Interpolation of sum and intersection spaces of L^q-type and applications to the Stokes problem in general unbounded domains

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In a general unbounded uniform C^2-domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and $1 \leq q \leq \infty$ consider the spaces $\tilde{L}^q(\Omega)$ defined by

$$\tilde{L}^q(\Omega) := \begin{cases} L^q(\Omega) + L^2(\Omega), & q < 2, \\ L^q(\Omega) \cap L^2(\Omega), & q \geq 2, \end{cases}$$

and corresponding subspaces of solenoidal vector fields, $\tilde{L}^q_\sigma(\Omega)$. By studying the complex and real interpolation spaces of these we derive embedding properties for fractional order spaces related to the Stokes problem and L^p-L^q-type estimates for the corresponding semigroup.

1 Introduction and main results

In the mathematical analysis of the Navier-Stokes equations or other equations from fluid mechanics the Helmholtz decomposition plays a crucial role. However, it has been pointed out by Bogovskij in [5] in 1986, that for certain unbounded domains Ω – no matter how smooth their boundaries $\partial\Omega$ – the Helmholtz decomposition fails to hold in spaces $L^q(\Omega)$, $q \neq 2$. Therefore, Farwig, Kozono and Sohr proposed in [9] to study slightly modified spaces of the form

$$\tilde{L}^q(\Omega) := \begin{cases} L^q(\Omega) + L^2(\Omega), & 1 < q < 2, \\ L^q(\Omega) \cap L^2(\Omega), & 2 \leq q < \infty, \end{cases}$$

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where functions behave locally like L^q-functions, but looking at decay at space infinity they behave like L^2-functions. In these spaces, the Helmholtz decomposition holds even in smooth unbounded domains Ω. They also showed in [8] certain solvability results on the Stokes and Navier-Stokes equations in unbounded domains in spaces of the type $\tilde{L}^q(\Omega)$. Moreover, they proved for instance that the Stokes operator $\tilde{A}_q$ generates an analytic semigroup $e^{-t\tilde{A}_q}$. This motivates the study of those sum and intersection spaces. In this paper the author proves the following estimates for the semigroup:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a uniform C^2-domain and let*

$$1 < q \leq r < \infty, \quad 0 \leq \alpha := \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r}\right).$$

Then, for every $0 < t < \infty$ the estimate

$$\|e^{-t\tilde{A}_q} f\|_{\tilde{L}^r(\Omega)} \leq C e^{\delta t} t^{-\alpha}(1+t)^{\alpha} \|f\|_{\tilde{L}^q(\Omega)}$$

holds for all $f \in \tilde{L}^q_\sigma(\Omega)$ with a constant C only depending on n, r, q, δ and the type type(Ω) *of Ω. The number $\delta > 0$ can be chosen arbitrarily small but positive. Moreover, the estimate*

$$\|\nabla e^{-t\tilde{A}_q} f\|_{\tilde{L}^r(\Omega)} \leq C e^{\delta t} t^{-\alpha-1/2}(1+t)^{\alpha+1/2} \|f\|_{\tilde{L}^q(\Omega)}$$

holds for all $f \in \tilde{L}^q_\sigma(\Omega)$ with a constant C as above.

For the precise meanings of all terms used here see below.

This can be used to find the following quite sharp estimate for the Stokes semigroup:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a uniform C^2-domain, $0 < T < \infty$, and*

$$1 < r < \infty,\ 1 < \gamma < q < \infty, \quad \frac{2}{r} + \frac{n}{q} = \frac{n}{\gamma}.$$

Then the estimate

$$\left(\int_0^T \|e^{-t\tilde{A}_q} f\|^r_{\tilde{L}^q(\Omega)} dt\right)^{1/r} \leq C\|f\|_{\tilde{L}^{\gamma,r}(\Omega)}$$

holds for all $f \in \tilde{L}^{\gamma,r}_\sigma(\Omega)$ with a constant C depending on n, q, r, T and the type type(Ω) *of Ω.*

Choosing $\gamma = n$ the exponents r and q are so-called Serrin exponents, i.e. they satisfy $2 < r < \infty$, $n < q < \infty$ and $\frac{2}{r} + \frac{n}{q} = 1$. If additionally $r \geq n$, we find the estimate

$$\left(\int_0^T \|e^{-t\tilde{A}_q} f\|^r_{\tilde{L}^q(\Omega)} dt\right)^{1/r} \leq C\|f\|_{\tilde{L}^n(\Omega)}$$

for all $f \in \tilde{L}^n_\sigma(\Omega)$.

Again we refer to the sections below for the precise definitions of all terms. In a forthcoming paper the author will use these results to develop the theory of very weak solutions to the Navier-Stokes equations in general unbounded domains.

