# Well-posedness for dislocation based gradient visco-plasticity II: monotone case

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**Abstract**

In this work we continue to investigate the well-posedness for infinitesimal dislocation based gradient viscoplasticity with linear kinematic hardening. We assume an additive split of the displacement gradient into non-symmetric elastic distortion and non-symmetric plastic distortion. The thermodynamic potential is augmented with a term taking the dislocation density tensor into account. The constitutive equations in the models we study are assumed to be only of monotone type. Based on the generalized version of Korn's inequality for incompatible tensor fields (the non-symmetric plastic distortion) due to Neff/Pauly/Witsch the existence of solutions of quasistatic initial-boundary value problems under consideration is shown using a time-discretization technique and a monotone operator method.

**Key words:** plasticity, gradient plasticity, viscoplasticity, rate dependent response, dislocations, plastic spin, Rothe's time-discretization method, maximal monotone method, Korn's inequality for incompatible tensor fields.

**AMS 2000 subject classification:** 35B65, 35D10, 74C10, 74D10, 35J25, 34G20, 34G25, 47H04, 47H05

## 1 Introduction

We study the existence of solutions of quasistatic initial-boundary value problems arising in gradient viscoplasticity. The models we study use rate-dependent constitutive equations with internal variables to describe the deformation behaviour of metals at infinitesimally small strain.

Our focus is on a phenomenological model on the macroscale not including the case of single crystal plasticity. From a mathematical point of view, the maze of equations, slip systems and physical mechanisms in single crystal plasticity is only obscuring the mathematical structure of the problem.

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Our model has been first presented in [20]. It is inspired by the early work of Menzel and Steinmann [17]. Contrary to more classical strain gradient approaches, the model features a non-symmetric plastic distortion field $p \in \mathcal{M}^3$ [4], a dislocation based energy storage based solely on $|\operatorname{Curl} p|$ and second gradients of the plastic distortion in the form of $\operatorname{Curl}\operatorname{Curl} p$ acting as dislocation based kinematical backstresses. We only consider energetic length scale effects and not higher gradients in the dissipation.

Uniqueness of classical solutions for rate-independent and rate-dependent formulations is shown in [19]. The existence question for the rate-independent model in terms of a weak reformulation is addressed in [20]. The rate-independent model with isotropic hardening is treated in [8]. First numerical results for a simplified rate-independent irrotational formulation (no plastic spin, symmetric plastic distortion $p$) are presented in [27]. In [9, 32] well-posedness for a rate-independent model of Gurtin and Anand [11] is shown under the decisive assumption that the plastic distortion is symmetric (the irrotational case), in which case we may really speak of a strain gradient plasticity model, since the gradient acts on the plastic strain.

In order to appreciate the simplicity and elegance of our model we sketch some of its ingredients. First, as is usual in plasticity theory, we split the total displacement gradient into non symmetric elastic and plastic distortions

$$\nabla u = e + p\,.$$

For invariance reasons, the elastic energy contribution may only depend on the elastic strains $\operatorname{sym} e = \operatorname{sym}(\nabla u - p)$. While $p$ is non-symmetric, a distinguishing feature of our model is that, similar to classical approaches, only the symmetric part $\varepsilon_p := \operatorname{sym} p$ of the plastic distortion appears in the local Cauchy stress $\sigma$, while the higher order stresses are non-symmetric. The reason for this is that we assume that $p$ has to obey the same transformation behavior as $\nabla u$ does, and thus the energy storage due to kinematical hardening should depend only on the plastic strains $\operatorname{sym} p$. For more on the basic invariance questions related to this issue dictating this type of behaviour, see [36, 18]. We assume as well plastic incompressibility $\operatorname{tr} p = 0$.

The thermodynamic potential of our model can therefore be written as

$$\int_\Omega \Big( \underbrace{\mathbb{C}[x](\operatorname{sym}(\nabla u - p))(\operatorname{sym}(\nabla u - p))}_{\text{elastic energy}} + \underbrace{\frac{C_1}{2}|\operatorname{dev}\operatorname{sym} p|^2}_{\text{kinematical hardening}} + \underbrace{\frac{C_2}{2}|\operatorname{Curl} p|^2}_{\text{dislocation storage}} + \underbrace{u \cdot b}_{\text{external volume forces}} \Big) dx$$

The positive definite elasticity tensor $\mathbb{C}$ is able to represent the elastic anisotropy of the material. The evolution equations for the plastic distortion $p$ are taken such that the stored energy is non-increasing along trajectories of $p$ at frozen displacement $u$, see [20]. Qualitatively, they have the form

$$\partial_t p \in g(\sigma - C_1 \operatorname{dev}\operatorname{sym} p - C_2 \operatorname{Curl}\operatorname{Curl} p)\,, \tag{1}$$

where $\sigma = \mathbb{C}[x]\operatorname{sym}(\nabla u - p)$ is the elastic symmetric Cauchy stress of the material and $g$ is a multivalued monotone flow function. Clearly, in the absence of energetic length scale effects ($C_2 = 0$), the plastic distortion remains symmetric and the model reduces to a classical plasticity model. Thus, the energetic length scale is solely responsible for the plastic spin in the model.

Regarding the boundary conditions necessary for the formulation of the higher order theory we assume that the boundary is a perfect conductor, this means that the tangential component of $p$ vanishes on $\partial\Omega$. In the context of dislocation dynamics these conditions express the requirement that there is no flux of the Burgers vector across a hard boundary. Gurtin [12] introduces the following different types of boundary conditions for the plastic distortion

$$\begin{aligned} \partial_t p \times n|_{\Gamma_{\text{hard}}} &= 0 \quad \text{"micro-hard" (perfect conductor)} \\ \partial_t p|_{\Gamma_{\text{hard}}} &= 0 \quad \text{"hard-slip"} \\ \operatorname{Curl} p \times n|_{\Gamma_{\text{hard}}} &= 0 \quad \text{"micro-free"}\,. \end{aligned} \tag{2}$$

We specify a sufficient condition for the micro-hard boundary condition, namely

$$p \times n|_{\Gamma_{\text{hard}}} = 0$$

and assume $\Gamma_{\text{hard}} = \partial\Omega$. This is the correct boundary condition for tensor fields in $H(\operatorname{Curl})$ which admits tangential traces.

We combine this with a new inequality extending Korn's inequality to incompatible tensor fields, namely

$$\begin{aligned} &\forall\, p \in H(\operatorname{Curl}) : \quad p \times n|_{\Gamma_{\text{hard}}} = 0 : \\ &\qquad \underbrace{\|p\|_{L^2(\Omega)}}_{\text{plastic distortion}} \le C(\Omega) \left( \underbrace{\|\operatorname{sym} p\|_{L^2(\Omega)}}_{\text{plastic strain}} + \underbrace{\|\operatorname{Curl} p\|_{L^2(\Omega)}}_{\text{dislocation density}} \right). \end{aligned} \tag{3}$$

Here, $\Gamma_{\text{hard}} \subset \partial\Omega$ with full two-dimensional surface measure and the domain $\Omega$ needs to be **sliceable**, i.e. cuttable into finitely many simply connected subdomains with Lipschitz boundaries. This inequality has been derived in [24, 25, 26, 23] and is precisely motivated by the well-posedness question for our model [20]. The inequality (3) expresses the fact that controlling the plastic strain $\operatorname{sym} p$ and the dislocation density $\operatorname{Curl} p$ in $L^2(\Omega)$ gives a control of the plastic distortion $p$ in $L^2(\Omega)$ provided the correct boundary conditions are specified: the micro-hard boundary condition.

It is worthy to note that with $g$ only monotone and not necessarily a subdifferential the powerful energetic solution concept [16, 9, 15] cannot be applied. In this contribution we face the combined challenge of a gradient plasticity model based on the dislocation density tensor $\operatorname{Curl} p$ involving the plastic spin, a general monotone flow-rule and a rate-dependent response.

**Setting of the problem.** Let $\Omega \subset \mathbb{R}^3$ be an open bounded set, the set of material points of the solid body, with a $C^1$-boundary. By $T_e$ we denote a positive number (time of existence), which can be chosen arbitrarily large, and for $0 < t \le T_e$

$$\Omega_t = \Omega \times (0, t).$$

The sets, $\mathcal{M}^3$ and $\mathcal{S}^3$ denote the sets of all $3\times3$–matrices and of all symmetric $3\times3$–matrices, respectively. Let $\mathfrak{sl}(3)$ be the set of all traceless $3\times3$–matrices, i.e.

$$\mathfrak{sl}(3) = \{v \in \mathcal{M}^3 \mid \operatorname{tr} v = 0\}.$$

Unknown in our small strain formulation are the displacement $u(x,t) \in \mathbb{R}^3$ of the material point $x$ at time $t$ and the non-symmetric infinitesimal plastic distortion $p(x,t) \in \mathfrak{sl}(3)$.

The model equations of the problem are

$$\begin{aligned} -\operatorname{div}_x \sigma(x,t) &= b(x,t), && (4)\\ \sigma(x,t) &= \mathbb{C}[x](\operatorname{sym}(\nabla_x u(x,t) - p(x,t))), && (5)\\ \partial_t p(x,t) &\in g\big(x, \Sigma^{\text{lin}}(x,t)\big), \quad \Sigma^{\text{lin}} = \Sigma^{\text{lin}}_{\text{e}} + \Sigma^{\text{lin}}_{\text{sh}} + \Sigma^{\text{lin}}_{\text{curl}}, && (6)\\ \Sigma^{\text{lin}}_{\text{e}} &= \sigma, \ \Sigma^{\text{lin}}_{\text{sh}} = -C_1 \operatorname{dev}\operatorname{sym} p, \quad \Sigma^{\text{lin}}_{\text{curl}} = -C_2 \operatorname{Curl}\operatorname{Curl} p\,, \end{aligned}$$

which must be satisfied in $\Omega \times [0, T_e)$. Here, $C_1, C_2 \geq 0$ are given material constants and $\Sigma^{\text{lin}}$ is the infinitesimal Eshelby stress tensor driving the evolution of the plastic distortion $p$. The initial condition and Dirichlet boundary condition are

$$\begin{aligned} p(x,0) &= p^{(0)}(x), && x \in \Omega, && (7)\\ p(x,t) \times n(x) &= 0, && (x,t) \in \partial\Omega \times [0,T_e), && (8)\\ u(x,t) &= 0, && (x,t) \in \partial\Omega \times [0,T_e)\,, && (9) \end{aligned}$$

where $n$ is a normal vector on the boundary $\partial\Omega$. For simplicity we consider only homogeneous boundary condition. The elasticity tensor $\mathbb{C}[x] : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, uniformly positive definite mapping. The mapping $x \mapsto \mathbb{C}[x] : \Omega \to \mathcal{S}^3$ is measurable. Classical linear kinematic hardening is included for $C_1 > 0$. Here, the nonlocal backstress contribution is given by the dislocation density motivated term $\Sigma^{\text{lin}}_{\text{curl}} = -C_2 \operatorname{Curl}\operatorname{Curl} p$ together with corresponding micro-hard boundary conditions.

