Two points, one limit: Homogenization techniques for two-point local algebras

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Dedicated to Wolfgang W. Wendland on the occasion of his 75th birthday.

Abstract

We study several algebras generated by convolution, multiplication and flip operators on $L^p(\mathbb{R})$, their Calkin counterparts and derive new isomorphism relations. We introduce a new class of homogenization strong limits, compatible with the flip operator and which explore the properties of the Fourier transform in $L^p(\mathbb{R})$, when $p \neq 2$.

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1 Introduction

In the study of approximation methods for any specific class of operators, a precise knowledge of the analytic properties of these operators is indispensable. A relevant part of these properties can be specified by describing the Banach algebra generated by the operators under consideration. In many cases (at least in those in the authors' main interest), the description of this algebra is given up to compact operators via algebra isomorphisms to a family of simpler algebras. The construction of such isomorphisms is often related with Allan's local principle [1] (see also [11, Section 2.2]), which creates a family of local representatives indexed by the points of the maximal ideal space of a central subalgebra.

The procedure described above was applied to sequence algebras generated by finite sections projections, operators of multiplication by piecewise continuous functions and operators of convolution by piecewise continuous Fourier multipliers in [10]. These operators were considered on L^p-spaces over the real line $\mathbb{R}$. In order to identify the corresponding local algebras and, thus, to obtain invertibility conditions for the local representatives, we used homomorphisms which are defined by certain strong limits. More precisely, given a sequence (A_n) of approximation operators, one multiplies A_n by certain shift operators V_n (which have to be specified in each context), and then the homomorphism maps the sequence (A_n) to the strong limit of the sequence $V_n^{-1} A_n V_n$. Homomorphisms of this form are widely used (see for instance the monographs and textbooks [2, 3, 4, 6, 7, 9, 11] and the papers cited there).

It is one striking advantage of homomorphisms of this special form that they allow one to master the inverse closedness problem, which typically arises when $p \neq 2$. The point is

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that many properties (e.g., the stability of a sequence, the Fredholm property of an operator) can be equivalently described as the invertibility of an associated object in a (typically large) algebra, whereas the available technical tools often allow one to study invertibility only in (small) subalgebras. If every element of the subalgebra which is invertible in the large one is also invertible in the subalgebra, then the subalgebra is called inverse closed. Note that there is no inverse closedness problem if $p = 2$: every C^*-subalgebra of a C^*-algebra is inverse closed.

It is unfortunately not possible to use the until now available homomorphisms when considering sequence algebras including the flip (defined as $Ju(x) := u(-x)$) in $L^p(\mathbb{R})$. The inclusion of the flip operator is important if one wants to study approximation methods for Hankel-type operators, for instance. When including the flip, the central subalgebra must be constituted by operators defined by even functions. This implies a "double point" localization at non-fixed points of the shift (which are 0 and ∞). A standard technique for dealing with this problem is doubling the dimension (see, e.g., [11, Section 1.1.5]), ending in operator matrices. When trying to extend the results of [10] to algebras with flip and to combine the usual homomorphisms with the doubling of the dimension, the authors were not able to overcome the occurring inverse closedness problem.

So, it is one purpose of this work to introduce two new families of homomorphisms, which are suitable to describe Banach operator algebras that include the flip *and* to tackle the inverse closedness problem. These homomorphisms possess, in some sense, a built-in doubling of dimension. Besides studying their properties, we use these homomorphism in the present paper to give a description of the Calkin algebra generated by convolution, multiplication and the flip operators, which is alternative to the description in [11, Section 5.7] and is interesting in its own right. The application of these homomorphism to sequence algebras and, thus, to study the stability will be the subject of a forthcoming paper.

Specifically, let $\mathcal{A}$ be the smallest closed subalgebra of $\mathcal{L}(L^p(\mathbb{R}))$ which contains multiplication operators aI and the Fourier convolution operators $W^0(b)$, with a and b piecewise continuous, and the flip operator J (proper definitions are given below, in Section 2). Our main result can then be stated as follows.

Theorem 1.1. *There is a family of algebra homomorphisms $\Upsilon_{s,t}$ labeled by the points in $([0, \infty] \times \{\infty\}) \cup (\{\infty\} \times [0, \infty])$ such that an operator $A \in \mathcal{A}$ is Fredholm on $L^p(\mathbb{R})$ if and only if all operators $\Upsilon_{s,t}(A)$ are invertible.*

The paper is organized as follows. In Section 2 we present technical background material. In Section 3, we derive an isomorphism between the algebra generated by the singular integral operator on the half-axis and the algebra of Toeplitz operators on $\mathbb{R}$. In Section 4 we introduce several auxiliary operators which are needed for the homogenization processes, and prove results on the interaction of these operators with convolution, multiplication and flip operators. We also introduce some families of strong limits which are later used to identify local algebras. In Section 5, the Calkin image of the algebra generated by convolution, multiplication and the flip operator is analyzed with the use of Allans's local principle. The resulting local algebras are identified via algebras isomorphisms in Section 6. In Section 7 we discuss some consequences of these results.

2 Notation

Throughout this paper, we will work on the Lebesgue space $L^p(\mathbb{R})$ with $1 < p < \infty$. Let $\mathcal{B} := \mathcal{B}(L^p(\mathbb{R}))$ denote the Banach algebra of all bounded linear operators on $L^p(\mathbb{R})$, and $\mathcal{K} := \mathcal{K}(L^p(\mathbb{R}))$ the closed ideal of the compact linear operators on $L^p(\mathbb{R})$. Given a subinterval Γ of the real axis, we consider $L^p(\Gamma)$ as a closed subspace of $L^p(\mathbb{R})$ in the natural way. In particular, we identify the identity operator on $L^p(\Gamma)$ with the operator $\chi_\Gamma I$ of multiplication by the characteristic function χ_Γ of the interval Γ, acting on $L^p(\mathbb{R})$. More general, each bounded linear operator A on $L^p(\Gamma)$ is identified with the operator $\chi_\Gamma A \chi_\Gamma I$ acting on $L^p(\mathbb{R})$. These identifications will often be used without further comment.

We write the *Fourier transform* F on the Schwartz space of rapidly decreasing infinite differentiable functions as

$$(Fu)(y) = \int_{-\infty}^{+\infty} e^{-2\pi \mathbf{i} yx} u(x)\, dx, \qquad y \in \mathbb{R}.$$

Then its inverse is given by

$$(F^{-1}v)(x) = \int_{-\infty}^{+\infty} e^{2\pi \mathbf{i} xy} v(y)\, dy, \qquad x \in \mathbb{R}.$$

It is well known that the operators F and F^{-1} can be extended continuously to bounded and unitary operators on the Hilbert space $L^2(\mathbb{R})$ and that F extends continuously to a bounded operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ where $q := p/(p-1)$ if $1 < p \leq 2$ (see, for instance, [12, Theorem 74]).

Let $\mathcal{M}_p$ denote the set of all *Fourier multipliers*, *i.e.*, the set of all functions $a \in L^\infty(\mathbb{R})$ with the following property: if $u \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$, then $F^{-1}aFu \in L^p(\mathbb{R})$, and there is a constant c_p independent of u such that $\|F^{-1}aFu\|_p \leq c_p \|u\|_p$. If $a \in \mathcal{M}_p$, then the operator $F^{-1}aF : L^2(\mathbb{R}) \cap L^p(\mathbb{R}) \to L^p(\mathbb{R})$ extends continuously to a bounded operator on $L^p(\mathbb{R})$. This extension is called a *(Fourier) convolution operator* , and we denote it by $W^0(a)$. The function a is called the *generating function* (sometimes also the symbol or pre-symbol) of $W^0(a)$. In particular, the convolution operator $W^0(\mathrm{sgn})$ can be identified the singular integral operator of Cauchy type,

$$(S_\mathbb{R} u)(t) := \frac{1}{\pi i} \int_\mathbb{R} \frac{u(s)}{s-t}\, ds, \qquad t \in \mathbb{R}.$$

This operator satisfies $S_\mathbb{R}^2 = I$. We denote the associated projections by $P_\mathbb{R} := (I + S_\mathbb{R})/2$ and $Q_\mathbb{R} := I - P_\mathbb{R}$. Further we write χ_+ and χ_- for the characteristic functions of the positive and negative half axis, respectively. Given $a \in \mathcal{M}_p$, the restriction of the operator $\chi_+ W^0(a) \chi_+ I$ onto $L^p(\mathbb{R}^+)$ is called a *Wiener-Hopf operator* and will be denoted by $W(a)$.

The set $\mathcal{M}_p$ of all Fourier multipliers forms a Banach algebra when equipped with the operations inherited from $L^\infty(\mathbb{R})$ and with the norm

$$\|a\|_{\mathcal{M}_p} := \|W^0(a)\|_{\mathcal{L}(L^p(\mathbb{R}))}.$$

We call a function $a \in L^\infty(\mathbb{R})$ *piecewise constant* (resp. *piecewise linear*) if there is a partition $-\infty = t_0 < t_1 < \ldots < t_n = +\infty$ of the real line such that a is constant (resp. linear) on each interval $[t_k, t_{k+1}]$. Stechkin's inequality (see for instance [5]) entails that the multiplier algebra $\mathcal{M}_p$ contains the (non-closed) algebras C_0 of all continuous and piecewise linear functions on

$\dot{\mathbb{R}}$ and PC_0 of all piecewise constant functions on $\mathbb{R}$. Let C_p and PC_p denote the closures of C_0 and PC_0 in $\mathcal{M}_p$, respectively.

We define the flip operator J acting on $L^p(\mathbb{R})$ by $(Ju)(x) := u(-x)$. Given a function $a \in L^\infty(\mathbb{R})$ we write $\tilde{a}I$ for the multiplication operator JaJ.

