon i}(1+|\underline{l}_\varepsilon|^2)^{\frac12}\sum_{k,l=1}^3 e^2_{kl}[\underline{v}_\varepsilon]\Big)\,dxdt-$$

$$-\mu\int\limits_0^T\int\limits_{\Omega_\varepsilon(t)}\sum_{i=1}^3\frac{\partial l_{\varepsilon i}}{\partial x_i}\sum_{k,l=1}^3 e^2_{kl}[\underline{v}_\varepsilon]\,dxdt+\mu\int\limits_0^T\int\limits_{\Omega_\varepsilon(t)}\sum_{i,k,l=1}^3\Big(\frac{\partial l_{\varepsilon i}}{\partial x_l}\frac{\partial v_{\varepsilon k}}{\partial x_i}+\frac{\partial l_{\varepsilon i}}{\partial x_k}\frac{\partial v_{\varepsilon l}}{\partial x_i}\Big)e_{kl}[\underline{v}_\varepsilon]\,dxdt =$$

$$= J^1_\varepsilon + J^2_\varepsilon + J^3_\varepsilon.$$

Due to (20), we have:

$$|J^2_\varepsilon|+|J^3_\varepsilon|\le C\|\nabla\underline{v}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)}.$$

Applying the divergence theorem to the integral J^1_ε and taking into account that the vector field $\hat{\underline{l}}_\varepsilon(\underline{x},t)(1+|\underline{l}_\varepsilon|^2)^{\frac12}\sum\limits_{k,l=1}^3 e^2_{kl}[\underline{v}_\varepsilon]$ is tangent to the lateral surface of the domain Ω_ε^T, we get:

$$J^1_\varepsilon = \mu\int\limits_{\Omega_\varepsilon(T)}\sum_{k,l=1}^3 e^2_{kl}[\underline{v}_\varepsilon(\underline{x},T)]\,dx-\mu\int\limits_{\Omega_\varepsilon(0)}\sum_{k,l=1}^3 e^2_{kl}[\underline{v}_\varepsilon(\underline{x},0)]\,dx.$$

Thus,

$$2\mu\int\limits_0^T\int\limits_{\Omega_\varepsilon(t)}\sum_{k,l=1}^3 e_{kl}[\underline{w}_\varepsilon]e_{kl}[\underline{v}_\varepsilon]\,dxdt\ge\mu\int\limits_{\Omega_\varepsilon(T)}\sum_{k,l=1}^3 e^2_{kl}[\underline{v}_\varepsilon(\underline{x},T)]\,dx-$$

$$-\mu\int\limits_{\Omega_\varepsilon(0)}\sum_{k,l=1}^3 e^2_{kl}[\underline{v}_{\varepsilon 0}(\underline{x})]\,dx - C\|\nabla\underline{v}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)}. \tag{32}$$

Combining now (25)-(32), we obtain:

$$\rho_f\Big(1-(3C+4)\delta\Big)\Big\|\frac{\partial\underline{v}_\varepsilon}{\partial t}\Big\|^2_{L_2(\Omega_\varepsilon^T)}+\mu\int\limits_{\Omega_\varepsilon(T)}\sum_{k,l=1}^3 e^2_{kl}[\underline{v}_\varepsilon(\underline{x},T)]\,dx\le\frac{\rho_f}{4\delta}|f_\varepsilon\|^2_{L_2(\Omega_T)}+\rho_s\|f_\varepsilon\|^2_{L_2(Q_\varepsilon^T)}+$$

$$+\Big(3\rho_f\big(\frac{C}{4\delta}+C^2\delta\big)+2C+1\Big)\|\nabla\underline{v}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)}+\rho_f\Big(\frac{1}{4\delta}+3\delta\Big)\|\underline{z}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)}+\|\nabla\underline{z}_\varepsilon\|^2_{L_2(\Omega_\varepsilon^T)}+$$

$$+\rho_s \|\underline{z}_\varepsilon\|^2_{L_2(Q^T_\varepsilon)} + 2C + \int\limits_{\Omega_\varepsilon(0)} \sum_{k,l=1}^{3} e^2_{kl}[\underline{v}_{\varepsilon 0}(\underline{x})]\, dx.$$

Taking here $\delta = \dfrac{1}{2(3C+4)}$, using previously obtained bounds (16) and (24), and taking into consideration condition 3.5), we get the required bound (17):

$$\rho_f \|\frac{\partial \underline{v}_\varepsilon}{\partial t}\|^2_{\mathbf{L_2}(\Omega^T_\varepsilon)} + 2\mu \int\limits_{\Omega_\varepsilon(T)} \sum_{k,l=1}^{3} e_{kl}[\underline{v}_\varepsilon(\underline{x}, T)]\, d\underline{x} \leq C,$$

where constant C does not depend on ε.

Before formulating the main result we introduce some definitions and assumptions.

4 Additional assumptions and the main result

Consider now the following auxiliary linear stationary problem in domain $\Omega_\varepsilon(t)$ (t is a parameter):

$$-\mu \triangle \underline{v}_\varepsilon = \nabla p_\varepsilon + \underline{F}_\varepsilon, \quad \mathrm{div}\underline{v}_\varepsilon = 0, \quad \underline{x} \in \Omega_\varepsilon, \tag{33}$$

$$\underline{v}_\varepsilon = \underline{a}^i_\varepsilon + \underline{b}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon), \quad \underline{b}^i_\varepsilon = P^d \underline{b}^i_\varepsilon, \quad \underline{x} \in \partial Q^i_\varepsilon, \tag{34}$$

$$\int\limits_{\partial Q^i_\varepsilon} \sigma[\underline{v}_\varepsilon]\nu\, ds = \int\limits_{Q^i_\varepsilon} \underline{F}_\varepsilon\, d\underline{x}, \tag{35}$$

$$P^d \int\limits_{\partial Q^i_\varepsilon} (\underline{x} - \underline{x}^i_\varepsilon) \times \sigma[\underline{v}_\varepsilon]\nu\, ds = P^d \int\limits_{Q^i_\varepsilon} (\underline{x} - \underline{x}^i_\varepsilon) \times \underline{F}_\varepsilon\, d\underline{x}, \tag{36}$$

$$\underline{v}_\varepsilon(\underline{x}) = 0, \quad \underline{x} \in \partial\Omega. \tag{37}$$