For the model we require that the nonlinear constitutive mapping $g(x,\cdot) : \mathcal{M}^3 \to 2^{\mathfrak{sl}(3)}$ is monotone[1] for a.e. $x \in \Omega$, i.e. it satisfies

$$\begin{aligned} 0 &\in g(x,0), && (10)\\ 0 &\leq (v_1 - v_2)\cdot(v_1^* - v_2^*), && (11) \end{aligned}$$

for all $v_i \in \mathcal{M}^3$, $v_i^* \in g(x, v_i)$, $i = 1,2$, and for a.e. $x \in \Omega$. The mapping $x \mapsto g(x,\cdot) : \Omega \to 2^{\mathfrak{sl}(3)}$ is measurable (see Section 2 for the definition of the measurability of multi-valued maps). Moreover, the function $g$ has the following property

$$g(x,v) \in \mathcal{S}^3 \text{ for any } v \in \mathcal{S}^3 \text{ and a.e. } x \in \Omega.$$

Given are the volume force $b(x,t) \in \mathbb{R}^3$ and the initial datum $p^{(0)}(x) \in \mathfrak{sl}(3)$.

*Remark* 1.1. *It is well known that classical viscoplasticity (without gradient effects) gives rise to a well-posed problem. We extend this result to our formulation of rate-dependent gradient plasticity. The presence of the classical linear kinematic hardening in our model is related to $C_1 > 0$ whereas the presence of the nonlocal gradient term is always related to $C_2 > 0$.*

In the recent work by the authors [28] the existence of solutions for the initial boundary problem (4) - (9) is studied under the assumption that the monotone function $g$ is a subdifferential of a proper lower-semicontinuous convex function

[1] Here $2^{\mathfrak{sl}(3)}$ denotes the power set of $\mathfrak{sl}(3)$.

$\phi : \mathcal{M}^3 \to \bar{\mathbb{R}}$ ($\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$), i.e. $g = \partial\phi$, and with the following different boundary condition

$$\operatorname{Curl} p(x,t) \times n(x) \quad = \quad 0, \qquad\qquad (x,t) \in \partial\Omega \times [0, T_e), \tag{12}$$

instead of (8). It is required there that the function $\phi$ satisfies the following two-sided estimate

$$a_0|v|^q - b_0 \le \phi(v) \le a_1|v|^q + b_1, \tag{13}$$

for positive $a_0$ and $a_1$, some $b_0$ and $b_1$ and any $v \in \mathcal{M}^3$. Based on methods of convex analysis the existence of weak solutions (see Definition 4.6) for the problem (4) - (7), (12) and (9) with $g = \partial\phi$ is obtained in [28] under the above restrictions on the function $g$. We note that the existence result derived recently in [28] is also valid for the new problem (4) - (9), i.e. with the boundary condition (8) instead of (12) and of course the subdifferential structural assumption on $g$. In this work, assuming $\Omega \subset \mathbb{R}^3$ is a sliceable domain with a $C^1$-boundary, we show the existence of strong solutions (see Definition 4.5) for the problem (4) - (9) with the monotone function $g$ belonging to the class $\mathcal{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$ defined in Section 4. The derivation of this result is based on the inequality (3), which is recently obtained in [21, 22] under the assumption that $\Omega$ is a sliceable domain, and on the monotonicity assumption for the function $g$. We note that in the case of the sliceable domain $\Omega$ the methods used in this work allow us to show the existence of strong solutions for (4) - (9) with $g = \partial\phi$, i.e. the weak solutions for (4) - (9) with $g = \partial\phi$ derived in [28] are the strong solutions in the sense of Definition 4.5 in this case. However, we do not know how to extend our results on the existence of strong solutions to domains $\Omega$ which are not sliceable. We note as well that the existence of strong solutions for the initial boundary problem formed by equations (4) - (7), (12) and (9) with $g \in \mathcal{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$ or $g = \partial\phi$ with $\phi$ satisfying (13) for any domain $\Omega$ is an open problem too.

**Notation.** Throughout the whole work we choose the numbers $q, q^*$ satisfying the following conditions

$$1 < q, q^* < \infty \quad \text{and} \quad 1/q + 1/q^* = 1,$$

and $|\cdot|$ denotes a norm in $\mathbb{R}^k$, $k \in \mathbb{N}$. Moreover, the following notations are used in this work. The space $W^{m,q}(\Omega, \mathbb{R}^k)$ with $q \in [1, \infty]$ consists of all functions in $L^q(\Omega, \mathbb{R}^k)$ with weak derivatives in $L^q(\Omega, \mathbb{R}^k)$ up to order $m$. If $m$ is not integer, then $W^{m,q}(\Omega, \mathbb{R}^k)$ denotes the corresponding Sobolev-Slobodecki space. We set $H^m(\Omega, \mathbb{R}^k) = W^{m,2}(\Omega, \mathbb{R}^k)$. The norm in $W^{m,q}(\Omega, \mathbb{R}^k)$ is denoted by $\|\cdot\|_{m,q,\Omega}$ ($\|\cdot\|_q := \|\cdot\|_{0,q,\Omega}$). The operator $\Gamma_0$ defined by

$$\Gamma_0 : v \in W^{1,q}(\Omega, \mathbb{R}^k) \mapsto W^{1-1/q,q}(\partial\Omega, \mathbb{R}^k)$$

denotes the usual trace operator. The space $W_0^{m,q}(\Omega, \mathbb{R}^k)$ with $q \in [1, \infty]$ consists of all functions $v$ in $W^{m,q}(\Omega, \mathbb{R}^k)$ with $\Gamma_0 v = 0$. One can define the bilinear form on the product space $L^q(\Omega, \mathcal{M}^3) \times L^{q^*}(\Omega, \mathcal{M}^3)$ by

$$(\xi, \zeta)_\Omega = \int_\Omega \xi(x) \cdot \zeta(x) dx.$$

The space

$$L^q_{\mathrm{Curl}}(\Omega,\mathcal{M}^3)=\{v\in L^q(\Omega,\mathcal{M}^3)\mid \operatorname{Curl} v\in L^q(\Omega,\mathcal{M}^3)\}$$

is a Banach space with respect to the norm

$$\|v\|_{q,\mathrm{Curl}}=\|v\|_q+\|\operatorname{Curl} v\|_q.$$

By $H(\mathrm{Curl})$ we denote the space of measurable functions in $L^2_{\mathrm{Curl}}(\Omega,\mathcal{M}^3)$, i.e. $H(\mathrm{Curl})=L^2_{\mathrm{Curl}}(\Omega,\mathcal{M}^3)$. The well known result on the generalized trace operator can be easily adopted to the functions with values in $\mathcal{M}^3$ (see [35, Section II.1.2]). Then, according to this result, there is a bounded operator $\Gamma_n$ on $L^q_{\mathrm{Curl}}(\Omega,\mathcal{M}^3)$

$$\Gamma_n: v\in L^q_{\mathrm{Curl}}(\Omega,\mathcal{M}^3)\mapsto \left(W^{1-1/q^*,q^*}(\partial\Omega,\mathcal{M}^3)\right)^*$$

with

$$\Gamma_n v=v\times n\big|_{\partial\Omega} \text{ if } v\in C^1(\bar\Omega,\mathcal{M}^3),$$

where $X^*$ denotes the dual of a Banach space $X$. Next,

$$L^q_{\mathrm{Curl},0}(\Omega,\mathcal{M}^3)=\{w\in L^q_{\mathrm{Curl}}(\Omega,\mathcal{M}^3)\mid \Gamma_n(w)=0\}.$$

We also define the space $Z^q_{\mathrm{Curl}}(\Omega,\mathcal{M}^3)$ by

$$Z^q_{\mathrm{Curl}}(\Omega,\mathcal{M}^3)=\{v\in L^q_{\mathrm{Curl},0}(\Omega,\mathcal{M}^3)\mid \operatorname{Curl}\operatorname{Curl} v\in L^q(\Omega,\mathcal{M}^3)\},$$

which is a Banach space with respect to the norm

$$\|v\|_{Z^q_{\mathrm{Curl}}}=\|v\|_{V^q}+\|\operatorname{Curl}\operatorname{Curl} v\|_q.$$

For functions $v$ defined on $\Omega\times[0,\infty)$ we denote by $v(t)$ the mapping $x\mapsto v(x,t)$, which is defined on $\Omega$. The space $L^q(0,T_e;X)$ denotes the Banach space of all Bochner-measurable functions $u:[0,T_e)\to X$ such that $t\mapsto\|u(t)\|^q_X$ is integrable on $[0,T_e)$. Finally, we frequently use the spaces $W^{m,q}(0,T_e;X)$, which consist of Bochner measurable functions having $q$-integrable weak derivatives up to order $m$.

# 2 Maximal monotone operators

In this section we recall some basics about monotone and maximal monotone operators. For more details see [3, 13, 30], for example.

Let $V$ be a reflexive Banach space with the norm $\|\cdot\|$, $V^*$ be its dual space with the norm $\|\cdot\|_*$. The brackets $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $V$ and $V^*$. Under $V$ we shall always mean a reflexive Banach space throughout this section. For a multivalued mapping $A:V\to 2^{V^*}$ the sets

$$D(A)=\{v\in V\mid Av\neq\emptyset\}$$

and

$$GrA=\{[v,v^*]\in V\times V^*\mid v\in D(A),\ v^*\in Av\}$$

are called the *effective domain* and the *graph* of $A$, respectively.