3 Isomorphy between algebras of SIOs in $L^p(\mathbb{R})$

In this section we will show that two algebras of singular integral operators which appear as "building blocks" of larger operator algebras are isometric. The first of these algebras is $\mathcal{E}_p$, the smallest closed subalgebra of $\mathcal{B}$ which contains the operators $\chi_+ I$ and $S_{\mathbb{R}^+} := \chi_+ W^0(\text{sgn})\chi_+ I$ or, equivalently, the operators $\chi_+ I$ and $P := \chi_+ W^0(\chi_+)\chi_+ I$. Likewise, one can consider $\mathcal{E}_p$ as the closed algebra of operators on $L^p(\mathbb{R}^+)$ which is generated by the identity operator and the operator $S_{\mathbb{R}^+}$, via the identification of $A \in \mathcal{B}(L^p(\mathbb{R}^+))$ with the operator $\chi_+ A \chi_+ I \in \mathcal{B}(L^p)$. The Hankel operator $H := \chi_+ W^0(\text{sgn}) J \chi_+ I$ is an element of $\mathcal{E}_p$ (see [11, Proposition 4.2.16]), and we let $\mathcal{N}_p$ denote the smallest closed ideal of $\mathcal{E}_p$ which contains H.

The second algebra, $\mathcal{E}_p^F$, is the smallest closed subalgebra of $\mathcal{B}$ which contains the operator $P_{\mathbb{R}}$ and the operator $T := P_{\mathbb{R}} \chi_+ P_{\mathbb{R}}$. If $p = 2$ it is obvious that $\mathcal{E}_2$ and $\mathcal{E}_2^F$ are isometrically isomorphic because each algebra is the Fourier image of the other, and the Fourier transform is a unitary operator on $L^2(\mathbb{R})$. If $p \neq 2$, the Fourier transform is not even bounded in general, and the isometry between these algebras is no longer obvious. The following result is certainly well-known to specialists, but we were not able to find an explicit reference in the literature.

Theorem 3.1. *There is a continuous isomorphism between the algebras $\mathcal{E}_p$ and $\mathcal{E}_p^F$.*

Proof. Let $\eta : L^p(\mathbb{R}) \to L_2^p(\mathbb{R}^+)$ be defined by

$$(\eta f)(x) := \begin{bmatrix} f(x) \\ f(-x) \end{bmatrix}$$

for $x \in \mathbb{R}^+$. By [11, Proposition 4.2.19] one has

$$\eta S_{\mathbb{R}} \eta^{-1} = \begin{bmatrix} S_{\mathbb{R}^+} & -H \\ H & -S_{\mathbb{R}^+} \end{bmatrix} \quad \text{and} \quad \eta \chi_+ \eta^{-1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

whence

$$\eta P_{\mathbb{R}} \eta^{-1} = \frac{1}{2} \begin{bmatrix} I + S_{\mathbb{R}^+} & -H \\ H & I - S_{\mathbb{R}^+} \end{bmatrix}$$

and

$$\eta T \eta^{-1} = \frac{1}{4} \begin{bmatrix} (I + S_{\mathbb{R}^+})^2 & -(I + S_{\mathbb{R}^+})H \\ H(I + S_{\mathbb{R}^+}) & -H^2 \end{bmatrix}.$$

Because $S_{\mathbb{R}^+}^2 - I = H^2$ by [11, Equation (4.40)], one can further write

$$\eta T \eta^{-1} = \frac{1}{4} \begin{bmatrix} (I + S_{\mathbb{R}^+})^2 & -(I + S_{\mathbb{R}^+})H \\ H(I + S_{\mathbb{R}^+}) & (I + S_{\mathbb{R}^+})(I - S_{\mathbb{R}^+}) \end{bmatrix} = \frac{I + S_{\mathbb{R}^+}}{2} \frac{1}{2} \begin{bmatrix} I + S_{\mathbb{R}^+} & -H \\ H & I - S_{\mathbb{R}^+} \end{bmatrix}$$

or, shortly,

$$\eta T\eta^{-1} = P \circ \eta P_{\mathbb{R}}\eta^{-1}$$

where the $\circ$ indicates that we multiply each entry of the matrix $\eta P_{\mathbb{R}}\eta^{-1}$ by P from the left-hand side.

Let now $q(z) := \sum_{j=0}^{n} q_j z^j$ be a polynomial. Then, because $P_{\mathbb{R}}$ is a projection and P commutes with the entries $S_{\mathbb{R}^+}$ and H of $\eta P_{\mathbb{R}}\eta^{-1}$,

$$\begin{aligned}\eta q(T)\eta^{-1} &= \sum_{j=0}^{n} q_j \left(\eta T\eta^{-1}\right)^j = \sum_{j=0}^{n} q_j \left(P \circ \eta P_{\mathbb{R}}\eta^{-1}\right)^j \\ &= \sum_{j=0}^{n} (q_j P^j) \circ \eta P_{\mathbb{R}}\eta^{-1}\end{aligned}$$

what finally results in

$$\eta q(T)\eta^{-1} = q(P) \circ \eta P_{\mathbb{R}}\eta^{-1}. \tag{1}$$

Consider the mapping

$$\mathsf{W} : \mathcal{B} \to \mathcal{L}\left(L^p(\mathbb{R}^+)\right), \qquad A \mapsto \begin{bmatrix} I & I \end{bmatrix} \eta A\eta^{-1} \begin{bmatrix} I \\ I \end{bmatrix}$$

which is evidently linear and continuous. Since the sum of the entries of the matrix $\eta P_{\mathbb{R}}\eta^{-1}$ equals I, we conclude from (1) that

$$\mathsf{W}(q(T)) = q(P)$$

for every polynomial q. Hence, W is a continuous homomorphism on the set of all polynomials in T. Since this set is dense in $\mathcal{E}_p^F$, the homomorphism W extends to a continuous homomorphism $\mathsf{W} : \mathcal{E}_p^F \to \mathcal{E}_p$.

Conversely, for A in $\mathcal{L}\left(L^p(\mathbb{R}^+)\right)$, define

$$\mathsf{V} : A \mapsto \eta^{-1}\left(A \circ \eta P_{\mathbb{R}}\eta^{-1}\right)\eta.$$

Clearly, V is linear and continuous. Moreover, by (1), we have for every polynomial q

$$\mathsf{V}\left(q(P)\right) = \eta^{-1}\left(q(P) \circ \eta P_{\mathbb{R}}\eta^{-1}\right)\eta = \eta^{-1}\left(\eta q(T)\eta^{-1}\right)\eta = q(T).$$

Hence, V is a homomorphism on the set of all polynomials in P. So it extends by continuity to a continuous homomorphism $\mathsf{V} : \mathcal{E}_p \to \mathcal{E}_p^F$.

Finally, on the sets of all polynomials in P (respectively in T) we have

$$\mathsf{WV} = I \quad \text{and} \quad \mathsf{VW} = I.$$

Since these sets are dense in $\mathcal{E}_p$ (respectively in $\mathcal{E}_p^F$), we obtain $\mathsf{W}^{-1} = \mathsf{V}$. Thus $\mathcal{E}_p$ is isomorphic to $\mathcal{E}_p^F$, with the isomorphism being given by W. ■

To obtain the image of the generator of the ideal $\mathcal{N}_p$ via the isomorphism V we note that

$$\eta J\eta^{-1} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

which implies

$$\eta\left(P_{\mathbb{R}}\mathrm{sgn}JP_{\mathbb{R}}\right)\eta^{-1}=\frac{1}{4}\begin{bmatrix}2H(I+S_{\mathbb{R}^+}) & -H^2+I-S_{\mathbb{R}^+}^2\\ S_{\mathbb{R}^+}^2-I+H^2 & 2H(I-S_{\mathbb{R}^+})\end{bmatrix}$$

and thus, because $S_{\mathbb{R}^+}H=HS_{\mathbb{R}^+}$,

$$\mathsf{W}\left(\eta\left(P_{\mathbb{R}}\mathrm{sgn}JP_{\mathbb{R}}\right)\eta^{-1}\right)=H.$$

So we arrive at the following corollary.

Corollary 3.2. *The operator $H^F:=P_{\mathbb{R}}\mathrm{sgn}JP_{\mathbb{R}}$ is the image of $H\in\mathcal{E}_p$ via the isomorphism V. Hence, $H^F\in\mathcal{E}_p^F$, and the smallest closed ideal $\mathcal{N}_p^F$ of $\mathcal{E}_p^F$ which contains H^F is isomorphic to the ideal $\mathcal{N}_p$ of $\mathcal{E}_p$.*

4 Projections, reflections, shifts, and related strong limits

In this section we are going to define several types of "shift" operators and introduce several strong limits associated to these shifts. These strong limits will be our main tool to identify local algebras in the following sections.

For every positive real number τ, we define the following operators acting on $L^p(\mathbb{R})$:

$$(P_\tau u)(x)=\begin{cases}u(x) & \text{if } |t|<\tau\\ 0 & \text{if } |t|>\tau\end{cases},\qquad Q_\tau=I-P_\tau, \tag{2}$$

$$(R_\tau u)(x)=\begin{cases}u(\tau-x) & \text{if } 0<x<\tau\\ u(-\tau-x) & \text{if } -\tau<x<0\\ 0 & \text{if } |x|>\tau\end{cases}, \tag{3}$$

$$(S_\tau u)(x)=\begin{cases}0 & \text{if } |t|<\tau\\ u(x-\tau) & \text{if } x>\tau\\ u(x+\tau) & \text{if } x<-\tau\end{cases},\qquad (S_{-\tau}u)(x)=\begin{cases}u(x+\tau) & \text{if } x>0\\ u(x-\tau) & \text{if } x<0\end{cases}. \tag{4}$$

For $t\in\mathbb{R}$ and $\tau>0$, we will further need the following operators on $L^p(\mathbb{R})$:

$$(U_tu)(x)=e^{-2\pi\mathbf{i}xt}u(x),\quad (V_tu)(x)=u(x-t),\quad (Z_\tau u)(x):=\tau^{-1/p}u(x/\tau).$$

Clearly, $U_t^{-1}=U_{-t}$, $V_t^{-1}=V_{-t}$, and $Z_\tau^{-1}=Z_{\tau^{-1}}$, and these operators have norm 1. We collect some elementary properties of these operators in a few lemmata, which we will use later without reference. The proofs of these facts are straightforward, and we omit them.