2 Notation and preliminaries

Definition 2.1. An open connected subset $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is called *uniform C^k-domain*, $k \in \mathbb{N}$, if there are finite constants $\alpha > 0$, $\beta > 0$, $K > 0$ such that for every boundary point $x_0 \in \partial\Omega$ there is a Cartesian coordinate system with origin at x_0 and coodinates $y = (y', y_n)$, $y' = (y_1, \dots, y_{n-1})$, and a C^k-function $h(y')$, $|y'| \leq \alpha$, with C^k-norm $\|h\|_{C^k} \leq K$ such that the neighborhood

$$U_{\alpha,\beta,h}(x_0) := \big\{(y', y_n) \in \mathbb{R}^n \colon h(y') - \beta < y_n < h(y') + \beta, |y'| < \alpha\big\}$$

of x_0 satisfies

$$\Omega \cap U_{\alpha,\beta,h} = U^-_{\alpha,\beta,h}(x_0) := \big\{(y', y_n) \in \mathbb{R}^n \colon h(y') - \beta < y_n < h(y'), |y'| < \alpha\big\},$$

and

$$\partial\Omega \cap U_{\alpha,\beta,h} = \big\{(y', h(y')) \colon |y'| < \alpha\big\}.$$

The triple (α, β, K) will be called the *type* of Ω and will be denoted by $\mathrm{type}(\Omega)$.

For two vector spaces X and Y both being embedded in a common topological vector space Ξ we can define the sum space $X + Y := \{z = x + y \in \Xi \colon x \in X, y \in Y\}$ with norm

$$\|z\|_{X+Y} := \inf\{\|x\|_X + \|y\|_Y \colon x \in X, y \in Y, x + y = z\}$$

and the intersection space $X \cap Y := \{z \in \Xi \colon z \in X, z \in Y\}$ with norm

$$\|z\|_{X\cap Y} := \max\{\|z\|_X, \|z\|_Y\}.$$

By [3, Theorem 8.III] the dual relations $(X + Y)' = X' \cap Y'$ and $(X \cap Y)' = X' + Y'$ hold, provided that $X \cap Y$ is dense both in X and in Y.

We let $[X, Y]_\theta$ denote the complex interpolation space and by $(X, Y)_{\theta,\rho}$ the real interpolation space for $0 < \theta < 1$, $1 \leq \rho \leq \infty$, cf. [4].

For any open set Ω, $k \in \mathbb{N}$ and $1 \leq q \leq \infty$ we denote by $L^q = L^q(\Omega)$ the usual Lebesgue spaces and by $W^{k,q} = W^{k,q}(\Omega)$ Sobolev spaces, see for example [1]. We will also use the space $W_0^{1,q} = W_0^{1,q}(\Omega)$ being the closure with respect to the norm of $W^{1,q}(\Omega)$ of the subspace $C_0^\infty(\Omega)$ consisting of smooth functions being compactly supported in Ω. By $L^{q,\rho}(\Omega)$, $1 \leq q, \rho \leq \infty$, we denote the usual Lorentz spaces as decribed e.g. in [4]. The Bochner-Lebesgue spaces will be denoted by $L^r(0, T; X)$ for some Banach space X, $0 < T \leq \infty$, $1 \leq r \leq \infty$. We also need the Bochner-Lorentz spaces $L^{r,\rho}(0, T; X)$, cf. [13] or [4].

From now on let $1 < q, r < \infty$, $0 < T < \infty$ and a uniform C^2-domain be fixed. Consider the space $C^\infty_{0,\sigma}(\Omega)$ consisting of $C^\infty(\Omega)$ functions u having compact support in Ω and satisfying $\operatorname{div} u = 0$. Its closure with respect to the L^q-norm is denoted by

$$L^q_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{L^q(\Omega)}}.$$

For $k \in \mathbb{N}$ and $1 < q < \infty$ we define

$$\tilde{W}^{k,q}(\Omega) := \begin{cases} W^{k,q}(\Omega) + W^{k,2}(\Omega), & q < 2, \\ W^{k,q}(\Omega) \cap W^{k,2}(\Omega), & q \geq 2, \end{cases}$$

and

$$\tilde{L}^{q,\rho}(\Omega) := \begin{cases} L^{q,\rho}(\Omega) + L^2(\Omega) & q < 2, \\ L^{q,\rho}(\Omega) \cap L^2(\Omega), & q > 2, \end{cases}$$

where we leave the case $q = 2$ undefined, and

$$\tilde{L}^q_\sigma(\Omega) := \begin{cases} L^q_\sigma(\Omega) + L^2_\sigma(\Omega), & q < 2, \\ L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega), & q \geq 2. \end{cases}$$

We define

$$\tilde{L}^{q,\rho}_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{\tilde{L}^{q,\rho}(\Omega)}}$$

for $1 < q < \infty$, $q \neq 2$, $1 \leq \rho < \infty$. Moreover, with $D_q := W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$ we let

$$\tilde{D}_q(\Omega) := \begin{cases} D_q(\Omega) + D_2(\Omega), & q < 2, \\ D_q(\Omega) \cap D_2(\Omega), & q \geq 2. \end{cases}$$

We collect now a number of results on the Helmholtz decomposition and the Stokes operator in the spaces $\tilde{L}^q_\sigma(\Omega)$. These have been obtained by Farwig, Kozono, Sohr and Kunstmann.

