A handy formula for the Fredholm index of Toeplitz plus Hankel operators

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Dedicated to the memory of Israel Gohberg

Abstract

We consider Toeplitz and Hankel operators with piecewise continuous generating functions on l^p-spaces and the Banach algebra generated by them. The goal of this paper is to provide a transparent symbol calculus for the Fredholm property and a handy formula for the Fredholm index for operators in this algebra.

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1 Introduction

Throughout this paper, let $1 < p < \infty$. For a non-empty subset $\mathbb{I}$ of the set $\mathbb{Z}$ of the integers, let $l^p(\mathbb{I})$ denote the complex Banach space of all sequences $x = (x_n)_{n\in\mathbb{I}}$ of complex numbers with norm $\|x\|_p = (\sum_{n\in\mathbb{I}} |x_n|^p)^{1/p} < \infty$. We consider $l^p(\mathbb{I})$ as a closed subspace of $l^p(\mathbb{Z})$ in the natural way and write $P_\mathbb{I}$ for the canonical projection from $l^p(\mathbb{Z})$ onto $l^p(\mathbb{I})$. For $\mathbb{I} = \mathbb{Z}^+$, the set of the non-negative integers, we write l^p and P instead of $l^p(\mathbb{I})$ and $P_\mathbb{I}$, respectively. By J we denote the operator on $l^p(\mathbb{Z})$ acting by $(Jx)_n := x_{-n-1}$, and we set $Q := I - P$.

For every Banach space X, let $L(X)$ stand for the Banach algebra of all bounded linear operators on X, and write $K(X)$ for the closed ideal of $L(X)$ of all compact operators. The quotient algebra $L(X)/K(X)$ is known as the Calkin algebra of X. Its importance in this paper stems from the fact that the invertibility of a coset $A + K(X)$ of an operator $A \in L(X)$ in this algebra is equivalent to the Fredholm property of A, i.e., to the finite dimensionality of the kernel $\ker A = \{x \in X : Ax = 0\}$ and the cokernel $\operatorname{coker} A = X/\operatorname{im} A$ of A, with $\operatorname{im} A = \{Ax : x \in X\}$ referring to the range of A. If A is a Fredholm operator then the difference $\operatorname{ind} A := \dim\ker A - \dim\operatorname{coker} A$ is known as the Fredholm index of A.

Our goal is a criterion for the Fredholm property and a formula for the Fredholm index for operators in the smallest closed subalgebra of $L(l^p)$ which contains all Toeplitz and Hankel operators with piecewise continuous generating function. The precise definition is as follows. Let $\mathbb{T}$ be the complex unit circle. For each function $a \in L^\infty(\mathbb{T})$, let $(a_k)_{k\in\mathbb{Z}}$ denote the sequence of its Fourier coefficients,

$$a_k := \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-ik\theta}\, d\theta.$$

The *Laurent operator* $L(a)$ associated with $a \in L^\infty(\mathbb{T})$ acts on the space $l^0(\mathbb{Z})$ of all finitely supported sequences on $\mathbb{Z}$ by $(L(a)x)_k := \sum_{m\in\mathbb{Z}} a_{k-m}x_m$. (For every $k \in \mathbb{Z}$, there are only finitely many non-vanishing summands in this sum.) We say that a is a multiplier on $l^p(\mathbb{Z})$ if $L(a)x \in l^p(\mathbb{Z})$ for every $x \in l^0(\mathbb{Z})$ and if

$$\|L(a)\| := \sup\{\|L(a)x\|_p : x \in l^0(\mathbb{Z}),\ \|x\|_p = 1\}$$

is finite. In this case, $L(a)$ extends to a bounded linear operator on $l^p(\mathbb{Z})$ which we denote by $L(a)$ again. The set M^p of all multipliers on $l^p(\mathbb{Z})$ is a Banach algebra under the norm $\|a\|_{M_p} := \|L(a)\|$. We let $M^{\langle p\rangle}$ stand for M^2 if $p = 2$ and for the set of all $a \in L^\infty(\mathbb{T})$ which belong to M^r for all r in a certain open neighborhood of p if $p \neq 2$.

It is well known that $M^2 = L^\infty(\mathbb{T})$. Moreover, every function a with bounded total variation $\mathrm{Var}(a)$ is in M^p for every p, and the Stechkin inequality

$$\|a\|_{M_p} \le c_p(\|a\|_\infty + \mathrm{Var}(a))$$

holds with a constant c_p independent of a. In particular, every trigonometric polynomial and every piecewise constant function on $\mathbb{T}$ are multipliers for every p. We denote the closure in M^p of the algebra $\mathcal{P}$ of all trigonometric polynomials and of the algebra $P\mathbb{C}$ of all piecewise constant functions by C_p and PC_p, respectively. Thus, C_p and PC_p are closed subalgebras of M^p for every p. Note that C_2 is just the algebra $C(\mathbb{T})$ of all continuous functions on $\mathbb{T}$, and PC_2 is the algebra $PC(\mathbb{T})$ of all piecewise continuous functions on $\mathbb{T}$. It is well known that $C_p \subseteq C(\mathbb{T})$ and $C_p \subseteq PC_p \subseteq PC(\mathbb{T})$ for every p. In particular, every multiplier $a \in PC_p$ possesses one-sided limits at every point $t \in \mathbb{T}$ (see [2] for these and further properties of multipliers). For definiteness, we agree that $\mathbb{T}$ is oriented counter-clockwise, and we denote the one-sided limit of a at t when approaching t from below (from above) by $a(t^-)$ (by $a(t^+)$).

Let $a \in M^p$. The operators $T(a) := PL(a)P$ and $H(a) := PL(a)QJ$, thought of as acting on $\operatorname{im} P = l^p$ are called the Toeplitz and Hankel operator with generating function a, respectively. It is well known that $\|T(a)\| = \|a\|_{M_p}$ and $\|H(a)\| \le \|a\|_{M_p}$ for every multiplier $a \in M_p$.

For a subalgebra A of M^p, we let $\mathsf{T}(A)$ and $\mathsf{TH}(A)$ stand for the smallest closed subalgebra of $L(l^p)$ which contains all operators $T(a)$ with $a \in A$ and all operators $T(a) + H(b)$ with $a, b \in A$, respectively. We will be mainly concerned with the algebras C_p, PC_p, and with their intersections with $M^{\langle p \rangle}$, in place of A. Now we can state the goal of the paper more precisely: we will state a criterion for the Fredholm property of operators in $\mathsf{TH}(PC_p)$ and derive a formula for the Fredholm index of operators $T(a) + H(b)$ with $a, b \in PC_p$.

The study of the Fredholm property of operators in $\mathsf{TH}(PC_p)$ has a long and involved history. We are going to mention only some of its main stages.

The Fredholm properties of operators in the algebra $\mathsf{T}(PC_p)$ are well understood thanks to the work of I. Gohberg/N. Krupnik and R. Duduchava; see [2] and the literature cited there. We will need these results later on; therefore we recall them in Section 2. Different approaches to these algebras were developed in [2] and [11]; our presentation will be mainly based on the latter.

The structure of the algebras $\mathsf{TH}(PC_p)$ is much more involved than that of $\mathsf{T}(PC_p)$. For instance, the Calkin image $\mathsf{T}^\pi(PC) := \mathsf{T}(PC)/K(l^2)$ of $\mathsf{T}(PC)$ is a commutative algebra, whereas that one of $\mathsf{TH}(PC)$ is not. The Calkin image of $\mathsf{TH}(PC)$ was first described by Power [16]. An alternative approach was developed by one of the authors in [21], where it was shown that the algebra $\mathsf{TH}^\pi(PC) := \mathsf{TH}(PC)/K(l^2)$ possesses a matrix-valued Fredholm symbol. In the present paper, we take up the approach from [21] in order to study the Fredholm properties of operators in $\mathsf{TH}(PC_p)$ for $p \neq 2$.

It should be mentioned that the algebras $\mathsf{TH}(PC_p)$ have close relatives which live on other spaces than l^p, such as the Hardy spaces $H^p(\mathbb{R})$ and the Lebesgue spaces $L^p(\mathbb{R}^+)$. The corresponding algebras were examined (with different methods) in the report [20], see also the recent monograph [19]. Despite these fairly complete results for the Fredholm property, a general, transparent and satisfying formula for the Fredholm index of operators in $\mathsf{TH}(PC_p)$ (or on related algebras) was not available until now. Among the particular results which hold under special assumptions we would like to emphasize the following. In [12], there is derived an index formula for operators of the form $\lambda I + H$ where $\lambda \in \mathbb{C}$ and H is a Hankel operator on $H^p(\mathbb{R})$. Already earlier, some classes of Wiener-Hopf plus Hankel operators were studied in connection with diffraction problems; see [13, 14]. Note also that the (very hard) invertibility problem for Toeplitz plus Hankel operators is treated in [1, 3].

Finally we would like to mention that algebras like $\mathsf{TH}(PC_p)$ can also be viewed of as subalgebras of algebras generated by convolution-type operators and Carleman shifts changing the orientation. First results in that direction were presented in [8, 9] where, in particular, a matrix-valued Fredholm symbol was constructed.

The goal of the present paper is to provide a transparent symbol calculus for the Fredholm property as well as a handy formula for the Fredholm index for operators in the algebra $\mathsf{TH}(PC_p)$. The techniques developed and used in this paper also allow to handle the corresponding questions for the related algebras on the spaces $H^p(\mathbb{R})$ and $L^p(\mathbb{R}^+)$.

2 The Fredholm property

In what follows, we fix $p \in (1, \infty)$ and consider all operators as acting on l^p unless stated otherwise.

As already mentioned, we start with recalling the basic results of the Fredholm theory of operators in the algebra $\mathsf{T}(PC_p)$, which are due Gohberg/Krupnik and Duduchava. The functions $f_{\pm 1}(t) := t^{\pm 1}$ are multipliers for every p. It is easy to check that the algebra generated by the Toeplitz operators $T(f_{\pm 1})$ contains a dense subalgebra of $K(l^p)$. Thus, the ideal $K(l^p)$ is contained in $\mathsf{T}(C_p)$, hence also in $\mathsf{T}(PC_p)$, and it makes sense to consider the quotient algebra $\mathsf{T}(PC_p)/K(l^p)$. Clearly, if $A \in \mathsf{T}(PC_p)$ and if the coset $A + L(l^p)$ is invertible in $\mathsf{T}(PC_p)/K(l^p)$, then it is also invertible in the Calkin algebra $L(l^p)/K(l^p)$, hence A is a Fredholm operator. The more interesting question is if the converse holds, i.e., if the invertibility of $A + L(l^p)$ in the Calkin algebra implies the invertibility of $A + K(l^p)$ in $\mathsf{T}(PC_p)/K(l^p)$. If this implication holds for every $A \in \mathsf{T}(PC_p)$, one says that $\mathsf{T}(PC_p)/K(l^p)$ is inverse closed in $L(l^p)/K(l^p)$.

Let $\overline{\mathbb{R}}$ denote the two-point compactification of the real line by the points $\pm\infty$ (thus $\overline{\mathbb{R}}$ is homeomorphic to a closed interval) and let the function $\mu_p : \overline{\mathbb{R}} \to \mathbb{C}$ be defined by

$$\mu_p(\lambda) := (1 + \coth(\pi(\lambda + i/p)))/2$$

if $\lambda \in \mathbb{R}$ and by $\mu_p(-\infty) = 0$ and $\mu_p(+\infty) = 1$. Note that when λ runs from $-\infty$ to ∞ then $\mu_p(\lambda)$ runs along a circular arc in $\mathbb{C}$ which joins 0 to 1 and passes through the point $(1 - i\cot(\pi/p))/2$. An easy calculation gives $\mu_p(-\lambda) = 1 - \mu_q(\lambda)$, where $1/p + 1/q = 1$. Thus, for fixed $t \in \mathbb{T}$, the values $\Gamma(T(a) + K(l^p))(t, \lambda)$ defined in the following theorem run from $a(t-0)$ to $a(t+0)$ along a circular arc when λ runs from $-\infty$ to ∞.

Theorem 1 *(a)* $\mathsf{T}(PC_p)/K(l^p)$ *is a commutative unital Banach algebra.*

(b) The maximal ideal space of $\mathsf{T}(PC_p)/K(l^p)$ *is homeomorphic with the cylinder* $\mathbb{T} \times \overline{\mathbb{R}}$*, provided with an exotic (non-Euclidean) topology.*

(c) The Gelfand transform $\Gamma : \mathsf{T}(PC_p)/K(l^p) \to C(\mathbb{T} \times \overline{\mathbb{R}})$ *of the coset* $T(a) + K(l^p)$ *with* $a \in PC_p$ *is*

$$\Gamma(T(a) + K(l^p))(t, \lambda) = a(t-0)(1 - \mu_q(\lambda)) + a(t+0)\mu_q(\lambda).$$

(d) $\mathsf{T}(PC_p)/K(l^p)$ *is inverse closed in* $L(l^p)/K(l^p)$.

The topology mentioned in assertion (b) will be explicitly described in Section 3. Note that this topology is independent of p. Since the cosets $T(a) + K(l^p)$ with $a \in PC_p$ generate the algebra $\mathsf{T}(PC_p)/K(l^p)$, the Gelfand transform on $\mathsf{T}(PC_p)/K(l^p)$ is completely described by assertion (c). Thus, if $A \in \mathsf{T}(PC_p)$, then the coset $A + K(l^p)$ is invertible in $\mathsf{T}(PC_p)/K(l^p)$ if and only if the function $\Gamma(A + K(l^p))$ does not vanish on $\mathbb{T} \times \overline{\mathbb{R}}$. Together with assertion (d) this shows that

$A \in \mathsf{T}(PC_p)$ is a Fredholm operator if and only if $\Gamma(A + K(l^p))$ does not vanish on $\mathbb{T} \times \overline{\mathbb{R}}$. It is therefore justified to call the function $\mathrm{smb}_p\, A := \Gamma(A + K(l^p))$ the *Fredholm symbol* of A.

The index of a Fredholm operator in $\mathsf{T}(PC_p)$ can be determined my means of its Fredholm symbol. First suppose that $a \in PC_p$ is a piecewise smooth function with only finitely many jumps. Then the range of the function

$$\Gamma(T(a) + K(l^p))(t,\, \lambda) = a(t^-)(1 - \mu_q(\lambda)) + a(t^+)\mu_q)(\lambda)$$

is a closed curve with a natural orientation, which is obtained from the (essential) range of a by filling in the circular arcs

$$\mathcal{C}_q(a(t^-),\, a(t^+)) := \{a(t^-)(1 - \mu_q(\lambda)) + a(t^+)\mu_q)(\lambda) : \lambda \in \overline{\mathbb{R}}\}$$

at every point $t \in \mathbb{T}$ where a has a jump. (If the function a is continuous at t, then $\mathcal{C}_q(a(t^-),\, a(t^+))$ reduces to the singleton $\{a(t)\}$.) If this curve does not pass through the origin, then we let $\mathrm{wind}\, \Gamma(T(a) + K(l^p))$ denote its winding number with respect to the origin, i.e., the integer $1/(2\pi)$ times the growth of the argument of $\Gamma(T(a) + K(l^p))$ when t moves along $\mathbb{T}$ in positive (= counter-clockwise) direction. If this condition is satisfied then $T(a)$ is a Fredholm operator, and

$$\mathrm{ind}\, T(a) = -\mathrm{wind}\, \Gamma(T(a) + K(l^p))$$

(see [2], Section 2.73 and Proposition 6.32 for details). Moreover, as in Section 5.49 of [2], one can extend both the definition of the winding number and the index identity to the case of an arbitrary Fredholm operator in $\mathsf{T}(PC_p)$. More precisely, one has the following.

