ON INSTABILITY OF THE EKMAN SPIRAL

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Abstract. The aim of this note is to present an elegant approach to linear and nonlinear instability of the Ekman spiral, the famous stationary geostrophic solution of the 3D Navier-Stokes equations in a rotating frame. As former approaches to the Ekman boundary layer problem, our result is based on the numerical existence of an unstable wave perturbation for Reynolds numbers large enough derived by Lilly in [15]. By the fact that this unstable wave is tangentially nondecaying at infinity, however, standard approaches (e.g. by cut-off techniques) to instability in standard function spaces (e.g. L^p) remain a technical and intricate issue. In spite of this fact, we will present a rather short proof of linear and nonlinear instability of the Ekman spiral in L^2. The results are based on a recently developed general approach to rotating boundary layer problems, which relies on Fourier transformed vector Radon measures, cf. [11].

1. Introduction

The rotational Navier Stokes equations in a three-dimensional halfspace $\mathbb{R}^3_+$ are given by

$$\left.\begin{array}{rcll} \partial_t u - \nu\Delta u + \Omega e_3 \times u + (u\cdot\nabla)u + \nabla p &=& 0, & x \in \mathbb{R}^3_+, t>0,\\ \operatorname{div} u &=& 0, & x\in\mathbb{R}^3_+, t>0,\\ u|_{\partial\mathbb{R}^3_+} &=& 0, & t>0,\\ u|_{t=0} &=& u_0, & x\in\mathbb{R}^3_+, \end{array}\right\} \quad (1.1)$$

where $u : \mathbb{R}^3_+ \times (0,\infty) \to \mathbb{R}^3$ denotes the velocity field and $p : \mathbb{R}^3_+ \times (0,\infty) \to \mathbb{R}$ the pressure. The constant $\nu > 0$ refers to the viscosity of the fluid and Ω is twice the angular velocity of rotation. For simplicity we assume rotation is about the x_3-axis, i.e. $e_3 = (0,0,1)^T$. It is well known that this problem has a stationary solution, given by

$$u_E(x) = u_\infty \begin{pmatrix} 1 - e^{-x_3/\delta}\cos(x_3/\delta) \\ e^{-x_3/\delta}\sin(x_3/\delta) \\ 0 \end{pmatrix},$$

$$p_E(x) = -\Omega u_\infty x_2,$$

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where $u_\infty > 0$. This solution, the *Ekman spiral*, is named after the Swedish oceanograph V.W. Ekman, see [6]. He discovered this steady state solution by observing that icebergs never follow the prevailing wind directly, but always shifted by a certain angle. It appears in rotating boundary layers in between a geostrophic flow and a solid wall where the no-slip boundary condition applies. The Ekman layer is considered as the region where friction dominates the flow field due to the no-slip condition and viscosity. Its thickness is defined by $\delta = \sqrt{2\nu/|\Omega|}$. For large x_3 the Ekman spiral converges to the geostrophic flow field above the surface, i.e.

$$u_E(x_3) \to (u_\infty, 0, 0), \quad \text{as } x_3 \to \infty.$$

Since it usually describes the situation in an ocean or the atmosphere, it seems natural to consider stability for this solution not in a halfspace, but in a layer $D = \mathbb{R}^2 \times (0, d)$ with $\delta < d < \infty$. So we replace the third equation in (1.1) by

$$u|_{\partial D} = u_E|_{\partial D}$$

and consider functions u solving the equation above and satisfying this modified boundary condition. (Note that $u|_{x_3=0} = 0$ still remains valid.) Defining

$$w = u - u_E, \qquad q = p - p_E$$

gives the following inital boundary value problem for the perturbed solution (w, q):

$$\left.\begin{array}{rcl} \partial_t w - \nu\Delta w + \Omega e_3 \times w + (u_E \cdot \nabla)w + w_3\partial_3 u_E + (w \cdot \nabla)w &=& -\nabla q, \\ \operatorname{div} w &=& 0, \\ w|_{\partial D} &=& 0, \\ w|_{t=0} &=& w_0, \end{array}\right\} \tag{1.2}$$

for $x \in D, t > 0$, where $w_0 = u_0 - u_E$.

The question for stability and instability is closely related to the value of the *Reynolds number*

$$Re = \frac{u_\infty \delta}{\nu}$$

of the stationary solution u_E. Physically, it is natural to expect that there is a critical value Re_c such that for Reynolds numbers $Re < Re_c$ the Ekman spiral is stable whereas it becomes unstable for $Re > Re_c$. In [3] stability of the Ekman spiral in $L^2(\mathbb{R}^3_+)$ is obtained, if Re is sufficiently small. This is improved to asymptotic stability in [14].

Mathematically, instability turns out to be a more intricate problem. There seems to be no rigorous analytic proof of linear or nonlinear instability available in the literature, even for arbitrarily large Reynolds number. In [15] Lilly derives a sufficient criterion for the existence of wave perturbations causing linear instability, namely the solvability of an ODE eigenvalue problem under certain constraints. For Reynolds numbers $Re > 55$, it is proved numerically that this ODE eigenvalue problem is solvable. Based on this numerical existence of an unstable wave, in this note we will prove results on linear and nonlinear instability in the spacial domain $\mathbb{R}^2 \times (0, d)$.

We emphasize that, given the unstable wave modes from [15], one might still be far from a proof of instability in a standard space like $L^2(\mathbb{R}^2 \times (0, d))$. This relies on the fact that waves do not decay at space infinity in tangential direction. Consequently, the unstable wave is at most an approximate eigenfunction in $L^2(\mathbb{R}^2 \times (0, d))$. One standard procedure to proof the unstable wave modes to be an approximate eigenfunction in $L^2(\mathbb{R}^2 \times (0, d))$ is to employ suitable cut-off techniques. In the Stokes and Navier-Stokes context this, however, always leads to a disturbance of the divergence free condition and the pressure gradient. This fact can make standard approaches a somewhat technical issue.