Lemma 4.1. (i) *$R_\tau P_\tau=P_\tau R_\tau=R_\tau$, $P_\tau=R_\tau^2$, $R_\tau^*=R_\tau$, $\|P_\tau\|=\|R_\tau\|=1$,*

(ii) *$S_\tau S_{-\tau}=Q_\tau$, $S_{-\tau}S_\tau=I$, $(S_\tau)^*=S_{-\tau}$, $\|S_\tau\|=\|S_{-\tau}\|=1$,*

(iii) *$JS_{\pm\tau}=S_{\pm\tau}J$, $JP_\tau=P_\tau J$, $JR_\tau=R_\tau J$,*

(iv) *$P_\tau\to I$, $S_{-\tau}\to 0$ strongly, and $R_\tau\to 0$, $S_\tau\to 0$ weakly as $\tau\to\infty$.*

Lemma 4.2. *If $a \in \mathcal{M}_p$ and $s \in \mathbb{R}$, then*

$$U_{-s}W^0(a)U_s = W^0(V_s a V_{-s}), \quad V_s W^0(a) V_{-s} = W^0(a), \quad JW^0(a)J = W^0(\tilde{a}).$$

Moreover, if $p = 2$, then

$$U_s F^{-1} = F^{-1} V_{-s}, \quad FU_s = V_{-s}F, \quad V_s F^{-1} = F^{-1} U_s, \quad FV_s u = U_s F.$$

Lemma 4.3. *Let J be the flip operator and F the Fourier transform. Then*

(i) *$JU_s = U_{-s}J$, $JV_s = V_{-s}J$ and $JZ_\tau = Z_\tau J$ for $s \in \mathbb{R}$ and $\tau \in \mathbb{R}^+$;*

(ii) *$FJ = JF$, $F^{-1}J = JF^{-1}$ and $F^{-1} = FJ$.*

It happens that when the operators P_τ, R_τ and $S_{\pm\tau}$ are restricted to the positive or negative half-axes, it is possible to describe them in terms of the shift V_t, multiplication by the characteristic functions of the positive or negative half-axes, and the flip J.

Lemma 4.4. *Let $\tau \in \mathbb{R}^+$. The following relations hold between the shift operators V_t and the operators defined in* (2) - (4)

$$\begin{array}{ll}
\chi_\pm P_\tau = P_\tau \chi_\pm = \chi_\pm V_{\pm\tau} \chi_\mp V_{\mp\tau}, & \chi_\pm Q_\tau = Q_\tau \chi_\pm = V_{\pm\tau} \chi_\pm V_{\mp\tau}, \\
\chi_\pm R_\tau = R_\tau \chi_\pm = J \chi_\mp V_{\mp\tau} \chi_\pm, & \\
\chi_\pm S_\tau = S_\tau \chi_\pm = V_{\pm\tau} \chi_\pm, & \chi_\pm S_{-\tau} = S_{-\tau} \chi_\pm = \chi_\pm V_{\mp\tau}.
\end{array}$$

A proof of the following lemma is in [11, Lemmas 4.2.5 and 4.2.12].

Lemma 4.5. *The operators V_t converge weakly to zero as $t \to \pm\infty$, and the $Z_\tau^{\pm 1}$ converge weakly to zero as $\tau \to \infty$.*

The next results characterize the application of the shifts to certain operators which will play an important role in what follows.

Proposition 4.6. *Let $c \in PC(\dot{\mathbb{R}})$. Then*

$$\begin{array}{rcll}
R_\tau \, cI \, R_\tau & \to & c(-\infty)\chi_- + c(+\infty)\chi_+ & \textit{as } \tau \to \infty, \\
R_\tau \, cI \, S_\tau & = & 0, & \\
S_{-\tau} \, cI \, R_\tau & = & 0, & \\
S_{-\tau} \, cI \, S_\tau & \to & c(-\infty)\chi_- + c(+\infty)\chi_+ & \textit{as } \tau \to \infty.
\end{array}$$

Proof. Let $u \in L^p(\mathbb{R})$. Then

$$(R_\tau \, cI \, R_\tau u)(x) = \begin{cases} c(\tau - x)u(x) & \text{if } \; 0 < x < \tau, \\ c(-\tau - x)u(x) & \text{if } \; -\tau < x < 0, \\ 0 & \text{if } \; |x| > \tau. \end{cases}$$

Assume that the support of c is contained in $\mathbb{R}^+$. Given any $u \in L^p(\mathbb{R})$, we must then prove that $\|(R_\tau \, cI \, R_\tau - c(+\infty)\chi_+)u\| \to 0$. Let $m = \max_x |c(x) - c(+\infty)|$. For any $\epsilon > 0$ there exists τ_1 such that

$$|c(\tau) - c(+\infty)| < \epsilon \quad \text{and} \quad \int_\tau^{+\infty} |u(x)|^p \, dx < \epsilon \quad \text{for } \tau \geq \tau_1.$$

Put $\tau_0 = 2\tau_1$. Then, for $\tau \geq \tau_0$,

$$
\begin{aligned}
&\|(R_\tau \, cI \, R_\tau - c(+\infty)\chi_+)u\|^p \\
&\quad = \int_0^\tau |(c(\tau - x) - c(+\infty))u(x)|^p \, dx + \int_\tau^{+\infty} |c(+\infty)u(x)|^p \, dx \\
&\quad = \int_0^{\tau-\tau_1} |(c(\tau - x) - c(+\infty))u(x)|^p \, dx + \int_{\tau-\tau_1}^{\tau} |(c(\tau - x) - c(+\infty))u(x)|^p \, dx \\
&\qquad + \int_\tau^{+\infty} |c(+\infty)u(x)|^p \, dx \leq \epsilon^p \int_0^{\tau-\tau_1} |u(x)|^p \, dx + (m^p + c(+\infty)^p)\epsilon,
\end{aligned}
$$

which finishes the proof of the first assertion. The proofs of the second and third assertion are immediate. For the last assertion, note that

$$
(S_{-\tau} \, cI \, S_\tau u)(x) = \begin{cases} c(x+\tau)u(x) & \text{if} \quad x > 0 \\ c(x-\tau)u(x) & \text{if} \quad x < 0 \end{cases}
$$

and use a reasoning similar to the one above. ■

Recall recall that $\tilde{a}(x) := a(-x)$. The assertions in the following proposition can be easily proved by writing the operators explicitly.

Proposition 4.7. *The following relations hold for $a \in PC_p(\dot{\mathbb{R}})$:*

$$
\begin{aligned}
R_\tau \chi_\pm W^0(a) \chi_\pm R_\tau &= P_\tau \chi_\pm W^0(\tilde{a}) \chi_\pm P_\tau, \\
R_\tau \chi_\pm W^0(a) \chi_\pm S_\tau &= P_\tau \chi_\pm J W^0(a) \chi_\pm, \\
S_{-\tau} \chi_\pm W^0(a) \chi_\pm R_\tau &= \chi_\pm W^0(a) J \chi_\pm P_\tau, \\
S_{-\tau} \chi_\pm W^0(a) \chi_\pm S_\tau &= \chi_\pm W^0(a) \chi_\pm,
\end{aligned}
$$

Proposition 4.8. *The following strong limits hold for $a \in PC_p(\dot{\mathbb{R}})$, when $\tau \to \infty$:*

$$
\begin{aligned}
R_\tau \chi_\pm W^0(a) \chi_\mp R_\tau &\to 0, \\
R_\tau \chi_\pm W^0(a) \chi_\mp S_\tau &\to 0, \\
S_{-\tau} \chi_\pm W^0(a) \chi_\mp R_\tau &\to 0, \\
S_{-\tau} \chi_\pm W^0(a) \chi_\mp S_\tau &\to 0.
\end{aligned}
$$

Proof. All assertions are similar, and we will only prove the first. Define the operator R'_τ on $L^p(\mathbb{R})$ by

$$
(R'_\tau u)(x) := \begin{cases} u(2\tau - x) & \text{if} \quad 0 < x < \tau \\ u(-2\tau - x) & \text{if} \quad -\tau < x < 0 \\ 0 & \text{if} \quad |x| > \tau \end{cases}
$$

It is easy to see that $\|R'_\tau\| = 1$ and $R'_\tau \to 0$.

If $x < 0$ or $x > \tau$ the function $R_\tau \chi_+ W^0(a) \chi_- R_\tau u$ gives the value 0. For $0 < x < \tau$ we have

$$\begin{aligned}
&(R_\tau \chi_+ W^0(a) \chi_- R_\tau u)(x) \\
&\quad = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i(\tau - x)\xi} a(\xi) \int_{-\tau}^{0} e^{i\xi y} u(-\tau - y)\, dy\, d\xi \\
&\quad = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i(-(2\tau - x))(-\xi)} a(\xi) \int_{-\tau}^{0} e^{i(-\xi)y'} u(y')\, dy'\, d\xi,
\end{aligned}$$

hence

$$R_\tau \chi_+ W^0(a) \chi_- R_\tau = R'_\tau J \chi_- W^0(\tilde{a}) \chi_- P_\tau$$

which converges strongly to zero. An alternate way to prove these results would be to use Lemma 4.4. That technique is used below, in the proof of Theorem 4.9. ■

Let $A \in \mathcal{L}(L^p(\mathbb{R}))$. If the strong limit

$$\operatorname*{s-lim}_{\tau \to +\infty} Z_\tau \begin{bmatrix} R_s \\ S_{-s} \end{bmatrix} A \begin{bmatrix} R_s & S_s \end{bmatrix} Z_\tau^{-1}$$

exists for some $s \in \mathbb{R}$, we denote it by $\mathsf{Y}_{s,\infty}(A)$. Further we write $a(s^+)$ (resp. $a(s^-)$) for the limit of the function a at s from the right-hand (resp. left-hand) side. By $a(-s^\pm)$ we denote the limits $a((-s)^\pm)$.