It was shown in [9] that the Helmholtz decomposition in $\tilde{L}^q_\sigma(\Omega)$ holds true and that the Helmholtz projection $\tilde{P}_q : u \mapsto u_0 \colon \tilde{L}^q(\Omega) \to \tilde{L}^q_\sigma(\Omega)$ is a well defined bounded linear operator. Moreover, as a consequence the authors obtained that $\tilde{L}^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{\tilde{L}^q(\Omega)}}$. In [11] the authors considered the Stokes operator $\tilde{A}_q \colon \tilde{D}_q \subset \tilde{L}^q_\sigma(\Omega) \to \tilde{L}^q_\sigma(\Omega)$ defined by $\tilde{A}_q u := -\tilde{P}_q \Delta u$, $u \in \tilde{D}_q$. They showed that it is a densely defined closed operator and that it generates an analytic semigroup $e^{-t\tilde{A}_q}$ in $\tilde{L}^q_\sigma(\Omega)$ with bound $\|e^{-t\tilde{A}_q} f\|_{\tilde{L}^q} \leq Me^{\delta t}\|f\|_{\tilde{L}^q}$, where $\delta > 0$ can be chosen arbitrarily small, but positive. Here $M > 0$ only depends on q, δ and type(Ω). In [10] the authors even proved maximal Sobolev regularity of the Stokes operator $\tilde{A}_q$. In [12] the author even proved that the operator $\varepsilon + \tilde{A}_q$, $\varepsilon > 0$, even admits a bounded H^∞-calculus and in particular bounded imaginary powers.

We will write $\tilde{D}^\alpha_q$, $0 \leq \alpha \leq 1$, for the domain of the fractional powers $(1 + \tilde{A}_q)^\alpha$. It is equipped by the norm $\|u\|_{\tilde{D}^\alpha_q} = \|(1 + \tilde{A}_q)^\alpha u\|_{\tilde{L}^q}$. For $-1 \leq \alpha < 0$ we let $\tilde{D}^\alpha_q$ be the closure of $\tilde{L}^q_\sigma(\Omega)$ with respect to the norm $\|(1 + \tilde{A}_q)^\alpha(\cdot)\|_{\tilde{L}^q_\sigma(\Omega)}$.

Then it holds that $\tilde{D}^\alpha_q = [\tilde{L}^q_\sigma(\Omega), \tilde{D}_q]_\alpha$, $0 < \alpha < 1$. Moreover, the dual relation $(\tilde{D}^\alpha_q)' = \tilde{D}^{-\alpha}_{q'}$, $-1 \leq \alpha \leq 1$ holds. These are consequences of the fact that $1 + \tilde{A}_q$ has a bounded inverse and admits bounded imaginary powers, cf. [2, Section V]. Moreover, by [12, Corollary 1.2], $\tilde{D}^{1/2}_q = \tilde{W}^{1,q}_0(\Omega) \cap \tilde{L}^q_\sigma(\Omega)$.

3 Interpolation of $\tilde{L}^q(\Omega)$ spaces

The main result in this section will be the following:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a uniform C^2-domain. Let $1 \le q, r, s \le \infty$, $0 < \theta < 1$, $1 \le \rho \le \infty$ such that*

$$\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{r}.$$

Then it holds that

$$[\tilde{L}^q(\Omega), \tilde{L}^r(\Omega)]_\theta = \tilde{L}^s(\Omega)$$

with equivalent norms. Moreover, in case $s \neq 2$, $q \neq r$, it holds that

$$(\tilde{L}^q(\Omega), \tilde{L}^r(\Omega))_{\theta,\rho} = \tilde{L}^{s,\rho}(\Omega).$$

The main ideas for the proof result from a very helpful private communication with M. Cwikel, [7]. For the proof we need the following notation: For $1 \le q \le \infty$ we write $\tilde{l}^q$ for the sequence space $l^q + l^2$, $q < 2$, or $l^q \cap l^2$, $q \ge 2$. A simple argument shows that $\tilde{l}^q = l^2$, but this notation will be helpful in the sequel. We shall also use the Lorentz-type sequence spaces $l^{q,\rho}$ and even $\tilde{l}^{q,\rho}$, which are defined by $l^{q,\rho} + l^2$, $q < 2$, and $l^{q,\rho} \cap l^2$, $q > 2$. Again it is not hard to see that $\tilde{l}^{q,\rho} = l^2$ for all $q \neq 2$, $1 \le \rho \le \infty$.

Note also that $\tilde{L}^q(0,1) = L^q(0,1)$, $1 \le q, \rho \le \infty$, and $\tilde{L}^{q,\rho}(0,1) = L^{q,\rho}(0,1)$, $q \neq 2$.

For any function $f \in L^1(M) + L^\infty(M)$, where M is a measure space, its nonincreasing equimeasurable rearrangement is denoted by f^*, cf. [4].

First we need a special case of a result due to Calderón, cf. [6, Theorem 1].

Proposition 3.2. *Let M_1 and M_2 be σ-finite measure spaces and let $f_i \in L^1(M_i) + L^\infty(M_i)$, $i = 1, 2$, respectively, be fixed functions. If they fulfill the estimate $f_2^* \le f_1^*$ almost everywhere on $(0, \infty)$, then there exists a linear map $L\colon L^1(M_1) + L^\infty(M_1) \to L^1(M_2) + L^\infty(M_2)$ with the property*

$$Lf_1 = f_2$$

and satisfying the estimates

$$\|Lu\|_{L^1(M_2)} \le \|u\|_{L^1(M_1)}, \quad \|Lu\|_{L^\infty(M_2)} \le \|u\|_{L^\infty(M_1)}$$

for all $u \in L^1(M_1)$ or $u \in L^\infty(M_1)$, respectively.

