A model for brittle fracture based on the hybrid phase field model

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Abstract

We propose a phase field model for crack propagation based on the hybrid model and justify the model by constructing a family of asymptotic solutions.

Dedicated to Ingo Müller on the occasion of his 75th anniversary.

1 The hybrid fracture model

We propose a model for crack propagation in an elastic solid, which is based on the hybrid phase field model introduced and studied in [1, 2, 3, 5]. First we formulate the model equations.

Let Ω be an open subset in $\mathbb{R}^3$. It represents the material points of a solid elastic body. We represent a crack in the material by a damaged region in Ω, where the elasticity tensor has a very small norm. The damaged and undamaged regions are charaterized by the values of a smooth parameter function $\varphi : [0,\infty) \times \Omega \to [0,1]$, which has the values 0 or 1 in the respective regions. This function, the displacement field $u : [0,\infty) \times \Omega \to \mathbb{R}^3$ and the stress field $T : [0,\infty) \times \Omega \to \mathcal{S}^3$, where $\mathcal{S}^3$ denotes the set of symmetric 3×3–matrices, are the unknowns. They must satisfy the model equations

$$
\begin{aligned}
-\mathrm{div}_x\, T &= \mathsf{b}, && (1.1)\\
T &= \chi(\varphi) D\varepsilon(\nabla_x u), && (1.2)\\
\partial_t \varphi &= -f\big(\psi_\varphi(\varepsilon(\nabla_x u), \varphi) - \nu \Delta_x \varphi\big)|\nabla_x \varphi|. && (1.3)
\end{aligned}
$$

Here $\nabla_x u$ denotes the 3×3–matrix of first order derivatives of u, the deformation gradient, and

$$\varepsilon(\nabla_x u) = \frac{1}{2}\big(\nabla_x u + (\nabla_x u)^T\big) \in \mathcal{S}^3$$

is the strain tensor, where $(\nabla_x u)^T$ denotes the transposed matrix. The elasticity tensor $D : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, positive definite mapping, $\mathsf{b} : [0,\infty) \times \Omega \to \mathbb{R}^3$ denotes the given volume force, and $\chi : [0,1] \to [0,1]$ is a positive, increasing function satisfying $\chi(0) = \kappa$ and $\chi(1) = 1$, with a small constant $0 < \kappa < 1$. Also ν is a small positive constant, and $\psi_\varphi = \frac{\partial}{\partial \varphi}\psi$ denotes the partial derivative of the function

$$\psi(\varepsilon, \varphi) = \chi(\varphi)\,\frac{1}{2}(\varepsilon : D\varepsilon) + \hat{\psi}(\varphi), \tag{1.4}$$

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with a double well potential $\hat{\psi} : \mathbb{R} \to \mathbb{R}$. Here the scalar product of two matrices is denoted by $A : B = \sum a_{ij} b_{ij}$. We thus have

$$\psi_\varphi(\varepsilon, \varphi) = \chi'(\varphi)\,\frac{1}{2}(\varepsilon : D\varepsilon) + \hat{\psi}'(\varphi). \tag{1.5}$$

ψ is a part of the free energy. The total free energy corresponding to the model (1.1) – (1.3) is given by the sum

$$\psi(\varepsilon, \varphi) + \frac{\nu}{2}|\nabla_x \varphi|^2.$$

For simplicity we require that the potential $\hat{\psi}$ satisfies $\hat{\psi}(0) = \hat{\psi}(1) = 0$ and $\hat{\psi}(\varphi) > 0$ for $0 < \varphi < 1$. However, it would neither be necessary that $\hat{\psi}$ vanishes at 0 and 1 nor that the values at these two points are the same. We refer to [4], where general potentials are considered for the related model for interface movement by interface diffusion. For the function χ we need that

$$\int_0^1 \chi(\vartheta)\big(1 - (1-\kappa)\vartheta\big)\, d\vartheta = \kappa. \tag{1.6}$$

The precise assumptions for $\hat{\psi}$ and χ are stated in Corollary 3.2. To define the nonlinear function $f : \mathbb{R} \to \mathbb{R}$ in (1.3) let c and λ be positive constants. The choice of these constants is discussed below. We set

$$f(r) = \begin{cases} 0, & r < \lambda, \\ c(r-\lambda), & r \geq \lambda. \end{cases} \tag{1.7}$$

(1.1), (1.2) are the equations of linear elasticity. In the region where $\varphi = 0$ the elasticity tensor in equation (1.2) is equal to κD and thus has a small norm. In this region the material is very soft; the behavior approximates the behavior of a fractured material. The evolution of the damage variable φ is governed by the degenerate parabolic evolution equation (1.3). The choice of this evolution equation and the necessity of the condition (1.6) are justified by the form of the sharp interface problem, which governs the behavior of φ in the limit $\nu \to 0$. This limit problem is determined in Section 3 by constructing a family of asymptotic solutions to the phase field model (1.1) – (1.3). Here we first introduce this limit problem and use it to explain the choice of the function f in (1.7).