The strategy we pursue here, however, is completely different. The fact that the unstable wave is nondecaying, suggests the development of a functional analytic frame such that a suitable class of nondecaying functions, like wave modes, are included. Such an approach recently was developed in [11]. There the space of Fourier transformed Radon measures FM is introduced which includes the unstable waves as eigenfunctions. So, instability in the space FM follows more or less at once by the existence of an unstable wave mode, as it is numerically verified in [15]. Moreover, the space FM displays a couple of further nice features in the treatment of rotating boundary layer problems. In fact, in FM one can obtain the same situation as in L^2 concerning multipliers. To be precise, merely boundedness of a symbol is enough to turn it into a multiplier. As is well known, this is usually not true in standard function classes, except for Hilbert spaces. It is exactly this fact, which provides an elegant way to come from FM back to L^2. Roughly speaking, we can derive an equality as

$$\|\mathcal{F}^{-1} m \mathcal{F}\|_{\mathcal{L}(\mathrm{FM})} = \|m\|_\infty = \|\mathcal{F}^{-1} m \mathcal{F}\|_{\mathcal{L}(L^2)} \tag{1.3}$$

for bounded and continuous symbols m and where $\mathcal{F}$ denotes the Fourier transform. (See Proposition 1 for a precise statement.) From this we immediately obtain instability in L^2, once it is proved in FM. Utilizing this strategy, under the assumption that an unstable wave mode exists, the proof of linear and nonlinear instability of the Ekman spiral becomes rather short (it is essentially the three pages of Section 4).

Note that (1.3) can also be used vice versa. That is, in order to prove estimates for (linearized) boundary layer problems in the space FM, it suffices to derive them in L^2. This method is utilized in [11] in order to develop a comprehensive approach to rotating boundary layer problems in classes containing nondecaying functions such as almost periodic functions. Please consult [11] for the details. For earlier approaches to rotating Navier-Stokes equations in FM-spaces see also [8], [9], and [10].

An earlier approach to nonlinear instability for the Ekman problem is given in [4]. There, based on the numerical results in [15], Desjardins and

Grenier proved nonlinear instability on the torus $\mathbb{T}^2 \times [0,1]$ in L^2 and L^∞. In a torus, however, the problem with the nondeaying structure of the unstable wave mode does, of course, not appear. An advantage of the approach performed in [4] is certainly given by the fact that it leads to a relatively precise description of an unstable solution until the instability time. This results in an exponential divergence rate. On the other hand, the assumptions required in [4] are much stronger than ours. In fact, we only need one single point in the unstable spectrum, which immediately follows from the numerical results in [15]. In [4] a whole open set in the unstable spectrum satisfying certain properties is assumed to be given. Additionally, some technical resolvent estimates need to be verified. So, by the just mentioned facts and since the methods seem to be completely different, we think that the two approaches, i.e. the one performed in [4] and the one given here, are not comparable.

We organized this paper as follows: in Section 2 we fix notation and explain our term of instability. We also sketch the approach in [15] and the connection between linear instability of the Ekman spiral and the ODE eigenvalue problem mentioned above. This enables us to fix the assumptions we use to state our main results on linear and nonlinear instability in FM and L^2. In Section 3 we briefly recall required results on Fourier transformed vector Radon measures from [11]. The proof of the instability results is carried out in Section 4.

2. Preliminaries and main results

Let $n \in \mathbb{N}$ and let $G \subset \mathbb{R}^n$ be a domain. We write $L^p(G)$ and $W^{k,p}(G)$ for the usual Lebesgue and Sobolev spaces, $1 \leq p \leq \infty$, $k \in \mathbb{N}$. Let $C_0^\infty(G)$ be the set of all compactly supported smooth functions in G. Its closure in $W^{k,p}(G)$ is denoted by $W_0^{k,p}(G)$. The spaces $L^p_\sigma(G)$, $W^{k,p}_\sigma(G) = W^{k,p}(G) \cap L^p_\sigma(G)$ and $W^{k,p}_{0,\sigma}(G) = W^{k,p}_0(G) \cap L^p_\sigma(G)$ represent the corresponding Lebesgue and Sobolev spaces of solenoidal functions for $1 < p < \infty$. The space of bounded and continuous functions and the space of bounded and uniformly continuous functions are denoted by $\mathrm{BC}(G)$ and $\mathrm{BUC}(G)$ respectively. we write $\mathcal{S}(\mathbb{R}^n)$ for the space of rapidly descreasing smooth functions. Its dual is denoted by $\mathcal{S}'(\mathbb{R}^n)$. The letters E, F, X, Y usually denote Banach spaces. We write $\mathcal{L}(E, F)$ for the space of bounded linear operators from E to F. In the case $E = F$ we abbreviate $\mathcal{L}(E)$. The E-valued versions of the L^p-spaces are written as $L^p(G, E)$ and $W^{k,p}(G, E)$ and so on. If we work with another measure than the Lebesgue measure we denote the corresponding spaces by $L^p(G, \mu, E)$.

We will write $J = (0, d)$ for $d \in (0, \infty)$. The set $\mathrm{M}_0(\mathbb{R}^2, L^2(J)^3)$ denotes the space of finite Radon measures with values in $L^2(J)^3$ and zero point mass at the origin. We set

$$\mathrm{FM}_0(\mathbb{R}^2, L^2(J)^3) = \{\mathcal{F}\mu : \mu \in \mathrm{M}_0(\mathbb{R}^2, L^2(J)^3)\}$$

(for a precise definition, see Section 3), where the Fourier transform $\mathcal{F}$ is defined by

$$\mathcal{F}u(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x)e^{-ix\xi}dx, \quad \xi \in \mathbb{R}^n, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

By Proposition 2(a) below, the Helmholtz projector P is a bounded operator on $\mathrm{FM}_0(\mathbb{R}^2, L^2(J)^3)$. So, we may define its solenoidal subspace by

$$\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(J)^3) := P\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(J)^3).$$