**Definition 2.1.** *A mapping* $A : V \to 2^{V^*}$ *is called* monotone *if and only if the inequality holds*

$$\langle v^* - u^*, v - u\rangle \geq 0 \quad \forall\ [v, v^*], [u, u^*] \in GrA.$$

*A monotone mapping* $A : V \to 2^{V^*}$ *is called* maximal monotone *iff the inequality*

$$\langle v^* - u^*, v - u\rangle \geq 0 \quad \forall\ [u, u^*] \in GrA$$

*implies* $[v, v^*] \in GrA$.

*A mapping* $A : V \to 2^{V^*}$ *is called* generalized pseudomonotone *iff the set* $Av$ *is closed, convex and bounded for all* $v \in D(A)$ *and for every pair of sequences* $\{v_n\}$ *and* $\{v_n^*\}$ *such that* $v_n^* \in Av_n$, $v_n \rightharpoonup v_0$, $v_n^* \rightharpoonup v_0^* \in V^*$ *and*

$$\limsup_{n\to\infty} \langle v_n^*, v_n - v_0\rangle \leq 0,$$

*we have that* $[v_0, v_0^*] \in GrA$ *and* $\langle v_n^*, v_n\rangle \to \langle v_0^*, v_0\rangle$.

*A mapping* $A : V \to 2^{V^*}$ *is called* strongly coercive *iff either* $D(A)$ *is bounded or* $D(A)$ *is unbounded and the condition*

$$\frac{\langle v^*, v - w\rangle}{\|v\|} \to +\infty \quad \text{as } \|v\| \to \infty, \quad [v, v^*] \in GrA,$$

*is satisfied for each* $w \in D(A)$.

It is well known ([30, p. 105]) that if $A$ is a maximal monotone operator, then for any $v \in D(A)$ the image $Av$ is a closed convex subset of $V^*$ and the graph $GrA$ is demi-closed.[2] A maximal monotone operator is also generalized pseudomonotone (see [3, 13, 30]).

*Remark* 2.2. We recall that the subdifferential of a lower semi-continuous and convex function is maximal monotone (see [31, Theorem 2.25]).

**Definition 2.3.** *The* duality mapping $J : V \to 2^{V^*}$ *is defined by*

$$J(v) = \{v^* \in V^* \mid \langle v^*, v\rangle = \|v\|^2 = \|v^*\|_*^2\ \}$$

*for all* $v \in V$.

Without loss of generality (due to Asplund's theorem) we can assume that both $V$ and $V^*$ are strictly convex, i.e. that the unit ball in the corresponding space is strictly convex. In virtue of [3, Theorem II.1.2], the equation

$$J(v_\lambda - v) + \lambda A v_\lambda \ni 0$$

has a solution $v_\lambda \in D(A)$ for every $v \in V$ and $\lambda > 0$ if $A$ is maximal monotone. The solution is unique (see [3, p. 41]).

**Definition 2.4.** *Setting*

$$v_\lambda = j_\lambda^A v \quad \text{and} \quad A_\lambda v = -\lambda^{-1} J(v_\lambda - v)$$

*we define two single valued operators: the* Yosida approximation $A_\lambda : V \to V^*$ *and the* resolvent $j_\lambda^A : V \to D(A)$ *with* $D(A_\lambda) = D(j_\lambda^A) = V$.

[2] A set $A \in V \times V^*$ is demi-closed if $v_n$ converges strongly to $v_0$ in $V$ and $v_n^*$ converges weakly to $v_0^*$ in $V^*$ (or $v_n$ converges weakly to $v_0$ in $V$ and $v_n^*$ converges strongly to $v_0^*$ in $V^*$) and $[v_n, v_n^*] \in GrA$, then $[v, v^*] \in GrA$

By the definition, one immediately sees that $A_\lambda v \in A\big(j_\lambda^A v\big)$. For the main properties of the Yosida approximation we refer to [3, 13, 30] and mention only that both are continuous operators and that $A_\lambda$ is bounded and maximal monotone.

Next, the maximality of the sum of two maximal monotone operators is given by the following result.

**Theorem 2.5.** *Let $V$ be a reflexive Banach space, and let $A$ and $B$ be maximal. Suppose that the condition*

$$D(A) \cap \operatorname{int} D(B) \neq \emptyset$$

*is fulfilled. Then the sum $A + B$ is a maximal monotone operator.*

*Proof.* See [30, Theorem III.3.6] or [3, Theorem II.1.7]). □

For deeper results on the maximality of the sum of two maximal monotone operators we refer the reader to the book [34]. The next surjectivity result plays an important role in the existence theory for monotone operators.

**Theorem 2.6.** *If $V$ is a (strictly convex) reflexive Banach space and $A : V \to 2^{V^*}$ is maximal monotone and coercive, then $A$ is surjective.*

*Proof.* See [30, Theorem III.2.10]. □

**Measurability of multi-valued mappings.** In this subsection we present briefly some facts about measurable multi-valued mappings. We assume that $V$, and hence $V^*$, is separable and denote the set of maximal monotone operators from $V$ to $V^*$ by $\mathcal{M}(V \times V^*)$. Further, let $(S, \Sigma(S), \mu)$ be a $\sigma-$finite $\mu-$complete measurable space.

**Definition 2.7.** *A function $A : S \to \mathcal{M}(V \times V^*)$ is measurable iff for every open set $U \in V \times V^*$ (respectively closed set, Borel set, open ball, closed ball),*

$$\{x \in S \mid A(x) \cap U \neq \emptyset\}$$

*is measurable in $S$.*

The next result states that the notion of measurability for maximal monotone mappins can be equivalently defined in terms of the measurability for appropriate single-valued mappings.

**Proposition 2.8.** *Let $A : S \to \mathcal{M}(V \times V^*)$, let $\lambda > 0$ and let $E$ be dense in $V$. The following are equivalent:*

*(a) $A$ is measurable;*

*(b) for every $v \in E$, $x \mapsto j_\lambda^{A(x)} v$ is measurable;*

*(c) $v \in E$, $x \mapsto A_\lambda(x) v$ is measurable.*

*Proof.* See [7, Proposition 2.11]. □

For further reading on measurable multi-valued mappings we refer the reader to [6, 13, 29].

**Canonical extensions of maximal monotone operators.** Given a mapping $A : S \to \mathcal{M}(V \times V^*)$, one can define a monotone graph from $L^p(S, V)$ to $L^q(S, V^*)$, where $1/p + 1/q = 1$, as follows:

**Definition 2.9.** *Let $A : S \to \mathcal{M}(V \times V^*)$, the canonical extension of $A$ from $L^p(S, V)$ to $L^q(S, V^*)$, where $1/p + 1/q = 1$, is defined by:*

$$Gr\mathcal{A} = \{[v, v^*] \in L^p(S, V) \times L^q(S, V^*) \mid [v(x), v^*(x)] \in GrA(x) \; for \; a.e. \; x \in S\}.$$

Monotonicity of $\mathcal{A}$ defined in Definition 2.9 is obvious, while its maximality follows from the next proposition.

**Proposition 2.10.** *Let $A : S \to \mathcal{M}(V \times V^*)$ be measurable. If $Gr\mathcal{A} \neq \emptyset$, then $\mathcal{A}$ is maximal monotone.*

*Proof.* See [7, Proposition 2.13]. □

We have to point out here that the maximality of $A(x)$ for almost every $x \in S$ does not imply the maximality of $\mathcal{A}$ as the latter can be empty ([7]): $S = (0, 1)$, and $GrA(x) = \{[v, v^*] \in \mathbb{R}^m \times \mathbb{R}^m \mid v^* = t^{-1/q}\}$.

# 3 Some properties of the Curl Curl-operator

In this section we present some results concerning the Curl Curl-operator, which are relevant to the further investigations. For the Curl Curl-operator with a slightly different domain of definition similar results are obtained in [28, Section 4]. Here we adopt the results in [28] to our purposes.

**Lemma 3.1** (Self-adjointness of Curl Curl)**.** *Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a Lipschitz boundary and $A : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3)$ be the linear operator defined by*

$$Av = \operatorname{Curl}\operatorname{Curl} v$$

*with* $\operatorname{dom}(A) = Z^2_{\operatorname{Curl}}(\Omega, \mathcal{M}^3)$. *The operator $A$ is selfadjoint and non-negative.*

*Proof.* Indeed, let us consider first the following linear closed operator $S : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3)$ defined by

$$Sv = \operatorname{Curl} v, \qquad v \in \operatorname{dom}(S) = L^2_{\operatorname{Curl},0}(\Omega, \mathcal{M}^3).$$

It is easily seen that its adjoint is given by

$$S^*v = \operatorname{Curl} v, \qquad v \in \operatorname{dom}(S^*) = L^2_{\operatorname{Curl}}(\Omega, \mathcal{M}^3).$$

Then, by Theorem V.3.24 in [14], the operator $S^*S$ with

$$\operatorname{dom}(S^*S) = \{v \in \operatorname{dom}(S) \mid Sv \in \operatorname{dom}(S^*)\},$$

which is exactly the operator $A$, is self-adjoint in $L^2(\Omega, \mathcal{M}^3)$. The non-negativity of $A$ follows from its representation by the operator $S$, i.e. $A = S^*S$, and the identity

$$(Av, u)_\Omega = (S^*Sv, u)_\Omega = (Sv, Su)_\Omega,$$

which holds for all $v \in \operatorname{dom}(A)$ and $u \in \operatorname{dom}(S)$. □

**Corollary 3.2.** *The operator* $A : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3)$ *defined in Lemma 3.1 is maximal monotone.*

*Proof.* According to the result of Brezis (see [5, Theorem 1]), a linear monotone operator $A$ is maximal monotone, if it is a densely defined closed operator such that its adjoint $A^*$ is monotone. The statement of the corollary follows then directly from Lemma 3.1 and the mentioned result of Brezis. □

**Boundary value problems.** Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a Lipschitz boundary. For every $v \in L^2(\Omega, \mathcal{M}^3)$ we define a functional $\Psi$ on $L^2(\Omega, \mathcal{M}^3)$ by

$$\Psi(v) = \begin{cases} \frac{1}{2}\int_\Omega |\operatorname{Curl} v(x)|^2 dx, & v \in L^2_{\operatorname{Curl},0}(\Omega, \mathcal{M}^3) \\ +\infty, & \text{otherwise} \end{cases}.$$

It is easy to check that $\Psi$ is proper, convex, lower semi-continuous. The next lemma gives a precise description of the subdifferential $\partial\Psi$.