The limit model governs the evolution of a smooth boundary $\Gamma(t) \subset \Omega$ of a crack $\mathfrak{C}(t) \subset \Omega$ with finite width. A possible form of the crack is depicted in Fig. 1. It is also possible that the crack lies completely in the interior of Ω and does not intersect the boundary $\partial\Omega$. In the sharp interface model we denote the strain and stress fields by $\hat{u}$ and $\hat{T}$, respectively. The function $\hat{\varphi}$ satisfies $\hat{\varphi}(t,x) = 0$ for $x \in \mathfrak{C}(t)$ and $\hat{\varphi}(t,x) = 1$ for $x \in \mathfrak{C}'(t) = \Omega \setminus \mathfrak{C}(t)$. The model consists of the equations

$$-\mathrm{div}_x \hat{T} = \mathsf{b}, \tag{1.8}$$

$$\hat{T} = \chi(\hat{\varphi}) D\varepsilon(\nabla_x \hat{u}), \tag{1.9}$$

$$s = f(n \cdot [\hat{C}]n), \tag{1.10}$$

$$[\hat{u}] = 0, \tag{1.11}$$

$$[\hat{T}]n = 0, \tag{1.12}$$

The equations (1.8), (1.9) must hold on $[0,\infty) \times \Omega$, the other three equations must be satisfied on the three dimensional manifold

$$\Gamma = \{(t,x) \mid x \in \Gamma(t)\}.$$

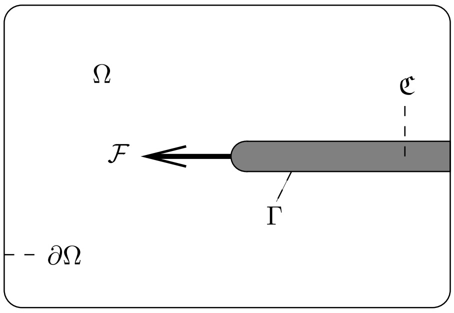

Figure 1: crack $\mathfrak{C}$, crack boundary Γ, driving force $\mathcal{F}$

By n we denote the unit normal vector field $x \mapsto n(t,x) : \Gamma(t) \to \mathbb{R}^3$, which points into the undamaged region $\mathfrak{C}'(t)$, and $s = s(t,x)$ is the normal speed of the crack surface Γ measured positive in direction of n. The square bracket $[w]$ stands for the jump of w across Γ:

$$[w](t,x) = w^+(t,x) - w^-(t,x), \quad (t,x) \in \Gamma,$$

with

$$w^\pm(t,x) = \lim_{\xi \searrow 0} w\big(t, x \pm \xi n(t,x)\big). \tag{1.13}$$

The Eshelby tensor is defined by

$$\hat{C}(\nabla_x \hat{u}, \hat{\varphi}) = \psi(\varepsilon(\nabla_x \hat{u}), \hat{\varphi})I - (\nabla_x \hat{u})^T \hat{T}, \tag{1.14}$$

where I is the 3×3–unit matrix and $(\nabla_x \hat{u})^T \hat{T}$ is the usual matrix product. The other notations are as in (1.1) – (1.3). For the kinetic relation (1.10) see also [6].

It is a noteworthy property of the phase field model (1.1) – (1.3) that the function f in the kinetic relation (1.10) of the limit model coincides with the constitutive function f in the evolution equation (1.3) of the phase field model, though this function is nonlinear. In the asymptotic solutions to the phase field model (1.1) – (1.3) constructed below the damage variable φ transists smoothly from 0 to 1 in a narrow neighborhood of the surface $\Gamma(t)$. This neighborhood has width proportional to $\nu^{1/2}$ and moves with the speed s of the surface $\Gamma(t)$. Therefore the region $\{x \in \Omega \mid \varphi(t,x) = 0\}$ representing the crack in the phase field model is almost equal to the fractured set $\mathfrak{C}(t)$ of the sharp interface model. Thus, if (1.8) – (1.12) can be used as a crack propagation model, then also the phase field model (1.1) – (1.3).

To see that (1.8) – (1.12) can be used as a crack propagation model observe that by (1.10) and by the definition (1.7) of f, in this model the crack does not move as long as the driving force $\mathcal{F} = n \cdot [\hat{C}]n$ is not larger than the limit value λ. If the driving force $\mathcal{F}$ exceeds this limit, the normal speed s is positive and the crack expands. Since f is a non-negative function, s is never negative. Therefore the crack length cannot decrease, no healing is possible.