Theorem 4.9. *Let $a \in PC_p(\dot{\mathbb{R}})$, $c \in PC(\dot{\mathbb{R}})$, and K a compact operator. Then the strong limits $\mathsf{Y}_{s,\infty}(cI)$, $\mathsf{Y}_{s,\infty}(W^0(a))$, and $\mathsf{Y}_{s,\infty}(K)$ exist for all $s > 0$ and*

$$\mathsf{Y}_{s,\infty}(cI) = \begin{bmatrix} c(-s^+)\chi_- + c(s^-)\chi_+ & 0 \\ 0 & c(-s^-)\chi_- + c(s^+)\chi_+ \end{bmatrix} \tag{5}$$

$$\mathsf{Y}_{s,\infty}(W^0(a)) = \begin{bmatrix} \chi_+ W^0(\tilde{a}_\infty)\chi_+ I + \chi_- W^0(\tilde{a}_\infty)\chi_- I & \chi_+ W^0(\tilde{a}_\infty)\chi_- J + \chi_- W^0(\tilde{a}_\infty)\chi_+ J \\ \chi_+ W^0(a_\infty)\chi_- J + \chi_- W^0(a_\infty)\chi_+ J & \chi_+ W^0(a_\infty)\chi_+ I + \chi_- W^0(a_\infty)\chi_- I \end{bmatrix} \tag{6}$$

$$\mathsf{Y}_{s,\infty}(J) = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}, \qquad \mathsf{Y}_{s,\infty}(K) = 0, \tag{7}$$

where $a_\infty := a(-\infty)\chi_- + a(+\infty)\chi_+$.

Proof. For the multiplication operator cI, the assertion follows from the identities

$$(Z_\tau R_s\, cI\, R_s Z_\tau^{-1} u)(x) = \begin{cases} c(s - x/\tau)u(x) & \text{if} \quad 0 < x < s\tau \\ c(-s - x/\tau)u(x) & \text{if} \quad -s\tau < x < 0 \\ 0 & \text{if} \quad |x| > s\tau \end{cases},$$

$$(Z_\tau S_{-s}\, cI\, S_s Z_\tau^{-1} u)(x) = \begin{cases} c(s + x/\tau)u(x) & \text{if} \quad x > 0 \\ c(-s + x/\tau)u(x) & \text{if} \quad x < 0 \end{cases}$$

in a similar way as in the proof of Proposition 4.6. For the result for the convolution operator $W^0(a)$, we use the decomposition

$$W^0(a) = \chi_+ W^0(a)\chi_+ I + \chi_+ W^0(a)\chi_- I + \chi_- W^0(a)\chi_+ I + \chi_- W^0(a)\chi_- I.$$

The values of $\mathsf{Y}_{s,\infty}(\chi_+ W^0(a)\chi_+ I)$ and $\mathsf{Y}_{s,\infty}(\chi_- W^0(a)\chi_- I)$ follow from propositions 4.7 and 4.8. Next we will prove that $\mathsf{Y}_{s,\infty}(\chi_+ W^0(a)\chi_- I) = 0$ and $\mathsf{Y}_{s,\infty}(\chi_- W^0(a)\chi_+ I) = 0$. Using the identities in Lemma 4.4, we get

$$\begin{aligned} R_s\chi_+ W^0(a)\chi_- R_s &= J\chi_- V_{-s}\chi_+ W^0(a)J\chi_+ V_s\chi_- \\ &= \chi_+ V_s\chi_- W^0(\tilde{a})\chi_+ V_s\chi_- \\ &= \chi_+ V_s\chi_- V_s W^0(\tilde{a})V_{-s}\chi_+ V_s\chi_- \\ &= \chi_+ V_s\chi_- V_s W^0(\tilde{a})\chi_- P_s \\ &= \chi_+ V_s\chi_- V_s Q_s W^0(\tilde{a})\chi_- P_s. \end{aligned}$$

Multiplying on the left and right by Z_τ and Z_τ^{-1}, respectively, we obtain

$$Z_\tau Q_s W^0(\tilde{a})\chi_- P_s Z_\tau^{-1} \to 0 \quad \text{strongly},$$

while the operators $Z_\tau\chi_+ V_s\chi_- V_s Z_\tau^{-1}$ are uniformly bounded. Thus the product converges strongly to zero. Similarly, one obtains

$$R_s\chi_+ W^0(a)\chi_- S_s = J\chi_- V_{-s}\chi_+ V_{-s}Q_s W^0(a)\chi_-,$$

and the same reasoning applies. Regarding $S_{-s}\chi_+ W^0(a)\chi_- R_s$, we write

$$S_{-s}\chi_+ W^0(a)\chi_- R_s = \chi_+ W^0(a)\chi_- Q_s V_{-s}\chi_- V_{-s}\chi_+ J.$$

Because $W^0(a)$ is an operator of local type, the operator $\chi_+ W^0(a)\chi_- Q_s$ is compact. Thus $S_{-s}\chi_+ W^0(a)\chi_- R_s$ is compact, and the result follows from Lemma 4.5 by [11, Lemma 1.4.6]. A similar reasoning can be used for

$$\begin{aligned} S_{-s}\chi_+ W^0(a)\chi_- S_s &= \chi_+ V_{-s} W^0(a) V_{-s}\chi_- \\ &= \chi_+ V_{-s}\chi_+ Q_s W^0(a) Q_s\chi_- V_{-s}\chi_-. \end{aligned}$$

Thus $\mathsf{Y}_{s,\infty}\big(\chi_+ W^0(a)\chi_- I\big) = 0$. The proof for $\mathsf{Y}_{s,\infty}\big(\chi_- W^0(a)\chi_+ I\big)$ is the same.

The third identity is immediate, and the fourth is easily proved using Lemma 4.5 and [11, Lemma 1.4.6]. ■

We introduce now the Fourier image equivalents of the operators defined in (3) and (4). For $\tau > 0$, define

$$R_\tau^F := F^{-1}R_\tau F, \quad S_\tau^F := F^{-1}S_\tau F \quad \text{and} \quad S_{-\tau}^F := F^{-1}S_{-\tau}F.$$

Proposition 4.10. *The operators R_τ^F, S_τ^F and $S_{-\tau}^F$ are uniformly bounded on $L^p(\mathbb{R})$, and the results of Lemma 4.1, with exception of those related to the norms, remain true if R_τ, S_τ and $S_{-\tau}$ are substituted by their corresponding Fourier images.*

Proof. Applying Lemma 4.4 one obtains

$$\begin{aligned} R_\tau^F &= F^{-1}R_\tau F \\ &= F^{-1}(\chi_- R_\tau + \chi_+ R_\tau)F \\ &= F^{-1}(J\chi_+ V_\tau\chi_- + J\chi_- V_{-\tau}\chi_+)F \\ &= JF^{-1}\chi_+ F\, F^{-1}V_\tau F\, F^{-1}\chi_- F + JF^{-1}\chi_- F\, F^{-1}V_{-\tau}F\, F^{-1}\chi_+ F \\ &= JW^0(\chi_+)U_{-\tau}W^0(\chi_-) + JW^0(\chi_-)U_\tau W^0(\chi_+). \end{aligned}$$

Since all involved operators are bounded on $L^p(\mathbb{R})$ and $\|U_\tau\| = 1$, the result follows. The proofs for S_τ^F and $S_{-\tau}^F$ are similar. The proof of the last assertion is straightforward. ∎

Let $A \in \mathcal{B}(L^p(\mathbb{R}))$. If the strong limit

$$\operatorname*{s-lim}_{\tau\to+\infty} Z_\tau^{-1} \begin{bmatrix} R_t^F \\ S_{-t}^F \end{bmatrix} A \begin{bmatrix} R_t^F & S_t^F \end{bmatrix} Z_\tau$$

exists for some $t \in \mathbb{R}$, we denote it by $\mathsf{Y}_{\infty,t}(A)$.

Theorem 4.11. *Let $a \in PC_p(\dot{\mathbb{R}})$, $c \in PC(\dot{\mathbb{R}})$ and K a compact operator. Then the strong limits $\mathsf{Y}_{\infty,t}(cI)$ and $\mathsf{Y}_{\infty,t}(W^0(a))$ exist for all $t > 0$, and*

$$\mathsf{Y}_{\infty,t}(cI) = \begin{bmatrix} \chi_+^F \tilde{c}_\infty \chi_+^F + \chi_-^F \tilde{c}_\infty \chi_-^F & \chi_+^F \tilde{c}_\infty \chi_-^F J + \chi_-^F \tilde{c}_\infty \chi_+^F J \\ \chi_+^F c_\infty \chi_-^F J + \chi_-^F c_\infty \chi_+^F J & \chi_+^F c_\infty \chi_+^F + \chi_-^F c_\infty \chi_-^F \end{bmatrix} \tag{8}$$

$$\mathsf{Y}_{\infty,t}(W^0(a)) = \begin{bmatrix} W^0\big(a(-t^+)\chi_- + a(t^-)\chi_+\big) & 0 \\ 0 & W^0\big(a(-t^-)\chi_- + a(t^+)\chi_+\big) \end{bmatrix} \tag{9}$$

$$\mathsf{Y}_{\infty,t}(J) = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}, \qquad \mathsf{Y}_{\infty,t}(K) = 0, \tag{10}$$

where $c_\infty = c(-\infty)\chi_- + c(+\infty)\chi_+$ and $\chi_\pm^F := F^{-1}\chi_\pm F$.

Proof. It is easy to check that $Z_\tau^{-1}F^{-1} = F^{-1}Z_\tau \tau^{1-2/p}$ and $FZ_\tau = \tau^{-1+2/p}Z_\tau^{-1}F$. Then the proofs for the assertions can be obtained as in the proof of Theorem 4.9, taking into account Lemma 4.3. ∎

5 Localization

Let $\tilde{C}(\dot{\mathbb{R}})$ and $\tilde{C}_p$ denote the sets of all even functions in $C(\dot{\mathbb{R}})$ and C_p, respectively, and write $\mathcal{L}$ for the set of all operators in $\mathcal{B}$ that commute with all operators fI and $W^0(g)$ where $f \in \tilde{C}(\dot{\mathbb{R}})$ and $g \in \tilde{C}_p$ modulo compact operators. Then $\mathcal{L}$ is a closed and inverse-closed subalgebra of $\mathcal{B}$, and $\mathcal{L}/\mathcal{K}$ is a closed and inverse-closed subalgebra of the Calkin algebra $\mathcal{B}/\mathcal{K}$, The cosets $\Phi(fI) := fI + \mathcal{K}$ and $\Phi(W^0(g)) := W^0(g) + \mathcal{K}$ with $f \in \tilde{C}(\dot{\mathbb{R}})$ and $g \in \tilde{C}_p$ belong to the center of $\mathcal{L}/\mathcal{K}$. Let $\mathcal{C}$ denote the smallest closed (necessarily commutative) subalgebra of $\mathcal{L}/\mathcal{K}$ generated by these cosets.