Proposition 2 *Let $A \in \mathsf{T}(PC_p)$ be a Fredholm operator. Then*

$$\mathrm{ind}\, A = -\mathrm{wind}\, \Gamma(A + K(l^p)).$$

We would like to emphasize an important point. The algebra $\mathsf{T}(PC_2)/K(l^2)$ is a commutative C^*-algebra, hence the Gelfand transform is an isometric *-isomorphism from $\mathsf{T}(PC_2)/K(l^2)$ onto $C(\mathbb{T} \times \overline{\mathbb{R}})$. In particular, the radical of $\mathsf{T}(PC_2)/K(l^2)$ is trivial, and the equality $\mathrm{smb}_2\, A = 0$ for some operator $A \in \mathsf{T}(PC_2)$ implies that A is compact. For general p it is not known if the radical of $\mathsf{T}(PC_p)/K(l^p)$ is still trivial; it is therefore not known if $\mathrm{smb}_p\, A = 0$ implies the compactness of A.

In order to state our results on the Fredholm property of operators in the Toeplitz+Hankel algebra $\mathsf{TH}(PC_p)/K(l^p)$ we need some notation. Let $\mathbb{T}_+$ be the set of all points in $\mathbb{T}$ with non-negative imaginary part and set $\mathbb{T}_+^0 := \mathbb{T}_+ \backslash \{-1,\, 1\}$. Further let the function $\nu_p : \overline{\mathbb{R}} \to \mathbb{C}$ be defined by

$$\nu_p(\lambda) := (2i\, \sinh(\pi(\lambda + i/p)))^{-1}$$

if $\lambda \in \mathbb{R}$ and by $\nu_p(\pm\infty) = 0$. Recall that $1/p + 1/q = 1$.

Theorem 3 *(a) Let $a,\, b \in PC_p$. Then the operator $T(a)+H(b)$ is Fredholm if and only if the matrix*

$$\mathrm{smb}_p\,(T(a)+H(b))(t,\,\lambda) := \tag{1}$$
$$\begin{pmatrix} a(t^+)\mu_q(\lambda)+a(t^-)(1-\mu_q(\lambda)) & (b(t^+)-b(t^-))\nu_q(\lambda) \\ (b(\bar{t}^-)-b(\bar{t}^+))\nu_q(\lambda) & a(\bar{t}^-)(1-\mu_q(\lambda))+a(\bar{t}^+)\mu_q(\lambda) \end{pmatrix}$$

is invertible for every $(t,\,\lambda) \in \mathbb{T}_+^0 \times \overline{\mathbb{R}}$ and if the number

$$\mathrm{smb}_p\,(T(a)+H(b))(t,\,\lambda) := \tag{2}$$
$$a(t^+)\mu_q(\lambda)+a(t^-)(1-\mu_q(\lambda))+it\,(b(t^+)-b(t^-))\nu_q(\lambda)$$

is not zero for every $(t,\,\lambda) \in \{\pm 1\} \times \overline{\mathbb{R}}$.

(b) The mapping smb_p defined in assertion (a) extends to a continuous algebra homomorphism from $\mathsf{TH}(PC_p)$ to the algebra $\mathcal{F}$ of all bounded functions on $\mathbb{T}_+ \times \overline{\mathbb{R}}$ with values in $\mathbb{C}^{2\times 2}$ on $\mathbb{T}_+^0 \times \overline{\mathbb{R}}$ and with values in $\mathbb{C}$ on $\{\pm 1\} \times \overline{\mathbb{R}}$. Moreover, there is a constant M such that

$$\|\mathrm{smb}_p\, A\| := \sup_{(t,\lambda)\in\mathbb{T}_+\times\overline{\mathbb{R}}} \|\mathrm{smb}_p\, A(t,\,\lambda)\|_\infty \le M \inf_{K\in K(l^p)} \|A+K\| \tag{3}$$

for every operator $A \in \mathsf{TH}(PC_p)$. Here, $\|B\|_\infty$ refers to the spectral norm of the matrix B.

(c) An operator $A \in \mathsf{TH}(PC_p)$ has the Fredholm property if and only if the function $\mathrm{smb}_p\, A$ is invertible in $\mathcal{F}$.

(d) The algebra $\mathsf{TH}(PC_p)/K(l^p)$ is inverse closed in $L(l^p)/K(l^p)$.

Before going into the details of the proof, we remark two consequences of Theorem 3 which will be needed in the next section.

Corollary 4 *Let $a,\, b \in PC_p$ and $T(a)+H(b)$ a Fredholm operator on l^p. Then*
(a) the function a is invertible in PC_p, and
(b) if b is continuous at ± 1, then $T(a)-H(b)$ is a Fredholm operator on l^p.

Proof. If $T(a)+H(b)$ is a Fredholm operator, then the diagonal matrices

$$\mathrm{smb}_p\,(T(a)+H(b))(t,\,\pm\infty) = \mathrm{diag}\,(a(t^\pm),\, a(\bar{t}^\pm))$$

are invertible for every $t \in \mathbb{T}_+^0$ and the numbers $\mathrm{smb}_p\,(T(a)+H(b))(1,\,\pm\infty) = a(1^\pm)$ and $\mathrm{smb}_p\,(T(a)+H(b))(-1,\,\pm\infty) = a((-1)^\pm)$ are not zero by assertion (a) of Theorem 3. Hence, a is invertible as an element of PC. Since the algebra PC_p is inverse closed in PC by Proposition 6.28 in [2], assertion (a) follows. The proof of assertion (b) is also immediate from the form of the symbol described in Theorem 3 (a). ∎

The remainder of this section is devoted to the proof of Theorem 3. We will need two auxiliary ingredients which we are going to recall first. Let $\mathcal{A}$ be a unital Banach algebra. The *center* of $\mathcal{A}$ is the set of all elements $a \in \mathcal{A}$ such that $ab = ba$ for all $b \in \mathcal{A}$. A *central* subalgebra of $\mathcal{A}$ is a closed subalgebra $\mathcal{C}$ of the center of $\mathcal{A}$ which contains the identity element. Thus, $\mathcal{C}$ is a commutative Banach algebra with compact maximal ideal space $M(\mathcal{C})$. For each maximal ideal x of $\mathcal{C}$, consider the smallest closed two-sided ideal $\mathcal{I}_x$ of $\mathcal{A}$ which contains x, and let Φ_x refer to the canonical homomorphism from $\mathcal{A}$ onto the quotient algebra $\mathcal{A}/\mathcal{I}_x$.

In contrast to the commutative setting, where $\mathcal{C}/x \cong \mathbb{C}$ for all $x \in M(\mathcal{C})$, the quotient algebras $\mathcal{A}/\mathcal{I}_x$ will depend on $x \in M(\mathcal{C})$ in general. In particular, it can happen that $\mathcal{I}_x = \mathcal{A}$ for certain maximal ideals x. In this case we *define* that $\Phi_x(a)$ is invertible in $\mathcal{A}/\mathcal{I}_x$ for every $a \in \mathcal{A}$.

Theorem 5 (Allan's local principle) *Let $\mathcal{C}$ be a central subalgebra of the unital Banach algebra $\mathcal{A}$. Then an element $a \in \mathcal{A}$ is invertible if and only if the cosets $\Phi_x(a)$ are invertible in $\mathcal{A}/\mathcal{I}_x$ for each $x \in M(\mathcal{C})$.*

Here is the second ingredient. Recall that an idempotent is an element p of an algebra such that $p^2 = p$.

Theorem 6 (Two idempotents theorem) *Let $\mathcal{A}$ be a Banach algebra with identity element e, let p and q be idempotents in $\mathcal{A}$, and let $\mathcal{B}$ denote the smallest closed subalgebra of $\mathcal{A}$ which contains p, q and e. Suppose that 0 and 1 belong to the spectrum $\sigma_{\mathcal{B}}(pqp)$ of pqp in $\mathcal{B}$ and that 0 and 1 are cluster points of that spectrum. Then*

(a) for each point $x \in \sigma_{\mathcal{B}}(pqp)$, there is a continuous algebra homomorphism $\Phi_x : \mathcal{B} \to \mathbb{C}^{2\times 2}$ which acts at the generators of $\mathcal{B}$ by

$$\Phi_x(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi_x(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Phi_x(q) = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}$$

where $\sqrt{x(1-x)}$ denotes any complex number with $(\sqrt{x(1-x)})^2 = x(1-x)$.

(b) an element $a \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if the matrices $\Phi_x(a)$ are invertible for every $x \in \sigma_{\mathcal{B}}(pqp)$.

(c) if $\sigma_{\mathcal{B}}(pqp) = \sigma_{\mathcal{A}}(pqp)$, then $\mathcal{B}$ is inverse closed in $\mathcal{A}$.

We proceed with the proof of Theorem 3, which we split into several steps.

Step 1: Localization. For every operator $A \in L(l^p)$, we denote its coset $A + K(l^p)$ in the Calkin algebra by A^π, and for every multiplier $a \in M^p$, we put $\tilde{a}(t) := a(1/t)$. The identities

$$T(ab) = T(a)T(b) + H(a)H(\tilde{b}) \quad \text{and} \quad H(ab) = T(a)H(b) + H(a)T(\tilde{b}), \tag{4}$$

which hold for arbitrary a, $b \in M^p$, together with the compactness of the Hankel operators $H(c)$ for $c \in C_p$ show that the set $\mathcal{C}_p$ of all cosets $T(c)^\pi$ with $c \in C_p$ and

$c = \tilde{c}$ forms a central subalgebra of the algebra $\mathsf{TH}(M^p)/K(l^p)$ and, in particular, of the algebra $\mathsf{TH}(PC_p)/K(l^p)$. One can, thus, reify Allan's local principle with $\mathsf{TH}(PC_p)/K(l^p)$ and $\mathcal{C}_p$ in place of $\mathcal{A}$ and $\mathcal{C}$, respectively. It is not hard to see that the maximal ideal space of $\mathcal{C}_p$ is homeomorphic to the arc $\mathbb{T}_+$, with $t \in \mathbb{T}_+$ corresponding to the maximal ideal $\{c \in \mathcal{C}_p : c(t) = 0\}$ of $\mathcal{C}_p$. We let $\mathcal{J}_t$ denote the smallest closed ideal of $\mathsf{TH}(PC_p)/K(l^p)$ which contains the maximal ideal t and write A_t^π for the coset $A^\pi + \mathcal{J}_t$ of $A \in \mathsf{TH}(PC_p)$. Instead of $T(a)_t^\pi$ and $H(b)_t^\pi$ we often write $T_t^\pi(a)$ and $H_t^\pi(b)$, respectively, and the local quotient algebra $(\mathsf{TH}(PC_p)/K(l^p))/\mathcal{J}_t$ is denoted by $\mathsf{TH}_t^\pi(PC_p)$ therefore. By Allan's local principle, we then have

$$\sigma_{\mathsf{TH}(PC_p)/K(l^p)}(A^\pi) = \cup_{t \in \mathbb{T}_+} \sigma_{\mathsf{TH}_t^\pi(PC_p)}(A_t^\pi) \tag{5}$$

for every $A \in \mathsf{TH}(PC_p)$.

Step 2: Local equivalence of multipliers. Let $a, b \in PC_p$ and $t \in \mathbb{T}_+$. We show that if $a(t^\pm) = b(t^\pm)$ and $a(\bar{t}^\pm) = b(\bar{t}^\pm)$, then $T_t^\pi(a) = T_t^\pi(b)$ and $H_t^\pi(a) = H_t^\pi(b)$. This fact will be used in what follows in order to replace multipliers by locally equivalent ones. It is clearly sufficient to prove that if $a \in PC_p$ satisfies $a(t^\pm) = a(\bar{t}^\pm) = 0$, then $T^\pi(a)$, $H^\pi(a) \in \mathcal{J}_t$. We will give this proof for $t \in \mathbb{T}_+^0$; the proof for for $t = \pm 1$ is similar.

Given $\varepsilon > 0$, let $f \in P\mathbb{C}$ such that $\|a - f\|_{M_p} < \varepsilon$. Then there is an open arc $U := (e^{-i\delta}t, e^{i\delta}t) \subset \mathbb{T}_+$ such that $|a(s)| < \varepsilon$ almost everywhere on $U \cup \overline{U}$ and such that f has at most one discontinuity in each of U and $\overline{U}$. Then $|f(s)| < 2\varepsilon$ for $s \in U \cup \overline{U}$. Now choose a real-valued function $\varphi_0 \in C^\infty(\mathbb{T})$ such that $\varphi_0(t) = 1$, the support of φ_0 is contained in U, and φ_0 is monotonously increasing on the arc $(e^{-i\delta}t, t)$ and monotonously decreasing on $(t, e^{i\delta}t)$. Set $\varphi := \varphi_0 + \widetilde{\varphi_0}$. Then $\varphi = \widetilde{\varphi}$, and

$$T^\pi(f) - T^\pi(f\varphi) = T^\pi(f(1-\varphi)) = T^\pi(f)T^\pi(1-\varphi) \in \mathcal{J}_t,$$

$$H^\pi(f) - H^\pi(f\varphi) = H^\pi(f(1-\varphi)) = H^\pi(f)T^\pi(1-\varphi) \in \mathcal{J}_t.$$

Since $\|f\varphi\|_\infty < 2\varepsilon$ and $\mathrm{Var}(f\varphi) < 8\varepsilon$, we conclude that $\|f\varphi\|_{M_p} < 10c_p\varepsilon$ from Stechkin's inequality. Thus, $\|T^\pi(f\varphi)\| < 10c_p\varepsilon$ and $\|H^\pi(f\varphi)\| < 10c_p\varepsilon$, with a constant c_p depending on p only. Thus, $T^\pi(a)$ differs from the element $T^\pi(f) - T^\pi(f\varphi) \in \mathcal{J}_t$ by the element $T^\pi(a-f) + T^\pi(f\varphi)$, which has a norm less than $(1 + 10c_p)\varepsilon$. Since $\varepsilon > 0$ is arbitrary and $\mathcal{J}_t$ is closed, this implies $T^\pi(a) \in \mathcal{J}_t$. Analogously, $H^\pi(a) \in \mathcal{J}_t$.