The operatorial form of system (1.2), regarded as a nonlinear evolution equation in the spaces $L^2_\sigma(D)$ or $\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(J)^3)$, then reads as

$$\begin{aligned} \dot{w}(t) + A_{SCE}w(t) &= -P(w(t)\cdot\nabla)w(t), \quad t > 0, \\ w(0) &= w_0, \end{aligned} \tag{2.1}$$

where the Stokes-Coriolis-Ekman operator A_{SCE} is given by

$$\begin{aligned} A_{SCE}w &:= A_S w + A_C w + A_E w \\ &:= -\nu P\Delta w + \Omega P(e_3 \times w) + P((u_E\cdot\nabla)w + w_3\partial_3 u_E), \end{aligned} \tag{2.2}$$

for $w \in \mathcal{D}(A_{SCE})$. In order to avoid confusion, occasionally we will write A_{SCE,L^2} or $A_{SCE,\mathrm{FM}}$. The Stokes operator A_{S,L^2} in an infinite layer with domain

$$\mathcal{D}(A_{S,L^2}) = W^{1,2}_{0,\sigma}(D) \cap W^{2,2}(D) \subset L^2_\sigma(D)$$

admits a bounded $\mathcal{H}^\infty$-calculus on $L^2_\sigma(D)$, cf. [1]. By the fact that A_C and A_E are lower order perturbations, a standard perturbation argument yields $\mathcal{D}(A_{SCE,L^2}) = \mathcal{D}(A_{S,L^2})$ and that also $\mu + A_{SCE,L^2}$ admits a bounded $\mathcal{H}^\infty$-calculus on $L^2_\sigma(D)$ for some $\mu > 0$. The domain of $A_{SCE,\mathrm{FM}}$ and its semigroup properties will be stated in Proposition 2.

Since the definition of instability varies significantly throughout literature we state precisely what we mean by nonlinear instability. The following definition is taken (and suitably adapted) from [13].

Definition 2.1 (Nonlinear instability of the Ekman spiral)**.** Let $d > \delta$, $J = (0, d)$, and $D = \mathbb{R}^2 \times J$. Let X be one of the spaces $L^2_\sigma(D)$ or $\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(J))$ and set $X^\gamma = \mathcal{D}((\mu + A_{SCE,X})^\gamma)$, $\gamma \in (0, 1)$, with norm $\|\cdot\|_\gamma := \|(\mu + A_{SCE,X})^\gamma \cdot\|_X$ for some μ in the resolvent set of A_{SCE}. The Ekman spiral is said to be nonlinearly unstable in X^γ if there is a constant $\varepsilon_0 > 0$ which meets the following:
for any $\delta > 0$ there exists $w_0 \in X^\gamma$ with $\|w_0\|_\gamma < \delta$ such that there is some finite time $t_0 > 0$ with

$$\|w(t_0, w_0)\|_\gamma \geq \varepsilon_0,$$

where $w(\cdot, w_0)$ denotes the solution of (2.1).

Remark 1. Reformulating this for the solution (u, p) of the original system (1.1) would mean the following: The Ekman spiral is said to be unstable in X^γ, if there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$ there exists a solution u to problem (1.1) with $\|u(0) - u_E\|_\gamma < \delta$ and a $t_0 > 0$ such that

$$\|u(t_0) - u_E\|_\gamma \geq \varepsilon_0.$$

In [15] the question for linear instability is addressed by the search for wave solutions to the linear version of (1.2) with growing amplitude. We briefly recall the derivation of the following ODE eigenvalue problem and related numerical results and refer to [15] for the details. Physically it is expected that perturbations that cause instability are two-dimensional. The domain is transformed suitably such that the perturbation becomes independent of the new x_1-variable. This allows for the introduction of a stream function. Then the following ODE of Orr-Sommerfeld type is obtained by a wave solution ansatz:

$$\left.\begin{aligned} \ddot{\ddot{\varphi}} - 2\alpha^2\ddot{\varphi} + \alpha^4\varphi - \mathrm{i}\alpha Re[(u'_{E,2} - c)(\ddot{\varphi} - \alpha^2\varphi) - \ddot{u}'_{E,2}\varphi] + 2\dot{\mu} &= 0, \\ \ddot{\mu} - \alpha^2\mu - \mathrm{i}\alpha Re[(u'_{E,2} - c)\mu + \dot{u}'_{E,1}\varphi] - 2\dot{\varphi} &= 0, \\ \varphi(z=0) = \dot{\varphi}(z=0) = \mu(z=0) &= 0, \\ \varphi(z=d') = \dot{\varphi}(z=d') = \mu(z=d') &= 0, \end{aligned}\right\} \tag{EVP}$$

where $d' = d/\delta$ is the scaled thickness of the layer, $Re = u_\infty\delta/\nu$ is the Reynolds number, $\alpha \in \mathbb{R}\backslash\{0\}$ the wave number, and $c \in \mathbb{C}$ the complex phase velocity. The dots refer to differentiation with respect to z. The functions $u'_{E,i}$ represent the non-dimensionalized and transformed components of the Ekman spiral, given by

$$u'_E(z) = \frac{1}{u_\infty}\begin{pmatrix} k_2 & -k_1 & 0 \\ k_1 & k_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} u_E(\delta z), \tag{2.3}$$

where $k = (k_1, k_2) \in \mathbb{R}^2$ is the normalized direction of the perturbation axis in the x_1x_2-plane.