**Lemma 3.3.** *We have that* $\partial\Psi = \operatorname{Curl}\operatorname{Curl}$ *with*

$$\operatorname{dom}(\partial\Psi) = Z^2_{\operatorname{Curl}}(\Omega, \mathcal{M}^3).$$

*Proof.* Let $A : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3)$ be the linear operator defined by

$$Av = \operatorname{Curl}\operatorname{Curl} v$$

and $\operatorname{dom}(A) = Z^2_{\operatorname{Curl}}(\Omega, \mathcal{M}^3)$. Due to Lemma 3.1, the following identity

$$\int_\Omega \operatorname{Curl}\operatorname{Curl} v(x) \cdot w(x) dx = \int_\Omega \operatorname{Curl} v(x) \cdot \operatorname{Curl} w(x) dx \tag{14}$$

holds for any $v, w \in Z^2_{\operatorname{Curl}}(\Omega, \mathcal{M}^3)$. Therefore, using (14) we obtain

$$\int_\Omega \operatorname{Curl}\operatorname{Curl} v \cdot (w - v) dx = \int_\Omega \operatorname{Curl} v \cdot (\operatorname{Curl} w - \operatorname{Curl} v) dx \le \Psi(w) - \Psi(v)$$

for every $v, w \in \operatorname{dom}(A)$. This shows that $A \subset \partial\Psi$. Since $A$ is maximal monotone (see Corollary 3.2) we conclude that $A = \partial\Psi$. □

# 4 Existence of strong solutions

In this section we prove the main existence result for (4) - (9). To show the existence of weak solutions a time-discretization method is used in this work. In the first step, we prove the existence of the solutions of the time-discretized problem in appropriate Hilbert spaces based on the Helmholtz projection in $L^2(\Omega, \mathcal{S}^3)$ (Section 4) and the monotone operator methods (Section 2). In the second step, we derive the uniform a priori estimates for the solutions of the time-discretized problem using the polynomial growth of the function $g$ (see Definition 4.1 below) and then we pass to the weak limit in the equivalent formulation of the time-discretized problem employing the weak lower semi-continuity of lower semi-continuous convex functions and the maximal monotonicity of $g$.

**Main result.** First, we define a class of maximal monotone functions we deal with in this work.

**Definition 4.1.** *For $m \in L^1(\Omega, \mathbb{R})$, $\alpha \in \mathbb{R}_+$ and $q > 1$, $\mathcal{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$ is the set of multi-valued functions $h : \Omega \times \mathbb{R}^k \to 2^{\mathbb{R}^k}$ with the following properties*

- *$v \mapsto h(x, v)$ is maximal monotone for almost all $x \in \Omega$,*
- *the mapping $x \mapsto j_\lambda(x, v) : \Omega \to \mathbb{R}^k$ is measurable for all $\lambda > 0$, where $j_\lambda(x, v)$ is the inverse of $v \mapsto v + \lambda h(x, v)$,*
- *for a.e. $x \in \Omega$ and every $v^* \in h(x, v)$*

$$\alpha \left( \frac{|v|^q}{q} + \frac{|v^*|^{q^*}}{q^*} \right) \leq (v, v^*) + m(x), \tag{15}$$

  *where $1/q + 1/q^* = 1$.*

*Remark* 4.2. We note that the condition (15) is equivalent to the following two inequalities

$$|v^*|^{q^*} \leq m_1(x) + \alpha_1 |v|^q, \tag{16}$$

$$(v, v^*) \geq m_2(x) + \alpha_2 |v|^q, \tag{17}$$

for a.e. $x \in \Omega$ and every $v^* \in h(x, v)$ and with suitable functions $m_1, m_2 \in L^1(\Omega, \mathbb{R})$ and numbers $\alpha_1, \alpha_2 \in \mathbb{R}_+$.

*Remark* 4.3. Visco-plasticity is typically included in the former conditions by choosing the function $g$ to be in Norton-Hoff form, i.e.

$$g(\Sigma) = [|\Sigma| - \sigma_{\mathrm{y}}]_+^r \frac{\Sigma}{|\Sigma|}, \quad \Sigma \in \mathcal{M}^3,$$

where $\sigma_{\mathrm{y}}$ is the flow stress and $r$ is some parameter together with $[x]_+ := \max(x, 0)$. If $g : \mathcal{M}^3 \mapsto \mathcal{S}^3$ then the flow is called irrotational (no plastic spin).

The main properties of the class $\mathcal{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$ are collected in the following proposition.

**Proposition 4.4.** *Let $\mathcal{H}$ be a canonical extension of a function $h : \mathbb{R}^k \to 2^{\mathbb{R}^k}$, which belongs to $\mathcal{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$. Then $\mathcal{H}$ is maximal monotone, surjective and $D(\mathcal{H}) = L^p(\Omega, \mathbb{R}^k)$.*

*Proof.* See Corollary 2.15 in [7]. □

Next, we define the following notion of solutions for the initial boundary value problem (4) - (9).

**Definition 4.5. (Strong solutions)** *A function $(u, \sigma, p)$ such that*

$$(u, \sigma) \in H^1(0, T_e; H_0^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3)), \quad \Sigma^{\mathrm{lin}} \in L^q(\Omega_{T_e}, \mathcal{M}^3),$$

$$p \in H^1(0, T_e; L^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)) \cap L^2(0, T_e; Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3))$$

*is called a* strong solution *of the initial boundary value problem (4) - (9), if for every $t \in [0, T_e]$ the function $(u(t), \sigma(t))$ is a weak solution of the boundary value problem (71) - (73) with $\hat{\varepsilon}_p = \operatorname{sym} p(t)$ and $\hat{b} = b(t)$, the evolution inclusion (6) and the initial condition (7) are satisfied pointwise.*

For the reader's convenience we give here also the definition of weak solutions for the problem (4) - (9) in the case when the monotone function $g$ is a subdifferential of a proper lower-semicontinuous convex function $\phi$, i.e. $g = \partial\phi$.

**Definition 4.6. (Weak solutions)** *A function* $(u, \sigma, p)$ *such that*

$$(u,\sigma) \in W^{1,q^*}(0,T_e; W_0^{1,q^*}(\Omega,\mathbb{R}^3) \times L^{q^*}(\Omega,\mathcal{S}^3)), \quad \Sigma^{\text{lin}} \in L^q(\Omega_{T_e},\mathcal{M}^3),$$

$$p \in W^{1,q^*}(0,T_e; L^{q^*}(\Omega,\mathcal{M}^3)) \cap L^{q^*}(0,T_e; Z^{q^*}_{\text{Curl}}(\Omega,\mathcal{M}^3))$$

*with*

$$(\sigma, \operatorname{dev}\operatorname{sym} p, \operatorname{Curl} p) \in L^\infty(0,T_e; L^2(\Omega, \mathcal{S}^3 \times \mathcal{M}^3 \times \mathcal{M}^3))$$

*is called a* weak solution *of the initial boundary value problem (4) - (9), if for every* $t \in [0,T_e]$ *the function* $(u(t),\sigma(t))$ *is a weak solution of the boundary value problem (71) - (73) with* $\hat{\varepsilon}_p = \operatorname{sym} p(t)$ *and* $\hat{b} = b(t)$*, the initial condition (7) is satisfied and the inequality*[3]

$$\frac{1}{2}\int_\Omega \mathbb{C}^{-1}[x]\sigma(x,t)\cdot\sigma(x,t)dx + C_1\|\operatorname{dev}\operatorname{sym} p(t)\|_2^2 + C_2\|\operatorname{Curl} p(t)\|_2^2$$

$$+\int_0^t\int_\Omega \big(\phi^*(x,\partial_s p(x,s)) + \phi(x,\Sigma^{\text{lin}}(x,s))\big)\,dxds \le \int_0^t (b(s),\partial_s u(s))_\Omega\,ds$$

$$+\frac{1}{2}\int_\Omega \mathbb{C}^{-1}[x]\sigma^{(0)}(x)\cdot\sigma^{(0)}(x)dx + C_1\|\operatorname{dev}\operatorname{sym} p^{(0)}\|_2^2 + C_2\|\operatorname{Curl} p^{(0)}\|_2^2$$

*holds for all* $t \in (0,T_e)$ *and with the function* $\sigma^{(0)} \in L^2(\Omega,\mathcal{S}^3)$ *determined by equations (71) - (73) for* $\hat{\varepsilon}_p = \operatorname{sym} p^{(0)}$ *and* $\hat{b} = b(0)$*.*

In our previous paper [28] it is shown that under some additional regularity the weak solutions of the problem (4) - (9) with $g = \partial\phi$ become strong solutions of (4) - (9) in the sense of Definition 4.5.

Next, we state the main result of this work.

**Theorem 4.7.** *Suppose that* $1 < q^* \le 2 \le q < \infty$*. Assume that* $\Omega \subset \mathbb{R}^3$ *is a sliceable domain with a* $C^1$*-boundary and* $\mathbb{C} \in L^\infty(\Omega,\mathcal{S}^3)$*. Let the functions* $b \in W^{1,q}(0,T_e; L^q(\Omega,\mathbb{R}^3))$ *and* $p^{(0)} \in Z^q_{\text{Curl}}(\Omega,\mathcal{M}^3)$ *be given. Assume that* $g \in \mathcal{M}(\Omega,\mathcal{M}^3,q,\alpha,m)$*. Then there exists a solution* $(u,\sigma,p)$ *of the initial boundary value problem (4) - (9).*

**The proof of Theorem 4.7.** In order to deal with the measurable elasticity tensor $\mathbb{C}$, we reformulate the problem (4) - (9) as follows:
Let the function $(\hat{v},\hat{\sigma}) \in W^{1,q}(0,T_e, W_0^{1,q}(\Omega,\mathbb{R}^3) \times L^q(\Omega,\mathcal{S}^3))$ be a solution of the linear elasticity problem formed by the equations

$$
\begin{aligned}
-\operatorname{div}_x \hat{\sigma}(x,t) &= b(x,t), && x \in \Omega, && (18)\\
\hat{\sigma}(x,t) &= \hat{\mathbb{C}}(\operatorname{sym}(\nabla_x \hat{v}(x,t)), && x \in \Omega, && (19)\\
\hat{v}(x,t) &= 0, && x \in \partial\Omega, && (20)
\end{aligned}
$$