Let $\bar{\mathbb{R}}^+$ denote the compactification of $\mathbb{R}^+$ by the point $\{\infty\}$, i.e., $\bar{\mathbb{R}}^+$ is homeomorphic to $[0, 1]$. Then the maximal ideal space of $\mathcal{C}$ is homeomorphic to the subset $M_\mathcal{C} := (\bar{\mathbb{R}}^+ \times \{\infty\}) \cup (\{\infty\} \times \bar{\mathbb{R}}^+)$ of the square $\bar{\mathbb{R}}^+ \times \bar{\mathbb{R}}^+$, see [11, Section 5.7]. The value of the Gelfand transform of an element $\Phi(fW^0(g)) \in \mathcal{C}$ at the point $(s, t) \in M_\mathcal{C}$ is $f(s)g(t)$.

The elements of $\mathcal{C}$ can be explicitly described as follows.

Proposition 5.1. *Every element of $\mathcal{C}$ can be uniquely written in the form $\Phi(\gamma I + fI + W^0(g))$ with $\gamma \in \mathbb{C}$ and $f \in \tilde{C}(\dot{\mathbb{R}})$ and $g \in \tilde{C}_p$ with $f(\infty) = g(\infty) = 0$. If $\gamma \neq 0$, then $\Phi(\gamma I + fI + W^0(g))$ can be written as a product $\Phi(f'W^0(g'))$ with $f' \in \tilde{C}(\dot{\mathbb{R}})$ and $g' \in \tilde{C}_p$.*

Proof. Let $\mathcal{C}'$ denote the set of all operators

$$\gamma I + fI + W^0(g) + K \tag{11}$$

where γ, f, g are as in the proposition and K is compact. Using Proposition 5.3.1 in [11] one deduces that $\mathcal{C}'$ is an algebra, whereas Lemma 5.4.2 in [11] can be used to show that $\mathcal{C}'$ is a closed algebra and that every element of that algebra has a unique representation in the form (11) with γ, f, g, K as mentioned. The argument runs as follows, with the notation from [11]: If A is an operator of the form (11), then the strong limit $L_U(A)$ of the sequence $U_n A U_{-n}$ exists and

$$L_U(\gamma I + fI + W^0(g) + K) = \gamma I + fI.$$

Moreover, $\|L_U(A)\| \leq \|A\|$. Thus, if $(\gamma_n I + f_n I + W^0(g_n) + K_n)$ is a Cauchy sequence in $\mathcal{C}'$, then $(\gamma_n I + f_n I)$ is a Cauchy sequence in $\tilde{C}(\dot{\mathbb{R}})$, hence convergent. Using the shift operators V_n from [11] in place of the U_n one gets the convergence of the sequence $(W^0(g_n))$, hence that of (K_n). The limits $\gamma I + f$, $W^0(g)$ and K of these sequences belong to the corresponding algebras; hence the limit $\gamma I + fI + W^0(g) + K$ of the sequence $(\gamma_n I + f_n I + W^0(g_n) + K_n)$ belongs to $\mathcal{C}'$.

Consequently, every element of $\mathcal{C}'/\mathcal{K}$ is of the form $\Phi(\gamma I + fI + W^0(g))$ with γ, f, g as in the proposition. The first assertion follows since $\mathcal{C}'/\mathcal{K} = \mathcal{C}$, as one easily checks.

Let now $\gamma \neq 0$ and f, g as above. Set $f' := \gamma^{-1}(f + \gamma)$ and $g' := g + \gamma$. Then

$$f'W^0(g') = \gamma^{-1}(f+\gamma)W^0(g+\gamma) = \gamma^{-1}(\gamma^2 I + \gamma fI + \gamma W^0(g) + fW^0(g)) = \gamma I + fI + W^0(g) + K$$

with a compact operator K due to Proposition 5.3.1 in [11] again. This settles the second assertion. ■

Given $(s,t) \in (\bar{\mathbb{R}}^+ \times \{\infty\}) \cup (\{\infty\} \times \bar{\mathbb{R}}^+)$, let $\mathcal{I}_{s,t}$ denote the smallest closed two-sided ideal of the quotient algebra $\mathcal{L}/\mathcal{K}$ which contains the maximal ideal corresponding to the point (s,t), and let $\Phi^{\mathcal{K}}_{s,t}$ refer to the canonical homomorphism from $\mathcal{L}/\mathcal{K}$ onto the quotient algebra $\mathcal{L}^{\mathcal{K}}_{s,t} := (\mathcal{L}/\mathcal{K})/\mathcal{I}_{s,t}$.

One cannot expect that the local algebras $\mathcal{L}^{\mathcal{K}}_{s,t}$ can be identified completely. But we will be able to identify the smallest closed subalgebra $\mathcal{A}^{\mathcal{K}}_{s,t}$ of $\mathcal{L}^{\mathcal{K}}_{s,t}$ which contains all cosets $(aI) + \mathcal{I}_{s,t}$ with $a \in PC(\dot{\mathbb{R}})$, $(W^0(b)) + \mathcal{I}_{s,t}$ with $b \in PC_p$ and $(J) + \mathcal{I}_{s,t}$, and this identification will be sufficient for our purposes.

We will identify the algebras $\mathcal{A}^{\mathcal{K}}_{s,t}$ by means of the family of the Y-homomorphisms. Note that, by (7) and (10), the operators $\mathsf{Y}_{\infty,y}(A)$ and $\mathsf{Y}_{x,\infty}(A)$ depend only on the coset of the operator A modulo $\mathcal{K}$. Thus, the quotient homomorphisms

$$A + \mathcal{K} \mapsto \mathsf{Y}_{\infty,t}(A) \quad \text{and} \quad A + \mathcal{K} \mapsto \mathsf{Y}_{s,\infty}(A)$$

are well defined. We denote them again by $\mathsf{Y}_{\infty,t}$ and $\mathsf{Y}_{s,\infty}$, respectively.

6 Identification of the local algebras

6.1 The algebras $\mathcal{A}^{\mathcal{K}}_{0,\infty}$, $\mathcal{A}^{\mathcal{K}}_{\infty,0}$ and $\mathcal{A}^{\mathcal{K}}_{\infty,\infty}$

The identification of the subalgebras $\mathcal{A}^{\mathcal{K}}_{0,\infty}$ and $\mathcal{A}^{\mathcal{K}}_{\infty,0}$ can be made as in [11, Section 5.7], using the homomorphisms

$$\mathsf{Y}_{0,\infty}(A) := \operatorname*{s-lim}_{\tau\to\infty} Z_\tau A Z_\tau^{-1};$$
$$\mathsf{Y}_{\infty,0}(A) := \operatorname*{s-lim}_{\tau\to\infty} Z_\tau^{-1} A Z_\tau.$$

Thus, the following descriptions of these algebras follow immediately from Propositions 5.7.2 and 5.7.3 in [11], whereas the identification of the algebra $\mathcal{A}^{\mathcal{K}}_{\infty,\infty}$ comes from Proposition 5.7.8 in [11].

Proposition 6.1. *The local algebra $\mathcal{A}^{\mathcal{K}}_{0,\infty}$ is isometrically isomorphic to the closed subalgebra* $\operatorname{alg}\{I, \chi_+ I, P_{\mathbb{R}}, J\}$ *of $\mathcal{L}(L^p(\mathbb{R}^+))$, with the isomorphism given by $\Phi^{\mathcal{K}}_{0,\infty}(A) \mapsto \mathsf{Y}_{0,\infty}(A)$. In particular, for $a \in PC(\dot{\mathbb{R}})$ and $b \in PC_p$,*

$$\begin{array}{lll} \Phi^{\mathcal{K}}_{0,\infty}(aI) & \mapsto & a(0^-)\chi_- I + a(0^+)\chi_+ I, \\ \Phi^{\mathcal{K}}_{0,\infty}(W(b)) & \mapsto & b(-\infty)W^0(\chi_-) + b(+\infty)W^0(\chi_+), \\ \Phi^{\mathcal{K}}_{0,\infty}(J) & \mapsto & J. \end{array}$$

Proposition 6.2. *The local algebra $\mathcal{A}^{\mathcal{K}}_{\infty,0}$ is isometrically isomorphic to the closed subalgebra* $\operatorname{alg}\{I, \chi_+ I, P_{\mathbb{R}}, J\}$ *of $\mathcal{L}(L^p(\mathbb{R}^+))$, and the isomorphism is given by $\Phi^{\mathcal{K}}_{\infty,0}(A) \mapsto \mathsf{Y}_{\infty,0}(A)$. In particular, for $a \in PC(\dot{\mathbb{R}})$ and $b \in PC_p$,*

$$\begin{array}{lll} \Phi^{\mathcal{K}}_{\infty,0}(aI) & \mapsto & a(-\infty)\chi_- I + a(+\infty)\chi_+ I, \\ \Phi^{\mathcal{K}}_{\infty,0}(W(b)) & \mapsto & b(0^-)W^0(\chi_-) + b(0^+)W^0(\chi_+), \\ \Phi^{\mathcal{K}}_{\infty,0}(J) & \mapsto & J. \end{array}$$

Proposition 6.3. *The local algebra $\mathcal{A}^{\mathcal{K}}_{\infty,\infty}$ is generated by the commuting projections $p = \Phi^{\mathcal{K}}_{\infty,\infty}(\chi_+ I)$ and $r = \Phi^{\mathcal{K}}_{\infty,\infty}(W^0(\chi_+))$ and by the flip $j = \Phi^{\mathcal{K}}_{\infty,\infty}(J)$. There is a symbol mapping which assigns with e, p, j and r a matrix-valued function on $\{0,1\}$ by*

$$(\operatorname{smb} e)(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (\operatorname{smb} p)(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$(\operatorname{smb} j)(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (\operatorname{smb} r)(x) = \begin{bmatrix} x & 0 \\ 0 & 1-x \end{bmatrix}.$$

Define the homomorphism $\mathsf{Y}_{\infty,\infty}$ as the composition $\operatorname{smb} \circ \Phi^{\mathcal{K}}_{\infty,\infty}$.