It is expected that the Ekman spiral becomes unstable for high Reynolds numbers Re. Having determined non-trivial solutions (φ, μ) it is possible to construct wave solutions to the linear problem associated to (1.2) with exponential growth rate $\operatorname{Im}\alpha c$ in time. So the link between (EVP) and linear instability is the question for non-trivial solutions (φ, μ) to (EVP) subject to $\operatorname{Im}\alpha c > 0$. In [15] for given Reynolds number $Re > 0$ sets of parameters $\alpha \in \mathbb{R}\backslash\{0\}$, $k \in \mathbb{R}^2$ with $\|k\|_2 = 1$ are defined to determine numerically the "eigenvalues" $c = c(\alpha, k) \in \mathbb{C}$, such that solutions (φ, μ) exist. The least such Reynolds number where positive growth rates are detected is numerically found to be $Re \approx 55$. Although some analytical works on problem (EVP) exist in the literature (see e.g. [12], [16]), an analytical proof for the existence of such solutions seems to be missing. The results mentioned above motivate the following assumption.

Assumption 1. For $d > \delta$ and $Re > 55$ there exist parameters $\alpha \in \mathbb{R} \setminus \{0\}$, $c \in \mathbb{C}$ such that $\operatorname{Im} \alpha c > 0$, $k \in \mathbb{R}^2$, and a nontrivial solution

$$(\varphi, \mu) \in [W^{4,2}(0, d') \cap W_0^{1,2}(0, d')] \times [W^{2,2}(0, d') \cap W_0^{1,2}(0, d')]$$

of (EVP), where $d' = d/\delta$.

We are now in position to formulate our main results.

Theorem 2.2 (Linear instability)**.** *Let $d > \delta$, $J = (0, d)$, $D = \mathbb{R}^2 \times J$, $Re > 55$, $X \in \{L^2_\sigma(D), \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(J)^3)\}$, and suppose Assumption 1 to hold true. Then there exist $C, \omega > 0$ such that the semigroup $(\exp(-tA_{SCE}))_{t\geq 0}$ generated by the Stokes-Coriolis-Ekman operator A_{SCE} as defined in (2.2) satisfies*

$$\| \exp(-tA_{SCE})\|_{\mathcal{L}(X)} \geq Ce^{wt} \quad (t \geq 0).$$

In particular, the Ekman spiral is linearly unstable in X.

Theorem 2.3 (Nonlinear instability)**.** *Let d, J, D, Re, X be given as in Theorem 2.2 and suppose Assumption 1 to hold true. Then the Ekman spiral is nonlinearly unstable in $X^\gamma = \mathcal{D}((\mu + A_{SCE,X})^\gamma)$, for $\gamma \in [5/8, 1)$ if $X = L^2_\sigma(D)$ and for $\gamma \in [1/2, 1)$ if $X = \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(J)^3)$.*

3. Spaces of Fourier transformed vector Radon measures

For the reader's convenience here we recall basic facts on Fourier transformed vector Radon measures and the main tools which will be applied in this paper. For basics on vector measures we refer to [5]. For the introduction of the FM-spaces and related results as well as their proofs, we refer to [11].

Definition 3.1. Let E be a Banach space, G be a set, $\mathcal{A}$ be a σ-algebra over G, and $\mu \colon \mathcal{A} \to E$ be a set function.

(a) The set function μ is called an E-valued measure, if $\mu(\emptyset) = 0$ and for all pairwise disjoint sets $A_j \in \mathcal{A}$, $j \in \mathbb{N}$ we have $\mu\left(\bigcup_{j=1}^\infty A_j\right) = \sum_{j=1}^\infty \mu(A_j)$.

(b) Let μ be an E-valued measure.

  (i) The variation $|\mu|$ of μ is defined by

$$|\mu|(O) := \sup \left\{ \sum_{A\in\pi} \|\mu(A)\|_E : \pi \subset \mathcal{A} \text{ finite decomposition of } O \right\}, \quad O \in \mathcal{A}.$$

  (ii) The quantity $|\mu|(G)$ is called the total variation of μ. If $|\mu|(G) < \infty$ then μ is called a finite E-valued measure.

  (iii) The E-valued measure μ is called an E-valued Radon measure, if $|\mu|$ is a Radon measure.

Let $n \in \mathbb{N}$ and E, F be Banach spaces with the Radon-Nikodým property, see [5] for a definition and classes having this property. We remark that for our purposes it is sufficient to know that reflexive spaces enjoy the Radon-Nikodým property. We define the space of finite E-valued Radon measures with zero point mass at the origin by

$$\mathrm{M}_0(\mathbb{R}^{n-1}, E) := \{\mu : \mu \ E\text{-valued Radon measure}, |\mu|(\mathbb{R}^{n-1}) < \infty,\ \mu(\{0\}) = 0\}.$$

Endowed with the norm $\|\cdot\|_{\mathrm{M}} := |\cdot|(\mathbb{R}^{n-1})$ the space $\mathrm{M}_0(\mathbb{R}^{n-1}, E)$ becomes a Banach space. The Radon-Nikodým property implies that for every $\mu \in \mathrm{M}_0(\mathbb{R}^{n-1}, E)$ there exists a $\nu_\mu \in L^1(\mathbb{R}^{n-1}, |\mu|, E)$ such that

$$\mu(O) = \int_O \nu_\mu d|\mu|, \quad O \in \mathcal{A}.$$

For $\psi \in \mathrm{BC}(\mathbb{R}^{n-1}\backslash\{0\}, \mathcal{L}(E, F)\}$ we define

$$\mu\lfloor\psi(O) := \int_O \psi\nu_\mu d|\mu| \in \mathrm{M}_0(\mathbb{R}^{n-1}, F), \quad O \in \mathcal{A}.$$

The norm can be calculated by

$$\|\mu\lfloor\psi\|_{\mathrm{M}} = (|\mu|\lfloor\|\psi\nu_\mu\|_F)(G) = \int_G \|\psi\nu_\mu\|_F d|\mu|. \tag{3.1}$$

It can also be shown that

$$\mathrm{M}_0(\mathbb{R}^{n-1}, E) \hookrightarrow \mathcal{L}(\mathcal{S}(\mathbb{R}^{n-1}), E) = \mathcal{S}'(\mathbb{R}^{n-1}, E).$$