Let K^y_h be a cube with the side length h ($\varepsilon \ll h \ll 1$) centered at $\underline{y} \in \Omega$. We assume that the edges of this cube are parallel to the coordinate axes. Let $J^{\widehat{\underline{\theta}}}_\varepsilon[K^y_h]$ be the following class of vector-functions:

$$J^{\widehat{\underline{\theta}}}_\varepsilon[K^y_h] = \{\underline{w}_\varepsilon \in H^1(K^y_h);\ \mathrm{div}\underline{w}_\varepsilon = 0;$$
$$\underline{w}_\varepsilon(\underline{x}) = \underline{w}^i_\varepsilon + [P^d \underline{\theta}^i_\varepsilon + (1 - P^d)\widehat{\underline{\theta}}] \times (\underline{x} - \underline{x}^i_\varepsilon),\ \underline{x} \in Q^i_\varepsilon \cap K^y_h\},$$

where $\underline{w}^i_\varepsilon$ and $\underline{\theta}^i_\varepsilon$ are arbitrary vectors, and $\widehat{\underline{\theta}}$ is a given vector. Consider a minimization problem in this class for the following functional (*mesocharacteristic*):

$$A^\gamma_{\varepsilon h}(\underline{w}_\varepsilon, \underline{y}, T) = E_{K^y_h}[\underline{w}_\varepsilon, \underline{w}_\varepsilon] +$$

$$+P^{\varepsilon h\gamma}_{K^y_h}\big[\underline{w}_\varepsilon(\underline{x})-\sum_{n,p=1}^{3}T_{np}\underline{\varphi}^{np}(\underline{x}-\underline{y}),\underline{w}_\varepsilon(\underline{x})-\sum_{q,r=1}^{3}T_{qr}\underline{\varphi}^{qr}(\underline{x}-\underline{y})\big],\tag{38}$$

where

$$E_G[\underline{u}_\varepsilon,\underline{v}_\varepsilon]=2\mu\int\limits_G\sum_{n,p=1}^{3}e_{np}[\underline{u}_\varepsilon]e_{np}[\underline{v}_\varepsilon]\,d\underline{x},\tag{39}$$

$$P^{\varepsilon h\gamma}_G[\underline{u}_\varepsilon(\underline{x}),\underline{v}_\varepsilon(\underline{x})]=h^{-2-\gamma}\int\limits_G\langle\underline{u}_\varepsilon(\underline{x}),\underline{v}_\varepsilon(\underline{x})\rangle\,dx,\tag{40}$$

$$\underline{\varphi}^{qr}(\underline{x})=\frac{1}{2}(x_r\underline{e}^q+x_q\underline{e}^r)-\frac{\delta_{qr}}{3}\sum_{n=1}^{3}x_n\underline{e}^n,\tag{41}$$

$e_{kl}[\underline{v}]=\dfrac{1}{2}\big(\dfrac{\partial v_k}{\partial x_l}+\dfrac{\partial v_l}{\partial x_k}\big)$, $T=\{T_{qr}\}$ is an arbitrary symmetric second rank tensor, $\{\underline{e}^n\}_{n=1}^3$ is an orthonormal basis in $\mathbb{R}^3$, and $0<\gamma<2$ is a penalty parameter.

This mesocharacteristic plays the crucial role in our consideration. Roughly speaking, it allows us to compute the energy of the suspension in some mesoscopic cube of size h ($\varepsilon\ll h\ll 1$), which is a so-called representative volume element. In other words, if a suspension can be described within the effective single medium approach, then the rheological properties of the suspension can be determined by calculation or measurements in some representative volume element of an intermediate mesoscale h, which is why we choose the cube K^y_h.

Next, observe that the first term (39) in (38) represents the energy of the suspension. The minimizer $\underline{w}_\varepsilon$ of (38) is "close", up to an additive constant, to the true global minimizer $\underline{u}_\varepsilon$ of the variational problem, which corresponds to (2)-(8) if the tensor T is chosen appropriately. Now one should choose T. If the single medium homogenized description is possible, then $\underline{u}_\varepsilon(\underline{x})$ is "close" to some smooth (homogenized) vector-function $\underline{u}(\underline{x})$, which depends only on the macroscopic variable $\underline{x}$ and does not depend on ε, so that it does not vary on the microscale ε. We then minimize the energy of the suspension, adding the constraint that the minimizer $\underline{w}_\varepsilon$ is "close" to the linear part (differential) of the global minimizer $\underline{u}$, so that $|\underline{w}_\varepsilon-\underline{u}|=o(h)\sim h^{1+\frac{\gamma}{2}}$ for some $\gamma>0$. This condition is imposed by introducing the penalty term (40).

It can be proved that there exists the unique vector-function which minimizes the functional (38); the minimal value of this functional is given by

$$\min_{\underline{w}_\varepsilon\in J^{\widehat{\theta}}_\varepsilon[K^y_h]}A^\gamma_{\varepsilon h}(\underline{w}_\varepsilon,\underline{y},T)=\sum_{n,p,q,r=1}^{3}a^{0,\gamma}_{npqr}(\underline{y},S^d,\varepsilon,h)T_{np}T_{qr}+$$

$$+2\sum_{n,p=1}^{3}\sum_{q=1}^{3}b^\gamma_{npq}(\underline{y},S^d,\varepsilon,h)T_{np}\widehat{\theta}_q+\sum_{q,r=1}^{3}c^\gamma_{qr}(\underline{y},S^d,\varepsilon,h)\widehat{\theta}_q\widehat{\theta}_r,\tag{42}$$

where $a^{0,\gamma}_{npqr}(\underline{y},S^d,\varepsilon,h)$, $b^\gamma_{npq}(\underline{y},S^d,\varepsilon,h)$ and $c^\gamma_{qr}(\underline{y},S^d,\varepsilon,h)$ are the components of the fourth-, third- and second-rank tensors respectively, defined as follows

$$a^{0,\gamma}_{npqr}(\underline{y}, S^d, \varepsilon, h) = E_{K^y_h}[\underline{w}^{np}, \underline{w}^{qr}] + P^{\varepsilon h\gamma}_{K^y_h}\big[\underline{w}^{np}(\underline{x}) - \underline{\varphi}^{np}(\underline{x} - \underline{y}), \underline{w}^{qr}(\underline{x}) - \underline{\varphi}^{qr}(\underline{x} - \underline{y})\big], \tag{43}$$

$$b^{\gamma}_{npq}(\underline{y}, S^d, \varepsilon, h) = E_{K^y_h}[\underline{w}^{np}, \underline{v}^{q}] + P^{\varepsilon h\gamma}_{K^y_h}\big[\underline{w}^{np}(\underline{x}) - \underline{\varphi}^{np}(\underline{x} - \underline{y}), \underline{v}^{q}(\underline{x})\big], \tag{44}$$

$$c^{\gamma}_{qr}(\underline{y}, S^d, \varepsilon, h) = E_{K^y_h}[\underline{v}^{q}, \underline{v}^{r}] + P^{\varepsilon h\gamma}_{K^y_h}\big[\underline{v}^{q}(\underline{x}), \underline{v}^{r}(\underline{x})\big]. \tag{45}$$

Here $\underline{w}^{np}(\underline{x})$ is the vector-function that minimizes the functional (38) in $J^0_\varepsilon[K^y_h]$ as $T = T^{np} = \frac{1}{2}(\underline{e}^n \otimes \underline{e}^p + \underline{e}^p \otimes \underline{e}^n)$, $\underline{v}^q(\underline{x})$ is the vector-function minimizing the functional (38) in $J^{\underline{e}^q}_\varepsilon[K^y_h]$ as $T = 0$, and $\underline{e}^n$ $(n = 1, 2, 3)$ form an orthonormal basis in $\mathbb{R}^3$.