6.2 The algebras $\mathcal{A}^{\mathcal{K}}_{s,\infty}$ for $s > 0$

We continue with describing the generators of the local algebras $\mathcal{A}^{\mathcal{K}}_{s,\infty}$ for $s > 0$.

Proposition 6.4. *The algebra $\mathcal{A}^{\mathcal{J}}_{s,\infty}$ is generated by the identity e, by the projections $p_1 := \Phi^{\mathcal{J}}_{s,\infty}(\chi_{]-s,0[}I)$, $p_2 := \Phi^{\mathcal{J}}_{s,\infty}(\chi_{]0,s[}I)$, $p_3 := \Phi^{\mathcal{J}}_{s,\infty}(\chi_{]s,\infty[}I)$, $r := \Phi^{\mathcal{J}}_{s,\infty}(W^0(\chi_+))$, and by the flip $j := \Phi^{\mathcal{J}}_{s,\infty}(J)$.*

Proof. For $c \in PC(\dot{\mathbb{R}})$, set

$$c' := c(-s^-) + \big(c(-s^+) - c(-s^-)\big)\chi_{]-s,0[} + \big(c(s^-) - c(-s^-)\big)\chi_{]0,s[} + \big(c(s^+) - c(-s^-)\big)\chi_{]s,+\infty[}.$$

The function $c - c'$ is continuous at the points $-s$ and s and takes the value 0 there. Given $\epsilon > 0$, choose a function $f_{s,\epsilon} \in \tilde{C}(\dot{\mathbb{R}})$ which takes the value 1 at the points $-s$, s and has its support in $]-s-\epsilon, -s+\epsilon\,[\, \cup \,]\, s-\epsilon, s+\epsilon\,[$. Then $\Phi^{\mathcal{J}}_{s,\infty}(f_{s,\epsilon}I)$ is the identity in the local algebra and

$$\Phi^{\mathcal{J}}_{s,\infty}(cI - c'I) = \Phi^{\mathcal{J}}_{s,\infty}(cI - c'I)\Phi^{\mathcal{J}}_{s,\infty}(f_{s,\epsilon}I) = \Phi^{\mathcal{J}}_{s,\infty}\big((cI - c'I)f_{s,\epsilon}I\big).$$

The norm of the latter expression can be made arbitrarily small by taking epsilon close to zero. It follows that $\Phi^{\mathcal{J}}_{s,\infty}(cI) = \Phi^{\mathcal{J}}_{s,\infty}(c'I)$ and

$$\Phi^{\mathcal{J}}_{s,\infty}(c'I) = c(-s^-)e + \big(c(-s^+) - c(-s^-)\big)p_1 + \big(c(s^-) - c(-s^-)\big)p_2 + \big(c(s^+) - c(-s^-)\big)p_3.$$

For $a \in PC_p(\dot{\mathbb{R}})$ one gets similarly $\Phi^{\mathcal{J}}_{s,\infty}(W^0(a)) = a(-\infty)(e - r) + a(+\infty)r$. ■

In order to use the homomorphisms $\mathsf{Y}_{s,\infty}$ to identify the local algebras, it is necessary to characterize their image and to prove that they are invertible. Let $\mathcal{Y}$ represent the set of all matrices

$$\begin{bmatrix} A_{11} A_{12} \\ A_{21} A_{22} \end{bmatrix} \quad \text{where} \quad \begin{array}{l} A_{11}, A_{22} \in \mathcal{E}_p + \mathcal{E}_pJ + J\mathcal{E}_p + J\mathcal{E}_pJ, \\ A_{12}, A_{21} \in \mathcal{N}_p + \mathcal{N}_pJ + J\mathcal{N}_p + J\mathcal{N}_pJ. \end{array} \tag{12}$$

It is easy to check that $\mathcal{Y}$ is an algebra and that its generators are in the image of the the homomorphism $\mathsf{Y}_{s,\infty}$.

Proposition 6.5. *The algebra $\mathcal{Y}$ is inverse-closed in $\mathcal{B}^{2\times 2}$.*

Proof. We identify $L^p_2(\mathbb{R})$ with $L^p_4(\mathbb{R}^+)$ via the mapping

$$(u, v) \mapsto (\chi_+ u, \chi_+ Ju, \chi_+ v, \chi_+ Jv).$$

The above mapping defines an isomorphism between $\mathcal{B}^{2\times 2}$ and $\mathcal{L}\big(L^p_4(\mathbb{R}^+)\big)$, which induces an isomorphism between the algebra $\mathcal{Y}$ and the algebra $\mathcal{Y}_1$ of all 4×4 matrices with the structure

$$\begin{bmatrix} E & E & N & N \\ E & E & N & N \\ N & N & E & E \\ N & N & E & E \end{bmatrix}$$

with E representing elements in $\mathcal{E}_p$ and N elements in $\mathcal{N}_p$. Because $\mathcal{E}_p$ is inverse-closed in $\mathcal{L}\big(L^p(\mathbb{R}^+)\big)$ (the generators have thin spectra, see [11, Corollary 1.2.32]), the algebra $\mathcal{E}_p^{4\times 4}$ is inverse-closed in $\mathcal{L}\big(L^p_4(\mathbb{R}^+)\big)$ by [11, Proposition 1.2.35]. That the algebra $\mathcal{Y}_1$ is inverse-closed in $\mathcal{E}_p^{4\times 4}$ can then be shown by employing the explicit formula for the inverse of a matrix with commuting entries via determinants. ∎

We proceed with finding an inverse for the homomorphism $\mathsf{Y}_{s,\infty}$. A natural candidate for this inverse is the mapping

$$\mathsf{Y}'_s : \mathcal{Y} \to \mathcal{B}, \quad A \mapsto \begin{bmatrix} R_s & S_s \end{bmatrix} A \begin{bmatrix} R_s \\ S_{-s} \end{bmatrix},$$

but it is not clear, *a priori*, if even the definition makes sense, because we are not dealing with an homomorphism. The next two results, prove that indeed this is the right choice.

Theorem 6.6. *The image of $\mathcal{Y}$ under the mapping Y'_s is in $\mathcal{L}$.*

Proof. Let A be an element of the form (12). Then

$$\mathsf{Y}'_s(A) = R_s A_{11} R_s + R_s A_{12} S_{-s} + S_s A_{21} R_s + S_s A_{22} S_{-s}. \tag{13}$$

We show that each of those four terms commutes with fI for $f \in \tilde{C}(\dot{\mathbb{R}})$ and with $W^0(g)$ for $g \in \tilde{C}_p$ modulo compact operators.

First consider the commutator $[fI, R_s A_{11} R_s]$. For $f \in \tilde{C}(\dot{\mathbb{R}})$ and $s > 0$, define

$$\widehat{f}_s(x) := \begin{cases} f(s-x) & \text{if } x \in [0,s], \\ f(-s-x) & \text{if } x \in [-s, 0[, \\ f(0) & \text{if } x > s. \end{cases}$$

Clearly, $\widehat{f}_s \in \tilde{C}(\dot{\mathbb{R}})$ and

$$R_s f I = \widehat{f}_s R_s, \qquad R_s \widehat{f}_s I = f_s R_s.$$

Because each of the terms in A_{11} is a operator that commutes modulo compact operators with functions in $\tilde{C}(\dot{\mathbb{R}})$ one has

$$f R_s A_{11} R_s = R_s \widehat{f}_s A_{11} R_s = R_s A_{11} \widehat{f}_s R_s + K = R_s A_{11} R_s f + K$$

where K is a compact operator.

For the commutator $[fI, S_s A_{22} S_{-s}]$, define

$$\check{f}_s(x) := \begin{cases} f(x+s) & \text{if } x \geq 0, \\ f(x-s) & \text{if } x < 0 \end{cases}$$

where $f \in \tilde{C}(\dot{\mathbb{R}})$ and $s > 0$. Then $\check{f}_s \in \tilde{C}(\dot{\mathbb{R}})$ and

$$S_{-s} f I = \check{f}_s S_{-s}, \qquad S_s \check{f}_s I = f_s S_s.$$

Hence, as above, $f S_s A_{22} S_{-s} - S_s A_{22} S_{-s} f +$ compact.

It remains to consider the "middle" terms in (13). For symmetry reasons it is sufficient to consider one of them, say the commutator $[fI, S_s A_{21} R_s]$. Write f as $f(\pm s) + \check{f}$ with

$\check{f}(\pm s) = 0$. Since $f(\pm s)I$ commutes with every operator, we can assume without loss of generality that f satisfies $f(\pm s) = 0$. Then, writing $\widehat{f}_s$ as above, $R_s f I = \widehat{f}_s R_s = \widehat{\widehat{f}}_s R_s$, where

$$\widehat{\widehat{f}}_s(x) := \begin{cases} \widehat{f}_s(x) & \text{if } x \in [-s, s], \\ \widehat{f}_s(s)(2s - |x|)/s & \text{if } |x| \in]s, 2s], \\ 0 & \text{if } |x| > 2s. \end{cases}$$

In particular, $\widehat{\widehat{f}}_s(0) = \widehat{\widehat{f}}_s(x)(\pm\infty) = 0$. From [11, Proposition 5.3.2(i)] we then infer that $\widehat{\widehat{f}}_s M^0(b)$ and $M^0(b)\widehat{\widehat{f}}_s$ are compact when $M^0(b)$ is a Mellin convolution with symbol $b \in \tilde{C}_p$ and $b(\pm\infty) = 0$, hence when $M^0(b)$ is an arbitrary element of $\mathcal{N}_p$ (see [11, Proposition 4.2.17(i)]). Consequently, if $f(\pm s) = 0$, then

$$S_s A_{21} R_s f I \;\in\; \mathcal{K}. \tag{14}$$

We will show that if $f(\pm s) = 0$ then

$$f S_s A_{21} R_s \;\in\; \mathcal{K}$$

also. Indeed, as above, $f S_s = S_s \check{f}_s I$. Write $\check{f}_s$ as a sum $\check{f}_s^o + \check{f}_s^{oo}$ where $\check{f}_s^o$ has a compact support and the support of $\check{f}_s^{oo}$ is contained in $]-\infty - 2s] \cup [2s, +\infty[$. Then $\check{f}_s^o(0) = \check{f}_s^o(\pm\infty) = 0$, and the same argument used for (14) gives

$$S_s \check{f}_s^o A_{21} R_s \;\in\; \mathcal{K}.$$

For $\check{f}_s^{oo}$ we argue as follows. By [11, Proposition 5.3.2(ii)-2],

$$S_s \check{f}_s^{oo} A_{21} R_s - S_s A_{21} \check{f}_s^{oo} R_s \;\in\; \mathcal{K}.$$

But $\check{f}_s^{oo} R_s = 0$, since $\operatorname{supp} \check{f}_s^{oo} \subseteq \mathbb{R} \setminus [-2s, 2s]$ and $\operatorname{supp} R_s u \subseteq [-s, s]$ for every $u \in L^p(\mathbb{R})$. Hence,

$$f I S_s A_{21} R_s = S_s \check{f}_s^o A_{21} R_s + S_s \check{f}_s^{oo} A_{21} R_s$$

is compact. We have thus so far proved that $[\mathsf{Y}(A), fI] \in \mathcal{K}$.