Since $\Gamma(t)$ is assumed to be smooth, the stress field has a finite value at the crack tip, but the smaller the radius of the crack tip, the higher the value of the stress at the tip. As will be shown later, a high value of the stress at the tip implies a large jump of the Eshelby tensor. Another feature of the models (1.8) – (1.12) and (1.1) – (1.3) is therefore that the more peaked the crack is, the smaller the loading is, which lets the crack grow.

This is different for phase field models based on Griffith's theory, cf. [7, 8, 9, 10, 11]. In this theory a crack grows when the work needed to generate new crack surface is smaller than the bulk energy released when the crack grows. Since the work needed to generate new surface is proportional to the surface area generated, and since this surface area is essentially independent of the radius of the crack tip, in such models the loading needed to let the crack grow is the same for more or less peaked cracks.

We note however, that in the phase field model (1.1) – (1.3) the smallest possible value of the radius of the crack tip is proportional to $\nu^{1/2}$, since the width of the transitional region, in which φ grows from 0 to 1, is proportional to $\nu^{1/2}$.

It remains to construct the family of asymptotic solutions for the model (1.1) – (1.3), from which it can be seen that (1.8) – (1.12) is the limit problem and in what sense the limit is attained. To this end we need to compute the jump of the Eshelby tensor in (1.10) as a function of the limit values of $\nabla_x \hat{u}$ on both sides of Γ. This computation is given in Section 2. The family of asymptotic solutions is constructed in Section 3.

2 The jump of the Eshelby tensor

In this section we compute the jump $n \cdot [\hat{C}]n$ in the kinetic relation (1.10). We start with some notations, definitions and assumptions, which we use in the remainder of the paper.

Let t_1, t_2 be given numbers with $0 \le t_1 < t_2$ and let $(\hat{u}, \hat{T}, \hat{\varphi}, \Gamma)$ be a solution of (1.8) – (1.12) in the domain

$$Q = [t_1, t_2] \times \Omega.$$

The set of all $(t, x) \in Q \setminus \Gamma$ with $\hat{\varphi}(t, x) = 0$ is denoted by $\mathfrak{C}$ and $\mathfrak{C}'$ denotes the set of all $(t, x) \in Q \setminus \Gamma$ with $\hat{\varphi}(t, x) = 1$. We assume that the crack boundary Γ is a three dimensional C^3–manifold embedded in Q. To avoid technicalities we assume moreover that the set Γ is a compact subset of Q and that the two dimensional manifold $\Gamma(t)$ does not have a boundary for all $t \in [t_1, t_2]$. This means that the crack $\mathfrak{C}(t) = \{x \in \Omega \mid (t, x) \in \mathfrak{C}\}$ is interior to the body for all t and does not intersect the boundary. Suppose that the function $(\hat{u}, \hat{T})$ belongs to the space $C^3(\mathfrak{C} \cup \mathfrak{C}') \times C^2(\mathfrak{C} \cup \mathfrak{C}')$ and that the derivatives of $\hat{u}$ up to order three and the derivatives of $\hat{T}$ up to order two have continuous extensions from $\mathfrak{C}$ to $\mathfrak{C} \cup \Gamma$ and from $\mathfrak{C}'$ to $\mathfrak{C}' \cup \Gamma$.

By the assumptions on Γ we can choose $\delta > 0$ sufficiently small such that

$$(t, \eta, \xi) \mapsto (t, x(t, \eta, \xi)) = (t, \eta + n(t, \eta)\xi) : \Gamma \times (-\delta, \delta) \to Q \tag{2.1}$$

is a C^2–parametrization of the sheet like region

$$\mathcal{U} = \{(t, \eta + n(t, \eta)\xi) \mid (t, \eta) \in \Gamma,\ |\xi| < \delta\} \subset Q,$$

which is the union of the C^2–parallel manifolds

$$\Gamma_\xi = \{(t, \eta + n(t, \eta)\xi) \mid (t, \eta) \in \Gamma\}, \quad -\delta < \xi < \delta.$$

Though (t, η) is a point on the manifold Γ, we say that the mapping (2.1) defines new coordinates (t, η, ξ) in $\mathcal{U}$. We set

$$\mathcal{U}(t) = \{x \in \mathbb{R}^3 \mid (t, x) \in \mathcal{U}\} \subseteq \mathbb{R}^3, \quad \Gamma_\xi(t) = \{x \in \Omega \mid (t, x) \in \Gamma_\xi\}.$$