Step 3: The local algebras at $t \in \mathbb{T}_+^0$. We start with describing the local algebras $\mathsf{TH}_t^\pi(PC_p)$ at points $t \in \mathbb{T}_+^0$. Let χ_t denote the characteristic function of the arc in $\mathbb{T}$ which connects t with $\bar{t}$ and runs through the point -1. Clearly, $\chi_t \in PC_p$. The crucial observation, which is a simple consequence of the identities (4), is that the operator $T(\chi_t) + H(\chi_t)$ is an idempotent. Further, let $\varphi_t \in C_p$ be any multiplier such that $0 \le \varphi_t \le 1$, $\varphi_t(t) = 1$, $\varphi(\bar{t}) = 0$ and $\varphi_t + \widetilde{\varphi}_t = 1$. Again by (4), the coset $T_t^\pi(\varphi_t)$ is an idempotent.

We claim that the idempotents $p_t := T_t^\pi(\varphi_t)$ and $q_t := T_t^\pi(\chi_t) + H_t^\pi(\chi_t)$ together with the identity element $e := I_t^\pi$ generate the local algebra $\mathsf{TH}_t^\pi(PC_p)$. Let a, $b \in PC_p$. Then, using step 2,

$$\begin{aligned} T_t^\pi(a) \;=\; & a(t^+)T_t^\pi(\chi_t\varphi_t) + a(t^-)T_t^\pi((1-\chi_t)\varphi_t) + a(\bar{t}^-)T_t^\pi(\chi_t(1-\varphi_t)) \\ & + a(\bar{t}^+)T_t^\pi((1-\chi_t)(1-\varphi_t)). \end{aligned} \tag{6}$$

It is not hard to check that

$$\begin{aligned} T_t^\pi(\chi_t\varphi_t) &= p_t q_t p_t, \\ T_t^\pi((1-\chi_t)\varphi_t) &= p_t(e-q_t)p_t, \\ T_t^\pi(\chi_t(1-\varphi_t)) &= (e-p_t)q_t(e-p_t), \\ T_t^\pi((1-\chi_t)(1-\varphi_t)) &= (e-p_t)(e-q_t)(e-p_t). \end{aligned} \tag{7}$$

Let us verify the first of these identities, for example. By definition,

$$p_t q_t p_t = T_t^\pi(\varphi_t)T_t^\pi(\chi_t)T_t^\pi(\varphi_t) + T_t^\pi(\varphi_t)H_t^\pi(\chi_t)T_t^\pi(\varphi_t).$$

Since $T(\varphi_t)$ commutes with $T(\chi_t)$ modulo compact operators and $H(\widetilde{\varphi}_t)$ is compact, we can use the identities (4) to conclude

$$T_t^\pi(\varphi_t)T_t^\pi(\chi_t)T_t^\pi(\varphi_t) = T_t^\pi(\chi_t)T_t^\pi(\varphi_t) = T_t^\pi(\chi_t\varphi_t).$$

Further, due to the compactness of $H(\varphi_t)$ and $H(\widetilde{\varphi}_t)$,

$$T_t^\pi(\varphi_t)H_t^\pi(\chi_t)T_t^\pi(\varphi_t) = H_t^\pi(\varphi_t\chi_t)T_t^\pi(\varphi_t) = H_t^\pi(\varphi_t\chi_t\widetilde{\varphi}_t).$$

Since $\varphi_t\chi_t\widetilde{\varphi}_t$ is a continuous function, $H_t^\pi(\varphi_t\chi_t\widetilde{\varphi}_t) = 0$. This gives the first of the identities (7). The others follow in a similar way. Thus, (6) and (7) imply that $T_t^\pi(a)$ belongs to the algebra generated by e, p_t and q_t. Similarly, we write

$$\begin{aligned} H_t^\pi(b) \;=\; & b(t^+)H_t^\pi(\chi_t\varphi_t) + b(t^-)H_t^\pi((1-\chi_t)\varphi_t) + b(\bar{t}^-)H_t^\pi(\chi_t(1-\varphi_t)) \\ & + b(\bar{t}^+)H_t^\pi((1-\chi_t)(1-\varphi_t)) \end{aligned} \tag{8}$$

and use the identities

$$\begin{aligned} H_t^\pi(\chi_t\varphi_t) &= p_t q_t(e-p_t), \\ H_t^\pi((1-\chi_t)\varphi_t) &= -p_t q_t(e-p_t), \\ H_t^\pi(\chi_t(1-\varphi_t)) &= (e-p_t)q_t p_t, \\ H_t^\pi((1-\chi_t)(1-\varphi_t)) &= -(e-p_t)q_t p_t \end{aligned} \tag{9}$$

to conclude that $H_t^\pi(b)$ also belongs to the algebra generated by e, p_t and q_t. Thus, the algebra $\mathsf{TH}_t^\pi(PC_p)$ is subject to the two idempotents theorem.

In order to apply this theorem we have to determine the spectrum of the coset $p_t q_t p_t = T_t^\pi(\chi_t \varphi_t)$ in that algebra. We claim that

$$\sigma_{\mathsf{TH}_t^\pi(PC_p)}(T_t^\pi(\chi_t\varphi_t)) = \{\mu_q(\lambda) : \lambda \in \overline{\mathbb{R}}\} \tag{10}$$

with $1/p + 1/q = 1$. Let $a_t \in PC_p$ be a multiplier with the following properties:

(a) a_t is continuous on $\mathbb{T} \setminus \{t\}$ and has a jump at $t \in \mathbb{T}$.

(b) $a_t(t^+) = \chi_t(t^+) = 1$ and $a_t(t^-) = \chi_t(t^-) = 0$.

(c) a_t takes values in $\{\mu_q(\lambda) : \lambda \in \overline{\mathbb{R}}\}$ only.

(d) a_t is zero on the arc joining $-t$ to t which contains the point 1.

Then, by Theorem 1, the essential spectrum of the Toeplitz operator $T(a_t)$ in each of the algebras $L(l^p)/K(l^p)$ and $\mathsf{T}(PC_p)/K(l^p)$ is equal to the arc $\{\mu_q(\lambda) : \lambda \in \overline{\mathbb{R}}\}$. Hence, the essential spectrum of $T(a_t)$, now considered as an element of the algebra $\mathsf{TH}(PC_p)/K(l^p)$, is also equal to this arc. Hence,

$$\sigma_{\mathsf{TH}_t^\pi(PC_p)}(T_t^\pi(a_t)) \subseteq \{\mu_q(\lambda) : \lambda \in \overline{\mathbb{R}}\}$$

by Allan's local principle. Since $T_t^\pi(a_t) = T_t^\pi(\chi_t\varphi_t)$, this settles the inclusion $\subseteq$ in (10). For the reverse inclusion, let $b_t \in PC_p$ be a multiplier with the following properties:

(a) b_t is continuous on $\mathbb{T} \setminus \{t\}$ and has a jump at $t \in \mathbb{T}$.

(b) $b_t(t^\pm) = \chi_t(t^\pm)$.

(c) b_t takes values not in $\{\mu_q(\lambda) : \lambda \in \mathbb{R}\}$ on the arc joining $-t$ to t which contains the point -1.

(d) b_t is zero on the arc joining $-t$ to t which contains the point 1.

Then, again by Theorem 1, the essential spectrum of the Toeplitz operator $T(b_t)$ in each of the algebras $L(l^p)/K(l^p)$ and $\mathsf{T}(PC_p)/K(l^p)$ is equal to the union of the arc $\{\mu_q(\lambda) : \lambda \in \overline{\mathbb{R}}\}$ and the range of b_t. Hence, the essential spectrum of $T(b_t)$, now considered as an element of the algebra $\mathsf{TH}(PC_p)/K(l^p)$, is also equal to this union. Since b_t is continuous on $\mathbb{T} \setminus \{t\}$ by property (a), we have

$$\sigma_{\mathsf{TH}_s^\pi(PC_p)}(T_s^\pi(b_t)) = \{b_t(s),\, b_t(\bar{s})\}$$

for $s \in \mathbb{T}_+^0 \setminus \{t\}$. Since the points $b_t(s)$ and $b_t(\bar{s})$ do not belong to $\{\mu_q(\lambda) : \lambda \in \mathbb{R}\}$ by property (c), we conclude that the open arc $\{\mu_q(\lambda) : \lambda \in \mathbb{R}\}$ is contained in the local spectrum of $T(b_t)$ at t. Since spectra are closed, this implies

$$\{\mu_q(\lambda) : \lambda \in \overline{\mathbb{R}}\} \subseteq \sigma_{\mathsf{TH}_t^\pi(PC_p)}(T_t^\pi(b_t)).$$

Since $T_t^\pi(b_t) = T_t^\pi(\chi_t\varphi_t)$ by property (b), this settles the inclusion $\supseteq$ in (10).

Since $\nu_q(\lambda)^2 = \mu_q(\lambda)(1-\mu_q(\lambda))$, we can choose $\sqrt{\mu_q(\lambda)(1-\mu_q(\lambda))} = \nu_q(\lambda)$. With this choice and identities (6) – (9) it becomes evident that the two idempotents theorem associates with the coset $T_t^\pi(a) + H_t^\pi(b)$ the matrix function

$$\lambda \mapsto \begin{pmatrix} a(t^+)\mu_q(\lambda) + a(t^-)(1-\mu_q(\lambda)) & (b(t^+) - b(t^-))\nu_q(\lambda) \\ (b(\bar{t}^-) - b(\bar{t}^+))\nu_q(\lambda) & a(\bar{t}^-)(1-\mu_q(\lambda)) + a(\bar{t}^+)\mu_q(\lambda) \end{pmatrix}$$

on $\overline{\mathbb{R}}$.

Step 4: The local algebra at $1 \in \mathbb{T}_+$. Next we are going to consider the local algebra $\mathsf{TH}_1^\pi(PC_p)$ at the fixed point 1 of the mapping $t \mapsto \bar{t}$. Let $f : \mathbb{T} \to \mathbb{C}$ denote the function $e^{is} \mapsto 1 - s/\pi$ where $s \in [0, 2\pi)$. This function belongs to PC_p, and it has its only jump at the point $1 \in \mathbb{T}$ where $f(1^\pm) = \pm 1$. Using ideas from [17], it was shown in [18] by one of the authors that the Hankel operator $H(f)$ belongs to the Toeplitz algebra $\mathsf{T}(PC_p)$ and that its essential spectrum is given by

$$\sigma_{ess}(H(f)) = \{2i\,\nu_q(\lambda) : \lambda \in \overline{\mathbb{R}}\}. \tag{11}$$

(in fact, this identity was derived in [18] with p in place of q, which makes no difference since $\nu_p(-\lambda) = \nu_q(\lambda)$ for every λ.) Let χ_+ denote the characteristic function of the upper half-circle $\mathbb{T}_+$. Since every coset $T_1^\pi(a)$ with $a \in PC_p$ is a linear combination of the cosets I_1^π and $T_1^\pi(\chi_+)$ and every coset $H_1^\pi(b)$ is a multiple of the coset $H_1^\pi(f)$, the local algebra $\mathsf{TH}_1^\pi(PC_p)$ is singly generated (as a unital algebra) by the coset $T_1^\pi(\chi_+)$. In particular, $\mathsf{TH}_1^\pi(PC_p)$ is a commutative Banach algebra, and its maximal ideal space is homeomorphic to the spectrum of its generating element. Similar to the proof of (10) one can show that

$$\sigma_{\mathsf{TH}_1^\pi(PC_p)}(T_t^\pi(\chi_+)) = \{\mu_q(\lambda) : \lambda \in \overline{\mathbb{R}}\} \tag{12}$$

It is convenient for our purposes to identify the maximal ideal space of the algebra $\mathsf{TH}_1^\pi(PC_p)$ with $\overline{\mathbb{R}}$. The Gelfand transform of $T_t^\pi(\chi_+)$ is then given by $\lambda \mapsto \mu_q(\lambda)$ due to (12). Let h denote the Gelfand transform of $H_1^\pi(f)$. From (4) we obtain

$$H_1^\pi(f)^2 = T_1^\pi(f\tilde{f}) - T_1^\pi(f)T_1^\pi(\tilde{f}).$$

The function $f\tilde{f}$ is continuous at $1 \in \mathbb{T}$ and has the value -1 there, and the function $f + \tilde{f}$ is continuous at $1 \in \mathbb{T}$ and has the value 0 there. Thus,

$$H_1^\pi(f)^2 = -I_1^\pi + T_1^\pi(f)^2.$$

Since $T_1^\pi(f) = T_1^\pi(2\chi_+ - 1) = 2T_1^\pi(\chi_+) - I_1^\pi$ we conclude that

$$h(\lambda)^2 = (2\mu_q(\lambda) - 1)^2 - 1 = (\sinh(\pi(\lambda + i/q)))^{-2}$$

if $\lambda \in \mathbb{R}$ and by $h(\pm\infty) = 0$. By (11), this equality necessarily implies that

$$h(\lambda) = (\sinh(\pi(\lambda + i/q)))^{-1} = 2i\nu_q(\lambda)$$

if $\lambda \in \mathbb{R}$ and $h(\pm\infty) = 0$. Combining these results we find that the Gelfand transform of $T_1^\pi(a) + H_1^\pi(b)$ is the function

$$\lambda \mapsto a(1^+)\mu_q(\lambda) + a(1^-)(1 - \mu_q(\lambda)) + i\,(b(1^+) - b(1^-))\nu_q(\lambda).$$

Step 5: The local algebra at $-1 \in \mathbb{T}_+$. It remains to examine the local algebra $\mathsf{TH}_{-1}^\pi(PC_p)$ at the point -1. Let $\Lambda : l^2 \to l^2$ denote the mapping $(x_n)_{n\geq 0} \mapsto$

$((-1)^n x_n)_{n\geq 0}$. Clearly, $\Lambda^{-1} = \Lambda$, and one easily checks (perhaps most easily on the level of the matrix entries, which are Fourier coefficients) that

$$\Lambda^{-1} T(a) \Lambda = T(\hat{a}) \quad \text{and} \quad \Lambda^{-1} H(a) \Lambda = -H(\hat{a})$$

for $a \in PC_p$, where $\hat{a}(t) := a(-t)$. Thus, the mapping $A \mapsto \Lambda^{-1} A \Lambda$ is an automorphism of the algebra $\mathsf{TH}(PC_p)$, which maps compact operators to compact operators and induces, thus, an automorphism of the algebra $\mathsf{TH}(PC_p)/K(l^p)$. The latter maps the local ideal at 1 to the local ideal at -1 and vice versa and induces, thus, an isomorphism between the local algebras $\mathsf{TH}_1^\pi(PC_p)$ and $\mathsf{TH}_{-1}^\pi(PC_p)$, which sends $T_1^\pi(\chi_+)$ to $T_{-1}^\pi(1-\chi_+)$ and $H_1^\pi(\chi_+)$ to $-H_{-1}^\pi(1-\chi_+) = H_{-1}^\pi(\chi_+)$, respectively.