Starting from the solution $\{\underline{v}_\varepsilon(\underline{x}),\, \underline{a}^i_\varepsilon,\, \underline{b}^i_\varepsilon(t) = P^d \underline{b}^i_\varepsilon(t),\, i = \overline{1, N_\varepsilon}\}$ of the problem (33) – (37) we construct the vector function

$$\tilde{\underline{v}}_\varepsilon(\underline{x}) = \chi_\varepsilon(\underline{x})\underline{v}_\varepsilon(\underline{x}) + \sum_{i=1}^{N_\varepsilon} \chi^i_\varepsilon(\underline{x})[\underline{a}^i_\varepsilon + \underline{b}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon)], \tag{46}$$

where $\chi_\varepsilon(\underline{x})$ is the characteristic function of the domain Ω_ε, filled with the fluid, and $\chi^i_\varepsilon(\underline{x})$ is the characteristic function of a particle Q^i_ε.

We assume that the following conditions hold:

4.1) for some real number $\gamma > 0$ the following limits exist heterogeneously at $\underline{x} \in \Omega$:

$$a)\ \lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}} \frac{a^{0,\gamma}_{npqr}(\underline{x}, S^d, \varepsilon, h)}{h^3} = \lim_{h\to 0}\underline{\lim_{\varepsilon\to 0}} \frac{a^{0,\gamma}_{npqr}(\underline{x}, S^d, \varepsilon, h)}{h^3} = a^0_{npqr}(\underline{x}, S^d),$$

$$b)\ \lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}} \frac{b^{\gamma}_{npq}(\underline{x}, S^d, \varepsilon, h)}{h^3} = \lim_{h\to 0}\underline{\lim_{\varepsilon\to 0}} \frac{b^{\gamma}_{npq}(\underline{x}, S^d, \varepsilon, h)}{h^3} = b_{npq}(\underline{x}, S^d),$$

$$c)\ \lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}} \frac{c^{\gamma}_{qr}(\underline{x}, S^d, \varepsilon, h)}{h^3} = \lim_{h\to 0}\underline{\lim_{\varepsilon\to 0}} \frac{c^{\gamma}_{qr}(\underline{x}, S^d, \varepsilon, h)}{h^3} = c_{qr}(\underline{x}, S^d),$$

where $\{a^0_{npqr}(\underline{x}, S^d)\}$, $\{b_{npq}(\underline{x}, S^d)\}$, $\{c_{qr}(\underline{x}, S^d)\}$ are continuous tensors (at $\underline{x} \in \Omega$).

4.2) the sequence $\underline{F}_\varepsilon(\underline{x})$ converges weakly in $L_2(\Omega)$ to a vector function $\underline{F}(\underline{x})$, as $\varepsilon \to 0$.

Note, that the existence of limits 4.1) is a general restriction on the spatial distributions of the particles. Since we do not require any spatial periodicity, we have to impose some conditions on these distributions.

Remark. If the limits in 4.1) exist for some $\gamma > 0$, then they exist for any $\gamma > 0$ and the limiting tensors do not depend on γ; moreover, $\{a^0_{npqr}(\underline{x}, S^d)\}$ and $\{c_{qr}(\underline{x}, S^d)\}$ are positive definite tensors (these facts can be proved analogously to [22]).

In our previous work [4] the following theorem was proved.

Theorem 2. *Let conditions 4.1)-4.2) hold. Then the sequence of vector-functions $\underline{\tilde{v}}_\varepsilon(\underline{x})$, defined by (46), converges strongly in $\mathbf{L_2(\Omega)}$ to a vector-function $\underline{v}(\underline{x},t)$, which is a solution of the following homogenized linear stationary problem:*

$$-\sum_{n,p,q,r=1}^{3}\frac{\partial}{\partial x_p}\Big[a^D_{npqr}(\underline{x})e_{qr}[v]+a^R_{npqr}(\underline{x})\omega_{qr}[v]\Big]\underline{e}^n=\nabla p+\underline{F},\quad \underline{x}\in\Omega, \tag{47}$$

$$\mathrm{div}\underline{v}=0,\quad \underline{x}\in\Omega, \tag{48}$$

$$\underline{v}(\underline{x})=0,\quad \underline{x}\in\partial\Omega. \tag{49}$$

Here

$$a^D_{npqr}=a^0_{npqr}+\frac{1}{2}\sum_{l=1}^{3}b_{qrl}\epsilon_{lnp},\quad a^R_{npqr}=\frac{1}{4}\sum_{l,m=1}^{3}c_{lm}\epsilon_{lnp}\epsilon_{mqr}+\frac{1}{2}\sum_{l=1}^{3}b_{npl}\epsilon_{lqr}, \tag{50}$$

$$\omega_{qr}[v]=\frac{1}{2}\big(\frac{\partial v_q}{\partial x_r}-\frac{\partial v_r}{\partial x_q}\big), \tag{51}$$

where $\{\epsilon_{lnp}\}$ is Levi-Civita permutation tensor.

The problem $(47)-(49)$ *has the unique solution.*

Let R^t_ε and R^t (t is a parameter) be the resolving operators of the problems (33)-(37) and (47)-(49), respectively ($\underline{v}_\varepsilon=R^t_\varepsilon\underline{F}_\varepsilon$ and $\underline{v}=R^t\underline{F}$). Analogously to [8] and [22], one can show that R^t_ε and R^t are compact and self-adjoint in $L_2(\Omega)$. Moreover, with the help of Theorem 2, it can be proved that for any $t\in[0,T]$ and any $\underline{f}\in L_2(\Omega)$

$$\lim_{\varepsilon\to 0}\|R^t_\varepsilon\underline{f}-R^t\underline{f}\|_{L_2(\Omega)}=0,\quad \|R^t_\varepsilon\underline{f}\|_{L_2(\Omega)}\le C, \tag{52}$$

where constant C depends neither on ε nor on t (for more details see [22]).

Assume now that the sequence $\sum_{i=1}^{N_\varepsilon}\chi^i_\varepsilon(\underline{x},t)=1-\chi_\varepsilon(\underline{x},t)$ converges as $\varepsilon\to 0$ $*$-weakly in $L^\infty(\Omega_T)$ to the function $0<C(\underline{x},t)<1$:

$$\sum_{i=1}^{N_\varepsilon}\chi^i_\varepsilon(\underline{x},t)\rightharpoonup C(\underline{x},t)\quad \big(*-\text{weakly in } L^\infty(\Omega_T)\big), \tag{53}$$

where $C_t(\underline{x},t)\in L_4\big(0,T;L_2(\Omega)\big)$, $\nabla C(\underline{x},t)\in L_2(\Omega_T)$, and the sequences $\underline{f}_\varepsilon(\underline{x},t)$ and $\underline{\tilde{v}}_{\varepsilon 0}(\underline{x})$ converge as $\varepsilon\to 0$ strongly in $L_2(\Omega_T)$ to $\underline{f}(\underline{x},t)$ and strongly in $L_2(\Omega)$ to $\underline{v}_0(\underline{x})$, respectively:

$$\lim_{\varepsilon\to 0}\|\underline{f}_\varepsilon-\underline{f}\|_{L_2(\Omega_T)}=0, \tag{54}$$

$$\lim_{\varepsilon\to 0}\|\underline{\tilde{v}}_{\varepsilon 0}-\underline{v}_0\|_{L_2(\Omega)}=0. \tag{55}$$