Hence, the space of Fourier-transformed finite E-valued Radon measures given as

$$\mathrm{FM}_0(\mathbb{R}^{n-1}, E) = \{\mathcal{F}\mu : \mu \in \mathrm{M}_0(\mathbb{R}^{n-1}, E)\}$$

is well-defined. With the induced norm $\|\cdot\|_{\mathrm{FM}} = \|\mathcal{F}^{-1}\cdot\|_{\mathrm{M}}$ the space $\mathrm{FM}_0(\mathbb{R}^{n-1}, E)$ becomes a Banach space. Observe that $\|\mathcal{F}\cdot\|_{\mathrm{M}} = \|\mathcal{F}^{-1}\cdot\|_{\mathrm{M}}$. Thus in the definition of the FM-spaces we might as well replace $\mathcal{F}$ by $\mathcal{F}^{-1}$. Let E_1 and E_2 be Banach spaces having the Radon-Nikodým property such that $E_1 \cdot E_2 \hookrightarrow E$. Then this multiplication structure carries over to the corresponding FM-spaces, i.e., we have

$$\mathrm{FM}_0(\mathbb{R}^{n-1}, E_1) \cdot \mathrm{FM}_0(\mathbb{R}^{n-1}, E_2) \hookrightarrow \mathrm{FM}_0(\mathbb{R}^{n-1}, E). \tag{3.2}$$

Furthermore, it can be proved that

$$\mathcal{F}L^1(\mathbb{R}^{n-1}, E) \hookrightarrow \mathrm{FM}_0(\mathbb{R}^{n-1}, E) \hookrightarrow \mathrm{BUC}(\mathbb{R}^{n-1}, E). \tag{3.3}$$

The following observation is crucial for our approach to instability of the Ekman problem. It will yield that unstable wave modes belong to a suitable FM-space.

Remark 2. The first inclusion in (3.3) is a strict one. This is due to the fact that $\delta_x \in \mathrm{M}_0(\mathbb{R}^{n-1}, E)\backslash L^1(\mathbb{R}^{n-1}, E)$ for $x \neq 0$, where δ_x denotes the

Dirac measure at $x \in \mathbb{R}^{n-1}$. Therefore the space $\mathrm{FM}_0(\mathbb{R}^{n-1}, E)$ contains almost periodic functions of the form

$$x \mapsto \sum_{j\in\mathbb{Z}} a_j e^{\lambda_j x},$$

where $(a_j)_{j\in\mathbb{Z}} \subset E$ is an absolutely summable sequence and $(\lambda_j)_{j\in\mathbb{Z}} \subset \mathbb{C}\backslash\{0\}$ is any sequence of complex frequencies.

Another very useful property of the spaces FM_0 is the following result on Fourier multipliers.

Proposition 1. [11, Proposition 2.13, Remark 2.14] *Let E, F be Banach spaces having the Radon-Nikoým property and assume that $\sigma \in \mathrm{BC}(\mathbb{R}^{n-1}\backslash\{0\}, \mathcal{L}(E,F))$. Then $Op(\sigma)$ defined by*

$$Op(\sigma)f := \mathcal{F}^{-1}((\mathcal{F}f)\lfloor\sigma), \quad f \in \mathrm{FM}_0(\mathbb{R}^{n-1}, E),$$

is bounded from $\mathrm{FM}_0(\mathbb{R}^{n-1}, E)$ to $\mathrm{FM}_0(\mathbb{R}^{n-1}, F)$ with

$$\|Op(\sigma)\|_{\mathcal{L}(\mathrm{FM}_0(\mathbb{R}^{n-1},E),\mathrm{FM}_0(\mathbb{R}^{n-1},F))} = \|\sigma\|_{L^\infty(\mathbb{R}^{n-1}\backslash\{0\},\mathcal{L}(E,F))}.$$

In the special situation when E, F are Hilbert spaces, Plancherel's theorem implies

$$\|Op(\sigma)\|_{\mathcal{L}(\mathrm{FM}_0(\mathbb{R}^{n-1},E),\mathrm{FM}_0(\mathbb{R}^{n-1},F))} = \|\sigma\|_\infty = \|Op(\sigma)\|_{\mathcal{L}(L^2(\mathbb{R}^{n-1},E),L^2(\mathbb{R}^{n-1},F))}.$$

Next, for $k \in \mathbb{N}$ we set

$$\mathrm{FM}_0^k(\mathbb{R}^{n-1}, E) := \{u \in \mathrm{FM}_0(\mathbb{R}^{n-1}, E) : \partial^\beta u \in \mathrm{FM}_0(\mathbb{R}^{n-1}, E), |\beta| \le k\}$$

with norm $\|\cdot\|_{\mathrm{FM}^k} := \sum_{|\beta|\le k} \|\partial^\beta \cdot\|_{\mathrm{FM}}$ and

$$\mathrm{FM}_0^\infty(\mathbb{R}^{n-1}, E) := \bigcap_{k\in\mathbb{N}} \mathrm{FM}_0^k(\mathbb{R}^{n-1}, E).$$

As a consequence of Proposition 1 many known facts in the L^p-setting transfer to the FM-setting, whenever operators possess a symbol representation. In [11] this is utilized in order to prove the following results.