Of course the map depends very crucially on the functions f_1 and f_2. This proposition can be used to prove the following powerful tool:

Theorem 3.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain and let $f \in L^1(\Omega) + L^\infty(\Omega)$ be a given and fixed function. Then there exist linear maps*

$$S_1\colon L^1(\Omega) + L^\infty(\Omega) \to L^1(0,1), \quad S_2\colon L^1(\Omega) + L^\infty(\Omega) \to l^\infty$$

and

$$T_1\colon L^1(0,1) \to L^1(\Omega) + L^\infty(\Omega), \quad T_2\colon l^\infty \to L^1(\Omega) + L^\infty(\Omega)$$

satisfying the equality

$$f = T_1 S_1 f + T_2 S_2 f$$

almost everywhere. Moreover, these maps satisfy the estimates

$$\|S_1 u\|_{L^p(0,1)} \le \|u\|_{L^p(\Omega)}, \quad \|S_2 u\|_{l^p} \le \|u\|_{L^p(\Omega)}$$

and

$$\|T_1 u\|_{L^p(\Omega)} \le \|u\|_{L^p(0,1)}, \quad \|T_2 u\|_{L^p(\Omega)} \le \|u\|_{l^p}$$

for all $1 \le p \le \infty$ and all u in the respective L^p-spaces.

Proof. First we define a linear operator $P\colon L^1(\Omega) + L^\infty(\Omega) \to L^1(0,\infty) + L^\infty(0,\infty)$ by choosing $f_1 := f$, $f_2 := f^*$ in Proposition 3.2. It thus satisfies $Pf = f^*$. We also define linear operators by

$$\begin{aligned} &V_1\colon L^1(0,\infty) + L^\infty(0,\infty) \to L^1(0,1), && u \mapsto u|_{(0,1)}, \\ &V_2\colon L^1(0,\infty) + L^\infty(0,\infty) \to l^\infty, && u \mapsto \left(\int_{n-1}^n u(s)ds\right)_{n\in\mathbb{N}}. \end{aligned}$$

Then the choice $S_1 := V_1 \circ P$ and $S_2 := V_2 \circ P$ defines the first operators. The estimates are easily checked for $p = 1$ and $p = \infty$ yielding the desired estimates for every $1 \le p \le \infty$ by the Riesz-Thorin theorem or complex interpolation.

We still have to construct T_1 and T_2. To this end we first define linear maps $W_1\colon L^1(0,1) \to L^1(0,\infty) + L^\infty(0,\infty)$ by

$$W_1(u)(t) := \begin{cases} u(t), & 0 < t < 1, \\ 0, & t \ge 1, \end{cases}$$

and $W_2\colon l^\infty \to L^1(0,\infty) + L^\infty(0,\infty)$ by

$$W_2((a_n)_{n\in\mathbb{N}})(t) \mapsto \begin{cases} 0, & 0 < t < 1, \\ a_n, & n \le t < n+1, n \in \mathbb{N}. \end{cases}$$

Then we define

$$g := W_1 V_1 P f + W_2 V_2 P f = W_1 S_1 f + W_2 S_2 f.$$

This means the following: $g = f^*$ identically on $(0,1)$ and $g = \int_{n-1}^n f^*(s)ds$ identically on the intervals $[n, n+1)$, $n \in \mathbb{N}$. Clearly $g(t) \ge f^*(t)$ for $0 < t < 1$ and for $n \le t \le n+1$ we can estimate $g(t) = \int_{n-1}^n f^*(s)ds \ge f^*(n) \ge f^*(t)$ for all $n \in \mathbb{N}$. Of course the monotonicity of f^* is crucial here. Altogether we get $g \ge f^*$ almost everywhere on $(0,\infty)$. Then it clearly also holds that $g^* \ge f^*$ and we can again use Proposition 3.2 to find a linear map $H\colon L^1(0,\infty) + L^\infty(0,\infty) \to L^1(\Omega) + L^\infty(\Omega)$ satisfying all needed estimates and having the property $Hg = f$. Now we set $T_1 := H \circ W_1$ and $T_2 := H \circ W_2$. Consequently we get

$$\begin{aligned} T_1 S_1 f + T_2 S_2 f &= H W_1 V_1 P f + H W_2 V_2 P f \\ &= H(W_1 V_1 P f + W_2 V_2 P f) = Hg = f. \end{aligned}$$

Moreover, all linear operators involved satisfy the necessary L^p type estimates with constants equal to 1. This is directly seen for $p = 1$ and $p = \infty$. Using the Riesz-Thorin theorem we get the estimates for all $1 \le q \le \infty$. □

Remark 3.4. Note that the operators S_1, S_2, T_1, T_2 in the above Theorem also satisfy the respective bounds in Lorentz spaces, i.e.

$$\|S_1 u\|_{L^{p,\rho}(0,1)} \le \|u\|_{L^{p,\rho}(\Omega)}, \quad \|S_2 u\|_{l^{p,\rho}} \le \|u\|_{L^{p,\rho}(\Omega)}$$

and

$$\|T_1 u\|_{L^{p,\rho}(\Omega)} \le \|u\|_{L^{p,\rho}(0,1)}, \quad \|T_2 u\|_{L^{p,\rho}(\Omega)} \le \|u\|_{l^{p,\rho}}$$

for all $1 < p < \infty$, $1 \le \rho \le \infty$. This is directly seen by real interpolation.