Theorem 3. *Let conditions 3.1)-3.5) are satisfied, the limits (53)-(55) exist, the limits in 4.1) exist for every $t \in [0,T]$ and the limiting tensors $\{a^0_{npqr}(\underline{x},t,S^d)\}$, $\{b_{npq}(\underline{x},t,S^d)\}$, $\{c_{np}(\underline{x},t,S^d)\}$ are continuous in Ω_T. Then the sequence of vector-functions $\underline{\tilde{v}}_\varepsilon(\underline{x},t)$, defined by (9), converges strongly in $L_2(\Omega_T)$ (and in $L_2(\Omega)$ uniformly with respect to t) to a vector-function $\underline{v}(\underline{x},t)$, which is a generalized solution of the following homogenized problem:*

$$\frac{\partial(\rho\underline{v})}{\partial t}+(\underline{v}\cdot\nabla)(\rho\underline{v})-\sum_{n,p,q,r=1}^{3}\frac{\partial}{\partial x_p}\Big[a^D_{npqr}(\underline{x},t)e_{qr}[v]+a^R_{npqr}(\underline{x},t)\omega_{qr}[v]\Big]\underline{e}^n=\nabla p+\underline{F},\quad \underline{x}\in\Omega_T, \quad (56)$$

$$\mathrm{div}\underline{v}(\underline{x},t)=0,\quad \underline{x}\in\Omega_T, \quad (57)$$

$$\underline{v}(\underline{x},t)=0,\quad (\underline{x},t)\in\partial\Omega\times[0,T], \quad (58)$$

$$\underline{v}(\underline{x},0)=\underline{v}_0(\underline{x}), \quad (59)$$

where

$$\rho(\underline{x},t)=\rho_f[1-C(\underline{x},t)]+\rho_s C(\underline{x},t),\quad \underline{F}(\underline{x},t)=\rho(\underline{x},t)\underline{f}(\underline{x},t). \quad (60)$$

Remark. A function $\underline{v}(\underline{x},t)\in\mathcal{L}(\Omega_T)=H^1(\Omega_T)\bigcap L_2\big(0,T;\overset{\circ}{J}(\Omega)\big)\bigcap L^\infty(\Omega_T)$, where $\overset{\circ}{J}(\Omega)$ is the class of divergence free vector functions from $\overset{\circ}{H^1}(\Omega)$, is said to be a generalized solution to the problem (56)-(59) if it satisfies the following integral identity

$$\int_0^\tau\int_\Omega\Big\{\big(-\rho\underline{v},\underline{\Phi}_t+(\underline{v}\cdot\nabla)\underline{\Phi}\big)+\sum_{n,p,q,r=1}^{3}[a^D_{npqr}(\underline{x},t)+a^R_{npqr}(\underline{x},t)]\frac{\partial v_n}{\partial x_p}\frac{\partial\Phi_q}{\partial x_r}\Big\}\,dxdt+$$

$$+\int_\Omega\big(\rho\underline{v},\Phi(\underline{x},\tau)\big)-\int_\Omega\big(\rho\underline{v},\Phi(\underline{x},0)\big)=\int_0^\tau\int_\Omega(\underline{F},\underline{\Phi})\,dxdt \quad (61)$$

for any $\underline{\Phi}(\underline{x},t)\in\mathcal{L}(\Omega_T)$ and τ $(0<\tau\le T)$.

Lemma 2. *Problem (56)-(59) can have at most one generalized solution from $\mathcal{L}(\Omega_T)$.*

Proof. Assume that there exist two solutions $\underline{v}'(\underline{x},t)\in\mathcal{L}(\Omega_T)$ and $\underline{v}''(\underline{x},t)\in\mathcal{L}(\Omega_T)$. Then the function $\underline{v}=\underline{v}'-\underline{v}''\in\mathcal{L}(\Omega_T)$ satisfies the following identity for any $\underline{\Phi}\in\mathcal{L}(\Omega_T)$:

$$-\int_0^\tau\int_\Omega\Big\{(\rho\underline{v},\underline{\Phi}_t)+\big(\rho\underline{v},(\underline{v}'\cdot\nabla)\underline{\Phi}\big)-\big(\rho\underline{v}'',(\underline{v}\cdot\nabla)\underline{\Phi}\big)\Big\}\,dxdt+$$

$$+\int_0^\tau\int_\Omega\sum_{n,p,q,r=1}^{3}[a^D_{npqr}(\underline{x},t)+a^R_{npqr}(\underline{x},t)]\frac{\partial v_n}{\partial x_p}\frac{\partial\Phi_q}{\partial x_r}\,dxdt+\int_\Omega\big(\rho(\underline{x},\tau)\underline{v}(\underline{x},\tau),\underline{\Phi}(\underline{x},\tau)\big)\,dx=0.$$

Choosing here $\underline{\Phi}(\underline{x},t) = \underline{v}(\underline{x},t)$, after obvious transformations we get that

$$\frac{1}{2}\int\limits_{\Omega} \rho(\underline{x},\tau)|\underline{v}(\underline{x},\tau)|^2\,dx + \int\limits_0^\tau\int\limits_\Omega \sum_{n,p,q,r=1}^{3} [a^D_{npqr}(\underline{x},t) + a^R_{npqr}(\underline{x},t)]\frac{\partial v_n}{\partial x_p}\frac{\partial v_q}{\partial x_r}\,dxdt \leq$$

$$\leq \frac{1}{2}\int\limits_0^\tau\int\limits_\Omega |\rho_t||\underline{v}|^2\,dxdt + \int\limits_0^\tau\int\limits_\Omega \rho\Big\{\big|\big(\underline{v},(\underline{v}'\cdot\nabla)\underline{v}\big)\big| + \big|\big(\underline{v}'',(\underline{v}\cdot\nabla)\underline{v}\big)\big|\Big\}\,dxdt.$$

Since for any fixed $t\in[0,\tau]$ $\underline{v}(\underline{x},t) \in \overset{\circ}{J}(\Omega)$, the following inequality holds (see [4]):

$$\int\limits_\Omega \big[a^D_{npqr} + a^R_{npqr}\big]\frac{\partial v_n}{\partial x_p}\frac{\partial v_q}{\partial x_r}\,d\underline{x} \geq \|\underline{v}\|^2_{H^1(\Omega)}, \tag{62}$$

where a^D_{npqr} and a^R_{npqr} are defined by (50). Using now inequality (62) and the fact that both the function $\rho(\underline{x},t)$ and the vector functions $\underline{v}'(\underline{x},t)$ and $\underline{v}''(\underline{x},t)$ are bounded, we obtain:

$$\max_{0\leq t\leq\tau}\int\limits_\Omega \rho(\underline{x},\tau)|\underline{v}(\underline{x},\tau)|^2\,dx + \int\limits_0^\tau\int\limits_\Omega |\nabla\underline{v}|^2\,dxdt \leq C\Big\{\int\limits_0^\tau\int\limits_\Omega |\rho_t||\underline{v}|^2\,dxdt + \int\limits_0^\tau\int\limits_\Omega |\underline{v}||\nabla\underline{v}|\,dxdt\Big\}, \tag{63}$$

where constant C does not depend on τ.