Proposition 2. [11, Lemma 3.1, Theorem 3.4, 3.5] *Let $1 < p < \infty$ and $n \in \mathbb{N}$.*

(a) *The Helmholtz decomposition of $\mathrm{FM}_0(\mathbb{R}^{n-1}, L^p(J)^n)$ exists, i.e., we have*

$$\mathrm{FM}_0(\mathbb{R}^{n-1}, L^p(J)^n) = \mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, L^p(J)^n) \oplus G_{\mathrm{FM}}.$$

with

$$\mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, L^p(J)^n) =$$

$$\overline{\left\{u \in \mathrm{FM}_0^\infty\left(\mathbb{R}^{n-1}, \bigcap_{k=0}^\infty W^{k,p}(J)^n\right); \operatorname{div} u = 0, \nu\cdot u|_{\partial(\mathbb{R}^{n-1}\times J)} = 0\right\}}^{\|\cdot\|_{\mathrm{FM}}}$$

and

$$G_{\mathrm{FM}} = \{\nabla p : p \in L^1_{loc}(\mathbb{R}^{n-1} \times J), \nabla p \in \mathrm{FM}_0(\mathbb{R}^{n-1}, L^p(J)^n)\}.$$

Hence, the associated Helmholtz projection P from $\mathrm{FM}_0(\mathbb{R}^{n-1}, L^p(J)^n)$ onto $\mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, L^p(J)^n)$ is bounded. Furthermore there exists a symbol $\sigma_P \in \mathrm{BC}(\mathbb{R}^{n-1}\backslash\{0\}, \mathcal{L}(L^p(J)^n))$ with $Pf = \mathcal{F}^{-1}((\mathcal{F}f)\lfloor\sigma_P)$.

(b) *The Stokes operator $A_S : \mathcal{D}(A_S) \to \mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, L^p(J)^n)$, $A_S u := -P\Delta u$ with*

$$\begin{aligned}\mathcal{D}(A_S) =&\{u \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, L^p(J)^n) : \\ &\partial^\beta u \in \mathrm{FM}_0(\mathbb{R}^{n-1}, L^p(J)^n), \beta \in \mathbb{N}_0^n, |\beta| \le 2, u|_{\partial(\mathbb{R}^{n-1}\times J)} = 0\}\end{aligned}$$

generates a bounded holomorphic C_0-semigroup on $\mathrm{FM}_{0,\sigma}(\mathbb{R}^{n-1}, L^2(J)^n)$. In case that $p = 2$, A_S is the generator of a semigroup of contractions. (Note that the trace $u|_{\partial(\mathbb{R}^{n-1}\times J)}$ is well-defined in view of (3.3).)

(c) *The Stokes-Coriolis-Ekman operator $A_{SCE} : \mathcal{D}(A_{SCE}) \to \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^p(J)^3)$, with A_{SCE} as given in (2.2) and*

$$\begin{aligned}\mathcal{D}(A_{SCE}) =&\{u \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(J)^3) : \\ &\partial^\beta u \in \mathrm{FM}_0(\mathbb{R}^2, L^2(J)^3), \beta \in \mathbb{N}_0^2, |\beta| \le 2, u|_{\partial(\mathbb{R}^2\times J)} = 0\}\end{aligned}$$

generates a holomorphic C_0-semigroup on $\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(J)^3)$. Moreover there exists a symbol $\sigma_{SCE}(t) \in \mathrm{BC}(\mathbb{R}^2\backslash\{0\}, \mathcal{L}(L^2(J)^3))$ such that

$$e^{-tA_{SCE}} f = \mathcal{F}^{-1}((\mathcal{F}f)\lfloor\sigma_{SCE}(t)) \quad (f \in \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, L^2(J)^3),\ t \ge 0).$$

4. Proof of instability

We turn to the proofs of Theorems 2.2 and 2.3. From now on let $Y = L^2(0,d)^3$.

4.1. **Linear instability (proof of Theorem 2.2).** Let $d > \delta$ and $Re > 55$. By Assumption 1 there exist parameters α, c, $k = (k_1, k_2)$ such that $\operatorname{Im}\alpha c > 0$ and a nontrivial solution $(\varphi, \mu) \in [W^{4,2}(0,d') \cap W_0^{1,2}(0,d')] \times [W^{2,2}(0,d') \cap W_0^{1,2}(0,d')]$ to problem (EVP). Let θ and (w, q) be defined by

$$\theta = \frac{\nu}{i\alpha\delta}(\alpha^2\dot\varphi - \dddot\varphi - 2\mu) + u_\infty((u'_{E,2} - c)\dot\varphi - \dot u'_{E,2}\varphi) \in W^{1,2}(0,d'),$$

$$\begin{pmatrix} w_1(x,t) \\ w_2(x,t) \\ w_3(x,t) \\ q(x,t) \end{pmatrix} = u_\infty \begin{pmatrix} k_2\mu(x_3/\delta) - k_1\dot\varphi(x_3/\delta) \\ -k_1\mu(x_3/\delta) - k_2\dot\varphi(x_3/\delta) \\ i\alpha\varphi(x_3/\delta) \\ \theta(x_3/\delta) \end{pmatrix} \exp\left(\frac{i\alpha}{\delta}(\langle k, x'\rangle - cu_\infty t)\right). \tag{4.1}$$

Then (w, q) solves

$$\begin{aligned} \partial_t w - \nu\Delta w + \Omega e_3 \times w + (u_E \cdot \nabla)w + w_3\partial_3 u_E + \nabla q &= 0, \quad x \in D, t > 0, \\ \operatorname{div} w &= 0, \quad x \in D, t > 0, \\ w|_{\partial D} &= 0, \quad t > 0. \end{aligned} \tag{4.2}$$

This can be seen by a straight forward calculation which we omit here. Because of $\operatorname{Im} \alpha c > 0$ the magnitude of w grows exponentially in time. This leads to linear instability in $\mathrm{FM}_{0,\sigma}(\mathbb{R}^2, Y)$ as can be seen as follows.