Let $\tau_1,\ \tau_2 \in \mathbb{R}^3$ be two orthogonal unit vectors tangent to $\Gamma_\xi(t)$ at $x \in \Gamma_\xi(t)$. For functions $w : \Gamma_\xi(t) \to \mathbb{R}$, $W : \Gamma_\xi(t) \to \mathbb{R}^3$ and $\hat{W} : \Gamma_\xi(t) \to \mathbb{R}^{3\times 3}$ we define the surface gradients

$$\nabla_{\Gamma_\xi} w = (\partial_{\tau_1} w)\tau_1 + (\partial_{\tau_2} w)\tau_2, \tag{2.2}$$

$$\nabla_{\Gamma_\xi} W = (\partial_{\tau_1} W) \otimes \tau_1 + (\partial_{\tau_2} W) \otimes \tau_2, \tag{2.3}$$

where for vectors $c, d \in \mathbb{R}^3$ we define a 3×3–matrix by

$$c \otimes d = (c_i d_j)_{i,j=1,2,3}\,.$$

Clearly, we have $\nabla_{\Gamma_\xi} w : \Gamma_\xi \mapsto \mathbb{R}^3$ and $\nabla_{\Gamma_\xi} W : \Gamma_\xi \mapsto \mathbb{R}^{3\times 3}$. Moreover, the splitting

$$\nabla_x W(t,x) = \partial_\xi W(t,\eta,\xi) \otimes n(t,\eta) + \nabla_{\Gamma_\xi} W(t,\eta,\xi), \tag{2.4}$$

holds, where $W(t,\eta,\xi) = W(t,\eta + n(t,\eta)\xi)$, as usual.

Let $\phi \in C^\infty(Q)$ be a function, which vanishes outside of the set $\mathcal{U}$ and is equal to one in a neighborhood of Γ. We set

$$\xi^+ = \begin{cases} \xi, & \xi \ge 0, \\ 0, & \xi < 0. \end{cases} \qquad 1^+(\xi) = \begin{cases} 1, & \xi \ge 0, \\ 0, & \xi < 0. \end{cases} \tag{2.5}$$

With the definition

$$u^*(t,\eta) = [\partial_\xi \hat{u}](t,\eta,0), \quad (t,\eta) \in \Gamma, \tag{2.6}$$

we decompose the function $\hat{u}$ in the form

$$\hat{u}(t,x) = u^*(t,\eta)\xi^+\phi(t,x) + v(t,x), \qquad (t,x) \in \mathcal{U}. \tag{2.7}$$

This defines the function $v : Q \to \mathbb{R}^3$. From our differentiability assumptions for Γ it follows by standard considerations that for $i + j \le 2$ and $i + j + l \le 3$ the derivatives $\partial_t^i \nabla_{\Gamma_\xi}^j \partial_\xi^l v$ exist in $\mathfrak{C} \cup \mathfrak{C}'$ and are bounded and continuous. For $i + j \le 2$ and $l \le 1$ these derivatives can be joined continuously across Γ, whence these derivatives exist in Q and are continuous.

Lemma 2.1 *The jump of $\nabla_x \hat{u}$ across the surface Γ satisfies*

$$[\nabla_x \hat{u}] = u^* \otimes n. \tag{2.8}$$

Proof: We use (2.4) to compute from (2.7) that in a neighborhood of Γ where $\phi = 1$

$$\nabla_x \hat{u} = \partial_\xi \hat{u} \otimes n + \nabla_{\Gamma_\xi} \hat{u} = (u^* \otimes n)1^+ + (\nabla_{\Gamma_\xi} u^*)\xi^+ + \nabla_x v, \tag{2.9}$$

with 1^+ defined in (2.5). Since $\nabla_x v$ is continuous, we have $[\nabla_x v] = 0$, which together with (2.9) yields $[\nabla_x \hat{u}] = u^* \otimes n$. ∎

With the definition of $(\nabla_x \hat{u})^\pm$ in (1.13) we obtain from (2.7) that

$$\varepsilon(\nabla_x \hat{u})^+ = \varepsilon\big((\nabla_x \hat{u})^+\big) = [\varepsilon(\nabla_x \hat{u})] + \varepsilon(\nabla_x \hat{u})^- = [\varepsilon(\nabla_x \hat{u})] + \varepsilon(\nabla_x v)^-. \tag{2.10}$$