Step 6: From local to global invertibility. We have identified the right-hand sides of (1) and (2) as the functions which are locally associated with the operator $T(a) + H(b)$ via the two idempotents theorem and via Gelfand theory for commutative Banach algebras, respectively. It follows from the two idempotents theorem and from Gelfand theory that the so-defined mappings $\text{smb}_p\,(t,\,\lambda)$ extend to a continuous homomorphism from $\mathsf{TH}(PC_p)$ to $\mathbb{C}^{2\times 2}$ or $\mathbb{C}$, respectively, which combine to a continuous homomorphism from $\mathsf{TH}(PC_p)$ to the algebra $\mathcal{F}$. Allan's local principle then implies that the coset $A + K(l^p)$ of an operator $A \in \mathsf{TH}(PC_p)$ is invertible in $\mathsf{TH}(PC_p)/K(l^p)$ if and only if its symbol does not vanish. The proof of estimate (3) will base on Mellin homogenization arguments. We therefore postpone it until Section 5; see estimate (26).

Step 7: Inverse closedness. It remains to show that $\mathsf{TH}(PC_p)/K(l^p)$ is an inverse closed subalgebra of the Calkin algebra $L(l^p)/K(l^p)$. We shall prove this fact by using a *thin spectra argument* as follows: If $\mathcal{A}$ is a unital closed subalgebra of a unital Banach algebra $\mathcal{B}$, and if the spectrum in $\mathcal{A}$ of every element in a dense subset of $\mathcal{A}$ is thin, i.e. if its interior with respect to the topology of $\mathbb{C}$ is empty, then $\mathcal{A}$ is inverse closed in $\mathcal{B}$. See, e.g., [19], Corollary 1.2.32, for a simple proof of this argument.

Let $\mathcal{A}_0$ be the set of all operators of the form

$$A := \sum_{i=1}^{l} \prod_{j=1}^{k} (T(a_{ij}) + H(b_{ij})) \quad \text{with } A_{ij},\, b_{ij} \in P\mathbb{C}, \tag{13}$$

and write $\sigma_{ess}^{TH}(A)$ for the spectrum of A in $\mathsf{TH}(PC_p)/K(l^p)$. Then $\mathcal{A}_0/K(l^p)$ is dense in $\mathsf{TH}(PC_p)/K(l^p)$, and the assertion will follow once we have shown that $\mathsf{TH}(PC_p)/K(l^p)$ is thin for every $A \in \mathcal{A}_0$.

Given A of the form (13), let Ω denote the set of all discontinuities of the functions a_{ij} and b_{ij}, and put $\widetilde{\Omega} := (\Omega \cup \overline{\Omega}) \cap \mathbb{T}_+$. Clearly, $\widetilde{\Omega}$ is a finite set. By what we have shown above,

$$\sigma_{ess}^{TH}(A) = \cup_{(t,\lambda)\in\mathbb{T}_+\times\overline{\mathbb{R}}}\, \sigma(\text{smb}_p\,(A)(t,\,\lambda))$$

where $\sigma(B)$ stands for the spectrum (= set of the eigenvalues) of the matrix B. We write $\sigma_{ess}^{TH}(A)$ as $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ where

$$\begin{aligned} \Sigma_1 &:= \cup_{(t,\lambda)\in\{-1,1\}\times\overline{\mathbb{R}}}\, \sigma(\mathrm{smb}_p\,(A)(t,\,\lambda)), \\ \Sigma_2 &:= \cup_{(t,\lambda)\in(\mathbb{T}_+^0\setminus\widetilde{\Omega})\times\overline{\mathbb{R}}}\, \sigma(\mathrm{smb}_p\,(A)(t,\,\lambda)), \\ \Sigma_3 &:= \cup_{(t,\lambda)\in(\widetilde{\Omega}\setminus\{-1,1\})\times\overline{\mathbb{R}}}\, \sigma(\mathrm{smb}_p\,(A)(t,\,\lambda)). \end{aligned}$$

It is clear that Σ_1 is a set of measure zero. It is also clear that each set

$$\Sigma_{2,t} := \cup_{\lambda\in\overline{\mathbb{R}}}\, \sigma(\mathrm{smb}_p\,(A)(t,\,\lambda)) \quad \text{with } t \in \mathbb{T}_+^0 \setminus \widetilde{\Omega}$$

has measure zero. Since the functions a_{ij} and b_{ij} are piecewise constant, the mapping $t \mapsto \Sigma_{2,t}$ is constant on each connected component of $\mathbb{T}_+^0 \setminus \widetilde{\Omega}$, and the number of components is finite. Thus, Σ_2 is actually a finite union of sets of measure zero. Since $\widetilde{\Omega}$ is finite, it remains to show that each of the sets

$$\Sigma_{3,t} := \cup_{\lambda\in\overline{\mathbb{R}}}\, \sigma(\mathrm{smb}_p\,(A)(t,\,\lambda)) \quad \text{with } t \in \widetilde{\Omega} \setminus \{-1,1\}$$

has measure zero. For this goal it is clearly sufficient to show that each set

$$\Sigma_{3,t}^0 := \cup_{\lambda\in\mathbb{R}}\, \sigma(\mathrm{smb}_p\,(A)(t,\,\lambda)) \quad \text{with } t \in \widetilde{\Omega} \setminus \{-1,1\}$$

has measure zero. Let $t \in \widetilde{\Omega} \setminus \{-1,1\}$, and write $\mathrm{smb}_p\,(A)(t,\,\lambda)$ as $(c_{ij}(\lambda))_{i,j=1}^2$. The eigenvalues of this matrix are $s_\pm(\lambda) = (c_{11}(\lambda) + c_{22}(\lambda))/2 \pm \sqrt{r(\lambda)}$ where

$$r(\lambda) = (a_{11}(\lambda) + a_{22}(\lambda))^2/4 - (a_{11}(\lambda)a_{22}(\lambda) - a_{12}(\lambda)a_{21}(\lambda))$$

and where $\sqrt{r(\lambda)}$ is any complex number the square of which is $r(\lambda)$. Since r is composed by the meromorphic functions coth and $1/\sinh$, the set of zeros of r is discrete. Hence, $\mathbb{R} \setminus \{\lambda \in \mathbb{R} : r(\lambda) = 0\}$ is an open set, which as the union of an at most countable family of open intervals. Let I be one of these intervals. Then I can be represented as the union of countably many compact subintervals I_n such that the intersection $I_n \cap I_m$ consists of at most one point whenever $n \neq m$ and each set $r(I_n)$ is contained in a domain where a continuous branch, say f_n, of the function $z \mapsto \sqrt{z}$ exists. Then $\pm f_n \circ r : I_n \to \mathbb{C}$ is a continuously differentiable function, which implies that $(\pm f_n \circ r)(I_n)$ is a set of measure zero. Consequently, the associated sets $s_\pm(I_n)$ of eigenvalues have measure zero, too. Since the countable union of sets of measure zero has measure zero, we conclude that each set $\Sigma_{3,t}^0$ has measure zero, which finally implies that $\sigma_{ess}^{TH}(A) = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ has measure zero and is, thus, thin. This settles the proof of the inverse closedness and concludes the proof of Theorem 3. ■

We would like to mention that there is another proof of the inverse closedness assertion in the previous theorem which is based on ideas from [5] and which works also in other situations.

3 An extended Toeplitz algebra

In the proof of the announced index formula for Toeplitz plus Hankel operators, we shall need an extension of the results of the previous section to certain matrix operators. For $k \in \mathbb{N}$ and X a linear space, we let X_k and $X_{k\times k}$ stand for the linear spaces of all vectors of length k and of all $k \times k$-matrices with entries in X, respectively. If X is an algebra, then $X_{k\times k}$ becomes an algebra under the standard matrix operations. If X is a Banach space, then X_k and $X_{k\times k}$ become Banach spaces with respect to the norms

$$\|(x_j)_{j=1}^k\| = \sum_{j=1}^k \|x_j\| \quad \text{and} \quad \|(a_{ij})_{i,j=1}^k\| = k \sup_{1\le i,j\le k} \|a_{ij}\|. \tag{14}$$

If, moreover, X is a Banach algebra, then $X_{k\times k}$ is a Banach algebra with respect to the introduced norm. Actually, any other norm on X_k and any other compatible matrix norm on $X_{k\times k}$ will do the same job. Note also that if X is a C^*-algebra there is a unique norm (different from the above mentioned) which makes $X_{k\times k}$ to a C^*-algebra. Since we will not employ C^*-arguments, the choice (14) will be sufficient for our purposes.

Let $\mathsf{T}^0(PC_p)$ denote the smallest closed subalgebra of $L(l^p(\mathbb{Z}))$ which contains the projection P and all Laurent operators $L(a)$ with $a \in PC_p$. The algebra $\mathsf{T}^0(PC_p)$ contains $\mathsf{T}(PC_p)$ in the sense that the operator $PL(a)P : \operatorname{im} P \to \operatorname{im} P$ can be identified with the Toeplitz operator $T(a)$. For $k \in \mathbb{N}$, the matrix algebra $\mathsf{T}^0(PC_p)_{k\times k}$ will be also denoted by $\mathsf{T}^0_{k\times k}(PC_p)$. One can characterize $\mathsf{T}^0_{k\times k}(PC_p)$ also as the smallest closed subalgebra of $L(l^p(\mathbb{Z})_k)$ which contains all operators of the form $L(a)\operatorname{diag} P + L(b)\operatorname{diag} Q$ with $a,\, b \in (PC_p)_{k\times k}$, where $Q := I - P$, $\operatorname{diag} A$ stands for the operator on $L(l^p(\mathbb{Z})_k)$ which has $A \in L(l^p(\mathbb{Z}))$ at each entry of its main diagonal and zeros at all other entries, and where $L(a) = (L(a_{ij}))_{i,j=1}^k$ refers to the matrix Laurent operator with generating function $a = (a_{ij})_{i,j=1}^k$. Note that $K(l^p(\mathbb{Z})_k)$ is contained in $\mathsf{T}^0_{k\times k}(PC_p)$.

The Fredholm theory for operators in $\mathsf{T}^0_{k\times k}(PC_p)$ is well known. We will present it in a form which is convenient for our purposes. Our main tools are again Allan's local principle (Theorem 5) and a matrix version of the two idempotents theorem (Theorem 6) due to [5]. Here is the result.

Theorem 7 *Let $a,\, b \in (PC_p)_{k\times k}$.*
(a) The operator $A := L(a)\operatorname{diag} P + L(b)\operatorname{diag} Q$ is Fredholm on $l^p(\mathbb{Z})_k$ if and only if the matrix

$$(\operatorname{smb}_p A)(t,\, \lambda) =$$
$$\begin{pmatrix} a(t^-) + (a(t^+) - a(t^-))\operatorname{diag} \mu_q(\lambda) & (b(t^+) - b(t^-))\operatorname{diag} \nu_q(\lambda) \\ (a(t^+) - a(t^-))\operatorname{diag} \nu_q(\lambda) & b(t^+) - (b(t^+) - b(t^-))\operatorname{diag} \mu_q(\lambda) \end{pmatrix}$$

is invertible for every pair $(t,\, \lambda) \in \mathbb{T} \times \overline{\mathbb{R}}$.

(*b*) *The mapping* smb_p *defined in assertion* (*a*) *extends to a continuous algebra homomorphism from* $\mathsf{T}^0_{k\times k}(PC_p)$ *to the algebra* $\mathcal{F}$ *of all bounded functions on* $\mathbb{T}\times\overline{\mathbb{R}}$ *with values in* $\mathbb{C}_{2k\times 2k}$. *Moreover, there is a constant* M *such that*

$$\|\mathrm{smb}_p A\| := \sup_{(t,\lambda)\in\mathbb{T}_+\times\overline{\mathbb{R}}} \|\mathrm{smb}_p A(t,\,\lambda)\|_\infty \le M \inf_{K\in K(l^p(\mathbb{Z})_k)} \|A+K\| \tag{15}$$

for every operator $A\in\mathsf{T}^0_{k\times k}(PC_p)$.
(*c*) *An operator* $A\in\mathsf{T}^0_{k\times k}(PC_p)$ *has the Fredholm property on* $l^p(\mathbb{Z})_k$ *if and only if the function* $\mathrm{smb}_p A$ *is invertible in* $\mathcal{F}$.
(*d*) *The algebra* $\mathsf{T}^0_{k\times k}(PC_p)/K(l^p(\mathbb{Z})_k)$ *is inverse closed in the Calkin algebra* $L(l^p(\mathbb{Z})_k)/K(l^p(\mathbb{Z})_k)$.
(*e*) *If* $A\in\mathsf{T}^0_{k\times k}(PC_p)$ *is a Fredholm operator, then*

$$\mathrm{ind}\,A = -\mathrm{wind}\,(\det\mathrm{smb}_p A(t,\,\lambda)/(\det a_{22}(t,\,\infty)\det a_{22}(t,\,-\infty)))$$

where $\mathrm{smb}_p A = (a_{ij})^2_{i,j=1}$ *with* $k\times k$*-matrix-valued functions* a_{ij}.

It is a non-trivial fact that the function

$$W:\mathbb{T}\times\overline{\mathbb{R}},\quad (t,\,\lambda)\mapsto \det\mathrm{smb}_p A(t,\,\lambda)/(\det a_{22}(t,\,\infty)\det a_{22}(t,\,-\infty))$$

forms a closed curve in the complex plane. Thus, the winding number of W is well defined if A is a Fredholm operator.

The remainder of this section is devoted to the proof of Theorem 7. We shall mainly make use of results from Sections 2.3 - 2.5 in [11] and Chapter 6 in [2]. We will be quite sketchy when the arguments are close to those from the proof of Theorem 3.

Step 1: Spline spaces. We start with recalling some facts about spline spaces and operators thereon from [11]. Let $\chi_{[0,1]}$ denote the characteristic function of the interval $[0,\,1]\subset\mathbb{R}$ and, for $n\in\mathbb{N}$, let S_n denote the smallest closed subspace of $L^p(\mathbb{R})$ which contains all functions

$$\varphi_{k,n}(t) := \chi_{[0,1]}(nt-k),\quad t\in\mathbb{R},$$

where $k\in\mathbb{Z}$. The space $l^p(\mathbb{Z})$ can be identified with each of the spaces S_n in the sense that a sequence (x_k) is in $l^p(\mathbb{Z})$ if and only if the series $\sum_{k\in\mathbb{Z}} x_k\varphi_{k,n}$ converges in $L^p(\mathbb{R})$ and that

$$\left\|\sum x_k\varphi_{k,n}\right\|_{L^p(\mathbb{R})} = n^{-1/p}\,\|(x_k)\|_{l^p(\mathbb{Z})}$$

in this case. Thus, the linear operator

$$E_n : l^p(\mathbb{Z})\to S_n\subset L^p(\mathbb{R}),\quad (x_k)\mapsto n^{1/p}\sum x_k\varphi_{k,n},$$

and its inverse $E_{-n} : L^p(\mathbb{R}) \supset S_n \to l^p(\mathbb{Z})$ are isometries for every n. Further we define operators

$$L_n : L^p(\mathbb{R}) \to S_n, \quad u \mapsto n \sum_{k\in\mathbb{Z}} \langle u, \varphi_{k,n}\rangle \varphi_{k,n}$$

with respect to the sesqui-linear form $\langle u, v\rangle := \int_{\mathbb{R}} u\bar{v}dx$, where $u \in L^p(\mathbb{R})$ and $v \in L^q(\mathbb{R})$ with $1/p + 1/q = 1$. It is easy to see that every L_n is a projection operator with norm 1 and that the L_n converge strongly to the identity operator on $L^p(\mathbb{R})$ as $n \to \infty$. Finally we set

$$Y_t : l^p(\mathbb{Z}) \to l^p(\mathbb{Z}), \quad (x_k) \mapsto (t^{-k}x_k) \quad \text{for } t \in \mathbb{T}.$$

Clearly, Y_t is an isometry, and $Y_t^{-1} = Y_{t^{-1}}$. One easily checks that $Y_t^{-1}L(a)Y_t = L(a_t)$ with $a_t(s) = a(ts)$ for every multiplier a, which implies in particular that $Y_t^{-1}\mathsf{T}^0(PC_p)Y_t = \mathsf{T}^0(PC_p)$.