Lemma 4.1. *Let $d > \delta$, $Re > 55$, and suppose Assumption 1 to hold true. Then there exist constants $C, \omega > 0$ such that*

$$\|e^{-tA_{SCE}}\|_{\mathcal{L}(\mathrm{FM}_{0,\sigma}(\mathbb{R}^2,Y))} \geq Ce^{\omega t} \quad (t \geq 0).$$

Proof. Let $x' = (x_1, x_2)$ and $V(x', t) := (w(x', \cdot, t), q(x', \cdot, t))^T$. By employing Remark 2, we may check that $V(\cdot, t) \in \mathrm{FM}_0(\mathbb{R}^2, L^2(0, d)^4)$ for all $t \geq 0$. To this end, we set

$$a = u_\infty \begin{pmatrix} k_2\mu(\cdot/\delta) - k_1\dot{\varphi}(\cdot\delta) \\ -k_1\mu(\cdot/\delta) - k_2\dot{\varphi}(\cdot/\delta) \\ i\alpha\varphi(\cdot/\delta) \\ \theta(\cdot/\delta) \end{pmatrix} \in L^2(0, d)^4, \quad g(t) := \exp(-i\alpha c u_\infty t/\delta), \quad \tilde{k} = \alpha k/\delta.$$

We can rewrite V in the following way:

$$V(\cdot, t) = g(t)e^{i\tilde{k}\cdot}a = 2\pi g(t)\mathcal{F}(\delta_{-\tilde{k}})a = 2\pi g(t)\mathcal{F}(\delta_{-\tilde{k}}\lfloor a) \in \mathrm{FM}_0(\mathbb{R}^2, Y) \quad (t \geq 0).$$

Since V is a solution to (4.2), $w = (V_1, V_2, V_3)^T$ solves the evolution equation

$$\dot{w}(t) + A_{SCE}w(t) = 0, \quad t > 0, \quad w(0) = ae^{i\tilde{k}\cdot}.$$

Thanks to (3.1) the norm of (V_1, V_2, V_3) can be calculated explicitly as

$$\begin{aligned} \|(V_1, V_2, V_3)(t)\|_{\mathrm{FM}} &= 2\pi\|\delta_{-\tilde{k}}\lfloor(a_1, a_2, a_3)\|_{\mathrm{M}}|g(t)| \\ &= 2\pi\|(a_1, a_2, a_3)\|_{L^2}e^{\omega t} \quad (t \geq 0), \end{aligned}$$

where $\omega = u_\infty\delta^{-1}\operatorname{Im}\alpha c > 0$. Thus, also the magnitude of the vector (V_1, V_2, V_3) in the $\mathrm{FM}_0(\mathbb{R}^2, Y)$-norm grows exponentially in time, which yields the result. □

By utilizing Proposition 1 we obtain the same result in $L^2_\sigma(D)$.

Lemma 4.2. *Let $d > \delta$, $Re > 55$, and suppose Assumption 1 to hold true. Then there exist constants $C, \omega > 0$ such that*

$$\|e^{-tA_{SCE}}\|_{\mathcal{L}(L^2_\sigma(D))} \geq Ce^{\omega t} \quad (t \geq 0).$$

Proof. According to Proposition 2 the Helmholtz projection P and the C_0-semigroup $e^{-tA_{SCE}}$ have symbols σ_P and $\sigma_{SCE}(t)$ respectively, both belonging to the class $\mathrm{BC}(\mathbb{R}^2\backslash\{0\},\mathcal{L}(Y))$. Thanks to Proposition 1 this gives us

$$\begin{aligned}\|e^{-tA_{SCE,L^2}}\|_{\mathcal{L}(L^2_\sigma(D))} &= \|e^{-tA_{SCE,L^2}}P\|_{\mathcal{L}(L^2(\mathbb{R}^2,Y))}\\ &= \|\sigma_{T_{SCE}}(t)\sigma_P\|_{L^\infty(\mathbb{R}^2\backslash\{0\},\mathcal{L}(Y))}\\ &= \|e^{-tA_{SCE,\mathrm{FM}}}P\|_{\mathcal{L}(\mathrm{FM}_0(\mathbb{R}^2,Y))}\\ &= \|e^{-tA_{SCE,\mathrm{FM}}}\|_{\mathcal{L}(\mathrm{FM}_{0,\sigma}(\mathbb{R}^2,Y))}\\ &\geq Ce^{\omega t} \quad (t\geq 0),\end{aligned}$$

hence the assertion follows. □

Lemmas 4.1 and 4.2 now imply Theorem 2.2. We also note the following consequence: since growth bound of an analytic semigroup and spectral bound of its generator coincide (see e.g. [7]) we have that

$$\sigma(-A_{SCE})\cap\mathbb{C}_+\neq\emptyset \tag{4.3}$$

for both, the L^2- and the FM-realization of A_{SCE}.

4.2. **Nonlinear instability (proof of Theorem 2.3).** We now consider the nonlinear evolution equation

$$\dot w(t)+A_{SCE}w(t)=-P(w(t)\cdot\nabla)w(t),\quad t>0,$$

and apply Henry's instability theorem [13], which we reformulate suitably for our purposes.

Proposition 3. [13, Corollary 5.1.6] *Let $-A$ be the generator of a holomorphic C_0-semigroup in a Banach space X and let $f\colon U\to X$, where U is an open neighborhood in X^γ for some $\gamma\in(0,1)$, be locally Lipschitz. Let $x_0\in\mathcal{D}(A)\cap U$ be an equilibrium point of*

$$\dot w(t)+Aw(t)=f(w(t)), \tag{4.4}$$

i.e. $Ax_0=f(x_0)$. Suppose

$$\begin{gathered}f(x_0+z)=f(x_0)+Bz+g(z),\quad g(0)=0,\\ \|g(z)\|=\mathcal{O}(\|z\|_\gamma^p),\quad \text{as } z\to 0 \text{ in } X^\gamma,\end{gathered}$$

for some $p>1$, $B\in\mathcal{L}(X^\gamma,X)$, and $\sigma(-A+B)\cap\mathbb{C}_+\neq\emptyset$. Then x_0 is nonlinearly unstable, i.e., there is a constant $\varepsilon_0>0$ which meets the following: for any $\delta>0$ there exists $x\in X^\gamma$ with $\|x-x_0\|_\gamma<\delta$ such that there is some finite time $t_0>0$ with

$$\|w(t_0,x)\|_\gamma\geq\varepsilon_0,$$

where $w(\cdot,x)$ denotes the solution of (4.4) with initial value $w(0,x)=x$.