Lemma 2.2 *$\nabla_x \hat{u}$ satisfies along Γ*

$$\big(D[\varepsilon(\nabla_x\hat{u})]\big) : [\varepsilon(\nabla_x\hat{u})] = -(1-\kappa)\big(D\varepsilon(\nabla_x\hat{u})^-\big) : [\varepsilon(\nabla_x\hat{u})], \tag{2.11}$$

whence $\big(D\varepsilon(\nabla_x\hat{u})^-\big) : [\varepsilon(\nabla_x\hat{u})] \le 0$.
The jump of the Eshelby tensor satisfies along Γ

$$\begin{aligned} n\cdot[\hat{C}]n &= \frac{1-\kappa}{2}\big(D\varepsilon(\nabla_x\hat{u})^-\big) : \varepsilon(\nabla_x\hat{u})^+ + [\hat{\psi}] \\ &= \frac{1-\kappa}{2}\big(D\varepsilon(\nabla_x\hat{u})^-\big) : \varepsilon(\nabla_x\hat{u})^- + \frac{1-\kappa}{2}\big(D\varepsilon(\nabla_x\hat{u})^-\big) : [\varepsilon(\nabla_x\hat{u})] + [\hat{\psi}] \\ &= \frac{1-\kappa}{2}\big(D\varepsilon(\nabla_x\hat{u})^+\big) : \varepsilon(\nabla_x\hat{u})^+ + \frac{\kappa}{2}\big(D[\varepsilon(\nabla_x\hat{u})]\big) : [\varepsilon(\nabla_x\hat{u})] + [\hat{\psi}]. \end{aligned} \tag{2.12}$$

Remark: Since we assume that $\hat{\psi}(0) = \hat{\psi}(1) = 0$, we have in fact $[\hat{\psi}] = \hat{\psi}(1) - \hat{\psi}(0) = 0$.

Proof: From (1.12) and (1.9) we obtain together with (2.10) that

$$0 = [\hat{T}]n = (T^+ - T^-)n = \Big(\chi(1)D\big([\varepsilon(\nabla_x\hat{u})] + \varepsilon(\nabla_x\hat{u})^-\big) - \chi(0)D\varepsilon(\nabla_x\hat{u})^-\Big)n,$$

hence, since $\chi(1) = 1$ and $\chi(0) = \kappa$,

$$\big(D[\varepsilon(\nabla_x\hat{u})]\big)n = -(1-\kappa)\big(D\varepsilon(\nabla_x\hat{u})^-\big)n,$$

and so, with u^* defined in (2.6),

$$u^*\cdot\big(D[\varepsilon(\nabla_x\hat{u})]\big)n = -(1-\kappa)u^*\cdot\big(D\varepsilon(\nabla_x\hat{u})^-\big)n.$$

This can be written as

$$\big(D[\varepsilon(\nabla_x\hat{u})]\big) : (u^*\otimes n) = -(1-\kappa)\big(D\varepsilon(\nabla_x\hat{u})^-\big) : (u^*\otimes n). \tag{2.13}$$

Using the symmetry of the expressions $D[\varepsilon(\nabla_x\hat{u})]$, $D\varepsilon(\nabla_x\hat{u})^-$ and the equation

$$[\varepsilon(\nabla_x\hat{u})] = \frac{1}{2}\big((u^*\otimes n) + (n\otimes u^*)\big),$$

which follows from (2.8), we infer from (2.13) that (2.11) holds.

To prove (2.12) we use the notation

$$\langle w\rangle = \frac{1}{2}\big(w^+ + w^-\big).$$

Note first that (1.12) and (2.8) imply

$$\begin{aligned} n\cdot[(\nabla_x\hat{u})^T\hat{T}]n &= n\cdot\Big([\nabla_x\hat{u}]^T\langle\hat{T}\rangle + \langle\nabla_x\hat{u}\rangle^T[\hat{T}]\Big)n = n\cdot[\nabla_x\hat{u}]^T\langle\hat{T}\rangle n \\ &= \big([\nabla_x\hat{u}]n\big)\cdot\langle\hat{T}\rangle n = (u^*\otimes n)n\cdot\langle\hat{T}\rangle n = u^*\cdot\langle\hat{T}\rangle n \\ &= \langle\hat{T}\rangle : (u^*\otimes n) = \langle\hat{T}\rangle : \frac{1}{2}\big((u^*\otimes n) + (n\otimes u^*)\big) = \langle\hat{T}\rangle : [\varepsilon(\nabla_x\hat{u})]. \end{aligned}$$