Step 2: Some homomorphisms. In Sections 2.3.3 and 2.5.2 of [11] it is shown that, for every $A \in \mathsf{T}^0(PC_p)$ and every $t \in \mathbb{T}$, the strong limit

$$\text{smb}_t A := \text{s-lim}_{n\to\infty} E_n Y_t^{-1} A Y_t E_{-n} L_n$$

exists and that the mapping smb_t is a bounded unital algebra homomorphism. This homomorphism can be extended in a natural way to the matrix algebra $\mathsf{T}^0_{k\times k}(PC_p)$. We denote this extension by $\text{smb}_t A$ again.

In order to characterize the range of the homomorphism smb_t, we have to introduce some operators on $L^p(\mathbb{R})$. Let χ_+ stand for the characteristic function of the interval $\mathbb{R}^+ = [0, \infty)$ and $\chi_+ I$ for the operator of multiplication by χ_+. Further, $S_{\mathbb{R}}$ refers to the singular integral operator

$$(S_{\mathbb{R}} f)(t) := \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(s)}{s-t}\, ds,$$

with the integral understood as a Cauchy principal value. Both $\chi_+ I$ and $S_{\mathbb{R}}$ are bounded on $L^p(\mathbb{R})$, and $S_{\mathbb{R}}^2 = I$. Thus, the operators $P_{\mathbb{R}} := (I + S_{\mathbb{R}})/2$ and $Q_{\mathbb{R}} := I - P_{\mathbb{R}}$ are bounded projections on $L^p(\mathbb{R})$. We let $\Sigma_k^p(\mathbb{R})$ stand for the smallest closed subalgebra of $L(L^p(\mathbb{R})_k)$ which contains the operators $\text{diag}\, \chi_+ I$, $\text{diag}\, S_{\mathbb{R}}$, and all operators of multiplication by constant $k \times k$-matrix-valued functions.

Theorem 8 *Let $t \in \mathbb{T}$. Then*
(*a*) $\text{smb}_t\, \text{diag}\, P = \text{diag}\, \chi_+ I$.
(*b*) $\text{smb}_t\, L(a) = a(t^+)\text{diag}\, Q_{\mathbb{R}} + a(t^-)\text{diag}\, P_{\mathbb{R}}$ *for $a \in (PC_p)_{k\times k}$.*
(*c*) $\text{smb}_t\, K = 0$ *for every compact operator K.*
(*d*) smb_t *maps the algebra $\mathsf{T}^0_{k\times k}(PC_p)$ onto $\Sigma_k^p(\mathbb{R})$.*
(*e*) *The algebra $\Sigma_k^p(\mathbb{R})$ is inverse closed in $L(L^p(\mathbb{R})_k)$.*

Assertion (c) of the previous theorem implies that every mapping smb_t induces a natural quotient homomorphism from $\mathsf{T}^0(PC_p)/K(l^p(\mathbb{Z}))$ to $\Sigma_1^p(\mathbb{R})$. We denote this quotient homomorphism by smb_t again. It now easily seen that the estimate (15) holds for every $A \in \mathsf{T}^0_{k\times k}(PC_p)$ (with the constant $M = 1$ for $k = 1$).

Step 3: The Fredholm property. Since the commutator $L(a)P - PL(a)$ is compact for every $a \in C_p$, the algebra $\mathcal{C}_p := \{\mathrm{diag}\, L(a) : a \in C_p\}/K(l^p(\mathbb{Z})_k)$ lies in the center of the algebra $\mathcal{A} := \mathsf{T}^0_{k\times k}(PC_p)/K(l^p(\mathbb{Z})_k)$. It is not hard to see that $\mathcal{C}_p$ is isomorphic to C_p; hence the maximal ideal space of $\mathcal{C}_p$ is homeomorphic to the unit circle $\mathbb{T}$. In accordance with Allan's local principle, we introduce the local ideals $\mathcal{J}_t$ and the local algebras $\mathcal{A}_t := \mathcal{A}/\mathcal{J}_t$ at $t \in \mathbb{T}$.

By Theorem 8 (b), the local ideal $\mathcal{J}_t$ lies in the kernel of smb_t. We denote the related quotient homomorphism by smb_t again. Thus, smb_t is an algebra homomorphism from $\mathcal{A}_t$ onto $\Sigma_k^p(\mathbb{R})$, which sends the local cosets containing the operators $\mathrm{diag}\, P$ and $L(a)$ with $a \in (PC_p)_{k\times k}$ to $\mathrm{diag}\, \chi_+ I$ and $a(t^+)\,\mathrm{diag}\, Q_\mathbb{R} + a(t^-)\,\mathrm{diag}\, P_\mathbb{R}$, respectively. By Theorem 2.3 in [11], this homomorphism is injective, i.e., it is an isomorphism between $\mathcal{A}_t$ and $\Sigma_k^p(\mathbb{R})$.

Since $P_\mathbb{R}$ and $\mathrm{diag}\, \chi_+ I$ are projections, the algebra $\Sigma_k^p(\mathbb{R})$ is subject to the two projections theorem with coefficients, as derived in [5]. Alternatively, this algebra can be described by means of the Mellin symbol calculus, see Section 2.1 in [11]. In each case, the result is that an operator of the form

$$(a^+\mathrm{diag}\, \chi_+ I + a^-\mathrm{diag}\, \chi_- I)\,\mathrm{diag}\, P_\mathbb{R} + (b^+\mathrm{diag}\, \chi_+ I + b^-\mathrm{diag}\, \chi_- I)\,\mathrm{diag}\, Q_\mathbb{R} \qquad (16)$$

where $\chi_- := 1 - \chi_+$ and $a^\pm,\, b^\pm \in \mathbb{C}_{k\times k}$ is invertible if and only if the $(2k)\times(2k)$-matrix-valued function

$$\lambda \mapsto \begin{pmatrix} a^+\mathrm{diag}\,(1-\mu_p(\lambda)) + a^-\mathrm{diag}\,\mu_p(\lambda) & (b^+ - b^-)\,\mathrm{diag}\,\nu_p(\lambda) \\ (a^+ - a^-)\,\mathrm{diag}\,\nu_p(\lambda) & b^+\mathrm{diag}\,\mu_p(\lambda) + b^-\mathrm{diag}\,(1-\mu_p(\lambda)) \end{pmatrix}$$

is invertible at each point $\lambda \in \overline{\mathbb{R}}$. Note that the function

$$\lambda \mapsto a^+\mathrm{diag}\,(1-\mu_p(\lambda)) + a^-\mathrm{diag}\,\mu_p(\lambda)$$

is continuous on $\overline{\mathbb{R}}$ and that this function connects a^+ with a^- if λ runs from $-\infty$ to $+\infty$. For the sake of index computation, one would prefer to work with a function which connects a^- with a^+ if λ increases. Since $\mu_p(-\lambda) = 1 - \mu_q(\lambda)$ and $\nu_p(-\lambda) = \nu_q(\lambda)$ with q satisfying $1/p + 1/q = 1$, we obtain that the operator A in (16) is invertible if and only if the matrix function

$$\lambda \mapsto \begin{pmatrix} a^+\mathrm{diag}\,\mu_q(\lambda) + a^-\mathrm{diag}\,(1-\mu_q(\lambda)) & (b^+ - b^-)\,\mathrm{diag}\,\nu_q(\lambda) \\ (a^+ - a^-)\,\mathrm{diag}\,\nu_q(\lambda) & b^+\mathrm{diag}\,(1-\mu_q(\lambda)) + b^-\mathrm{diag}\,\mu_q(\lambda) \end{pmatrix}$$

is invertible on $\overline{\mathbb{R}}$. This observation, together with the local principle, implies that the coset $L(a)\mathrm{diag}\, P + L(b)\mathrm{diag}\, Q + K(l^p(\mathbb{Z})_k)$ is invertible in the quotient algebra $\mathsf{T}^0_{k\times k}(PC_p)/K(l^p(\mathbb{Z})_k)$ if and only if the matrix function in assertion (a) of Theorem 7 is invertible. In particular, this gives the "if"-part of assertion (a). The "only if"-part of this assertion follows from the inverse closedness assertion (d), which can be proved using ideas from [5], where inverse closedness issues of two projections algebras with coefficients are studied. The proof of assertions (b) and (c) of Theorem Theorem 7 is then standard.

Step 4: The index formula. It remains to prove the index formula (e). First we have to equip the cylinder $\mathbb{T}\times\overline{\mathbb{R}}$ with a suitable topology, which will be different from the usual product topology. We provide $\mathbb{T}$ with the counter-clockwise orientation and $\overline{\mathbb{R}}$ with the natural orientation given by the order $<$. Then the desired topology is determined by the system of neighborhoods $U(t_0,\,\lambda_0)$ of the point $(t_0,\,\lambda_0)\in\mathbb{T}\times\overline{\mathbb{R}}$, defined by

$$U(t_0,\,-\infty)=\{(t,\,\lambda)\in\mathbb{T}\times\overline{\mathbb{R}}:|t-t_0|<\delta, t\prec t_0\}\cup\{(t_0,\,\lambda)\in\mathbb{T}\times\overline{\mathbb{R}}:\lambda<\varepsilon\},$$

$$U(t_0,\,+\infty)=\{(t,\,\lambda)\in\mathbb{T}\times\overline{\mathbb{R}}:|t-t_0|<\delta, t_0\prec t\}\cup\{(t_0,\,\lambda)\in\mathbb{T}\times\overline{\mathbb{R}}:\varepsilon<\lambda\}$$

if $\lambda_0=\pm\infty$ and by

$$U(t_0,\lambda_0)=\{(t_0,\,\lambda)\in\mathbb{T}\times\overline{\mathbb{R}}:\lambda_0-\delta_1<\lambda<\lambda_0+\delta_2\}$$

if $\lambda_0\in\mathbb{R}$, where $\varepsilon\in\mathbb{R}$ and δ, δ_1,δ_2 are sufficiently small positive numbers, and where $t\prec s$ means that t precedes s with respect to the chosen orientation of $\mathbb{T}$. Note that the cylinder $\mathbb{T}\times\overline{\mathbb{R}}$, provided with the described topology, is just a homeomorphic image of the cylinder $\mathbb{T}\times[0,\,1]$, provided with the Gohberg-Krupnik topology. The latter has been shown by Gohberg and Krupnik to be (homeomorphic to) the maximal ideal space of the commutative Banach algebra $\mathsf{T}(PC_p)/K(l^p)$; see [6] and [2], Proposition 6.28. If one identifies $\mathbb{T}\times[0,\,1]$ with $\mathbb{T}\times\overline{\mathbb{R}}$, then the Gelfand transform of a coset $A+K(l^p)$ of $A\in\mathsf{T}(PC_p)$ is just the function $\Gamma(A)$ defined in Theorem 1.

It is an important point to mention that while the function $\mathrm{smb}_p\,A$ for $A\in\in\mathsf{T}^0_{k\times k}(PC_p)$ is *not* continuous on $\mathbb{T}\times\overline{\mathbb{R}}$ (just consider the south-east entry of $\mathrm{smb}_p\,(L(a)P+L(b)Q)$), the function

$$(t,\,\lambda)\mapsto\det\mathrm{smb}_p\,A(t,\,\lambda)/(\det a_{22}(t,\,\infty)\det a_{22}(t,\,-\infty)$$

is continuous on $\mathbb{T}\times\overline{\mathbb{R}}$. This non-trivial fact was observed by Gohberg and Krupnik in a similar situation when studying the Fredholm theory for singular integral operators with piecewise continuous coefficients (see [7]; an introduction to this topic is also in Chapter V of [15]).

We will establish the index formula by employing a method which also goes back to Gohberg and Krupnik and is known as linear extension. This method has found its first applications in the Fredholm theory of one-dimensional singular integral equations; see [10, 15]. We will use this method in the slightly different context of Toeplitz plus Hankel operators. Therefore, and for the readers' convenience, we recall it here.

Let $\mathcal{B}$ be a unital ring with identity element e. With every $h \times r$-matrix $\beta := (b_{jl})_{j,l=1}^{h,r}$ with entries in $\mathcal{B}$, we associate the element

$$\mathrm{el}(\beta) = \sum_{j=1}^{h} b_{j1} \dots b_{jr} \in \mathcal{B} \tag{17}$$

generated by β and call the b_{jl} the generators of $\mathrm{el}(\beta)$. For each element of this form, there is a canonical matrix $\mathrm{ext}(\beta) \in \mathcal{B}_{s\times s}$ with $s = h(r+1)+1$ with entries in the set $\{0, e, b_{jk} : 1 \le j \le h, 1 \le k \le r\}$ and with the property that $\mathrm{el}(\beta)$ is invertible in $\mathcal{B}$ if and only if $\mathrm{ext}(\beta)$ is invertible in $\mathcal{B}_{s\times s}$. Actually, a matrix with this property can be constructed as follows. Let

$$\mathrm{ext}(\beta) := \begin{pmatrix} Z & X \\ Y & 0 \end{pmatrix} = \begin{pmatrix} e_{h(r+1)} & 0 \\ W & e \end{pmatrix} \begin{pmatrix} e_{h(r+1)} & 0 \\ 0 & \mathrm{el}(\beta) \end{pmatrix} \begin{pmatrix} Z & X \\ 0 & e \end{pmatrix} \tag{18}$$

where e_l denotes the unit element of $\mathcal{B}_{l\times l}$,

$$Z := e_{h(r+1)} + \begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & B_r \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with $B_j := \mathrm{diag}\,(b_{1j}, b_{2j}, \dots, b_{hj})$, X is the column $-(0, \dots, 0, e, \dots, e)^T$ with hr zeros followed by h identity elements, Y is the row $(e, \dots, e, 0, \dots, 0)$ with h identity elements followed by hr zeros, and $W := (M_0, M_1, \dots, M_r)$ with $M_0 := (e, \dots, e)$ consisting of h identity elements and

$$M_j := (b_{11}b_{12} \dots b_{1j}, b_{21}b_{22} \dots b_{2j}, \dots, b_{h1}b_{h2} \dots b_{hj})$$

for $j = 1, \dots, r$. The matrix $\mathrm{ext}(\beta)$ in (18) is called the linear extension of $\mathrm{el}(\beta)$.