[3]Here $\phi^*$ is the Legendre-Fenchel conjugate of $\phi$.

where $\hat{\mathbb{C}} : \mathcal{S}^3 \to \mathcal{S}^3$ is any positive definite linear mapping independent of $(x,t)$. Such a function $(\hat{v},\hat{\sigma})$ exists (see Section 4). Then the solution $(u,\sigma,p)$ of the initial boundary value problem (4) - (9) has the following form

$$(u,\sigma,p) = (\tilde{v}+\hat{v}, \tilde{\sigma}+\hat{\sigma}, p),$$

where the function $(\tilde{v},\tilde{\sigma},p)$ solves the problem

$$\begin{aligned}
-\operatorname{div}_x \tilde{\sigma}(x,t) &= 0, && (21)\\
\tilde{\sigma}(x,t) &= \mathbb{C}[x](\operatorname{sym}(\nabla_x \tilde{v}(x,t) - p(x,t))) && (22)\\
&\qquad + (\mathbb{C}[x]-\hat{\mathbb{C}})(\operatorname{sym}(\nabla_x \hat{v}(x,t)), &&\\
\partial_t p(x,t) &\in g\big(x, \Sigma^{\text{lin}}(x,t)\big), \quad \Sigma^{\text{lin}} = \Sigma^{\text{lin}}_e + \Sigma^{\text{lin}}_{\text{sh}} + \Sigma^{\text{lin}}_{\text{curl}}, && (23)\\
\Sigma^{\text{lin}}_{\text{e}} &= \tilde{\sigma}+\hat{\sigma}, \ \Sigma^{\text{lin}}_{\text{sh}} = -C_1 \operatorname{dev}\operatorname{sym} p, \quad \Sigma^{\text{lin}}_{\text{curl}} = -C_2 \operatorname{Curl}\operatorname{Curl} p\,, &&\\
p(x,0) &= p^{(0)}(x), \qquad x\in\Omega, && (24)\\
p(x,t)\times n(x) &= 0, \qquad (x,t)\in\partial\Omega\times[0,T_e), && (25)\\
\tilde{v}(x,t) &= 0, \qquad (x,t)\in\partial\Omega\times[0,T_e)\,. && (26)
\end{aligned}$$

Here, the function $(\hat{v},\hat{\sigma})$ given as the solution of (18) - (20) is considered as known. Next, we show that the problem (21) - (26) has a solution. This will prove the existence of solutions for (4) - (9).

*Proof.* We show the existence of solutions using the Rothe method (a time-discretization method, see [33] for details). In order to introduce a time-discretized problem, let us fix any $m\in\mathbb{N}$ and set

$$h := \frac{T_e}{2^m},\ p^0_m := p^{(0)},\ \hat{\sigma}^n_m := \frac{1}{h}\int_{(n-1)h}^{nh} \hat{\sigma}(s)ds \in L^q(\Omega,\mathbb{R}^3), \quad n=1,...,2^m.$$

Then we are looking for functions $u^n_m \in H^1(\Omega,\mathbb{R}^3)$, $\sigma^n_m \in L^2(\Omega,\mathcal{S}^3)$ and $p^n_m \in Z^2_{\text{Curl}}(\Omega,\mathcal{M}^3)$ with

$$\Sigma^{\text{lin}}_{n,m} := \sigma^n_m + \hat{\sigma}^n_m - C_1 \operatorname{dev}\operatorname{sym} p^n_m - \frac{1}{m} p^n_m - C_2 \operatorname{Curl}\operatorname{Curl} p^n_m \in L^q(\Omega,\mathcal{M}^3)$$

solving the following problem

$$\begin{aligned}
-\operatorname{div}_x \sigma^n_m(x) &= 0, && (27)\\
\sigma^n_m(x) &= \mathbb{C}[x](\operatorname{sym}(\nabla_x u^n_m(x) - p^n_m(x))) && (28)\\
&\quad + (\mathbb{C}[x]-\hat{\mathbb{C}})(\hat{\mathbb{C}})^{-1}\hat{\sigma}^n_m(x), &&\\
\frac{p^n_m(x)-p^{n-1}_m(x)}{h} &\in g\big(\Sigma^{\text{lin}}_{n,m}(x)\big), && (29)
\end{aligned}$$

together with the boundary conditions

$$\begin{aligned}
p^n_m(x)\times n(x) &= 0, \qquad x\in\partial\Omega, && (30)\\
u^n_m(x) &= 0, \quad x\in\partial\Omega\,. && (31)
\end{aligned}$$

Next, we adopt the reduction technique proposed in [1] to the above equations. Let $(u^n_m, \sigma^n_m, p^n_m)$ be a solution of the boundary value problem (27) - (31). The equations (27) - (28), (31) form a boundary value problem for the solution $(u^n_m, \sigma^n_m)$ of the problem of linear elasticity. Due to linearity of this problem we can write these components of the solution in the form

$$(u^n_m, \sigma^n_m) = (\tilde{u}^n_m, \tilde{\sigma}^n_m) + (\bar{v}^n_m, \bar{\sigma}^n_m),$$

with the solution $(\bar{v}^n_m, \bar{\sigma}^n_m)$ of the Dirichlet boundary value problem (71) - (73) to the data $\hat{b} = 0$, $\hat{\varepsilon}_p = (\mathbb{C} - \hat{\mathbb{C}})(\hat{\mathbb{C}})^{-1}\hat{\sigma}^n_m$, and with the solution $(\tilde{u}^n_m, \tilde{\sigma}^n_m)$ of the problem (71) - (73) to the data $\hat{b} = 0$, $\hat{\varepsilon}_p = \mathrm{sym}(p^n_m)$. We thus obtain

$$\mathrm{sym}(\nabla_x u^n_m) - \mathrm{sym}(p^n_m) = (P_2 - I)\mathrm{sym}(p^n_m) + \mathrm{sym}(\nabla_x \bar{v}^n_m).$$

where the operator $P_2$ is defined in Definition 4.8. We insert this equation into (28) and get that (29) can be rewritten in the following form

$$\frac{p^n_m - p^{n-1}_m}{h} \in \mathcal{G}\big( - M_m p^n_m - C_2 \,\mathrm{Curl}\,\mathrm{Curl}\, p^n_m + (\hat{\sigma}^n_m + \bar{\sigma}^n_m)\big), \qquad (32)$$

$$p^n_m(x) \times n(x) = 0, \qquad x \in \partial\Omega, \qquad (33)$$

where

$$M_m := (\mathbb{C}Q_2 + L)\,\mathrm{sym} + \frac{1}{m} I : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3)$$

with the Helmholtz projection $Q_2$ and the operator $L$ defined by (74). Here $\mathcal{G}$ denotes the canonical extension of $g$. Next, the problem (32) - (33) reads

$$\Psi(p^n_m) \ni \hat{\sigma}^n_m + \bar{\sigma}^n_m, \qquad (34)$$

where

$$\Psi(v) = \mathcal{G}^{-1}\Big(\frac{v - p^{n-1}_m}{h}\Big) + M_m(v) + \partial\Phi(v).$$

Here, the functional $\Phi : L^2(\Omega, \mathcal{M}^3) \to \bar{\mathbb{R}}$ is given by

$$\Phi(v) := \begin{cases} \frac{1}{2}\int_\Omega |\,\mathrm{Curl}\, v(x)|^2 dx, & v \in L^2_{\mathrm{Curl},0}(\Omega, \mathcal{M}^3) \\ +\infty, & \text{otherwise} \end{cases},$$

respectively. The facts that $\Phi$ is a proper convex lower semi-continuous functional and that $\mathrm{Curl}\,\mathrm{Curl} = \partial\Phi$ are proved in Section 3. Since $M_m$ is bounded, self-adjoint and positive definite (see Corollary 4.10 and the definition of $M_m$), it is maximal monotone by Theorem II.1.3 in [3]. The last thing which we have to verify is whether the following operator

$$\Psi = \mathcal{G}^{-1} + M_m + \partial\Phi$$

is maximal monotone. Since $g \in \mathcal{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$, using the boundness of $M_m$ we conclude that the domains of $\mathcal{G}^{-1}$ and $M_m$ are equal to the whole space $L^2(\Omega, \mathcal{M}^3)$. Therefore, Theorem 2.5 guarantees that the sum $\mathcal{G}^{-1} + M_m + \partial\Phi$ is maximal monotone with

$$\mathrm{dom}(\Psi) = \mathrm{dom}(\partial\Phi) := Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3).$$

Since $M_m$ is coercive in $L^2(\Omega, \mathcal{M}^3)$, which obviously yields the coercivity of $\Psi$, the operator $\Psi$ is surjective by Theorem 2.6. Thus, we conclude that equation (34) as well as the problem (32) - (33) have the solutions with the required regularity, i.e. $p_m^n \in Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$. By the constructions this implies that the boundary value problem (27) - (31) is solvable as well (for more details we refer the reader to [1]).

**Rothe approximation functions:** For any family $\{\xi_m^n\}_{n=0,\dots,m}$ of functions in a reflexive Banach space $X$, we define *the piecewise affine interpolant* $\xi_m \in C([0, T_e], X)$ by

$$\xi_m(t) := \left(\frac{t}{h} - (n-1)\right) \xi_m^n + \left(n - \frac{t}{h}\right) \xi_m^{n-1} \quad \text{for } (n-1)h \le t \le nh \tag{35}$$

and *the piecewise constant interpolant* $\bar{\xi}_m \in L^\infty(0, T_e; X)$ by

$$\bar{\xi}_m(t) := \xi_m^n \text{ for } (n-1)h < t \le nh,\ n = 1, ..., 2^m, \text{ and } \bar{\xi}_m(0) := \xi_m^0. \tag{36}$$

For the further analysis we recall the following property of $\bar{\xi}_m$ and $\xi_m$:

$$\|\xi_m\|_{L^q(0,T_e;X)} \le \|\bar{\xi}_m\|_{L^q(-h,T_e;X)} \le \left(h\|\xi_m^0\|_X^q + \|\bar{\xi}_m\|^q_{L^q(0,T_e;X)}\right)^{1/q}, \tag{37}$$

where $\bar{\xi}_m$ is formally extended to $t \le 0$ by $\xi_m^0$ and $1 \le q \le \infty$ (see [33]).