Using Hölder's inequality, we can write

$$\int\limits_0^\tau\int\limits_\Omega |\rho_t||\underline{v}|^2\,dxdt \leq \Big\{\int\limits_0^\tau\Big(\int\limits_\Omega |\underline{v}|^q\,dx\Big)^{\frac{r}{q}}\,dt\Big\}^{\frac{2}{r}}\Big\{\int\limits_0^\tau\Big(\int\limits_\Omega |\rho_t|^{q'}\,dx\Big)^{\frac{r'}{q'}}\,dt\Big\}^{\frac{1}{r'}} =$$

$$= \|\underline{v}\|^2_{L_r\left(0,\tau;L_q(\Omega)\right)}\|\rho_t\|_{L_{r'}\left(0,\tau;L_{q'}(\Omega)\right)}, \tag{64}$$

where $q' = \dfrac{q}{q-2}$, $r' = \dfrac{r}{r-2}$ $(q\geq 2,\ r\geq 2)$.

Introduce now the following space of vector functions:

$$\overset{\circ}{V}_2(\Omega_\tau) = \Big\{\underline{v}(\underline{x},t):\ \|\underline{v}\|^2_{\Omega_\tau} = \operatorname{ess}\max_{0\leq t\leq\tau}\int\limits_\Omega |\underline{v}(\underline{x},t)|^2\,dx + \int\limits_0^\tau\int\limits_\Omega |\nabla\underline{v}(\underline{x},t)|^2\,dxdt < \infty;$$

$$\underline{v}(\underline{x},t) = 0,\ (\underline{x},t)\in\partial\Omega\times[0,T]\Big\}.$$

For any vector function $\underline{v}(\underline{x},t) \in \overset{\circ}{V}_2(\Omega_\tau)$ the following inequality holds (see [19]):

$$\|\underline{v}\|_{L_{\overline{r}}\left(0,\tau;L_{\overline{q}}(\Omega)\right)} \leq \beta\|\underline{v}\|_{\Omega_\tau}, \tag{65}$$

where $\overline{q}$ and $\overline{r}$ are arbitrary constants such that $\dfrac{1}{\overline{r}}+\dfrac{3}{2\overline{q}}=\dfrac{3}{4}$, $\overline{r}\geq 2$, $2\leq\overline{q}\leq 6$ and $\beta=4^{2/\overline{r}}$. Note that due to the Embedding theorems (see [19]) $\mathcal{L}(\Omega_T)\subset\overset{\circ}{V}_2(\Omega_T)$.

Choosing in (64) $q=4$ and $r=\dfrac{8}{3}$ and taking into account (65), we get

$$\int_0^\tau\int_\Omega|\rho_t||\underline{v}|^2\,dxdt\leq\beta^2\|\underline{v}\|^2_{\Omega_\tau}\|\rho_t\|_{L_4\left(0,\tau;L_2(\Omega)\right)}. \tag{66}$$

Next, using Hölder's inequality, we have

$$\int_0^\tau\int_\Omega|\underline{v}|\,|\nabla\underline{v}|\,dxdt\leq\|\underline{v}\|_{\Omega_\tau}\Big\{\int_0^\tau\int_\Omega|\underline{v}|^2\,dxdt\Big\}^{\frac{1}{2}}\leq(\tau)^{\frac{1}{r'}}(\text{meas}\,\Omega)^{\frac{1}{q'}}\|\underline{v}\|_{\Omega_\tau}\|\underline{v}\|_{L_r\left(0,\tau;L_q(\Omega)\right)},$$

where $r'=\dfrac{2r}{r-2}$ and $q'=\dfrac{2q}{q-2}$. Choosing, as before, $q=4$ and $r=\dfrac{8}{3}$ and using inequality (65), we get

$$\int_0^\tau\int_\Omega|\underline{v}|\,|\nabla\underline{v}|\,dxdt\leq\beta(\tau)^{\frac{1}{8}}(\text{meas}\,\Omega)^{\frac{1}{4}}\|\underline{v}\|^2_{\Omega_\tau}. \tag{67}$$

From (63), (66) and (67) it follows that

$$\min\{\rho_f,\rho_s,1\}\|\underline{v}\|^2_{\Omega_\tau}\leq\delta(\tau)\|\underline{v}\|^2_{\Omega_\tau}, \tag{68}$$

where $\delta(\tau)=\beta^2\|\rho_t\|_{L_4\left(0,\tau;L_2(\Omega)\right)}+\beta(\tau)^{\frac{1}{8}}(\text{meas}\,\Omega)^{\frac{1}{4}}$. Taking now τ so that $\delta(\tau)<\min\{\rho_f,\rho_s,1\}$, from (68) we conclude that $\underline{v}(\underline{x},t)=0$. If $\tau<T$, then partitioning interval $[0,T]$ into subintervals of length τ_k (for which $\underline{v}(\underline{x},t)=0$) and repeating the above arguments, after a final number of steps one can show that $\underline{v}(\underline{x},t)=\underline{v}'(\underline{x},t)-\underline{v}''(\underline{x},t)\equiv 0$ in Ω_T. The Lemma is proved.

5 Proof of Theorem 3

In accordance with the bounds (16)-(17) and conditions 3.1)-3.3) the sequence of vector functions $\{\underline{v}_\varepsilon(\underline{x},t),\,\varepsilon>0\}$ is bounded in $H^1(\Omega_T)$ and, thus, it is weakly compact in $H^1(\Omega_T)$ (and in $\overset{\circ}{H}{}^1(\Omega)$ for any $t\in[0,T]$). Due to the Embedding theorem, this sequence is compact in $L_2(\Omega_T)$ (and in $L_2(\Omega)$ for any $t\in[0,T]$). Hence, there exists a subsequence $\{\underline{v}_{\varepsilon_k}(\underline{x},t),\,\varepsilon_k>0\}$ which converges weakly in $H^1(\Omega_T)$ (and in $H^1(\Omega)$ for any $t\in[0,T]$) and strongly in $L_2(\Omega_T)$ (and in $L_2(\Omega)$ for any $t\in[0,T]$) to some vector function $\underline{v}(\underline{x},t)$. Using conditions 3.3)-3.4), we conclude that $\underline{v}(\underline{x},t)\in\mathcal{L}(\Omega_T)$.

As it is shown below, the limiting vector function $\underline{v}(\underline{x},t)$ is a solution of the problem (56)-(59). Since this problem, due to Lemma 2, has a unique generalized solution from $\mathcal{L}(\Omega_T)$, the entire sequence $\{\underline{v}_\varepsilon(\underline{x},t),\,\varepsilon>0\}$ also converges to $\underline{v}(\underline{x},t)$.

Introduce now the following vector function

$$\underline{W}_\varepsilon(\underline{x},t)=\rho_f\Big[\frac{\partial \underline{v}_\varepsilon}{\partial t}+(\underline{v}_\varepsilon\cdot\nabla)\underline{v}_\varepsilon\Big]\chi_\varepsilon(\underline{x},t)+\rho_s\sum_{i=1}^{N_\varepsilon}\Big[\frac{\partial \underline{V}^i_\varepsilon}{\partial t}+(\underline{V}^i_\varepsilon\cdot\nabla)\underline{V}^i_\varepsilon\Big]\chi^i_\varepsilon(\underline{x},t), \tag{69}$$

where $\underline{V}^i_\varepsilon(\underline{x},t)=\underline{\dot{u}}^i_\varepsilon(t)+\underline{\dot{\theta}}^i_\varepsilon(t)\times(\underline{x}-\underline{x}^i_\varepsilon)$.