In the second last step we used the symmetry of $\langle\hat{T}\rangle$. With this equation we obtain from (1.14), (1.4) and (1.9) that

$$
\begin{aligned}
n\cdot[\hat{C}]n &= [\psi] - \langle\hat{T}\rangle : [\varepsilon(\nabla_x\hat{u})] \\
&= [\hat{\psi}] + \big[\chi(\hat{\varphi})\tfrac{1}{2}\varepsilon(\nabla_x\hat{u}) : D\varepsilon(\nabla_x\hat{u})\big] - \langle\hat{T}\rangle : [\varepsilon(\nabla_x\hat{u})] \\
&= [\hat{\psi}] + \frac{1}{2}\big([\varepsilon(\nabla_x\hat{u})] + \varepsilon(\nabla_x\hat{u})^-\big) : T^+ - \frac{1}{2}\varepsilon(\nabla_x\hat{u})^- : T^- - \frac{1}{2}(T^+ + T^-) : [\varepsilon(\nabla_x\hat{u})] \\
&= [\hat{\psi}] + \frac{1}{2}\varepsilon(\nabla_x\hat{u})^- : T^+ - \frac{1}{2}\big([\varepsilon(\nabla_x\hat{u})] + \varepsilon(\nabla_x\hat{u})^-\big) : T^- \\
&= [\hat{\psi}] + \frac{1}{2}\varepsilon(\nabla_x\hat{u})^- : D\varepsilon(\nabla_x\hat{u})^+ - \kappa\frac{1}{2}\varepsilon(\nabla_x\hat{u})^+ : D\varepsilon(\nabla_x\hat{u})^-.
\end{aligned}
$$

This equation yields the first equality in (2.12). The second equality is obtained by insertion of

$$\varepsilon(\nabla_x\hat{u})^+ = \varepsilon(\nabla_x\hat{u})^- + [\varepsilon(\nabla_x\hat{u})]$$

into the first equality. To get the third equality, we use (2.11) to compute

$$
\begin{aligned}
\frac{1-\kappa}{2}\,\varepsilon(\nabla_x\hat{u})^- : D\varepsilon(\nabla_x\hat{u})^+ &= \frac{1-\kappa}{2}\,\big(\varepsilon(\nabla_x\hat{u})^+ - [\varepsilon(\nabla_x\hat{u})]\big) : D\varepsilon(\nabla_x\hat{u})^+ \\
&= \frac{1-\kappa}{2}\,\Big(\varepsilon(\nabla_x\hat{u})^+ : D\varepsilon(\nabla_x\hat{u})^+ - [\varepsilon(\nabla_x\hat{u})] : D\big([\varepsilon(\nabla_x\hat{u})] + \varepsilon(\nabla_x\hat{u})^-\big)\Big) \\
&= \frac{1-\kappa}{2}\,\Big(\varepsilon(\nabla_x\hat{u})^+ : D\varepsilon(\nabla_x\hat{u})^+ - [\varepsilon(\nabla_x\hat{u})] : D[\varepsilon(\nabla_x\hat{u})] \\
&\qquad + \frac{1}{1-\kappa}[\varepsilon(\nabla_x\hat{u})] : D[\varepsilon(\nabla_x\hat{u})]\Big) \\
&= \frac{1-\kappa}{2}\,\varepsilon(\nabla_x\hat{u})^+ : D\varepsilon(\nabla_x\hat{u})^+ + \frac{\kappa}{2}[\varepsilon(\nabla_x\hat{u})] : D[\varepsilon(\nabla_x\hat{u})].
\end{aligned}
$$

We combine this equation with the first equality in (2.12) and get the third. ∎

3 The asymptotic solutions

In this section we construct asymptotic solutions to the model (1.1) – (1.3) for $\nu \to 0$ using the results from the previous section. That is, we construct a family of functions $\{(u^{(\nu)}, T^{(\nu)}, \varphi^{(\nu)})_\nu$, which satisfy the equations (1.1) – (1.3) up to an error, which tends to zero for $\nu \to 0$. The asymptotic solutions are similar to the asymptotic solutions of the hybrid phase field model for phase interfaces in solids constructed in [5]. Since the construction is explained there in detail, we give here a more concise presentation. We also refer the reader to [4], where a related construction of asymptotic solutions for a hybrid phase field model for interface motion by interface diffusion is given.

Let $(\hat{u}, \hat{T}, \hat{\varphi}, \Gamma)$ be a solution of the sharp interface model (1.8) – (1.12) satisfying the assumptions stated a the beginning of Section 2. With ϕ and v from (2.7) we make for the asymptotic solution the ansatz

$$u^{(\nu)}(t,x) = \nu^{1/2}u_0\big(t,\eta,\frac{\xi}{\nu^{1/2}}\big)\,\phi(t,x) + v(t,x), \tag{3.1}$$

$$\varphi^{(\nu)}(t,x) = \varphi_0\big(t,\eta,\frac{\xi}{\nu^{1/2}}\big)\,\phi(t,x) + \hat{\varphi}(t,x)\,\big(1-\phi(t,x)\big), \tag{3.2}$$

$$T^{(\nu)}(t,x) = \chi(\varphi^{(\nu)}(t,x))D\varepsilon(\nabla_x u^{(\nu)}(t,x)), \tag{3.3}$$

with the function u_0 given by

$$u_0(t,\eta,\zeta) = u^*(t,\eta)\varphi_0^{(-1)}(t,\eta,\zeta). \tag{3.4}$$

Here we use the notation

$$\varphi_0^{(-1)}(t,\eta,\zeta) = \int_{-\infty}^{\zeta} \varphi_0(t,\eta,\vartheta)d\vartheta. \tag{3.5}$$