**A-priori estimates.** Multiplying (27) by $(u_m^n - u_m^{n-1})/h$ and then integrating over $\Omega$ we get

$$\left(\sigma_m^n, \mathrm{sym}(\nabla_x(u_m^n - u_m^{n-1})/h)\right)_\Omega = 0.$$

Equations (28), (29) implies that for a.e. $x \in \Omega$

$$\sigma_m^n \cdot \Big(\mathrm{sym}(\nabla_x(u_m^n - u_m^{n-1})/h) - \mathbb{C}^{-1}[x](\sigma_m^n - \sigma_m^{n-1})/h\Big)$$

$$\sigma_m^n \cdot \Big(\mathbb{C}^{-1}[x](\mathbb{C}[x] - \hat{\mathbb{C}})(\hat{\mathbb{C}})^{-1}(\hat{\sigma}_m^n - \hat{\sigma}_m^{n-1})/h\Big)$$

$$-\frac{p_m^n - p_m^{n-1}}{h} \cdot \Big(C_1 \,\mathrm{dev\,sym}\, p_m^n + \frac{1}{m} p_m^n + C_2 \,\mathrm{Curl\,Curl}\, p_m^n\Big)$$

$$+\frac{p_m^n - p_m^{n-1}}{h} \cdot \hat{\sigma}_m^n = g^{-1}\Big(\frac{p_m^n - p_m^{n-1}}{h}\Big) \cdot \Big(\frac{p_m^n - p_m^{n-1}}{h}\Big).$$

After integrating the last identity over $\Omega$, the above computations imply

$$\frac{1}{h}\Big(\mathbb{C}^{-1}\sigma_m^n, \sigma_m^n - \sigma_m^{n-1}\Big)_\Omega + C_1\frac{1}{h}\Big(\mathrm{dev\,sym}(p_m^n - p_m^{n-1}), \mathrm{dev\,sym}\, p_m^n\Big)_\Omega$$

$$+\frac{1}{m}\frac{1}{h}\Big(p_m^n - p_m^{n-1}, p_m^n\Big)_\Omega + C_2\frac{1}{h}\Big(\mathrm{Curl}(p_m^n - p_m^{n-1}), \mathrm{Curl}\, p_m^n\Big)_\Omega$$

$$+\alpha\left\|\frac{p_m^n - p_m^{n-1}}{h}\right\|_\Omega^{q^*} \le \frac{1}{h}\left(\sigma_m^n, \bar{\mathbb{C}}(\hat{\sigma}_m^n - \hat{\sigma}_m^{n-1})\right)_\Omega + \frac{1}{h}\left(\hat{\sigma}_m^n, p_m^n - p_m^{n-1}\right)_\Omega,$$

where $\bar{\mathbb{C}} := \mathbb{C}^{-1}(\mathbb{C} - \hat{\mathbb{C}})(\hat{\mathbb{C}})^{-1}$. Multiplying by $h$ and summing the obtained relation for $n = 1, ..., l$ for any fixed $l \in [1, 2^m]$ we derive the following inequality (here $\mathbb{B} := \mathbb{C}^{-1}$)

$$\frac{1}{2}\Big(\|\mathbb{B}^{1/2}\sigma_m^l\|_2^2 + C_1\|\operatorname{dev}\operatorname{sym} p_m^l\|_2^2 + \frac{1}{m}\|p_m^l\|_2^2 + C_2\|\operatorname{Curl} p_m^l\|_2^2\Big)$$
$$+h\alpha\sum_{n=1}^{l}\left\|\frac{p_m^n - p_m^{n-1}}{h}\right\|_\Omega^{q^*} \le C^{(0)} \tag{38}$$
$$+h\sum_{n=1}^{l}\left(\sigma_m^n, \bar{\mathbb{C}}\frac{\hat\sigma_m^n - \hat\sigma_m^{n-1}}{h}\right)_\Omega + h\sum_{n=1}^{l}\left(\hat\sigma_m^n, \frac{p_m^n - p_m^{n-1}}{h}\right)_\Omega,$$

where[4]

$$2C^{(0)} := \|\mathbb{B}^{1/2}\sigma^{(0)}\|_2^2 + C_1\|\operatorname{dev}\operatorname{sym} p^{(0)}\|_2^2 + \frac{1}{m}\|p^{(0)}\|_2^2 + C_2\|\operatorname{Curl} p^{(0)}\|_2^2.$$

Since $\hat\sigma_m^n \in L^q(\Omega, \mathcal{S}^3)$, using Young's inequality with $\epsilon > 0$ we get that

$$\left(\hat\sigma_m^n, \frac{p_m^n - p_m^{n-1}}{h}\right)_\Omega \le \|\hat\sigma_m^n\|_q \|(p_m^n - p_m^{n-1})/h\|_{q^*}$$
$$\le C_\epsilon\|\hat\sigma_m^n\|_q^q + \epsilon\|(p_m^n - p_m^{n-1})/h\|_{q^*}^{q^*}, \tag{39}$$

where $C_\epsilon$ is a positive constant appearing in the Young inequality. Analogically, we obtain

$$\left(\sigma_m^n, \bar{\mathbb{C}}\frac{\hat\sigma_m^n - \hat\sigma_m^{n-1}}{h}\right)_\Omega \le \epsilon\|\sigma_m^n\|_2^2 + C_\epsilon\|(\hat\sigma_m^n - \hat\sigma_m^{n-1})/h\|_2^2 \tag{40}$$

with some other constant $C_\epsilon$. Combining the inequalities (38), (39) and (40), and choosing an appropriate value for $\epsilon > 0$ we obtain the following estimate

$$\frac{1}{2}\Big(\|\mathbb{B}^{1/2}\sigma_m^l\|_2^2 + C_1\|\operatorname{dev}\operatorname{sym} p_m^l\|_2^2 + \frac{1}{m}\|p_m^l\|_2^2 + C_2\|\operatorname{Curl} p_m^l\|_2^2\Big)$$
$$+h\hat{C}_\epsilon\sum_{n=1}^{l}\left\|\frac{p_m^n - p_m^{n-1}}{h}\right\|_\Omega^{q^*} \le C^{(0)} + h\epsilon\sum_{n=1}^{l}\|\sigma_m^n\|_2^2 \tag{41}$$
$$+h\tilde{C}_\epsilon\sum_{n=1}^{l}\Big(\|\hat\sigma_m^n\|_q^q + \|(\hat\sigma_m^n - \hat\sigma_m^{n-1})/h\|_2^2\Big),$$

where $\tilde{C}, \tilde{C}_\epsilon$ and $\hat{C}_\epsilon$ are some positive constants. Now, taking Remark 8.15 in [33] and the definition of Rothe's approximation functions into account we

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[4] Here we use the following inequality

$$\sum_{n=1}^{l}(\phi_m^n - \phi_m^{n-1}, \phi_m^n)_\Omega = \frac{1}{2}\sum_{n=1}^{l}\Big(\|\phi_m^n\|_2^2 - \|\phi_m^{n-1}\|_2^2\Big)$$
$$+\frac{1}{2}\sum_{n=1}^{l}\|\phi_m^n - \phi_m^{n-1}\|_2^2 \ge \frac{1}{2}\|\phi_m^l\|_2^2 - \frac{1}{2}\|\phi_m^0\|_2^2$$

for any family of functions $\phi_m^0, \phi_m^1, ..., \phi_m^m$.

rewrite (41) as follows

$$
\begin{aligned}
&\|\mathbb{B}^{1/2}\bar{\sigma}_m(t)\|_2^2 + C_1\|\operatorname{dev}\operatorname{sym}\bar{p}_m(t)\|_2^2 + \frac{1}{m}\|\bar{p}_m(t)\|_2^2 + C_2\|\operatorname{Curl}\bar{p}_m(t)\|_2^2 \\
&+2\hat{C}_\epsilon\int_0^{T_e}\int_\Omega \big|\partial_t p_m(x,t)\big|^{q^*}dxdt \qquad (42)\\
&\le 2C^{(0)} + \epsilon\|\sigma_m\|^2_{2,\Omega\times(0,t)} + 2\tilde{C}_\epsilon\|\hat{\sigma}\|^q_{W^{1,q}(0,T_e;L^q(\Omega,\mathcal{S}^3))}.
\end{aligned}
$$

From (42) we get immediately that

$$
\begin{aligned}
&\bar{C}_\epsilon\|\sigma_m\|^2_{2,\Omega\times(0,t)} + C_1\|\operatorname{dev}\operatorname{sym}\bar{p}_m(t)\|_2^2 + \frac{1}{m}\|\bar{p}_m(t)\|_2^2 + C_2\|\operatorname{Curl}\bar{p}_m(t)\|_2^2 \\
&+2\hat{C}_\epsilon\|\partial_t p_m\|^{q^*}_{q^*,\Omega\times(0,t)} \le 2C^{(0)} + 2\tilde{C}_\epsilon\|\hat{\sigma}\|^q_{W^{1,q}(0,T_e;L^q(\Omega,\mathcal{S}^3))}, \qquad (43)
\end{aligned}
$$

where $\bar{C}_\epsilon$ is some other constant depending on $\epsilon$. Altogether, from estimate (43) we get that

$$\{p_m\}_m \text{ is uniformly bounded in } W^{1,q^*}(0,T_e;L^{q^*}(\Omega,\mathcal{M}^3)), \qquad (44)$$

$$\{\operatorname{dev}\operatorname{sym}\bar{p}_m\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{M}^3)), \qquad (45)$$

$$\{\sigma_m\}_m, \text{ is uniformly bounded in } L^2(0,T_e;L^2(\Omega,\mathcal{S}^3)), \qquad (46)$$

$$\{\operatorname{Curl}\bar{p}_m\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{M}^3)), \qquad (47)$$

$$\left\{\frac{1}{\sqrt{m}}\bar{p}_m\right\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{M}^3)). \qquad (48)$$

In particular, the uniform boundedness of the sequences in (44) - (51) yields

$$\{u_m\}_m \text{ is uniformly bounded in } W^{1,q^*}(0,T_e;W_0^{1,q^*}(\Omega,\mathbb{R}^3)), \qquad (49)$$

$$\{\operatorname{Curl}\operatorname{Curl}\bar{p}_m\}_m \text{ is uniformly bounded in } L^2(0,T_e;L^2(\Omega,\mathcal{M}^3)), \qquad (50)$$

$$\left\{\bar{\Sigma}_m^{\text{lin}}\right\}_m \text{ is uniformly bounded in } L^q(0,T_e;L^q(\Omega,\mathcal{M}^3)). \qquad (51)$$