Lemma 3. *The sequence of vector functions $\{\underline{W}_\varepsilon(\underline{x},t),\ \varepsilon>0\}$ is weakly compact in $L_2(\Omega_T)$. So, there exists a subsequence $\{\underline{W}_{\varepsilon_k}(\underline{x},t),\ \varepsilon_k>0\}$ which converges as $\varepsilon_k\to 0$ weakly in $L_2(\Omega_T)$ to the limiting vector function $\underline{W}(\underline{x},t)$ which is given by*

$$\underline{W}(\underline{x},t)=\frac{\partial(\rho\underline{v})}{\partial t}+(\underline{v}\cdot\nabla)(\rho\underline{v}), \tag{70}$$

where $\rho(\underline{x},t)$ is determined by equation (60) and $\underline{v}(\underline{x},t)$ is a limit of the subsequence $\{\underline{v}_{\varepsilon_k}(\underline{x},t),\ \varepsilon_k\to 0\}$.

Proof. From the bounds (16)-(17) and conditions 3.3)-3.4) it follows that the sequence $\{\underline{W}_\varepsilon(\underline{x},t),\ \varepsilon>0\}$ is bounded in $L_2(\Omega_T)$ and, hence, it is weakly compact. Let $\underline{\varphi}(\underline{x},t)\in C^1_0(\Omega_T)$ be an arbitrary smooth vector function with a compact support. Then

$$\int_0^T\int_\Omega(\underline{W}_{\varepsilon_k},\underline{\varphi})\,dxdt=\rho_f\int_0^T\int_{\Omega_\varepsilon(t)}\Big(\frac{\partial \underline{v}_{\varepsilon_k}}{\partial t}+(\underline{v}_{\varepsilon_k}\cdot\nabla)\underline{v}_{\varepsilon_k},\underline{\varphi}\Big)\,dxdt+$$

$$+\rho_s\sum_{i=1}^{N_\varepsilon}\int_0^T\int_{Q^i_\varepsilon(t)}\Big(\frac{\partial \underline{V}^i_{\varepsilon_k}}{\partial t}+(\underline{V}^i_{\varepsilon_k}\cdot\nabla)\underline{V}^i_{\varepsilon_k},\underline{\varphi}\Big)\,dxdt=\rho_f\int_0^T\int_\Omega\Big(\frac{\partial \underline{v}_{\varepsilon_k}}{\partial t}+(\underline{v}_{\varepsilon_k}\cdot\nabla)\underline{v}_{\varepsilon_k},\underline{\varphi}\Big)\,dxdt+$$

$$+(\rho_s-\rho_f)\sum_{i=1}^{N_\varepsilon}\int_0^T\int_{Q^i_\varepsilon(t)}\Big(\frac{\partial \underline{V}^i_{\varepsilon_k}}{\partial t}+(\underline{V}^i_{\varepsilon_k}\cdot\nabla)\underline{V}^i_{\varepsilon_k},\underline{\varphi}\Big)\,dxdt=J^1_{\varepsilon_k}+J^2_{\varepsilon_k}. \tag{71}$$

Since $\underline{v}_{\varepsilon_k}(\underline{x},t)$ converges weakly in $H^1(\Omega_T)$ and strongly in $L_2(\Omega_T)$ to $\underline{v}(\underline{x},t)$,

$$\lim_{\varepsilon_k\to 0}J^1_{\varepsilon_k}=\rho_f\int_0^T\int_\Omega\Big(\frac{\partial \underline{v}}{\partial t}+(\underline{v}\cdot\nabla)\underline{v},\underline{\varphi}\Big)\,dxdt. \tag{72}$$

The second integral in (71) we can write in the form

$$J^2_{\varepsilon_k}=(\rho_s-\rho_f)\sum_{i=1}^{N_\varepsilon}\int_0^T\int_{Q^i_\varepsilon(t)}\sum_{j=0}^{3}\frac{\partial}{\partial x_j}[(\underline{V}^i_{\varepsilon_k},\underline{\varphi})\hat{V}^i_{\varepsilon_k,j}]\,dxdt-$$

$$-(\rho_s-\rho_f)\sum_{i=1}^{N_\varepsilon}\int_0^T\int_{Q^i_\varepsilon(t)}\sum_{j=0}^{3}\hat{V}^i_{\varepsilon_k,j}\Big(\underline{V}^i_{\varepsilon_k},\frac{\partial\underline{\varphi}}{\partial x_j}\Big)\,dxdt=J^{2,1}_{\varepsilon_k}+J^{2,2}_{\varepsilon_k},$$

where $x_0 \equiv t$, $\hat{V}^i_{\varepsilon_k,0} \equiv 1$, $\hat{V}^i_{\varepsilon_k,j} = V^i_{\varepsilon_k,j}(\underline{x},t)$ $(j=\overline{1,3})$. Since the vector $\underline{\hat{V}}^i_{\varepsilon_k} = \{\hat{V}^i_{\varepsilon_k,j}\}_{j=0}^3$ is tangent to the lateral surface of $[0,T] \times Q^i_{\varepsilon_k}(t)$ $(i=\overline{1,N_{\varepsilon_k}})$ and the vector function $\underline{\varphi}(\underline{x},t)$ has a compact support in Ω_T, with the help of the divergence theorem we conclude that $J^{2,1}_{\varepsilon_k} = 0$.

Using (53) and taking into account that $\underline{V}^i_{\varepsilon_k}(\underline{x},t) = \underline{v}_{\varepsilon_k}(\underline{x},t)$ at $(\underline{x},t) \in Q^i_{\varepsilon_k}(t) \times [0,T]$ and $\underline{v}_{\varepsilon_k}(\underline{x},t)$ converges strongly in $L_2(\Omega_T)$ to $\underline{v}(\underline{x},t)$, we obtain

$$\lim_{\varepsilon_k \to 0} J^{2,2}_{\varepsilon_k} = -(\rho_s - \rho_f) \int_0^T \int_\Omega C(\underline{x},t) \Big[\Big(\underline{v}, \frac{\partial \underline{\varphi}}{\partial t} \Big) + \sum_{j=1}^3 v_j \Big(\underline{v}, \frac{\partial \underline{\varphi}}{\partial x_j} \Big) \Big] \, dx dt.$$

Thus,

$$\lim_{\varepsilon_k \to 0} J^{2}_{\varepsilon_k} = -(\rho_s - \rho_f) \int_0^T \int_\Omega C(\underline{x},t) \Big[\Big(\underline{v}, \frac{\partial \underline{\varphi}}{\partial t} \Big) + \sum_{j=1}^3 v_j \Big(\underline{v}, \frac{\partial \underline{\varphi}}{\partial x_j} \Big) \Big] \, dx dt =$$

$$= (\rho_s - \rho_f) \int_0^T \int_\Omega \Big(\frac{\partial (C\underline{v})}{\partial t} + (\underline{v} \cdot \nabla)(C\underline{v}), \underline{\varphi} \Big) \, dx dt. \qquad (73)$$

Here, the integration by parts is justified, because $\underline{v}(\underline{x},t) \in H^1(\Omega_T) \bigcap L^\infty(\Omega_T)$, $\rho(\underline{x},t) \in H^1(\Omega_T) \bigcap L^\infty(\Omega_T)$ and $\underline{\varphi}(\underline{x},t) \in C^1_0(\Omega_T)$.