Proof of Theorem 3.1. First we treat the complex interpolation space. Let first $f \in \tilde{L}^s(\Omega)$ and let the linear maps S_1, S_2, T_1 and T_2 be the maps from Theorem 3.3, for the function f. Then

$$S_1 f \in \tilde{L}^s(0,1) = L^s(0,1) = [L^q(0,1), L^r(0,1)]_\theta = [\tilde{L}^q(0,1), \tilde{L}^r(0,1)]_\theta.$$

By interpolation theory, we thus have $T_1 S_1 f \in [\tilde{L}^q(\Omega), \tilde{L}^r(\Omega)]_\theta$. Similarly we have

$$S_2 f \in \tilde{l}^s = l^2 = [l^2, l^2]_\theta = [\tilde{l}^q, \tilde{l}^r]_\theta$$

and hence $T_2 S_2 f \in [\tilde{L}^q(\Omega), \tilde{L}^r(\Omega)]_\theta$. Together, this implies $f = T_1 S_1 f + T_2 S_2 f \in [\tilde{L}^q(\Omega), \tilde{L}^r(\Omega)]_\theta$ and we obtain the inequality $\|f\|_{[\tilde{L}^q(\Omega), \tilde{L}^r(\Omega)]_\theta} \le C_1 \|f\|_{\tilde{L}^s(\Omega)}$ with a constant C_1 only depending on q, r and s.

For the reverse implication let $f \in [\tilde{L}^q(\Omega), \tilde{L}^r(\Omega)]_\theta$ and let again S_1, S_2, T_1 and T_2 be chosen as above for the function f. Then we get by interpolation that

$$S_1 f \in [\tilde{L}^q(0,1), \tilde{L}^r(0,1)]_\theta = [L^q(0,1), L^r(0,1)]_\theta = L^s(0,1) = \tilde{L}^s(0,1)$$

and that

$$S_2 f \in [\tilde{l}^q, \tilde{l}^r]_\theta = [l^2, l^2]_\theta = l^2 = \tilde{l}^s.$$

This leads to $T_1 S_1 f, T_2 S_2 f \in \tilde{L}^s(\Omega)$ and by $f = T_1 S_1 f + T_2 S_2 f$ this implies $f \in \tilde{L}^s(\Omega)$ and the inequality $\|f\|_{\tilde{L}^s(\Omega)} \le C_2 \|f\|_{[\tilde{L}^q(\Omega), \tilde{L}^r(\Omega)]_\theta}$ with a constant C_2 only depending on q, r and s. This finishes the proof for the complex interpolation spaces.

Now we treat the real interpolation spaces. The proof will be similar. Let $f \in (\tilde{L}^q(\Omega), \tilde{L}^r(\Omega))_{\theta,\rho}$ and define by Theorem 3.3 the maps S_1, S_2, T_1 and T_2 for f. Then

$$S_1 f \in (\tilde{L}^q(0,1), \tilde{L}^r(0,1))_{\theta,\rho} = (L^q(0,1), L^r(0,1))_{\theta,\rho} = L^{s,\rho}(0,1) = \tilde{L}^{s,\rho}(0,1),$$

and hence $T_1S_1f \in \tilde{L}^{s,\rho}(\Omega)$. Concerning the second term we get $S_2f \in (l^2, l^2)_{\theta,\rho} = l^2 = \tilde{l}^{s,\rho}$ implying that $T_2S_2f \in \tilde{L}^{s,\rho}(\Omega)$. Together we see that $f = T_1S_1f + T_2S_2f$ is an element of $\tilde{L}^{s,\rho}(\Omega)$.

For the reverse inclusion let $f \in \tilde{L}^{s,\rho}(\Omega)$. Then

$$S_1f \in \tilde{L}^{s,\rho}(0,1) = L^{s,\rho}(0,1) = (L^q(0,1), L^r(0,1))_{\theta,\rho} = (\tilde{L}^q(0,1), \tilde{L}^r(0,1))_{\theta,\rho},$$

yielding that $T_1S_1f \in (\tilde{L}^q(\Omega), \tilde{L}^r(\Omega))_{\theta,\rho}$. Moreover,

$$S_2f \in \tilde{l}^{s,\rho} = l^2 = (l^2, l^2)_{\theta,\rho} = (\tilde{l}^q, \tilde{l}^r)_{\theta,\rho}$$

and hence $T_2S_2f \in (\tilde{L}^q(\Omega), \tilde{L}^r(\Omega))_{\theta,\rho}$, proving also this inclusion. The proof is finished. □

By density arguments and using the projection operator $\tilde{P}_q$ we find the following corollary.