Employing (37) the estimates (44) - (51) further imply that the sequences $\{\sigma_m\}_m$, $\{\operatorname{dev}\operatorname{sym}p_m\}_m$, $\{\operatorname{Curl}p_m\}_m$, $\{p_m/\sqrt{m}\}_m$, $\left\{\Sigma_m^{\text{lin}}\right\}_m$ and $\{\operatorname{Curl}\operatorname{Curl}p_m\}_m$ are also uniformly bounded in the corresponding spaces. As a result, we have

$$\{p_m\}_m \text{ is uniformly bounded in } L^{q^*}(0,T_e;Z^{q^*}_{\operatorname{Curl}}(\Omega,\mathcal{M}^3)). \qquad (52)$$

Furthermore, due to inequality (3) and (45), (47) and (50) we obtain that

$$\{\bar{p}_m\}_m, \{p_m\}_m \text{ are uniformly bounded in } L^2(0,T_e;Z^2_{\operatorname{Curl}}(\Omega,\mathcal{M}^3)). \qquad (53)$$

**Additional regularity of discrete solutions.** To this end, let us assume that

$$p^{(0)} \in Z^q_{\operatorname{Curl}}(\Omega,\mathcal{M}^3).$$

Then, in virtue of our uniform estimates, the condition $g \in \mathcal{M}(\Omega,\mathcal{M}^3,q,\alpha,m)$ and the regularity of $p^{(0)}$ one easily gets that

$$g(\Sigma^{\text{lin}}_{0,m}) \le C \qquad (54)$$

with a positive constant $C$ independent of $m$. Let us now multiply (29) by $-(\Sigma^{\text{lin}}_{n,m} - \Sigma^{\text{lin}}_{(n-1),m})/h$. Then using the monotonicity of $g$ we obtain that

$$\frac{1}{m}\left\|\frac{p^n_m - p^{n-1}_m}{h}\right\|^2_\Omega + \left\|\operatorname{dev}\operatorname{sym}\left(\frac{p^n_m - p^{n-1}_m}{h}\right)\right\|^2_\Omega + \left\|\operatorname{Curl}\left(\frac{p^n_m - p^{n-1}_m}{h}\right)\right\|^2_\Omega$$
$$\le \left(\frac{p^n_m - p^{n-1}_m}{h}, \frac{\sigma^n_m - \sigma^{n-1}_m}{h}\right)_\Omega + \left(\frac{p^n_m - p^{n-1}_m}{h}, \frac{\hat\sigma^n_m - \hat\sigma^{n-1}_m}{h}\right)_\Omega.$$

With (27) and (28) the last inequality can be rewritten as follows

$$\frac{1}{m}\left\|\frac{p^n_m - p^{n-1}_m}{h}\right\|^2_\Omega + \left\|\operatorname{dev}\operatorname{sym}\left(\frac{p^n_m - p^{n-1}_m}{h}\right)\right\|^2_\Omega$$
$$+\left\|\operatorname{Curl}\left(\frac{p^n_m - p^{n-1}_m}{h}\right)\right\|^2_\Omega + \left(\mathbb{C}^{-1}\frac{\sigma^n_m - \sigma^{n-1}_m}{h}, \frac{\sigma^n_m - \sigma^{n-1}_m}{h}\right)_\Omega$$
$$\le \left(\frac{p^n_m - p^{n-1}_m}{h}, \frac{\hat\sigma^n_m - \hat\sigma^{n-1}_m}{h}\right)_\Omega + \left(\frac{\sigma^n_m - \sigma^{n-1}_m}{h}, \bar{\mathbb{C}}\frac{\hat\sigma^n_m - \hat\sigma^{n-1}_m}{h}\right)_\Omega.$$

Multiplying by $h$ and summing then for $n = 1, ..., 2^m$ we get the estimate

$$\frac{1}{m}\|\partial_t p_m\|^2_{\Omega_{T_e}} + \|\operatorname{dev}\operatorname{sym}(\partial_t p_m)\|^2_{\Omega_{T_e}} + \|\operatorname{Curl}(\partial_t p_m)\|^2_{\Omega_{T_e}}$$
$$+C\|\partial_t\sigma_m\|^2_{\Omega_{T_e}} \le C\|\partial_t\hat\sigma_m\|_{2,\Omega_{T_e}}(\|\partial_t\sigma_m\|_{2,\Omega_{T_e}} + \|\partial_t p_m\|_{2,\Omega_{T_e}}). \tag{55}$$

Using now inequality (3) and Young's inequality with $\epsilon > 0$ in (55) we obtain that

$$\frac{1}{m}\|\partial_t p_m\|^2_{\Omega_{T_e}} + C_\epsilon\|\partial_t p_m\|^2_{\Omega_{T_e}} + C_\epsilon\|\partial_t\sigma_m\|^2_{\Omega_{T_e}} \le C\|\partial_t\hat\sigma_m\|^2_{2,\Omega_{T_e}}. \tag{56}$$

Since $\hat\sigma_m$ is uniformly bounded in $W^{1,q}(\Omega_{T_e}, \mathcal{S}^3)$, estimates (55) and (56) imply

$$\{\operatorname{dev}\operatorname{sym}\partial_t p_m\}_m \text{ is uniformly bounded in } L^2(0,T_e;L^2(\Omega,\mathcal{M}^3)), \tag{57}$$
$$\{\partial_t\sigma_m\}_m \text{ is uniformly bounded in } L^2(0,T_e;L^2(\Omega,\mathcal{M}^3)), \tag{58}$$
$$\{\operatorname{Curl}\partial_t p_m\}_m \text{ is uniformly bounded in } L^2(0,T_e;L^2(\Omega,\mathcal{M}^3)), \tag{59}$$
$$\left\{\frac{1}{\sqrt{m}}\partial_t p_m\right\}_m \text{ is uniformly bounded in } L^2(0,T_e;L^2(\Omega,\mathcal{M}^3)), \tag{60}$$
$$\{p_m\}_m \text{ is uniformly bounded in } H^1(0,T_e;L^2_{\operatorname{Curl}}(\Omega,\mathcal{M}^3)). \tag{61}$$

**Existence of solutions.** By estimates (44) - (53), (57) - (61) and at the expense of extracting a subsequence, we have that the sequences in (44) - (53), (57) - (61) converge with respect to weak and weak-star topologies in corresponding spaces, respectively. Next, we claim that weak limits of $\{\bar p_m\}_m$ and $\{p_m\}_m$ coincide. Indeed, using (44) this can be shown as follows

$$\|p_m - \bar p_m\|^2_{2,\Omega_{T_e}} = \sum_{n=1}^{m}\int_{(n-1)h}^{nh}\left\|(p^n_m - p^{n-1}_m)\frac{t-nh}{h}\right\|^2_2 dt$$

$$= \frac{h^{2+1}}{2+1} \sum_{n=1}^{m} \left\| \frac{p_m^n - p_m^{n-1}}{h} \right\|_2^2 = \frac{h^2}{2+1} \left\| \frac{dp_m}{dt} \right\|_{2,\Omega_{T_e}}^2 ,$$

which implies that $\bar{p}_m - p_m$ converges strongly to 0 in $L^2(\Omega_{T_e}, \mathcal{M}^3)$. The proof of the fact that the difference $\bar{\sigma}_m - \sigma_m$ converges weakly to 0 in $L^2(\Omega_{T_e}, \mathcal{S}^3)$ can be performed as in [33, p. 210]. For the reader's convenience we reproduce here the reasoning from there. Let us choose some appropriate number $d \in \mathbb{N}$ and then fix any integer $n_0 \in [1, 2^d]$. Let $h_0 = T_e/2^{n_0}$. Consider functions $I_{[h_0(n_0-1), h_0 n_0]} v$ with $v \in L^2(\Omega, \mathcal{S}^3)$, where $I_K$ denotes the indicator function of a set $K$. We note that, according to [33, Proposition 1.36], the linear combinations of all such functions are dense in $L^2(\Omega_{T_e}, \mathcal{S}^3)$. Then for any $h \le h_0$[5]

$$\left(\sigma_m - \bar{\sigma}_m, I_{[h_0(n_0-1), h_0 n_0]} v\right)_{\Omega_{T_e}} = \int_{h_0(n_0-1)}^{h_0 n_0} (\sigma_m(t) - \bar{\sigma}_m(t), v)_\Omega \, dt$$

$$= \sum_{n=h_0(n_0-1)/h+1}^{h_0 n_0/h} \int_{(n-1)h}^{nh} \left( (\sigma_m^n - \sigma_m^{n-1}) \frac{t - nh}{h}, v \right)_\Omega dt$$

$$= -\frac{h}{2} \left( \sigma_m^{h_0 n_0/h} - \sigma_m^{h_0(n_0-1)/h}, v \right)_\Omega = -\frac{h}{2} \left( \bar{\sigma}_m(h_0 n_0) - \bar{\sigma}_m(h_0(n_0-1)), v \right)_\Omega .$$

Employing (46) we get that $\bar{\sigma}_m - \sigma_m$ converges weakly to 0 in $L^2(\Omega_{T_e}, \mathcal{S}^3)$. Next, by (48) the sequence $\{p_m/m\}_m$ converges strongly to 0 in $L^2(\Omega_{T_e}, \mathcal{M}^3)$. Summarizing all observations made above we may conclude that the limit functions denoted by $\tilde{v}, \tilde{\sigma}, p$ and $\Sigma^{\text{lin}}$ have the following properties

$$(\tilde{v}, \tilde{\sigma}) \in H^1(0, T_e; H_0^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3)), \quad \Sigma^{\text{lin}} \in L^q(\Omega_{T_e}, \mathcal{M}^3),$$

$$p \in H^1(0, T_e; L^2_{\text{Curl}}(\Omega, \mathcal{M}^3)) \cap L^2(0, T_e; Z^2_{\text{Curl}}(\Omega, \mathcal{M}^3)).$$