Relation (4.3) implies that in our situation the assumptions of Proposition 3 for system (2.1) are satisfied for $A = A_{SCE}$, $B = 0$, and $p = 2$, since we have a quadratic nonlinearity.

Next, we determine the range for $\gamma \in (0,1)$ such that the nonlinearity estimate is well-defined. First consider the case $X = \mathrm{FM}_{0,\sigma}(\mathbb{R}^2, Y)$. Note that Sobolev's embedding theorem implies $W^{1,2}(J) \hookrightarrow L^\infty(J)$, hence we have that $W^{1,2}(J) \cdot Y \hookrightarrow Y$. In view of (3.2) this yields

$$\|P(w\cdot\nabla)w\|_{\mathrm{FM}_0(\mathbb{R}^2,Y)} \leq C\|w\|_{\mathrm{FM}_0(\mathbb{R}^2,Y^1)}(\|w\|_{\mathrm{FM}_0^1(\mathbb{R}^2,Y)} + \|w\|_{\mathrm{FM}_0(\mathbb{R}^2,Y^1)}),$$

where we abbreviated $Y^1 := W^{1,2}(J)^3$. We will show that

$$\mathcal{D}((\mu + A_{SCE,\mathrm{FM}})^\gamma) \hookrightarrow \mathrm{FM}_0(\mathbb{R}^2, Y^1) \cap \mathrm{FM}_0^1(\mathbb{R}^2, Y) \tag{4.5}$$

for $\gamma \geq 1/2$. For this purpose, let $\sigma_{A_{SCE}}$ denote the operator-valued symbol of A_{SCE} which is formally obtained by applying tangential Fourier transform to representation (2.2) of A_{SCE}, see [11] for the details. Since $\mu + A_{SCE}$ admits a bounded $\mathcal{H}^\infty$-calculus on $L^2_\sigma(D)$ for $\mu > 0$ sufficiently large, we have

$$\begin{aligned}\mathcal{D}((\mu + A_{SCE,L^2})^\gamma) &= \big[L^2_\sigma(D),\, \mathcal{D}((\mu + A_{SCE,L^2}))\big]_\gamma \\ &\hookrightarrow \big[L^2(D),\, W^{2,2}(D)\big]_\gamma \hookrightarrow W^{2\gamma,2}(D) \quad (\gamma \in (0,1)),\end{aligned} \tag{4.6}$$

where $[\cdot,\cdot]_\gamma$ denotes the complex interpolation functor and where we refer to [2] for the last embedding. Proposition 1 then implies

$$\begin{aligned}&\|(\mu + A_{SCE})^{-1/2}\|_{\mathcal{L}(\mathrm{FM}_{0,\sigma}(\mathbb{R}^2,Y),\mathrm{FM}_{0,\sigma}(\mathbb{R}^2,Y^1))} \\ &\qquad = \|(\mu + A_{SCE})^{-1/2}P\|_{\mathcal{L}(\mathrm{FM}_0(\mathbb{R}^2,Y),\mathrm{FM}_0(\mathbb{R}^2,Y^1))} \\ &\qquad = \|(\mu + \sigma_{A_{SCE}}(\cdot))^{-1/2}\sigma_P(\cdot)\|_{L^\infty(\mathbb{R}^2\setminus\{0\},\mathcal{L}(L^2(\mathbb{R}^2,Y),L^2(\mathbb{R}^2,Y^1)))} \\ &\qquad = \|(\mu + A_{SCE})^{-1/2}P\|_{\mathcal{L}(L^2(\mathbb{R}^2,Y),L^2(\mathbb{R}^2,Y^1))} \\ &\qquad = \|(\mu + A_{SCE})^{-1/2}\|_{\mathcal{L}(L^2_\sigma(D),L^2(\mathbb{R}^2,Y^1))} \\ &\qquad < \infty.\end{aligned}$$

We write ∇' for the gradient in tangential direction and calculate in the same fashion

$$\begin{aligned}\|\nabla'(\mu + A_{SCE})^{-1/2}\|_{\mathcal{L}(\mathrm{FM}_{0,\sigma}(\mathbb{R}^2,Y),\mathrm{FM}_0(\mathbb{R}^2,Y))} &= \|\nabla'(\mu + A_{SCE})^{-1/2}\|_{\mathcal{L}(L^2_\sigma(D),L^2(D))} \\ &< \infty.\end{aligned}$$

Thus, (4.5) follows for $\gamma \geq 1/2$.

Now consider $X = L^2_\sigma(D)$. In this case we aim for an estimate as

$$\|P(w\cdot\nabla)w\|_2 \leq \|w\|_{2p}\|\nabla w\|_{2q} \leq \|w\|_{2p}\|w\|_{W^{1,2q}} \leq K\|w\|^2_{W^{1+\varepsilon,2}},$$

for a certain $\varepsilon > 0$ and where $1/p + 1/q = 1$. To this end, it is necessary and sufficient to have

$$W^{1+\varepsilon,2} \hookrightarrow L^{2p} \cap W^{1,2q}.$$

By Sobolev's embedding theorem (see e.g. [2]) the continuous inclusions $W^{1+\varepsilon,2} \hookrightarrow L^{2p}$ and $W^{\varepsilon,2} \hookrightarrow L^{2q}$ hold true if

$$2p \le \frac{6}{1-2\varepsilon} \quad \text{and} \quad 2q \le \frac{6}{3-2\varepsilon}.$$

Plugging in $q = (p-1)/p$ and solving the resulting inequalities for p yields

$$\frac{3}{2\varepsilon} \le p \le \frac{3}{1-2\varepsilon},$$

which implies $\varepsilon \ge 1/4$. Thus, by virtue of (4.6), the map $f\colon X^{\gamma}_{L^2} \to L^2_\sigma(D), w \mapsto -P(w \cdot \nabla)w$ is well-defined for $\gamma \ge 5/8$, which proves the L^2-assertion. The proof of Theorem 2.3 is now complete.

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