Corollary 3.5. *Let $1 < q, r < \infty$, $0 < \theta < 1$, and let s be defined by $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{r}$. Then,*

$$[\tilde{L}^q_\sigma(\Omega), \tilde{L}^r_\sigma(\Omega)]_\theta = \tilde{L}^s_\sigma(\Omega).$$

Assume that also $1 \le \rho < \infty$ is given and that $s \neq 2$, $q \neq r$. Then,

$$(\tilde{L}^q_\sigma(\Omega), \tilde{L}^r_\sigma(\Omega))_{\theta,\rho} = \tilde{L}^{s,\rho}_\sigma(\Omega).$$

The Sobolev embedding theorem can be carried over to the context of $\tilde{L}^q$-spaces:

Proposition 3.6. *Let $m \in \mathbb{N}$, $1 \le q < \infty$ and $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain. Then the embedding*

$$\tilde{W}^{m,q}(\Omega) \hookrightarrow \tilde{L}^r(\Omega)$$

holds, i.e.

$$\|u\|_{\tilde{L}^r(\Omega)} \le C\|u\|_{\tilde{W}^{m,q}(\Omega)}$$

for $u \in \tilde{W}^{m,q}(\Omega)$ holds with the following choice of exponents:

1. *$q \le r \le \infty$ if $mq > n$,*
2. *$q \le r < \infty$ if $mq = n$,*
3. *$q \le r \le \frac{nq}{n-mq}$ if $mq < n$.*

The constant C above only depends on q, r, m, n and the type type(Ω) *of Ω.*

Proof. Assume first that $q \ge 2$. This implies that $r \ge 2$. Then we obtain for $f \in \tilde{W}^{m,q}(\Omega)$ that

$$\|f\|_{\tilde{L}^r} \le \|f\|_{L^r} + \|f\|_{L^2} \le C\|f\|_{W^{m,q}} + \|f\|_{W^{m,2}} \le C\|f\|_{\tilde{W}^{m,q}},$$

where we use the classical Sobolev embedding, cf. [1, Theorem 4.12], yielding a constant C of the desired type.

Now consider the case $q < 2$, $r \geq 2$. Let $f \in \tilde{W}^{m,q}(\Omega)$ and let $f_1 \in W^{m,q}(\Omega)$, $f_2 \in W^{m,2}(\Omega)$, $f = f_1 + f_2$, $\|f_1\|_{W^{m,q}(\Omega)} + \|f_2\|_{W^{m,2}(\Omega)} \leq \|f\|_{\tilde{W}^{m,q}} + \epsilon$, $\varepsilon > 0$. Now observe that $\|f_1\|_{L^2} \leq C\|f_1\|_{W^{m,q}}$ and $\|f_1\|_{L^r} \leq C\|f_1\|_{W^{m,q}}$ and $\|f_2\|_{L^2} \leq C\|f_2\|_{W^{m,2}}$ and $\|f_2\|_{L^r} \leq C\|f_1\|_{W^{m,2}}$. This implies that $f_1, f_2 \in \tilde{L}^r = L^2 \cap L^r$ and

$$\begin{aligned}\|f\|_{\tilde{L}^r} &\leq \|f_1\|_{\tilde{L}^r} + \|f_2\|_{\tilde{L}^r} \\ &\leq \|f_1\|_{L^r} + \|f_1\|_{L^2} + \|f_2\|_{L^r} + \|f_2\|_{L^2} \\ &\leq C(\|f_1\|_{W^{m,q}} + \|f_2\|_{W^{m,2}}) \\ &\leq C(\|f\|_{\tilde{W}^{m,q}} + \varepsilon)\end{aligned}$$

with a constant as above. Since $\varepsilon > 0$ can be chosen arbitrarily small, this finishes this case.

For $q < 2$, $r < 2$ let $f \in \tilde{W}^{m,q}(\Omega)$ and let $f_1 \in W^{m,q}(\Omega)$ and $f_2 \in W^{m,2}(\Omega)$ satisfy $f_1 + f_2 = f$ and $\|f_1\|_{W^{m,q}(\Omega)} + \|f_2\|_{W^{m,2}(\Omega)} \leq \|f\|_{\tilde{W}^{m,q}} + \epsilon$, where $\epsilon > 0$. Then we have

$$\|f\|_{\tilde{L}^r} \leq \|f_1\|_{L^r} + \|f_2\|_{L^2} \leq C\|f_1\|_{W^{m,q}} + \|f_2\|_{W^{m,2}} \leq C(\|f\|_{\tilde{W}^{m,q}} + \epsilon)$$

with a constant as above. This proves the result, since ϵ can be chosen arbitrarily small. □

Now we are in the position to prove the following important embedding estimates.

Proposition 3.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain, $n \geq 3$. Let $0 \leq \alpha \leq 1$ and $1 < q \leq r < \infty$ satisfy*

$$\frac{1}{r} = \frac{1}{q} - \frac{2\alpha}{n}.$$

Then we have the embedding property

$$\|u\|_{\tilde{L}^r(\Omega)} \leq C\|(1 + \tilde{A}_q)^\alpha u\|_{\tilde{L}^q(\Omega)}$$

for all $u \in \tilde{D}_q^\alpha$ with a constant $C = C(\mathrm{type}(\Omega), n, q, \alpha)$.

Proof. We need three steps.