Combining now (71)-(73) and taking into account (60), we finally get

$$\int_0^T \int_\Omega \big(\underline{W}(\underline{x},t), \underline{\varphi}(\underline{x},t) \big) \, dx dt = \lim_{\varepsilon_k \to 0} \int_0^T \int_\Omega \big(\underline{W}_{\varepsilon_k}(\underline{x},t), \underline{\varphi}(\underline{x},t) \big) \, dx dt =$$

$$= \int_0^T \int_\Omega \Big(\frac{\partial (\rho\underline{v})}{\partial t} + (\underline{v} \cdot \nabla)(\rho\underline{v}), \underline{\varphi} \Big) \, dx dt,$$

whence the statement of the lemma follows. □

Denote by F_ε the RHS of problem (2)-(5):

$$F_\varepsilon(\underline{x},t) = \big[\rho_f \chi_\varepsilon(\underline{x},t) + \rho_s (1 - \chi_\varepsilon(\underline{x},t) \big] \underline{f}_\varepsilon(\underline{x},t). \qquad (74)$$

Recalling now (69) and taking into account that

$$\frac{\partial \underline{V}^i_\varepsilon}{\partial t} + (\underline{V}^i_\varepsilon \cdot \nabla) \underline{V}^i_\varepsilon = \underline{\ddot{u}}^i_\varepsilon(t) + \underline{\ddot{\theta}}^i_\varepsilon(t) \times \big(\underline{x} - \underline{x}^i_\varepsilon(t) \big) + \underline{\dot{\theta}}^i_\varepsilon(t) \times \big[\underline{\dot{\theta}}^i_\varepsilon(t) \times \big(\underline{x} - \underline{x}^i_\varepsilon(t) \big) \big], \qquad (75)$$

we can represent the solution $\underline{v}_\varepsilon(\underline{x},t)$ of the problem (2)-(8) in the form $\underline{v}_\varepsilon(\underline{x},t) = R^t_\varepsilon \Phi^t_\varepsilon[\underline{x}]$, where R^t_ε is the resolving operator of the problem (33)-(37) in the domain $\Omega_\varepsilon(t)$ (t is a parameter), and $\Phi^t_\varepsilon[\underline{x}] = \underline{F}_\varepsilon(\underline{x},t) - \underline{W}_\varepsilon(\underline{x},t)$.

Since for every t the operator R^t_ε is self-adjoint in $L_2(\Omega)$, for any continuous vector function $\underline{\varphi}(\underline{x},t) \equiv \underline{\varphi}^t[\underline{x}]$ we have:

$$\int\limits_0^T \int\limits_\Omega (\underline{v}_\varepsilon, \underline{\varphi})\, dxdt = \int\limits_0^T \int\limits_\Omega (R^t_\varepsilon \underline{\Phi}^t_\varepsilon, \underline{\varphi}^t)\, dxdt = \int\limits_0^T \int\limits_\Omega (\underline{\Phi}^t_\varepsilon, R^t_\varepsilon \underline{\varphi}^t)\, dxdt =$$

$$= \int\limits_0^T \int\limits_\Omega (\underline{\Phi}^t_\varepsilon, R^t \underline{\varphi}^t)\, dxdt + \int\limits_0^T \int\limits_\Omega (\underline{\Phi}^t_\varepsilon, R^t_\varepsilon \underline{\varphi}^t - R^t \underline{\varphi}^t)\, dxdt. \tag{76}$$

Using Lemma 3 and conditions (53)-(54), we conclude that the subsequence $\{\underline{\Phi}^t_{\varepsilon_k}[\underline{x}] \equiv \underline{\Phi}_{\varepsilon_k}(\underline{x},t),\ \varepsilon_k > 0\}$ converges weakly in $L_2(\Omega_T)$ to the vector function $\underline{\Phi}(\underline{x},t) = \underline{F}(\underline{x},t) - \underline{W}(\underline{x},t) \equiv \underline{\Phi}^t[\underline{x}]$, where the vector functions $\underline{F}$ and $\underline{W}$ are defined in (60) and (70), respectively. Thus,

$$\lim_{\varepsilon_k \to 0} \int\limits_0^T \int\limits_\Omega (\underline{\Phi}^t_{\varepsilon_k}, R^t \underline{\varphi}^t)\, dxdt = \lim_{\varepsilon_k \to 0} \int\limits_0^T \int\limits_\Omega (\underline{\Phi}_{\varepsilon_k}(\underline{x},t), R^t \underline{\varphi}(\underline{x},t))\, dxdt = \int\limits_0^T \int\limits_\Omega (\underline{\Phi}(\underline{x},t), R^t \underline{\varphi}(\underline{x},t))\, dxdt =$$

$$= \int\limits_0^T \int\limits_\Omega (\underline{\Phi}^t, R^t \underline{\varphi}^t)\, dxdt = \int\limits_0^T \int\limits_\Omega (R^t \underline{\Phi}^t, \underline{\varphi}^t)\, dxdt = \int\limits_0^T \int\limits_\Omega (R^t \underline{\Phi}^t[\underline{x}], \underline{\varphi}(\underline{x},t))\, dxdt. \tag{77}$$

Here we took into consideration that for almost all $t \in [0,T]$ $\underline{\Phi}(\underline{x},t) \in L_2(\Omega)$ and the operator R^t is self-adjoint in $L_2(\Omega)$. Next,

$$\Big| \int\limits_0^T \int\limits_\Omega (\underline{\Phi}^t_\varepsilon, R^t_\varepsilon \underline{\varphi}^t - R^t \underline{\varphi}^t)\, dxdt \Big| \le \|\underline{\Phi}_\varepsilon\|_{L_2(\Omega_T)} \Big\{ \int\limits_0^T \|R^t_\varepsilon \underline{\varphi}^t - R^t \underline{\varphi}^t\|^2_{L_2(\Omega)}\, dt \Big\}^{\frac{1}{2}}.$$

Since the sequence $\{\underline{\Phi}_\varepsilon,\ \varepsilon > 0\}$ is bounded in $L_2(\Omega_T)$ uniformly with respect to ε, with the help of (52) we get:

$$\lim_{\varepsilon \to 0} \Big| \int\limits_0^T \int\limits_\Omega (\underline{\Phi}^t_\varepsilon, R^t_\varepsilon \underline{\varphi}^t - R^t \underline{\varphi}^t)\, dxdt \Big| = 0. \tag{78}$$

Combining now (76)-(78), we obtain:

$$\lim_{\varepsilon_k \to 0} \Big| \int\limits_0^T \int\limits_\Omega (\underline{v}_\varepsilon, \underline{\varphi})\, dxdt \Big| = \int\limits_0^T \int\limits_\Omega (R^t \underline{\Phi}^t[\underline{x}], \underline{\varphi}^t)\, dxdt.$$