The real valued function φ_0 must be determined such that $(u^{(\nu)}, T^{(\nu)}, \varphi^{(\nu)})$ satisfies (1.1) – (1.3) asymptotically and such that $\varphi^{(\nu)}$ is a transition profile connecting the state $\varphi^{(\nu)} = 0$ to the state $\varphi^{(\nu)} = 1$. To satisfy the last condition we require that there exist functions $a : \Gamma \to (-\infty, 0)$ and $b : \Gamma \to (0, \infty)$ such that

$$\varphi_0\big(t,\eta,\frac{\xi}{\nu^{1/2}}\big) = \begin{cases} 0, & (t,x(t,\eta,\xi)) \in \mathcal{U},\ \xi \le \nu^{1/2}\, a(t,\eta), \\ 1, & (t,x(t,\eta,\xi)) \in \mathcal{U},\ \xi \ge \nu^{1/2}\, b(t,\eta). \end{cases} \tag{3.6}$$

We only consider values of the parameter $\nu > 0$, which are sufficiently small such that $-\delta < \nu^{1/2}a(t,\eta) < \nu^{1/2}b(t,\eta) < \delta$. From (3.2) we see that if such functions a and b exist, then

$$\Gamma[\nu] = \{(t,x(t,\eta,\xi)) \mid (t,\eta) \in \Gamma,\ \nu^{1/2}a(t,\eta) \le \xi \le \nu^{1/2}b(t,\eta)\} \subseteq \mathcal{U} \tag{3.7}$$

is the transitional region, where the order parameter $\varphi^{(\nu)}$ changes from 0 to 1. The width of the transitional region decreases like $\nu^{1/2}$ for $\nu \to 0$. For fixed ν the width is not constant but depends on the point $(t,\eta) \in \Gamma$. Because of the coordinate transformation (2.1) we identify $\Gamma[\nu]$ with the set

$$\{(t,\eta,\xi) \mid (t,\eta) \in \Gamma,\ \nu^{1/2}a(t,\eta) \le \xi \le \nu^{1/2}b(t,\eta)\} \subseteq \Gamma \times (-\delta,\delta).$$

The equations (3.6) and (3.2) imply that

$$\varphi_0(t,\eta,\zeta) = \begin{cases} 0, & \text{for } \zeta \le a(t,\eta), \\ 1, & \text{for } \zeta \ge b(t,\eta), \end{cases} \tag{3.8}$$

$$\varphi^{(\nu)}(t,x) = \hat{\varphi}(t,x), \quad \text{for } (t,x) \in Q \setminus \Gamma[\nu]. \tag{3.9}$$

It remains to determine the functions a and b and to determine the function φ_0 on the set

$$\Gamma[a,b] = \{(t,\eta,\zeta) \mid (t,\eta) \in \Gamma,\ a(t,\eta) \le \zeta \le b(t,\eta)\}. \tag{3.10}$$

To this end we insert the asymptotic solution (3.1) – (3.3) into the differential equations (1.1) – (1.3) and collect terms with the same power of ν. The resulting expansions starts with an absolute term, which is independent of ν. Setting this term equal to zero yields

$$\tilde{\psi}_\varphi\big(t,\eta,\varphi_0(t,\eta,\zeta)\big) - \varphi_0''(t,\eta,\zeta) = 0, \qquad \zeta \in [a(t,\eta), b(t,\eta)], \tag{3.11}$$

where we use the notation $\varphi_0'' = \partial_\zeta^2\varphi_0$ and where $\tilde{\psi}_\varphi = \partial_\varphi\tilde{\psi}$ is the partial derivative of the modified double well potential

$$\tilde{\psi}(t,\eta,\theta) = \int_0^\theta \psi_\varphi\big(\vartheta[\varepsilon(\nabla_x\hat{u})] + \varepsilon(\nabla_x\hat{u})^-, \vartheta\big)d\vartheta - (n\cdot[\hat{C}]n)\,\theta. \tag{3.12}$$

(3.11) is an ordinary differential equation for the function $\zeta \mapsto \varphi_0(t,\eta,\zeta)$ in the interval $[a(t,\eta), b(t,\eta)]$ for all $(t,\eta) \in \Gamma$. Since the differential equation is of second order, we can prescribe two boundary conditions. However, (3.8) implies that φ_0 must satisfy the two Dirichlet boundary conditions $\varphi(t,\eta,a(t,\eta)) = 0$, $\varphi(t,\eta,b(t,\eta)) = 1$ and the additonal conditions

$$\varphi'(t,\eta,a(t,\eta)) = \varphi'(t,\eta,b(t,\eta)) = 0.$$

These four boundary conditions and the differential equation (3.11) serve to determine the functions φ_0, a and b.