Now we consider the commutators with the Fourier convolution $W^0(g)$. First we show

$$R_s W^0(g) - W^0(\tilde{g}) R_s \;\in\; \mathcal{K}, \tag{15}$$

where $\tilde{g}(x) = g(-x)$. Write

$$R_s W^0(g) = R_s W^0(g) R_s R_s + R_s W^0(g) S_s S_{-s}. \tag{16}$$

Now decompose the part $R_s W^0(g) R_s$ as

$$\begin{aligned} R_s W^0(g) R_s = {} & R_s \chi_+ W^0(g) \chi_+ R_s + R_s \chi_+ W^0(g) \chi_- R_s \\ & + R_s \chi_- W^0(g) \chi_+ R_s + R_s \chi_- W^0(g) \chi_- R_s. \end{aligned} \tag{17}$$

The operators $\chi_\pm W^0(g) \chi_\mp = \chi_\pm W^0(g) J \chi_\pm$ are Hankel type operators with a continuous generating function in $\tilde{C}_p$ and thus compact. So

$$\begin{aligned} R_s W^0(g) R_s &= R_s \chi_+ W^0(g) \chi_+ R_s + R_s \chi_- W^0(g) \chi_- R_s \quad + \text{ compact} \\ &= P_s \chi_+ W^0(\tilde{g}) \chi_+ P_s + P_s \chi_- W^0(\tilde{g}) \chi_- P_s \quad + \text{ compact}. \end{aligned}$$

Using a decomposition as in (17) again, one finally obtains

$$R_s W^0(g) R_s - P_s W^0(\tilde{g}) P_s \in \mathcal{K}. \tag{18}$$

Next consider the operators $R_s W^0(g) S_s$,

$$\begin{aligned} R_s W^0(g) S_s = & R_s \chi_+ W^0(g) \chi_+ S_s + R_s \chi_- W^0(g) \chi_- S_s \\ & + R_s \chi_- W^0(g) \chi_+ S_s + R_s \chi_+ W^0(g) \chi_- S_s. \end{aligned}$$

The operators in the second line are compact again. For the operators in the first line one has

$$R_s \chi_+ W^0(g) \chi_+ S_s = R_s W(g) S_s = P_s H(\tilde{g}),$$

which again is compact. Hence, $R_s W^0(g) S_s$ is compact. So, from (16) and (18) we conclude that

$$R_s W^0(g) - P_s W^0(\tilde{g}) R_s \in \mathcal{K}. \tag{19}$$

Moreover, multiplying (16) by $Q_s = I - P_s$, we get that $R_s W^0(g) Q_s$ is compact, whence $R_s W^0(g) = R_s W^0(g) P_s + \text{compact}$. As everything works in $L^p(\mathbb{R})$ for every $1 < p < \infty$, one can take adjoints and obtain

$$W^0(g) R_s - P_s W^0(g) R_s \in \mathcal{K},$$

in particular

$$P_s W^0(\tilde{g}) R_s - W^0(\tilde{g}) R_s \in \mathcal{K}.$$

Together with (19), this gives (15).

Finally, consider $W^0(g) S_s$. We claim that

$$W^0(g) S_s - S_s W^0(g) \in \mathcal{K}. \tag{20}$$

Write

$$W^0(g) S_s = R_s R_s W^0(g) S_s + S_s S_{-s} W^0(g) S_s.$$

The operator $R_s W^0(g) S_s$ is compact, as seen above. For the second term we get (using similar arguments from above)

$$\begin{aligned} S_s S_{-s} W^0(g) S_s &= S_s S_{-s} \chi_+ W^0(g) \chi_+ S_s + S_s S_{-s} \chi_- W^0(g) \chi_- S_s \quad + \text{ compact} \\ &= S_s \chi_+ W^0(g) \chi_+ + S_s \chi_- W^0(g) \chi_- \quad + \text{ compact} \\ &= S_s W^0(g) \quad + \text{ compact}. \end{aligned}$$

So we proved (20), and by taking adjoints we obtain

$$W^0(g) S_{-s} - S_{-s} W^0(g) \in \mathcal{K}. \tag{21}$$

Consider now $\mathsf{Y}'_s(A)$ as in (13). The identities (15), (20) and (21) allow to commute $W^0(g)$ R_s, S_s and S_{-s} when $g \in \tilde{C}_p$. Hence it remains to show that $W^0(g)$ commutes with the entries of the matrix A. This follows from the fact that the commutators $[\chi_+ W^0(g) \chi_+, M^0(b)]$ are compact for $g \in \tilde{C}_p$ and $b \in C_p(\overline{R})$ by [11, Proposition 5.3.4 (ii)]. ∎

Theorem 6.7. *Let $s > 0$. The homomorphism $\mathsf{Y}_{s,\infty}$ is an isomorphism between $\mathcal{A}^{\mathcal{K}}_{s,\infty}$ and the algebra $\mathcal{Y}$.*

Proof. The mapping $\mathsf{Y}'_s : \mathcal{Y} \to \mathcal{L}$ is not an homomorphism. In fact, it easy to see that

$$\begin{aligned} &\mathsf{Y}'_{s,\infty}(A)\mathsf{Y}'_{s,\infty}(B) - \mathsf{Y}_{s,\infty}(AB) \\ &\quad = R_sEQ_sER_s + R_sEQ_sNS_{-s} + S_sNQ_sER_s + S_sNQ_sNS_{-s} \end{aligned} \tag{22}$$

where, in each appearance, E and N represent (possibly different) elements in $\mathcal{E}_p + \mathcal{E}_pJ + J\mathcal{E}_p + J\mathcal{E}_pJ$ and $\mathcal{N}_p + \mathcal{N}_pJ + J\mathcal{N}_p + J\mathcal{N}_pJ$, respectively. We prove next that the above operators are in the ideal $\mathcal{I}_{s,\infty} + \mathcal{K}$.

Consider first $E = \chi_+ I$. In this case we immediately obtain

$$R_sEQ_s = 0 \quad \text{and} \quad Q_sER_s = 0. \tag{23}$$

For $E = W(\mathrm{sgn})$, let $f_s \in \tilde{C}(\dot{\mathbb{R}})$ denote a function which takes the value 1 at the points $\{-s, s\}$ and the value 0 in a neighborhood of 0, with $f_s = f_s^+ + f_s^-$ where $f_s^{\pm}$ have support in the positive/negative half-axis, respectively. The multiplication operator f_sI belongs to the identity coset in the local algebra $\mathcal{L}^{\mathcal{K}}_{s,\infty}$. It is also well known that $R_sW(a)Q_s = P_sR_s\chi_+W^0(a)\chi_+S_sS_{-s} = P_s\chi_+W^0(a)\chi_-JS_{-s}$ for $a \in \mathcal{M}_p$ (see for instance [3]). Then

$$\begin{aligned} f_sR_sW(\mathrm{sgn})Q_s &= f_sP_s\chi_+W^0(\mathrm{sgn})\chi_-JS_{-s} \\ &= P_sf_s\chi_+W^0(\mathrm{sgn})\chi_-JS_{-s} \\ &= P_s\chi_+f_s^+W^0(\mathrm{sgn})\chi_-JS_{-s} \\ &= P_s\chi_+W^0(\mathrm{sgn})f_s^+\chi_-JS_{-s} + K = K \end{aligned} \tag{24}$$

with K a compact operator. Thus, by (23) and (24), R_sEQ_s is in the ideal $\mathcal{I}_{s,\infty} + \mathcal{K}$ for any $E \in \mathcal{E}_p$. Because J commutes with R_s and Q_s, the result also holds for $E \in \mathcal{E}_p + \mathcal{E}_pJ + J\mathcal{E}_p + J\mathcal{E}_pJ$.

Now we turn our attention to the operator Q_sNS_{-s}. We consider the more general operator Q_sES_{-s} with $E \in \mathcal{E}_p + \mathcal{E}_pJ + J\mathcal{E}_p + J\mathcal{E}_pJ$. It is necessary that the function f_s, besides the properties above, satisfies $f_s(x) = 0$ for $|x| \geq 2s$. Then

$$Q_sES_{-s}f_s = Q_sE\tilde{f}_sS_{-s} = Q_s\tilde{f}_sES_{-s} + K = K$$

with K a compact operator.

We have thus proved that elements of the form (22) with E and N substituted by generators of the algebra $\mathcal{E}_p$ belong to the ideal $\mathcal{I}_{s,\infty} + \mathcal{K}$. Note now that, for any $A, B \in \mathcal{E}_p$ one has

$$\begin{aligned} R_sABQ_s &= R_sA(R_sR_s + Q_s)BQ_s = R_sAR_s(R_sBQ_s) + (R_sAQ_s)BQ_s \\ Q_sABS_{-s} &= Q_sAS_{-s}S_sBS_{-s} = (Q_sAS_{-s})S_sBS_{-s}. \end{aligned}$$

By induction the result is valid for any polynomial of the generators. As any element $E \in \mathcal{E}_p$ can be approximated by polynomials of the generators and the ideal is closed, the result is true for any element of the form (22).