On the other hand, since the subsequence $\{\underline{v}_{\varepsilon_k}(\underline{x},t) \equiv \underline{v}^t_{\varepsilon_k}[\underline{x}],\ \varepsilon_k > 0\}$ converges as $\varepsilon_k \to 0$ strongly in $L_2(\Omega_T)$ (and in $L_2(\Omega)$ uniformly with respect to $t \in [0,T]$) to the vector function $\underline{v}(\underline{x},t) \equiv \underline{v}^t[\underline{x}]$, we have:

$$\lim_{\varepsilon_k \to 0} \int_0^T \int_\Omega \big(\underline{v}_\varepsilon(\underline{x},t), \underline{\varphi}(\underline{x},t)\big)\, dxdt = \int_0^T \int_\Omega (\underline{v}^t, \underline{\varphi}^t)\, dxdt.$$

So, for all $t \in [0,T]$ $\underline{v}(\underline{x},t) \equiv \underline{v}^t[\underline{x}] = R^t \underline{\Phi}^t[\underline{x}] = R^t[\underline{F}(\underline{x},t) - \underline{W}(\underline{x},t)]$. Moreover, with the aid of condition (55), we conclude that $\underline{v}(\underline{x},0) = \underline{v}_0$.

Recalling the definition of the resolving operator R^t and the form of the vector function $\underline{W}(\underline{x},t)$ (see (70)), we conclude that $\underline{v}(\underline{x},t)$ is the generalized solution of the problem (56)-(59). Theorem 3 is proved.

References

[1] Aero E.L., Bulygin A.N., Kuvshinsky E.V., *Asymmetric hydrodynamics*, J. Appl. Math. Mech. (PMM), **29** (1965), 297-308 (in Russian).

[2] Grégory Allaire, *Homogenization of the Navier-Stokes Equations in Open Sets Perforated with Tiny Holes. I. Abstract Framework, a Volume Distribution of Holes. II. Noncritical sizes of the holes for a volume distribution and surface distribution of holes.*, Arch. Rational Mech. Anal., **113** (1991), 209–259, 261–298.

[3] T. Belytschko, W. K. Liu, and B. Moran, "Nonlinear Finite Elements for Continua and Structures," Wiley, 2000.

[4] M. Berezhnyi and E. Khruslov, *Asymmetric hydrodynamics of suspensions subjected to the influence of strong external magnetic fields*, Journal of Multiscale Modeling, **4** (2012), 89–109.

[5] (MR2206053) M. A.Berezhnyy and L. V.Berlyand, *Continuum limit for three-dimensional mass-spring networks and discrete Korn's inequality*, Journal of the Mechanics and Physics of Solids, **54** (2006), 635–669.

[6] (MR2338184) M. A. Berezhnyi, *The asymptotic bahavior of viscous incompressible fluid small oscillations with solid interacting particles*, Journal of Mathematical Physics, Analysis, Geometry, **3** (2007), 135–156.

[7] (MR2448943) M. Berezhnyi, L. Berlyand and E. Khruslov, *The homogenized model of small oscillations of complex fluids*, NHM, Networks and heterogeneous media, **3** (2008), 835–869.

[8] M. Berezhnyi, "Homogenized Models of Complex Fluids," PhD Thesis, ILTPE, 2009 (in ukrainian), 159 p. URL: http://www.dlib.com.ua/useredneni-modeli-strukturovanykh-ridyn.html.

[9] M. Berezhnyi and E. Khruslov, *Non-standard dynamics of elastic composites*, NHM, Networks and heterogeneous media, **6** (2011), 89–109.

[10] M. Berezhnyi, *Discrete model of the nonsymmetric theory of elasticity*, Ukrainian mathematical journal, **63** (2011), 891–913.

[11] (MR0688245) L. V. Berlyand and A. D. Okhotsimskii, *Averaged description of an elastic medium with a large number of small absolutely rigid inclusions*, Dokl. Akad. Nauk SSSR, **268** (1983), 317–320 (in Russian).

[12] (MR2068451) L. Berlyand and E. Khruslov, *Homogenized non-Newtonian viscoelastic rheology of a suspension of interacting particles in a viscous Newtonian fluid*, SIAM, Journal of Applied Mathematics, **64** (2004), 1002–1034.

[13] E. Cosserat et F. Cosserat, "Théorie des Corps Deformables," Hermann, Paris, 1909.

[14] (MR0196422) V. A. Ditkin and A. P. Prudnikov, "Integral Transforms and Operational Calculus," Oxford; New York: Pergamon, 1965, 529 p.

[15] (MR0204005) Eringen A.C., *Theory of micropolar fluids*, J. Math. Mech., **16.1** (1966), 1-18.

[16] (MR0115342) G. Grioli, *Ellasticá asymmetrica*, Annali di matematica pura ed applicata, **4** (1960), 389–418.

[17] U. Hornung, "Homogenization and Porous Media," Interdisciplinary Applied Mathematics, Springer, New York, 1996.

[18] (MR1335452) T. Kato, "Perturbation Theory for Linear Operators," Springer, 1995, 652 p.

[19] (MR0241822) O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'tseva, "Linear and Quasilinear Equations of Parabolic Type," Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1967.

[20] (MR0093319) L. D. Landau and E. M. Lifshitz, "Course of Theoretical Physics. Quantum Mechanics. Non-relativistic Theory," London: Pergamon, 1958, 515 p.

[21] (MR2428102) A. I. Leonov, *Algebraic theory of linear viscoelastic nematodynamics*, Mathematical Physics, Analysis and Geometry, **11** (2008), 87–116.

[22] (MR2182441) V. Marchenko and E. Khruslov, "Homogenization of Partial Differential Equations," Birkhäuser, Boston, 2006, 401 p.

[23] (MR0708893) A. I. Marcushevich, "Theory of Analytic Functions: Brief Course," Mir, Moscow, 1983.

[24] (MR0144513) R. D. Mindlin and H. F. Tiersten, *Effects of couple-stresses in linear elasticity*, Archive for Rational Mechanics and Analysis, **11** (1962), 415–448.

[25] (MR1195131) O. A. Oleinic, A. S. Shamaev and G. A. Iosif'yan, "Mathematical Problems in Elasticity and Homogenization," in "Studies in Mathematics and its Applications," **26**, North-Holland Publishing Co., Amsterdam, 1992, 398 p.

[26] E. Sanchez-Palencia, "Non-homogeneous Media and Vibration Theory," Springer-Verlag, New York, 1980.

[27] I. Y. Smolin, P. V. Makarov, D. V. Shmick and I. V. Savlevich, *A micropolar model of plastic deformation of polycrystals at the mesolevel*, Computational Materials Science, **19** (2000), 133–142.

[28] T. Takahashi, *Existence of strong solutions for the problem of a rigid-fluid system*, C.R. Acad. Sci. Paris, Ser. I 336 (2003), 453–458.

[29] X. Zhang and P. Sharma, *Inclusions and inhomogeneities in strain gradient elasticity with couple stresses and related problems*, International Journal of Solids and Structures, **42** (2005), 3833–3851.