Step I. First we consider the case $\alpha \leq \frac{1}{2}$ and $q < n$. We can express the domains of the fractional powers of $1 + \tilde{A}_q$ as complex interpolation spaces, i.e.

$$\tilde{D}_q^\alpha = [\tilde{L}_\sigma^q(\Omega), \tilde{D}_q^1]_\alpha.$$

By reiteration, we can – because of the assumption $\alpha \leq \frac{1}{2}$ – also write

$$\tilde{D}_q^\alpha = [\tilde{L}_\sigma^q, \tilde{D}_q^{1/2}]_{2\alpha}.$$

Since $\tilde{D}_q^{1/2} = \tilde{W}_0^{1,q} \cap \tilde{L}_\sigma^q$ we obtain

$$\tilde{D}_q^\alpha \hookrightarrow [\tilde{L}^q, \tilde{W}^{1,q}]_{2\alpha}.$$

Now let γ be defined by $\frac{1}{r} = \frac{1-2\alpha}{q} + \frac{2\alpha}{\gamma}$ or, equivalently, $\gamma = \frac{nq}{n-q}$. Here we need the assumption $q < n$. We can now use the Sobolev embedding theorem, cf. Proposition 3.6, to see that $\tilde{W}^{1,q} \hookrightarrow \tilde{L}^\gamma$. We get then

$$[\tilde{L}^q, \tilde{W}^{1,q}]_{2\alpha} \hookrightarrow [\tilde{L}^q, \tilde{L}^\gamma]_{2\alpha} = \tilde{L}^r(\Omega),$$

using Theorem 3.1, which proves the embedding for $\alpha \leq \frac{1}{2}$.

Step II. Now we consider the case $\frac{1}{2} < \alpha \leq 1$ and still $q < n$. In this case we define ρ by

$$\frac{1}{\rho} - \frac{1}{n} = \frac{1}{r}$$

and find that $q \leq \rho < n$ by $r < \infty$. Hence we can use the result just proved to get that

$$\|u\|_{\tilde{L}^r} \leq C\|(1 + \tilde{A}_\rho)^{\frac{1}{2}} u\|_{\tilde{L}^\rho}.$$

By the definition of ρ it holds furthermore that

$$\frac{1}{\rho} = \frac{1}{q} - \frac{2(\alpha - \frac{1}{2})}{n}$$

and we can again use the result from above, implying that

$$\|(1 + \tilde{A}_\rho)^{\frac{1}{2}} u\|_{\tilde{L}^\rho} \leq C\|(1 + \tilde{A}_q)^{\alpha - \frac{1}{2}}(1 + \tilde{A}_\rho)^{\frac{1}{2}} u\|_{\tilde{L}^q} = C\|(1 + \tilde{A}_q)^\alpha u\|_{\tilde{L}^q},$$

and this finishes the proof also in the case $\frac{1}{2} < \alpha \leq 1$ and $q < n$.

Step III. We still need to consider $q \geq n$. In this case we use duality. We let $\phi \in C^\infty_{0,\sigma}(\Omega)$ and first of all calculate

$$|(u, \phi)_\Omega| \leq \|(1 + \tilde{A}_q)^\alpha u\|_{\tilde{L}^q} \|(1 + \tilde{A}_{q'})^{-\alpha} \phi\|_{\tilde{L}^{q'}}.$$

We abbreviate $v := (1 + \tilde{A}_{q'})^{-\alpha}\phi$ and note furthermore that

$$r' \leq q' \leq n' < n, \quad \frac{1}{q'} = \frac{1}{r'} - \frac{2\alpha}{n}.$$

Here the assumption $n \geq 3$ is needed. Consequently, Step I. (in case $\alpha \leq \frac{1}{2}$) or Step II. (in case $\alpha \geq \frac{1}{2}$) can be used to find

$$\|(1 + \tilde{A}_{q'})^{-\alpha}\phi\|_{\tilde{L}^{q'}} = \|v\|_{\tilde{L}^{q'}} \leq C\|(1 + \tilde{A}_{r'})^\alpha v\|_{\tilde{L}^{r'}} = C\|\phi\|_{\tilde{L}^{r'}}.$$

Combining the estimates we get the duality estimate

$$|(u, \phi)_\Omega| \leq C\|(1 + \tilde{A}_q)^\alpha u\|_{\tilde{L}^q} \|\phi\|_{\tilde{L}^{r'}} \text{ for all } \phi \in C^\infty_{0,\sigma}(\Omega),$$

and since $C^\infty_{0,\sigma}(\Omega)$ is dense in $\tilde{L}^{r'}_\sigma(\Omega)$, the estimate holds for every $\phi \in \tilde{L}^{r'}_\sigma$. This implies that $u \in \tilde{L}^r_\sigma$ and the desired estimate

$$\|u\|_{\tilde{L}^r} \leq C\|(1 + \tilde{A}_q)^\alpha u\|_{\tilde{L}^q}.$$

This finishes Step III. and the proof of the proposition. □

4 Proofs of the main results

Proof of Theorem 1.1. Let $f \in \tilde{L}^q_\sigma(\Omega)$. Assume first that $\alpha \le 1$. Then

$$\|e^{-t\tilde{A}_q} f\|_{\tilde{L}^r} \le C\|(1+\tilde{A}_q)^\alpha e^{-t\tilde{A}_q} f\|_{\tilde{L}^q} \le C\|(1+\tilde{A}_q)e^{-t\tilde{A}_q} f\|^\alpha_{\tilde{L}^q} \|e^{-t\tilde{A}_q} f\|^{1-\alpha}_{\tilde{L}^q}$$