Since the outer factors on the right-hand side of (18) are invertible, it follows indeed that $\mathrm{el}(\beta)$ is invertible in $\mathcal{B}$ if and only if its linear extension $\mathrm{ext}(\beta)$ is invertible in $\mathcal{B}_{s\times s}$. As a special case we obtain that if the b_{jl} are bounded linear operators on some Banach space B, then $\mathrm{el}(\beta)$ is a Fredholm operator on B if and only if $\mathrm{ext}(\beta)$ is a Fredholm operator on $L(B)_{s\times s} = L(B_s)$. Moreover, $\mathrm{ind}\,\mathrm{el}(\beta) = \mathrm{ind}\,\mathrm{ext}(\beta)$ is this case.

We shall apply this observation for $B = l^p(\mathbb{Z})_k$ and for the generating operators

$$b_{jl} := L(c_{jl})\,\mathrm{diag}\,P + L(d_{jl})\,\mathrm{diag}\,Q \quad \text{with } c_{jl},\, d_{jl} \in (PC_p)_{k\times k}. \tag{19}$$

Put $\beta := (b_{jl})_{j,l=1}^{h,r}$, $\gamma := (L(c_{jl}))_{j,l=1}^{h,r}$ and $\delta := (L(d_{jl}))_{j,l=1}^{h,r}$. The linear extensions of γ and δ are Laurent operators again; thus $\text{ext}(\gamma) = L(c)$ and $\text{ext}(\delta) = L(d)$ with piecewise continuous multipliers c and d. Moreover,

$$\text{ext}(\beta) = L(c)\,\text{diag}\,P + L(d)\,\text{diag}\,Q. \tag{20}$$

If $\text{el}(\beta)$ is a Fredholm operator then, by Theorem 7 (a), the matrices $c(t^{\pm})$ and $d(t^{\pm})$ are invertible for every $t \in \mathbb{T}$. Hence, c and d are invertible in $(PC_p)_{ks\times ks}$. This fact together with the above observation implies that the operator $\text{el}(\beta)$ is Fredholm on $l^p(\mathbb{Z})_k$ if and only if its linear extension $\text{ext}(\beta)$ is Fredholm on $l^p(\mathbb{Z})_{ks}$, which on its hand holds if and only if the Toeplitz operator $T(d^{-1}c)$ is Fredholm on l^p_{ks}, and that the Fredholm indices of the operators $\text{el}(\beta)$, $\text{ext}(\beta)$ and $T(d^{-1}c)$ coincide in this case. The symbol of the Toeplitz operator $T(d^{-1}c)$ is the function

$$\text{smb}_p\,(T(d^{-1}c))(t,\,\lambda) = (d^{-1}c)(t^+)\text{diag}\,\mu_q(\lambda) + (d^{-1}c)(t^-)\text{diag}\,(1-\mu_q(\lambda))$$

(which stems from the matrix-version of Theorem 1), and $\text{smb}_p\,(\text{ext}(\beta)) =: (a_{ij})_{i,j=1}^2$ is related with $\text{smb}_p\,(T(d^{-1}c))$ via

$$\det \text{smb}_p\,(T(d^{-1}c))(t,\,\lambda) = \det(\text{smb}_p\,\text{ext}(\beta))(t,\lambda)/(\det a_{22}(t,\,\infty)\det a_{22}(t,-\infty))$$

as can be checked directly; see [10, 15] for details. This fact can finally be used to derive the index formula for Fredholm operators of the form $\text{el}(\beta)$ with the entries of β given by (19). For details we refer to [10, 15] again, where a similar setting is considered.

Since the operators $\text{el}(\beta)$ lie dense in $\mathsf{T}^0_{k\times k}(PC_p)$, the index formula for a Fredholm operator in this algebra follows by a standard approximation argument. To carry out this argument one has to use the estimate

$$\|\text{smb}_p\,\text{el}(\beta)\| \le M \inf_{K\in K(l^p(\mathbb{Z})_k)} \|\text{el}(\beta)+K\|$$

with M independent of β, which is an immediate consequence of (15). ∎

4 The index formula for $T+H$-operators

Our next goal is to provide an index formula for Fredholm operators of the form $T(a) + H(b)$ on l^p where a, b are multipliers in PC_p with a finite set of discontinuities. We start with a couple of lemmata.

Lemma 9 *If $a \in C(\mathbb{T}) \cap M^{\langle p\rangle}$, then $H(a)$ is compact on l^p.*

Proof. It is shown in Proposition 2.45 in [2] that $C(\mathbb{T}) \cap M^{\langle p\rangle} \subseteq C_p$ (in fact it is shown there that the closure of $C(\mathbb{T}) \cap M^{\langle p\rangle}$ in the multiplier norm equals C_p)

and in Theorem 2.47 that $H(a)$ is compact on l^p if $a \in C_p$. ■

For a subset Ω of $\mathbb{T}$, let $PC(\Omega)$ stand for the set of all piecewise continuous functions which are continuous on $\mathbb{T} \setminus \Omega$, and put $PC_{\langle p\rangle}(\Omega) := PC(\Omega) \cap M^{\langle p\rangle}$. Thus, $C_{\langle p\rangle} := PC_{\langle p\rangle}(\emptyset) = C(\mathbb{T}) \cap M^{\langle p\rangle}$. From 6.27 in [2] one concludes that $PC_{\langle p\rangle}(\Omega) \subseteq PC_p$ if Ω is finite.

In what follows, we specify $\Omega_0 := \{\tau_1, \ldots, \tau_m\}$ to be a finite subset of $\mathbb{T}\setminus\{\pm 1\}$ and put $\Omega := \Omega_0 \cup \{\pm 1\}$. Let $\varphi_0 \in C_{\langle p\rangle}$ be a multiplier which satisfies $\varphi = \tilde{\varphi}$, takes its values in $[0, 1]$, and is identically 1 on a certain neighborhood of $\{-1, 1\}$ and identically 0 on a certain neighborhood of $\Omega_0 \cup \overline{\Omega_0}$. Moreover, we suppose that $\varphi_0^2 + \varphi_1^2 = 1$ where $\varphi_1 := 1 - \varphi_0$.

Lemma 10 *Let $c \in PC_{\langle p\rangle}(\{-1, 1\})$ and $d \in PC_{\langle p\rangle}(\Omega_0)$. Then the operators $H(c)T(d) - H(cd\varphi_0)$ and $T(c)H(d) - H(cd\varphi_1)$ are compact on l^p.*

Proof. We write $H(c)T(d) = H(c)T(d)T(\varphi_0) + H(c)T(d)T(\varphi_1)$ with

$$\begin{aligned} H(c)T(d)T(\varphi_0) &= H(c)\left(T(d\varphi_0) - H(d)H(\widetilde{\varphi_0})\right) \\ &= H(cd\varphi_0) - T(c)H(\widetilde{d\varphi_0}) - H(c)H(d)H(\widetilde{\varphi_0}), \end{aligned}$$

$$\begin{aligned} H(c)T(d)T(\varphi_1) &= H(c)T(\varphi_1)T(d) + H(c)\left(T(d)T(\varphi_1) - T(\varphi_1)T(d)\right) \\ &= \left(H(c\varphi_1) - T(c)H(\widetilde{\varphi_1})\right)T(d) \\ &\qquad + H(c)\left(H(d)H(\widetilde{\varphi_1}) - H(\varphi_1)H(\tilde{d})\right). \end{aligned}$$

The operators $H(\widetilde{d\varphi_0})$, $H(\widetilde{\varphi_0})$, $H(c\varphi_1)$, $H(\varphi_1)$ and $H(\widetilde{\varphi_1})$ are compact by Lemma 9, which gives the first assertion. The proof of the second assertion proceeds similarly. ■

Lemma 11 *Let $a_0, b_0 \in PC_{\langle p\rangle}(\{-1, 1\})$ and $a_1, b_1 \in PC_{\langle p\rangle}(\Omega_0)$. Then the operator*

$$(T(a_0) + H(b_0))(T(a_1) + H(b_1)) - (T(a_0a_1) + H(a_1b_0\varphi_0) + H(a_0b_1\varphi_1))$$

is compact on l^p.

Proof. We write $(T(a_0) + H(b_0))(T(a_1) + H(b_1))$ as

$$\begin{aligned} &T(a_0)T(a_1) + T(a_0)H(b_1) + H(b_0)T(a_1) + H(b_0)H(b_1) \\ &\quad = T(a_0a_1) + K_1 + H(a_0b_1\varphi_1) + K_2 + H(b_0a_1\varphi_0) + K_3 + K_4 \end{aligned}$$

where $K_1 := T(a_0)T(a_1) - T(a_0a_1)$ and $K_4 := H(b_0)H(b_1) = T(b_0)T(\widetilde{b_1}) - T(b_0\widetilde{b_1})$ are compact on l^p by Proposition 6.29 in [2], and $K_2 := T(a_0)H(b_1) - H(a_0b_1\varphi_1)$ and $K_3 := H(b_0)T(a_1) - H(b_0a_1\varphi_0)$ are compact by Lemma 10. ■

The following proposition provides us with a key observation; it will allow us to separate the discontinuities in Ω_0 and $\{-1, 1\}$.

Proposition 12 *Let a, $b \in PC_{\langle p \rangle}(\Omega)$. If the operator $T(a) + H(b)$ is Fredholm on l^p, then there are functions a_0, $b_0 \in PC_{\langle p \rangle}(\{-1, 1\})$ and a_1, $b_1 \in PC_{\langle p \rangle}(\Omega_0)$ such that $T(a_0) + H(b_0)$ and $T(a_1) + H(b_1)$ are Fredholm operators on l^p and the difference*

$$(T(a_0) + H(b_0))(T(a_1) + H(b_1)) - (T(a) + H(b))$$

is compact.

Proof. If $T(a)+H(b)$ is Fredholm on l^p, then a is invertible in PC_p by Corollary 4 (a). Since the maximal ideal space of PC_p is independent on p and $a \in PC_{\langle p \rangle}$, one even has $a^{-1} \in PC_{\langle p \rangle}$.

Let U and V be open neighborhoods of $\{-1, 1\}$ and $\Omega_0 \cup \overline{\Omega_0}$, respectively, such that $\operatorname{clos} U \cap \operatorname{clos} V = \emptyset$. We will assume moreover that $U = U_{-1} \cup U_1$ is the union of two open arcs such that $\pm 1 \in U_{\pm 1}$, and that $V = V_+ \cup V_-$ is the union of two open arcs such that $V_+ \subseteq \mathbb{T}_+^0$ and $V_- \subseteq \mathbb{T} \setminus \mathbb{T}_+^0$. Note that these conditions imply that $\operatorname{clos} U_{-1} \cap \operatorname{clos} U_1 = \emptyset$.

Now we choose a continuous piecewise (with respect to a finite partition of $\mathbb{T}$) linear function c on $\mathbb{T}$ which is identically 1 on $\operatorname{clos} V$, coincides with a on ∂U, and does not vanish on $\mathbb{T} \setminus U$. This function is of bounded total variation; thus $c \in C(\mathbb{T}) \cap M^{\langle p \rangle}$, whence $c \in C_p$ as mentioned in the proof of Lemma 9. Put $a_0 := a\chi_U + c\chi_{\mathbb{T}\setminus U}$. Then $a_0 \in PC_{\langle p \rangle}$ and $a_0^{-1} \in PC_{\langle p \rangle}$. Further, set $a_1 := a_0^{-1}a$. The function a_1 is identically 1 on U and coincides with a on V. Since $PC_{\langle p \rangle}$ is an algebra, a_1 belongs to $PC_{\langle p \rangle}$. Finally, set $b_0 := b\varphi_0$ and $b_1 := b\varphi_1$, with φ_0 and φ_1 as in front of Lemma 10.

The above construction guarantees that a_0, $b_0 \in PC_{\langle p \rangle}(\{-1, 1\})$ and a_1, $b_1 \in PC_{\langle p \rangle}(\Omega_0)$, and the operator

$$(T(a_0) + H(b_0))(T(a_1) + H(b_1)) - (T(a_0a_1) + H(a_1b_0\varphi_0) + H(a_0b_1\varphi_1))$$

is compact on l^p by Lemma 11. The functions $(a_1 - 1)b_0\varphi_0$ and $(a_0 - 1)b_1\varphi_1$ vanish identically on a certain neighborhood of Ω by their construction. Hence, the Hankel operators $H((a_1-1)b_0\varphi_0)$ and $H((a_0-1)b_1\varphi_1)$ are compact by Lemma 9, which implies that the operator

$$(T(a_0) + H(b_0))(T(a_1) + H(b_1)) - (T(a_0a_1) + H(b_0\varphi_0) + H(b_1\varphi_1))$$

is compact. Since $a_0a_1 = a$ and $b_0\varphi_0 + b_1\varphi_1 = b(\varphi_0^2 + \varphi_1^2) = b$, and since $T(a_0) + H(b_0)$ and $T(a_1) + H(b_1)$ are Fredholm operators on l^p by Theorem 3, the assertion follows. ∎

By the previous proposition,

$$\operatorname{ind}(T(a) + H(b)) = \operatorname{ind}(T(a_0) + H(b_0)) + \operatorname{ind}(T(a_1) + H(b_1)).$$

Since $H(b_0) \in \mathsf{T}(PC_p)$ as already mentioned, and since an index formula for Fredholm operators in $\mathsf{T}(PC_p)$ is known (see, e.g., 6.40 in [2]), the determination of $\operatorname{ind}(T(a_0) + H(b_0))$ is no serious problem. The following theorem provides us with a basic step on the way to compute the index of $T(a_1) + H(b_1)$.

Theorem 13 *Let* $a, b \in PC_{\langle p\rangle}(\Omega_0)$*. If one of the operators* $T(a) \pm H(b)$ *is Fredholm on* l^p*, then the other one is Fredholm on* l^p*, too, and the Fredholm indices of these operators coincide.*

Proof. By Corollary 4 (*b*), the operators $T(a) + H(b)$ and $T(a) - H(b)$ are Fredholm operators on l^p only simultaneously. It remains to prove that their indices coincide. Recall from the introduction that $T(a) = PL(a)P$ and $H(a) = PL(a)QJ$. Thus, the index equality will follow once we have constructed a Fredholm operator D such that the difference

$$D(PL(a)P + PL(b)QJ + Q) - (PL(a)P - PL(b)QJ + Q)D \tag{21}$$

is compact. The following construction of D is a modification of an idea in [12]. (Note that the compactness of the operator (21) also provides an alternate proof of the simultaneous Fredholm property of the operators $T(a) \pm H(b)$.)