Before passing to the weak limit, we note that the Rothe approximation functions satisfy the equations

$$-\operatorname{div}_x \bar{\sigma}_m(x,t) = \bar{b}_m(x,t), \tag{62}$$

$$\sigma_m(x,t) = \mathbb{C}(\text{sym}(\nabla_x u_m(x,t) - p_m(x,t))) + (\mathbb{C}[x] - \hat{\mathbb{C}})(\hat{\mathbb{C}})^{-1} \hat{\sigma}_m(x), \tag{63}$$

$$\partial_t p_m(x,t) \in g\big(\bar{\Sigma}_m^{\text{lin}}(x,t)\big), \tag{64}$$

together with the initial and boundary conditions

$$p_m(x,0) = p^{(0)}(x), \quad x \in \Omega, \tag{65}$$

$$p_m(x,t) \times n(x) = 0, \quad x \in \partial\Omega, \tag{66}$$

$$u_m(x,t) = 0, \quad x \in \partial\Omega . \tag{67}$$

Passing to the weak limit in (62), (63) and (65) - (67) we obtain that the limit functions $\tilde{v}, \tilde{\sigma}, p$ and $\Sigma^{\text{lin}}$ satisfy equations (21), (22) and (24) - (26). To show

[5] We recall that $h$ is chosen to be equal to $T_e/2^m$ for some $m \in \mathbb{N}$.

that the limit functions satisfy also (23) we proceed as follows:
As above, the system (62) - (67) can be rewritten as

$$\int_0^{T_e}\int_\Omega \big(g^{-1}(\partial_t p_m(x,t))\cdot\partial_t p_m(x,t))\big)\,dxdt = -\left(\frac{d\sigma_m}{dt},\mathbb{C}^{-1}\bar\sigma_m\right)_{\Omega_{T_e}}$$
$$-C_1\left(\frac{dp_m}{dt},\operatorname{dev}\operatorname{sym}\bar p_m\right)_{\Omega_{T_e}} - \frac{1}{m}\left(\frac{dp_m}{dt},\bar p_m\right)_{\Omega_{T_e}} \tag{68}$$
$$-C_2\left(\frac{dp_m}{dt},\operatorname{Curl}\operatorname{Curl}\bar p_m\right)_{\Omega_{T_e}} + (\hat\sigma_m,\partial_t p_m)_{\Omega_{T_e}} + \big(\bar{\mathbb{C}}\bar\sigma_m,\partial_t\hat\sigma_m\big)_{\Omega_{T_e}}\,.$$

Due to (57) - (61) and Lemma 4.11 we can pass to the weak limit inferior in (68) to get the following inequality

$$\limsup_{m\to\infty}\int_0^{T_e}\int_\Omega \big(g^{-1}(\partial_t p_m(x,t))\cdot\partial_t p_m(x,t))\big)\,dxdt \tag{69}$$
$$\le (\partial_t p,\tilde\sigma+\hat\sigma-\operatorname{dev}\operatorname{sym} p-\operatorname{Curl}\operatorname{Curl} p)_{\Omega_{T_e}}\,.$$

Let $\mathcal{G}$ denote the canonical extension of $g$. Then (69) reads as follows

$$\limsup_{m\to\infty}\big(\mathcal{G}^{-1}(\partial_t p_m),\partial_t p_m\big)_{\Omega_{T_e}} \le (\partial_t p,\tilde\sigma+\hat\sigma-\operatorname{dev}\operatorname{sym} p-\operatorname{Curl}\operatorname{Curl} p)_{\Omega_{T_e}}\,. \tag{70}$$

Since $\mathcal{G}^{-1}$ is pseudo-monotone, inequality (70) yields that for a.e. $(x,t)\in\Omega_{T_e}$

$$\partial_t p(x,s)\in g(\tilde\sigma(x,t)+\hat\sigma(x,t)-\operatorname{dev}\operatorname{sym} p(x,t)-\operatorname{Curl}\operatorname{Curl} p(x,t)).$$

Therefore, we conclude that the limit functions $\tilde v,\tilde\sigma,p$ and $\Sigma^{\text{lin}}$ satisfy equations (21) - (26) and the existence of strong solutions is herewith established.

This completes the proof of Theorem 4.7. □

# Appendix A: Helmholtz's projection

In this section we present some results concerning projection operators to spaces of tensor fields, which are symmetric gradients and to spaces of tensor fields with vanishing divergence. For details the reader is referred to [2].

In the linear elasticity theory it is well known (see [10, Theorem 10.15]) that a Dirichlet boundary value problem formed by the equations

$$-\operatorname{div}_x\sigma(x) = \hat b(x), \qquad x\in\Omega, \tag{71}$$
$$\sigma(x) = \mathbb{C}[x](\operatorname{sym}(\nabla_x u(x))-\hat\varepsilon_p(x)), \qquad x\in\Omega, \tag{72}$$
$$u(x) = 0, \qquad x\in\partial\Omega, \tag{73}$$

to given $\hat b\in W^{-1,q}(\Omega,\mathbb{R}^3)$ and $\hat\varepsilon_p\in L^q(\Omega,\mathcal{S}^3)$ has a unique weak solution $(u,\sigma)\in W_0^{1,q}(\Omega,\mathbb{R}^3)\times L^q(\Omega,\mathcal{S}^3)$ provided the open set $\Omega$ has a $C^1$-boundary and $\mathbb{C}\in C(\bar\Omega,\mathcal{S}^3)$. Here the number $q$ satisfies $1<q<\infty$. For $\hat b=0$ the solution of (71) - (73) satisfies the inequality

$$\|\operatorname{sym}(\nabla_x u)\|_q \le C\|\hat\varepsilon_p\|_q$$

with some positive constant $C$.

**Definition 4.8.** *For every* $\hat{\varepsilon}_p \in L^q(\Omega, \mathcal{S}^3)$ *we define a linear operator* $P_q : L^q(\Omega, \mathcal{S}^3) \to L^q(\Omega, \mathcal{S}^3)$ *by*

$$P_q \hat{\varepsilon}_p := \mathrm{sym}(\nabla_x u),$$

*where* $u \in W_0^{1,q}(\Omega, \mathbb{R}^3)$ *is the unique weak solution of (71) - (73) to the given function* $\hat{\varepsilon}_p$ *and* $\hat{b} = 0$.

Next, a subset $\mathcal{G}^q$ of $L^q(\Omega, \mathcal{S}^3)$ is defined by

$$\mathcal{G}^q = \{\mathrm{sym}(\nabla_x u) \mid u \in W_0^{1,q}(\Omega, \mathbb{R}^3)\}.$$

The main properties of $P_q$ are stated in the following lemma.

**Lemma 4.9.** *For every* $1 < q < \infty$ *the operator* $P_p$ *is a bounded projector onto the subset* $\mathcal{G}^q$ *of* $L^q(\Omega, \mathcal{S}^3)$*. The projector* $(P_q)^*$ *adjoint with respect to the bilinear form* $[\xi, \zeta]_\Omega := (\xi, \zeta)_\Omega$ *on* $L^q(\Omega, \mathcal{S}^3) \times L^{q^*}(\Omega, \mathcal{S}^3)$ *satisfies*

$$(P_q)^* = P_{q^*}, \quad \textit{where} \quad \frac{1}{q^*} + \frac{1}{q} = 1.$$

Due to Lemma 4.9 the following projection operator

$$Q_q = (I - P_q) : L^q(\Omega, \mathcal{S}^3) \to L^q(\Omega, \mathcal{S}^3)$$

is well-defined and generalizes the classical Helmholtz projection.

Let $L : \mathcal{S}^3 \to \mathcal{S}^3$ be the linear, positive semi-definite mapping given by

$$Lv = C_1 \,\mathrm{dev}\, v. \tag{74}$$

The next result is needed for the subsequent analysis.

**Corollary 4.10.** *Let* $(\mathbb{C}P_q + L)^*$ *be the operator adjoint to* $\mathbb{C}P_q + L : L^q(\Omega, \mathcal{S}^3) \to L^q(\Omega, \mathcal{S}^3)$ *with respect to the bilinear form* $(\xi, \zeta)_\Omega$ *on the product space* $L^q(\Omega, \mathcal{S}^3) \times L^{q^*}(\Omega, \mathcal{S}^3)$*. Then* $(\mathbb{C}P_q + L)^* = \mathbb{C}P_{q^*} + L$*. Moreover, the operator* $\mathbb{C}Q_2 + L$ *is non-negative and self-adjoint.*

For the proof of this result the reader is referred to [1].

# Appendix B

In this appendix we prove the following lemma.

**Lemma 4.11.** *Let* $X$ *be a reflexive Banach space embedded continuously into a Hilbert space* $H$*, the functions* $\phi_m, \bar{\phi}_m$ *be defined by (35) and (36) for any family of functions* $\phi_m^0, \phi_m^1, ..., \phi_m^m$*, respectively, and* $\phi$ *be a weak limit of* $\phi_m$*. Then the following inequality*

$$\limsup_{m \to \infty} \left\langle \frac{d\phi_m}{dt}, \bar{\phi}_m \right\rangle_{L^q(X^*), L^p(X)} \geq \left\langle \frac{d\phi}{dt}, \phi \right\rangle_{L^q(X^*), L^p(X)}$$

*holds, where* $\langle \cdot, \cdot \rangle_{L^q(X^*), L^p(X)}$ *denotes the the dual pairing between* $L^p(X)$ *and* $L^q(X^*)$.

*Proof.* The last inequality results from the next line by taking lim sup from both side and using the lower semi-continuity of the norm

$$\left\langle \frac{d\phi_m}{dt}, \bar{\phi}_m \right\rangle_{L^q(X^*),L^p(X)} = \sum_{n=1}^{m} \int_{h(n-1)}^{hn} \left\langle \frac{\phi_m^n - \phi_m^{n-1}}{h}, \phi_m^n \right\rangle_{X^*,X} dt$$

$$= \sum_{n=1}^{m} \left\langle \phi_m^n - \phi_m^{n-1}, \phi_m^n \right\rangle_{X^*,X} = \sum_{n=1}^{m} \frac{1}{2}\|\phi_m^n\|_H^2 - \frac{1}{2}\|\phi_m^{n-1}\|_H^2$$

$$+\frac{1}{2}\|\phi_m^n - \phi_m^{n-1}\|_H^2 \geq \frac{1}{2}\|\phi_m^m\|_H^2 - \frac{1}{2}\|\phi_m^0\|_H^2.$$

The proof is completed by the application of the generalized integration-by-parts formula.

□

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