Let us denote by the same symbol Y'_s the composition of $\mathsf{Y}'_s : \mathcal{Y} \to \mathcal{L}$ with the canonical homomorphism $\mathcal{L} \to \mathcal{L}^{\mathcal{K}}_{s,\infty}$. We proved the mapping $\mathsf{Y}'_s : \mathcal{Y} \to \mathcal{L}^{\mathcal{K}}_{s,\infty}$ is an homomorphism.

It is easy to see that Y'_s indeed maps the generators of $\mathcal{Y}$ to the generators of $\mathcal{A}^{\mathcal{K}}_{s,\infty}$. Then $\mathsf{Y}'_s(\mathsf{Y}_{s,\infty}(A)) = A$ for any $A \in \mathcal{A}^{\mathcal{K}}_{s,\infty}$, and the result is proved. ∎

6.3 The algebras $\mathcal{A}^{\mathcal{K}}_{\infty,t}$ for $t > 0$

The structure of this section is similar to the previous one, and many results in this section can be proved as their counterparts. So we often omit the details.

We start with describing the generators of the local algebras. The proof runs parallel to that of Proposition 6.4.

Proposition 6.8. *The algebra $\mathcal{A}^{\mathcal{J}}_{\infty,t}$ is generated by the identity e, the projections $p := \Phi^{\mathcal{J}}_{\infty,t}(\chi_+ I)$, $r_1 := \Phi^{\mathcal{J}}_{\infty,t}(W^0(\chi_{]-t,0[}))$, $r_2 := \Phi^{\mathcal{J}}_{\infty,t}(W^0(\chi_{]0,t[}))$, $r_3 := \Phi^{\mathcal{J}}_{\infty,t}(W^0(\chi_{]t,\infty[}))$ and the flip $j := \Phi^{\mathcal{J}}_{\infty,t}(J)$.*

The image of the homomorphism $\mathsf{Y}_{\infty,t}$ can be characterized as follows. Consider the algebras $\mathcal{E}^F_p$ introduced in Section 2, and let $\mathcal{Y}^F$ represent the set of all matrices

$$\begin{bmatrix} A_{11} A_{12} \\ A_{21} A_{22} \end{bmatrix} \quad \text{where} \quad \begin{aligned} A_{11}, A_{22} &\in \mathcal{E}^F_p + \mathcal{E}^F_p J + J\mathcal{E}^F_p + J\mathcal{E}^F_p J, \\ A_{12}, A_{21} &\in \mathcal{N}^F_p + \mathcal{N}^F_p J + J\mathcal{N}^F_p + J\mathcal{N}^F_p J. \end{aligned} \tag{25}$$

It is easy to check that $\mathcal{Y}^F$ is an algebra and that its generators are in the image of the homomorphism $\mathsf{Y}_{\infty,t}$. By Theorem 3.1 and its corollary, the algebra $\mathcal{Y}^F$ is isomorphic to the algebra $\mathcal{Y}$ studied in the previous section.

Our next goal is to find the inverse of the homomorphism $\mathsf{Y}_{\infty,t}$. The natural candidate for this inverse is the mapping

$$\mathsf{X}^F_t : \mathcal{Y}^F \to \mathcal{B}, \quad A \mapsto \begin{bmatrix} R^F_t & S^F_t \end{bmatrix} A \begin{bmatrix} R^F_t \\ S^F_{-t} \end{bmatrix}$$

Theorem 6.9. *Let $t > 0$. Then the image of the mapping X^F_t is in $\mathcal{L}$.*

Proof. Let A be an element of the form (25). Then

$$\mathsf{X}^F_t(A) = R^F_t A_{11} R^F_t + R^F_t A_{12} S^F_{-t} + S^F_t A_{21} R^F_t + S^F_t A_{22} S^F_{-t}.$$

We have to show that each of the four terms in this sum commutes with fI for $f \in \tilde{C}(\dot{\mathbb{R}})$ and with $W^0(g)$ for $g \in \tilde{C}_p$ modulo compact operators. Consider first the case $p = 2$. Then

$$\mathsf{X}^F_t(A) = F^{-1}\left(R_t F A_{11} F^{-1} R_t + R_t F A_{12} F^{-1} S_{-t} + S_t F A_{21} F^{-1} R_t + S_t F A_{22} F^{-1} S_{-t}\right) F,$$

with each element $FA_{ij}F^{-1}$ belonging to $\mathcal{E}_2$. But these operators are then the same that appear in the proof of Theorem 6.6. And because $\tilde{C}_2 = \tilde{C}(\dot{\mathbb{R}})$, the operators fI and $W^0(fI) = F^{-1}fF$ are Fourier images of one another. So one can use the same proofs as in Theorem 6.6 to show that the commutators are compact and the result for $p = 2$ follows. Consider now $1 < p < \infty$. Let g be a piecewise linear function on $\dot{\mathbb{R}}$ with bounded variation. Then the commutators

$$f\mathsf{X}^F_t(A) - \mathsf{X}^F_t(A) fI \quad \text{and} \quad W^0(g)\mathsf{X}^F_t(A) - \mathsf{X}^F_t(A) W^0(g)$$

are bounded for any p and compact for $p = 2$. Then, by the Krasnoselskii interpolation theorem (see [8, Section 3.4]), these commutators are compact for any p. As any function $g \in PC_p$ can be approximated in the $\mathcal{M}_p$-norm by piecewise linear functions, the result follows. ■

The following result can be proved in the same spirit by referring to Theorem 6.7 and employing Krasnoselskii's interpolation theorem.

Theorem 6.10. *Let $t > 0$. The homomorphism $\mathsf{Y}_{\infty,t}$ is an isomorphism between $\mathcal{A}^{\mathcal{K}}_{\infty,t}$ and the algebra $\mathcal{Y}^F$.*

7 Discussion and Conclusions

Let $1 < p < \infty$ and $\mathcal{A}$ be the smallest closed subalgebra of $\mathcal{L}(L^p(\mathbb{R}))$ containing all operators of multiplication aI with $a \in PC(\dot{\mathbb{R}})$, all convolution operators $W^0(b)$ with $b \in PC_p$ and the flip J. We proved the following result:

Theorem 7.1. *If $A \in \mathcal{A}$, then A is Fredholm on $L^p(\mathbb{R})$ if and only if all operators $\mathsf{Y}_{s,t}(A)$, $(s,t) \in ([0,\infty] \times \{\infty\}) \cup (\{\infty\} \times [0,\infty])$, are invertible.*

In [11, Propositions 5.7.5 and 5.7.6] it was proved that the local algebras $\mathcal{A}^{\mathcal{K}}_{s,\infty}$ and $\mathcal{A}^{\mathcal{K}}_{\infty,t}$ are isomorphic to the matrix algebra $[\mathrm{alg}\{I,\ \chi_+ I,\ P_{\mathbb{R}}\}]^{2\times 2}$. This was done by using homomorphisms that are not well defined on the algebra containing the flip J, and then the flip was included by doubling the dimension. The new homomorphisms introduced in the present paper, which are compatible with the flip, could thus be argued to give nothing new, regarding the analysis of the Calkin algebra $\mathcal{A}$. In fact, the homomorphisms presented and studied here are a new tool that can be used to study more involved algebras. So we hope that this paper can provide a solid basis for some future work.

In addition, in the present study of the Banach algebra $\mathcal{A}$, we have discussed isomorphisms between the following pairs of algebras: $\mathcal{E}_p$ and $\mathcal{E}_p^F$, $\mathcal{A}^{\mathcal{K}}_{s,\infty}$ and $\mathcal{Y}$, and $\mathcal{A}^{\mathcal{K}}_{\infty,t}$ and $\mathcal{Y}^F$. Joining these results to [11, Propositions 5.7.5 and 5.7.6], we arrive at the following Corollary.

Corollary 7.2. *The matrix algebras $\mathcal{Y}$, $\mathcal{Y}^F$ and $[\mathrm{alg}\{I,\ \chi_+ I,\ P_{\mathbb{R}}\}]^{2\times 2}$ are isomorphic.*

References

[1] G.R. Allan. Ideals of vector-valued functions. *Proc. London Math. Soc.*, 18(2):193–216, 1968.

[2] A. Böttcher and B. Silbermann. *Introduction to large truncated Toeplitz matrices.* Universitext. Springer-Verlag, New York, 1999.

[3] A. Böttcher and B. Silbermann. *Analysis of Toeplitz operators.* Springer-Verlag, Berlin, second edition, 2006.

[4] V. Didenko and B. Silbermann. *Approximation of additive convolution-like operators: real C^*-algebra approach.* Birkhäuser, Basel, Boston, Berlin, 2008.

[5] R. Duduchava. *Integral equations with fixed singularities.* B.G. Teubner Verlaggesellschaft, Leipzig, 1979.

[6] R. Hagen, S. Roch, and B. Silbermann. *Spectral theory of approximation methods for convolution equations.* Birkhäuser, Basel, 1995.

[7] R. Hagen, S. Roch, and B. Silbermann. *C^*-algebras and numerical analysis.* Marcel Dekker, Inc., New York, 2001.

[8] M.A. Krasnoselskii, P.P. Zabreiko, E.I. Pustylnik, and P.E. Sobolevskii. *Integral operators in spaces of summable functions.* Noordhoff International Publishing, Leiden, 1976.

[9] S. Prössdorf and B. Silbermann. *Numerical analysis for integral and related operator equations.* Birkhäuser-Verlag, Basel, 1991.

[10] S. Roch, P.A. Santos, and B. Silbermann. A sequence algebra of finite sections, convolution and multiplication operators on $L^p(\mathbb{R})$. *Numer. Funct. Anal. Optim.*, 31(1):45–77, 2010.

[11] S. Roch, P.A. Santos, and B. Silbermann. *Non-commutative Gelfand theories.* Springer, 2011.

[12] E.C. Titchmarsh. *Introduction to the theory of Fourier integrals.* Oxford University Press, Oxford, 1967. (first edition, 1948).

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