using Proposition 3.7 and we continue by noting that

$$\begin{aligned}\|(1+\tilde{A}_q)e^{-t\tilde{A}_q} f\|_{\tilde{L}^q} &\le \|e^{-t\tilde{A}_q} f\|_{\tilde{L}^q} + \|\tilde{A}_q e^{-t\tilde{A}_q} f\|_{\tilde{L}^q} \\ &\le (Me^{\delta t} + Mt^{-1}e^{\delta t})\|f\|_{\tilde{L}^q} \\ &= Mt^{-1}(1+t)e^{\delta t}\|f\|_{\tilde{L}^q},\end{aligned}$$

where we used the analyticity of the semigroup, cf. [2, Remark 5.1.2]. Combining the estimates we get

$$\|e^{-t\tilde{A}_q} f\|_{\tilde{L}^r} \le Ct^{-\alpha}(1+t)^\alpha e^{\delta t}\|f\|_{\tilde{L}^q},$$

proving the first part of the theorem for $\alpha \le 1$. If $1 < \alpha < 2$, we write $e^{-t\tilde{A}_q} = e^{-t\tilde{A}_q/2} e^{-t\tilde{A}_q/2}$ and apply the argument as above twice. Similarly, we can argue for any $\alpha \ge 0$, repeating the arguments sufficiently often.

To prove the second part assume first $\alpha \le 1/2$. Note that

$$\|\nabla e^{-t\tilde{A}_q} f\|_{\tilde{L}^r} \le C\|(1+\tilde{A}_r)^{1/2} e^{-t\tilde{A}_q} f\|_{\tilde{L}^r},$$

since $D^{1/2}_q = \tilde{W}^{1,q}_0(\Omega) \cap \tilde{L}^q_\sigma(\Omega)$ with equivalent norms. Applying Proposition 3.7 we find that

$$\|\nabla e^{-t\tilde{A}_q} f\|_{\tilde{L}^r} \le C\|(1+\tilde{A}_q)^{\alpha+1/2} e^{-t\tilde{A}_q} f\|_{\tilde{L}^q}$$

and then we continue as above for the proof of the desired estimate as long as $\alpha \le 1/2$. For $\alpha > 1/2$ we need to again repeat the argument finitely many times as above. □

Proof of Theorem 1.2. We will need real interpolation for this proof. We define a linear map B by

$$f \mapsto e^{-t\tilde{A}_q} f.$$

Assume first that $f \in \tilde{L}^q_\sigma(\Omega)$. Then the bound for the semigroup yields for all $0 < t < T$ the estimate

$$\|Bf(t)\|_{\tilde{L}^q} \le C\|f\|_{\tilde{L}^q}$$

with a constant C which depends on $T < \infty$, showing that

$$B\colon \tilde{L}^q_\sigma(\Omega) \to L^\infty(0,T;\tilde{L}^q_\sigma(\Omega))$$

as a bounded linear operator.

Choose any $1 < p < \gamma$. Then Theorem 1.1 yields for all $f \in \tilde{L}^p_\sigma(\Omega)$ the bound

$$\|Bf(t)\|_{\tilde{L}^q} \le Ct^{-\alpha}\|f\|_{\tilde{L}^p}$$

with $\alpha = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right)$ showing that

$$B\colon \tilde{L}^p_\sigma(\Omega) \to L^{1/\alpha,\infty}(0,T;\tilde{L}^q_\sigma(\Omega))$$

as a bounded linear operator.

Real interpolation theory thus shows that

$$B\colon (\tilde{L}^q_\sigma(\Omega), \tilde{L}^p_\sigma(\Omega))_{\theta,r} \to \Big(L^\infty(0,T;\tilde{L}^q_\sigma(\Omega)), L^{1/\alpha,\infty}(0,T;\tilde{L}^q_\sigma(\Omega))\Big)_{\theta,r}$$

as a linear bounded linear operator, where $0 < \theta < 1$ is chosen such that $\frac{1}{\gamma} = \frac{1-\theta}{q} + \frac{\theta}{p}$.

By Corollary 3.5 we find that $(\tilde{L}^q_\sigma(\Omega), \tilde{L}^p_\sigma(\Omega))_{\theta,r} = \tilde{L}^{\gamma,r}_\sigma(\Omega)$. On the other hand [13, Theorem 1.18.6.2] implies that

$$\begin{aligned}\Big(L^\infty(0,T;\tilde{L}^q_\sigma(\Omega)), L^{1/\alpha,\infty}(0,T;\tilde{L}^q_\sigma(\Omega))\Big)_{\theta,r} &= L^{r,r}(0,T;\tilde{L}^q_\sigma(\Omega)) \\ &= L^r(0,T;\tilde{L}^q_\sigma(\Omega)),\end{aligned}$$

which proves that B maps $\tilde{L}^{\gamma,r}_\sigma(\Omega)$ continuously into $L^r(0,T;\tilde{L}^q_\sigma(\Omega))$, which finishes the proof of the first assertion.

For the second assertion assume that $\gamma = n$ and that $r \geq n$. In that case, it is readily seen that $\tilde{L}^n_\sigma(\Omega) \subset \tilde{L}^{\gamma,r}_\sigma(\Omega)$, from which the rest of the proof follows.

□

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