A function $c \in M_p$ is called even (resp. odd) if $c = \tilde{c}$ (resp. $c = -\tilde{c}$) or, equivalently, if $JL(c)J = L(c)$ (resp. $JL(c)J = -L(c)$). Every function $c \in C_p$ can be written as a sum of an even and an odd function in a unique way: $c = (c + \tilde{c})/2 + (c - \tilde{c})/2$. Let θ_o and θ_e be an odd and an even function in $C(\mathbb{T}) \cap M^{\langle p\rangle}$, respectively, and assume that θ_e vanishes at all points of Ω_0 (and, hence, at all points of $\overline{\Omega_0}$). Put

$$D := PL(\theta_o + \theta_e)P + QL(\theta_o - \theta_e)Q. \tag{22}$$

We will later specify the functions θ_o and θ_e such that D becomes a Fredholm operator. First note that

$$JPL(\theta_o + \theta_e)PJ = -QL(\theta_o - \theta_e)Q, \quad JQL(\theta_o - \theta_e)QJ = -PL(\theta_o + \theta_e)P,$$

whence $JDJ = -D$ and $JD + DJ = 0$. Next we show that D commutes with the operator $PL(a)P + PL(b)Q + Q$ up to a compact operator. Since the Toeplitz operators $PL(\theta_o + \theta_e)P$ and $PL(a)P$ commute modulo a compact operator, it remains to show that D commutes with $PL(b)Q$ up to a compact operator. The latter fact follows easily from the identity

$$\begin{aligned} &DPL(b)Q - PL(b)QD \\ &\quad = PL(\theta_o + \theta_e)PL(b)Q - PL(b)QL(\theta_o - \theta_e)Q \\ &\quad = PL(\theta_o + \theta_e)L(b)Q - PL(\theta_o + \theta_e)QL(b)Q \\ &\qquad - PL(b)L(\theta_o - \theta_e)Q + PL(b)PL(\theta_o - \theta_e)Q \\ &\quad = 2PL(\theta_e b)Q - PL(\theta_o + \theta_e)QL(b)Q + PL(b)PL(\theta_o - \theta_e)Q \end{aligned}$$

and from the compactness of the operators $PL(\theta_e b)Q$ and $PL(\theta_o \pm \theta_e)Q$ by Lemma 9 (note that $\theta_e b \in C(\mathbb{T}) \cap M^{\langle p\rangle}$). The compactness of the operator (21) is then a consequence of the identity

$$\begin{aligned} &D(PL(a)P + PL(b)QJ + Q) - (PL(a)P - PL(b)QJ + Q)D \\ &\quad = DPL(a)P - PL(a)PD + DPL(b)QJ + PL(b)QJD \\ &\quad = DPL(a)P - PL(a)PD + (DPL(b)Q - PL(b)QD)J \end{aligned}$$

and of the compactness of the commutators $[D, PL(a)P]$ and $[D, PL(b)Q]$.

Finally we show that the functions θ_e and θ_o can be specified such that the operator D in (22) is a Fredholm operator on l^p. Set $\hat{\theta}_o(t) := |t^2 - 1|^2$ for $t \in \mathbb{T}$. Then $\hat{\theta}_o$ is an even function in $C^\infty(\mathbb{T})$ and $\theta_o := \chi_{\mathbb{T}_+}\hat{\theta}_o - \chi_{\mathbb{T}_-}\hat{\theta}_o$ is an odd function in $C(\mathbb{T}) \cap M^{\langle p \rangle}$. Further,

$$\theta_e(t) := i \prod_{j=1}^{m} |t - \tau_j|^2 |t - \overline{\tau_j}|^2, \quad t \in \mathbb{T}$$

defines an even function $\theta_e \in C(\mathbb{T}) \cap M^{\langle p \rangle}$ which vanishes at the points of Ω_0. Since θ_o and $i\theta_e$ are real-valued functions, we conclude that $\theta_o \pm \theta_e$ are invertible in $C(\mathbb{T}) \cap M^{\langle p \rangle}$, which implies that D is a Fredholm operator as desired. ■

Now we are in a position to derive an index formula for a Fredholm operator of the form $T(a) + H(b)$ with $a, b \in PC_{\langle p \rangle}(\Omega_0)$. We make use of the well-known identity

$$\begin{aligned} &\begin{pmatrix} PL(a)P + PL(b)QJ + Q & 0 \\ 0 & PL(a)P - PL(b)QJ + Q \end{pmatrix} \\ &\quad = \frac{1}{2} \begin{pmatrix} I & J \\ I & -J \end{pmatrix} \begin{pmatrix} PL(a)P + Q & PL(b)Q \\ JPL(b)QJ & J(PL(a)P + Q)J \end{pmatrix} \begin{pmatrix} I & I \\ J & -J \end{pmatrix}, \quad (23) \end{aligned}$$

where the outer factors in (23) are the inverses of each other. Thus, if one of the operators $T(a) \pm H(b) = PL(a)P \pm PL(b)QJ$ is a Fredholm operator, then so is the other, and the Fredholm indices of these operators coincide. Hence the middle factor

$$\begin{pmatrix} PL(a)P + Q & PL(b)Q \\ JPL(b)QJ & J(PL(a)P + Q)J \end{pmatrix} = \begin{pmatrix} PL(a)P + Q & PL(b)Q \\ QL(\tilde{b})P & QL(\tilde{a})Q + P \end{pmatrix}$$

in (23) is a Fredholm operator, and

$$\begin{aligned} \operatorname{ind}(T(a) + H(b)) &= \frac{1}{2} \operatorname{ind} \begin{pmatrix} PL(a)P + Q & PL(b)Q \\ QL(\tilde{b})P & QL(\tilde{a})Q + P \end{pmatrix} \\ &= \frac{1}{2} \operatorname{ind} \begin{pmatrix} PL(a)P & PL(b)Q \\ QL(\tilde{b})P & QL(\tilde{a})Q \end{pmatrix}. \end{aligned}$$

For the latter identity note that the operator

$$A := \begin{pmatrix} PL(a)P + Q & PL(b)Q \\ QL(\tilde{b})P & QL(\tilde{a})Q + P \end{pmatrix} \in L(l^p(\mathbb{Z})_2)$$

has the complementary subspaces $L_1 := \{(Qx_1,\, Px_2) : (x_1,\, x_2) \in l^p(\mathbb{Z})_2\}$ and $L_2 := \{(Px_1,\, Qx_2) : (x_1,\, x_2) \in l^p(\mathbb{Z})_2\}$ of $l^p(\mathbb{Z})_2$ as invariant subspaces and that A acts on L_1 as the identity operator and on L_2 as the operator

$$A_0 := \begin{pmatrix} PL(a)P & PL(b)Q \\ QL(\tilde{b})P & QL(\tilde{a})Q \end{pmatrix}.$$

Let the function $W : \mathbb{T} \times \overline{\mathbb{R}} \to \mathbb{C}$ be defined by

$$W(t,\, \lambda) = \det \mathrm{smb}_p\, A_0(t,\, \lambda)/(\tilde{a}(t,\, \infty)\tilde{a}(t, -\infty)).$$

Since $T(a)+H(b)$ is Fredholm, W does not pass through the origin, and Theorem 7 entails that $\mathrm{ind}\, A_0 = -\mathrm{wind}\, W$. Thus,

$$\mathrm{ind}\,(T(a) + H(b)) = -\frac{1}{2}\mathrm{wind}\, W.$$

We are going to show that actually

$$\mathrm{ind}\,(T(a) + H(b)) = -\mathrm{wind}_{\,\mathbb{T}_+} W, \tag{24}$$

where the right-hand side is defined as follows. The compression of W onto $\mathbb{T}_+ \times \overline{\mathbb{R}}$ is a continuous function the values of which form a closed oriented curve in $\mathcal{C}$ which starts and ends at $1 \in \mathbb{C}$ and does not contain the origin. The winding number of this curve is denoted by $\mathrm{wind}_{\,\mathbb{T}_+} W$. Analogously, we define $\mathrm{wind}_{\,\mathbb{T}_-} W$.

For the proof of (24) we suppose for simplicity that a and b have jumps only at the points t_1 and $\overline{t_1}$ where $t_1 \in \mathbb{T}_+^0$. If t moves along $\mathbb{T}_+$ from 1 to t_1 (resp. on $\mathbb{T}_-$ from 1 to $\overline{t_1}$), then the values of $W(t,\, \lambda) = a(t)/\tilde{a}(t) = a(t)/a(\bar{t})$ move continuously from 1 to $a(t_1^-)/a(\overline{t_1}^+)$ (resp. from 1 to $a(\overline{t_1}^+)/a(t_1^-)$). Using that $W(t,\, \lambda) = W(\bar{t}, \lambda)^{-1}$ for $t \in \mathbb{T} \setminus \{-1,\, 1\}$, one easily concludes that

$$[\arg W]_{1 \to t_1 \subset \mathbb{T}_+} = [\arg W]_{\overline{t_1} \to 1 \subset \mathbb{T}_-}$$

where the numbers on the left- and right-hand side stand for the increase of the argument of W if t moves in positive direction along the arc from 1 to t_1 in $\mathbb{T}_+$ and along the arc from $\overline{t_1}$ to 1 in $\mathbb{T}_-$, respectively. Analogously,

$$[\arg W]_{-1 \to \overline{t_1} \subset \mathbb{T}_-} = [\arg W]_{t_1 \to -1 \subset \mathbb{T}_+}.$$

Consider

$$\begin{aligned} &W(t_1, \lambda)/(a(\overline{t_1}^+)a(\overline{t_1}^-)) \\ &\quad = [a(t_1^+)\mu_q(\lambda) + a(t_1^-)(1-\mu_q(\lambda))]\, [a(\overline{t_1}^+)\mu_q(\lambda) + a(\overline{t_1}^-)(1-\mu_q(\lambda))] \\ &\qquad -(b(t_1^+) - b(t_1^-))(b(\overline{t_1}^+) - b(\overline{t_1}^-))\mu_q(\lambda)(1-\mu_q(\lambda)) \end{aligned}$$

and the related expression for $W(\overline{t_1}, \lambda)/(a(t_1^+)a(t_1^-))$. Then

$$[\arg W]_{\mathcal{C}_q(a(t_1^-),\, a(t_1^+))} = [\arg W]_{\mathcal{C}_q(a(\overline{t_1}^-),\, a(\overline{t_1}^+))}$$

because $W(t_1,\, \lambda)/(a(\overline{t_1}^+)a(\overline{t_1}^-)) = W(\overline{t_1}, \lambda)/(a(t_1^+)a(t_1^-))$. So we arrive at the equality $\mathrm{wind}_{\mathbb{T}_+} W = \mathrm{wind}_{\mathbb{T}_-} W$, whence (24) follows.

Now suppose that $a,\, b \in PC_{\langle p \rangle}$ are continuous on $\mathbb{T} \setminus \{-1,\, 1\}$. Then we define a function $W : \mathbb{T}_+ \times \overline{\mathbb{R}}$ by

$$W(t,\, \lambda) = \big(a(t^+)\mu_q(\lambda) + a(t^-)(1 - \mu_q(\lambda)) + it(b(t^+) - b(t^-))\nu_q(\lambda)\big)\, a^{-1}(\pm 1^{\mp})$$

if $t = \pm 1$ and by $W(t,\, \lambda) = a(t)/a(\bar{t})$ if $t \in \mathbb{T}_+^0$. The function W is continuous and determines a closed curve which starts and ends at $1 \in \mathbb{C}$. If $T(a) + H(b)$ is a Fredholm operator, then this curve does not pass through the origin and possesses, thus, a well defined winding number.

Since $T(a)+H(b)$ is in $\mathsf{T}(PC_p)$ and the symbol $V : \mathbb{T} \times \overline{\mathbb{R}} \to \mathbb{C}$ of this operator relative to the algebra $\mathsf{T}(PC_p)$ is known (it is just given by

$$V(t,\, \lambda) = a(t^+)\mu_q(\lambda) + a(t^-)(1 - \mu_q(\lambda)) + it(b(t^+) - b(t^-))\nu_q(\lambda)$$

if $t = \pm 1$ and by $V(t,\, \lambda) = a(t)$ if $t \in \mathbb{T} \setminus \{-1,\, 1\}$) and since $\mathrm{ind}\, T(a) = -\mathrm{wind}_{\mathbb{T}} V$, one can again prove that $\mathrm{wind}_{\mathbb{T}} V = \mathrm{wind}_{\mathbb{T}_+} W$ by comparing the increments of the arguments as above.

Now we look at the factorization given by Proposition 12 and denote by W_0 and W_1 the above defined function $W : \mathbb{T}_+ \times \overline{\mathbb{R}}$ for the operators $T(a_0) + H(b_0)$ and $T(a_1) + H(b_1)$, respectively. It is easy to see that $W_0 W_1$ coincides with the function W for the operator $T(a) + H(b)$. Summarizing, we get

Theorem 14 *Let $a,\, b \in PC_{\langle p \rangle}$ and $T(a)+H(b)$ a Fredholm operator on l^p. Then*

$$\mathrm{ind}\,(T(a) + H(b)) = -\mathrm{wind}_{\mathbb{T}_+} W_0 - \mathrm{wind}_{\mathbb{T}_+} W_1 = -\mathrm{wind}_{\mathbb{T}_+} W$$

with W, W_0 and W_1 defined as above.

5 The general case

In this section we want to sketch an approach to derive an index formula for an arbitrary Fredholm operator $A \in \mathsf{TH}(PC_p)$. With A, we associate the function $W(A) : \mathbb{T}_+ \times \overline{\mathbb{R}} \to \mathbb{C}$ defined by

$$W(A)(t,\, \lambda) = \begin{cases} \mathrm{smb}_p\, A(t,\, \lambda)/\mathrm{smb}_p\, A(t, \mp\infty) & \text{if } t = \pm 1 \\ \det \mathrm{smb}_p\, A(t,\, \lambda)/(a_{22}(t,\, \infty)a_{22}(t,\, -\infty)) & \text{if } t \neq \pm 1 \end{cases}$$

where we wrote $\mathrm{smb}_p\, A(t,\, \lambda) = (a_{ij}(t,\, \lambda))_{i,j=1}^2$ for $t \in \mathbb{T}_+^0$. For $A = T(a) + H(b)$, this definition coincides with that one from the previous section.

Theorem 15 *If $A \in \mathsf{TH}(PC_p)$ is a Fredholm operator, then*

$$\mathrm{ind}\, A = -\mathrm{wind}_{\mathbb{T}_+} W(A). \tag{25}$$

The remainder of this section is devoted to the proof of this theorem. It will become evident from this proof that $W(A)$ traces out a closed oriented curve which does not pass through the origin; so the winding number of $W(A)$ is well defined.

We start with the observation that Theorem 3 remains true for matrix-valued multipliers $a,\, b \in (PC_p)_{k\times k}$: just replace μ_q, $1-\mu_q$ and ν_q by the corresponding $k \times k$-diagonal matrices $\operatorname{diag}\mu_q$, $\operatorname{diag}(1-\mu_q)$ and $\operatorname{diag}\nu_q$, respectively. Also Proposition 2 holds in the matrix setting: If

$$T(a)+H(b) := (\operatorname{diag} P)L(a)(\operatorname{diag} P) + (\operatorname{diag} P)L(b)(\operatorname{diag} QJ)$$

is a Fredholm operator, then the identity

$$\operatorname{ind}(T(a)+H(b)) = -\operatorname{wind} W(T(a)+H(b))$$

still holds if one replaces in the above definition of W all scalars by the determinants of the corresponding matrices. These facts follow in a similar way as their scalar counterparts.

Now let $a_{jl},\, b_{jl} \in PC_p$, consider the $h \times r$-matrix $\beta := (T(a_{jl})+H(b_{jl}))_{j,l=1}^{h,r}$, and associate with β the operator

$$A := \operatorname{el}(\beta) = \sum_{j=1}^{h} (T(a_{j1})+H(b_{j1}))\dots(T(a_{jr})+H(b_{jr})) \in \mathsf{TH}(PC_p)$$

as in (17). Further set $\gamma := (L(a_{jl}))_{j,l=1}^{h,r}$ and $\delta := (L(b_{jl}))_{j,l=1}^{h,r}$. The linear extensions of γ and δ are Laurent operators again; thus $\operatorname{ext}(\gamma) = L(a)$ and $\operatorname{ext}(\delta) = L(b)$ with certain multipliers $a,\, b \in (PC_p)_{s\times s}$ with $s = h(r+1)+1$. Moreover, these extensions are related with the extension of β by

$$\operatorname{ext}(\beta) = T(\operatorname{ext}(\gamma)) + H(\operatorname{ext}(\delta)) = T(a)+H(b) \in L(l^p_s)$$

(note that $H(1)=0$). In Section 3 we noticed that if $\operatorname{el}(\beta)$ is Fredholm, then (and only then) $\operatorname{ext}(\beta)$ is Fredholm and $\operatorname{ind}\operatorname{el}(\beta) = \operatorname{ind}\operatorname{ext}(\beta)$. Further, if $\operatorname{el}(\beta)$ is a Fredholm operator, then the matrices $a(t^\pm)$ are invertible for every $t \in \mathbb{T}$. Hence, a is invertible in $(PC_p)_{s\times s}$. Now consider

$$\operatorname{smb}_p \operatorname{el}(\beta) = \sum_{j=1}^{h} \operatorname{smb}_p(T(a_{j1})+H(b_{j1}))\dots\operatorname{smb}_p(T(a_{jr})+H(b_{jr})).$$

Let $t \neq \pm 1$. Then $\operatorname{smb}_p(T(a)+H(b))(t,\lambda)$ is a matrix of size $2s \times 2s$. We put the rows and columns of this matrix in a new matrix according to the following rules: If $j \le h(r+1)+1$, then the j th row of the old matrix becomes the $2j-1$ th row of the new one, whereas if $j > h(r+1)+1$, the j th row of the old matrix becomes the $2(j-h(r+1)-1)$ th row of the new matrix. The columns of

$\mathrm{smb}_p\,(T(a)+H(b))(t,\,\lambda)$ are re-arranged in the same way. The matrix obtained in this way is just $\mathrm{smb}_p\,\mathrm{el}(\beta)(t,\,\lambda)$. By these manipulations,

$$\mathrm{smb}_p\,\mathrm{el}(\beta)(t,\,\lambda) = \mathcal{P}\mathrm{smb}_p\,(T(a)+H(b))(t,\,\lambda)\mathcal{P}^T$$

with a certain permutation matrix $\mathcal{P}$ and its transpose $\mathcal{P}^T$. Hence,

$$\det \mathrm{smb}_p\,(T(a)+H(b))(t,\,\lambda) = \det \mathrm{smb}_p\,(\mathrm{el}(\beta))(t,\,\lambda)$$

for $t \neq \pm 1$. For $t = \pm 1$ we do not change the matrix $\mathrm{smb}_p\,(T(a)+H(b))(t,\,\lambda)$.

For $t \neq \pm 1$, we write $\mathrm{smb}_p\,(T(a)+H(b)(t,\lambda) = (a_{mn}(t,\,\lambda))_{m,n=1}^2$ and

$$\mathrm{smb}_p\,(T(a_{jl})+H(b_{jl}))(t,\,\lambda)) = (a_{mn}^{jl}(t,\,\lambda))_{m,n=1}^2.$$

Then

$$\mathrm{smb}_p\,\mathrm{el}(\beta)(t,\,\pm\infty) = \sum_{j=1}^{h}\prod_{l=1}^{r}\begin{pmatrix} a_{11}^{jl}(t,\,\pm\infty) & 0 \\ 0 & a_{22}^{jl}(t,\,\pm\infty)\end{pmatrix},$$

and it follows that

$$\det a_{22}(t,\,\pm\infty) = \det \mathrm{ext}(\rho(t,\,\pm\infty))$$

where $\rho(t,\,\pm\infty) := (a_{22}^{jl}(t,\,\pm\infty))_{j,l=1}^{hr}$. It is now easy to see that

$$W(\mathrm{el}(\beta))(t,\,\lambda) = W(T(a)+H(b))(t,\,\lambda) = W(\mathrm{ext}(\beta))(t,\,\lambda)$$

for all $(t,\,\lambda) \in \mathbb{T}_+ \times \overline{\mathbb{R}}$, which implies that $\mathrm{ind}\,\mathrm{el}(\beta) = -\mathrm{wind}_{\,\mathbb{T}_+} W(\mathrm{el}(\beta))$ and, thus, settles the proof of the index formula (25) for a dense subset of Fredholm operators in $\mathsf{TH}(PC_p)$.

Finally, we are going to prove estimate (3), i.e., we will show that there is a constant M such that

$$\|\mathrm{smb}_p\,A\|_\infty \leq M \inf\{\|A+K\| : K \text{ compact}\} \tag{26}$$

for every operator $A \in \mathsf{TH}(PC_p)$. Once this estimate is shown, the validity of the index formula (25) for an arbitrary Fredholm operator in $\mathsf{TH}(PC_p)$ will follow by standard approximation arguments as at the end of Section 3.

To prove (26), we consider $\mathsf{TH}(PC_p)$ as a subalgebra of the smallest closed subalgebra $\mathsf{T}_J^0(PC_p)$ of $L(l^p(\mathbb{Z}))$ which contains all Laurent operators $L(a)$ with $a \in PC_p$, the projection P, and the flip J. The homomorphism smb_t defined in Section 3 cannot be extended to the algebra $\mathsf{T}_J^0(PC_p)$ unless $t = \pm 1$. Instead, we are going to use ideas from [4] and introduce a related family of homomorphisms $\mathrm{smb}_{t,\bar{t}}$ with $t \in \mathbb{T}_+^0$ from $\mathsf{T}_J^0(PC_p)$ onto $(\Sigma_1^p(\mathbb{R}))_{2\times 2}$. A crucial observation ([4]) is that the strong limit

$$\mathrm{smb}_{t,\bar{t}}\,A := \text{s-lim}_{n\to\infty}\begin{pmatrix} A_{t,n,0,0} & A_{t,n,0,1} \\ A_{t,n,1,0} & A_{t,n,1,1}\end{pmatrix} \tag{27}$$

with $A_{t,n,i,j} := E_n Y_t^{-1} L(\chi_{\mathbb{T}^+}) J^i A J^j L(\chi_{\mathbb{T}^+}) Y_t E_{-n} L_n$ exists for every operator $A \in \mathsf{T}_J^0(PC_p)$ and every $t \in \mathbb{T}_+^0$.

Theorem 16 *Let* $t \in \mathbb{T}_+^0$. *Then the mapping* $\mathrm{smb}_{t,\bar{t}}$ *is a bounded homomorphism from* $\mathsf{T}_J^0(PC_p)$ *onto* $(\Sigma_1^p(\mathbb{R}))_{2\times 2}$. *In particular,*
(*a*) $\mathrm{smb}_{t,\bar{t}}\, P = \mathrm{diag}\,(\chi_+ I,\, \chi_- I)$ *with* $\chi_- = 1 - \chi_+$,
(*b*) $\mathrm{smb}_{t,\bar{t}}\, L(a) = \mathrm{diag}\,(a(t^+)Q_\mathbb{R} + a(t^-)P_\mathbb{R},\, a(\bar{t}^-)Q_\mathbb{R} + a(\bar{t}^+)P_\mathbb{R})$ *for* $a \in PC_p$,
(*c*) $\mathrm{smb}_{t,\bar{t}}\, K = 0$ *for every compact operator* K,
(*d*) $\mathrm{smb}_{t,\bar{t}}\, J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

Sketch of the proof. The existence of the strong limits of the operators in (a) - (d) and their actual values follow by straightforward computation. Let us check assertion (a), for instance. For $A = P$, the strong limits of the diagonal elements of the matrix (27) exist and are equal to $\chi_+ I$ and $\chi_- I$ by Theorem 8 (a) and since $JPJ = Q$. Now consider the 01-entry of that matrix. It is $L(\chi_{\mathbb{T}^+})PJ = JL(\chi_{\mathbb{T}^-})Q$ and thus

$$\begin{aligned} &E_n Y_t^{-1} L(\chi_{\mathbb{T}^+}) PJL(\chi_{\mathbb{T}^+}) Y_t E_{-n} L_n \\ &\quad = \left(E_n Y_t^{-1} J Y_t E_{-n}\right) \left(E_n Y_t^{-1} L(\chi_{\mathbb{T}^-}) Q L(\chi_{\mathbb{T}^+}) Y_t E_{-n} L_n\right). \end{aligned} \tag{28}$$

The first factor on the right-hand side is uniformly bounded with respect to n, whereas the second one tends strongly to 0 by Theorem 8 (note that $\chi_{\mathbb{T}^-}(t) = 0$ for $t \in \mathbb{T}_+^0$). Thus, the sequence of the operators (28) tends strongly to zero. The strong convergence of the 10-entry to zero follows analogously.

Another straightforward calculation shows that the mappings $\mathrm{smb}_{t,\bar{t}}$ are algebra homomorphisms and that these mappings are uniformly bounded with respect to $t \in \mathbb{T}_+^0$. Thus, the mappings $\mathrm{smb}_{t,\bar{t}}$ are well-defined on a dense subalgebra of $\mathsf{T}_J^0(PC_p)$, and they extend to (uniformly bounded with respect to t) homomorphisms on all of $\mathsf{T}_J^0(PC_p)$ by continuity. ■

By assertion (c) of the previous theorem, every mapping $\mathrm{smb}_{t,\bar{t}}$ induces a quotient homomorphism on $\mathsf{T}_J^0(PC_p)/K(l^p(\mathbb{Z}))$ in a natural way. We denote this homomorphism by $\mathrm{smb}_{t,\bar{t}}$ again.

Now we are ready for the last step. Let $t \in \mathbb{T}_+^0$ and $a,\, b \in PC_p$. From Theorem 16 we conclude that then the operator $\mathrm{smb}_{t,\bar{t}}(T(a) + H(b))$ is given by the matrix

$$\begin{pmatrix} \chi_+(a(t^+)Q_\mathbb{R} + a(t^-)P_\mathbb{R})\chi_+ I & \chi_+(b(t^+)Q_\mathbb{R} + b(t^-)P_\mathbb{R})\chi_- I \\ \chi_-(b(\bar{t}^-)Q_\mathbb{R} + b(\bar{t}^+)P_\mathbb{R})\chi_+ I & \chi_-(a(\bar{t}^-)Q_\mathbb{R} + a(\bar{t}^+)P_\mathbb{R})\chi_- I \end{pmatrix}$$

acting on $L^p(\mathbb{R})_2$. This matrix operator has the complementary subspaces

$$L_1 := \{(\chi_- f_1,\, \chi_+ f_2) : f_1,\, f_2 \in L^p(\mathbb{R})\},\ L_2 := \{(\chi_+ f_1,\, \chi_- f_2) : f_1,\, f_2 \in L^p(\mathbb{R})\}$$

of $L^p(\mathbb{R})_2$ as invariant subspaces, and it acts as the zero operator on L_1. So we can identify $\mathrm{smb}_{t,\bar{t}}(T(a) + H(b))$ with its restriction to L_2, which we denote by A_0 for brevity.

The space L_2 can be identified with $L^p(\mathbb{R})$ in a natural way. Under this identification, the operator A_0 can be identified with the operator

$$\begin{aligned} A_1 := {} & \chi_+(a(t^+)Q_\mathbb{R} + a(t^-)P_\mathbb{R})\chi_+ I + \chi_+(b(t^+)Q_\mathbb{R} + b(t^-)P_\mathbb{R})\chi_- I \\ & + \chi_-(b(\bar{t}^-)Q_\mathbb{R} + b(\bar{t}^+)P_\mathbb{R})\chi_+ I + \chi_-(a(\bar{t}^-)Q_\mathbb{R} + a(\bar{t}^+)P_\mathbb{R})\chi_- I \end{aligned}$$

which belongs to $\Sigma^p(\mathbb{R})$. It is well known (see Section 4.2 in [19]) and not hard to check that the algebra $\Sigma^p(\mathbb{R})$ is isomorphic to $\Sigma^p_{2\times 2}(\mathbb{R}_+)$, where the isomorphism η acts on the generating operators of $\Sigma^p(\mathbb{R})$ by

$$\eta(S_\mathbb{R}) = \begin{pmatrix} S_{\mathbb{R}_+} & H_\pi \\ -H_\pi & -S_{\mathbb{R}_+} \end{pmatrix} \quad \text{and} \quad \eta(\chi_+ I) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

with H_π referring to the Hankel operator

$$(H_\pi \varphi)(s) := \frac{1}{\pi i} \int_{\mathbb{R}_+} \frac{\varphi(t)}{t+s}\, dt$$

on $L^p(\mathbb{R}_+)$. The entries of the matrix $\eta(A_1)$ are Mellin operators, and the value of the Mellin symbol of $\eta(A_1)$ at $(t,\lambda) \in \mathbb{T}^0_+ \times \overline{\mathbb{R}}$ is the matrix

$$\begin{pmatrix} a(t^+)\mu_q(\lambda) + a(t^-)(1-\mu_q(\lambda)) & (b(t^+) - b(t^-))\nu_q(\lambda) \\ (b(\bar{t}^-) - b(\bar{t}^+))\nu_q(\lambda) & a(\bar{t}^-)(1-\mu_q(\lambda)) + a(\bar{t}^+)\mu_q(\lambda) \end{pmatrix},$$

which evidently coincides with $\operatorname{smb}_p(T(a)+H(b))(t,\lambda)$ given in (1). Summarizing the above arguments we conclude that the homomorphisms

$$A + K(l^p) \mapsto (\operatorname{smb}_p A)(t,\lambda)$$

are uniformly bounded with respect to $(t,\lambda) \in \mathbb{T}^0_+ \times \overline{\mathbb{R}}$, which finally implies the estimate (26). ■

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