The differential equation (3.11) has solutions in the form of a transition profile if the modified double well potential $\tilde{\psi}(t,\eta,\theta)$ vanishes for $\theta = 0$ and $\theta = 1$ and is positive in between. These properties of $\tilde{\psi}$ are investigated in the following lemma:

Lemma 3.1 *Assume that* $\chi \in C^1([0,1],[0,1])$ *is a positive, increasing function satisfying* $\chi(0) = \kappa$, $\chi(1) = 1$ *and*

$$\int_0^1 \chi(\vartheta)\big(1 - (1-\kappa)\vartheta\big)\, d\vartheta = \kappa. \tag{3.13}$$

Assume further that $\hat{\psi} \in C([0,1])$ *satisfies* $\hat{\psi}(0) = \hat{\psi}(1) = 0$. *Then for all* $(t,\eta) \in \Gamma$ *the function* $\tilde{\psi}$ *satisfies* $\tilde{\psi}(t,\eta,0) = \tilde{\psi}(t,\eta,1) = 0$ *and*

$$\tilde{\psi}(t,\eta,\theta) \geq \hat{\psi}(\theta) - \min\{\theta, h(\theta)\}(n \cdot [\hat{C}]n), \tag{3.14}$$

with the function

$$h(\theta) = \frac{1}{1-\kappa} \int_\theta^1 \chi'(\vartheta)\Big(\vartheta + \frac{\kappa+1}{\kappa}(1-\vartheta)\Big) d\vartheta. \tag{3.15}$$

Proof: Since $\chi(0) = \kappa$, it follows from (3.12) that $\tilde{\psi}(t,\eta,0) = 0$. To verify that $\tilde{\psi}(t,\eta,1) = 0$, we insert (1.5) into (3.12) and use that $\hat{\psi}(0) = 0$ to obtain

$$\begin{aligned} \tilde{\psi}(t,\eta,\theta) \;&=\; \hat{\psi}(\theta) - (n \cdot [\hat{C}]n)\,\theta \\ &\quad + \int_0^\theta \chi'(\vartheta)\frac{1}{2}\big(\vartheta[\varepsilon(\nabla_x \hat{u})] + \varepsilon(\nabla_x \hat{u})^-\big) : D\big(\vartheta[\varepsilon(\nabla_x \hat{u})] + \varepsilon(\nabla_x \hat{u})^-\big) d\vartheta. \end{aligned} \tag{3.16}$$

From (2.11) we conclude

$$\begin{aligned} &\frac{1}{2}\big(\vartheta[\varepsilon(\nabla_x \hat{u})] + \varepsilon(\nabla_x \hat{u})^-\big) : D\big(\vartheta[\varepsilon(\nabla_x \hat{u})] + \varepsilon(\nabla_x \hat{u})^-\big) \\ &\quad = \vartheta^2\,\frac{1}{2}[\varepsilon(\nabla_x \hat{u})] : D[\varepsilon(\nabla_x \hat{u})] + \vartheta\,[\varepsilon(\nabla_x \hat{u})] : D\varepsilon(\nabla_x \hat{u})^- + \frac{1}{2}\varepsilon(\nabla_x \hat{u})^- : D\varepsilon(\nabla_x \hat{u})^- \\ &\quad = (\vartheta - \frac{1-\kappa}{2}\vartheta^2)\,[\varepsilon(\nabla_x \hat{u})] : D\varepsilon(\nabla_x \hat{u})^- + \frac{1}{2}\varepsilon(\nabla_x \hat{u})^- : D\varepsilon(\nabla_x \hat{u})^-. \end{aligned}$$

We insert this equation and the second equality of (2.12) into (3.16) and obtain

$$\begin{aligned} \tilde{\psi}(t,\eta,\theta) &= \hat{\psi}(\theta) + \frac{1}{2}\big(\varepsilon(\nabla_x \hat{u})^- : D\varepsilon(\nabla_x \hat{u})^-\big)\big(\chi(\theta) - \kappa - (1-\kappa)\theta\big) \\ &\quad + \Big([\varepsilon(\nabla_x \hat{u})] : D\varepsilon(\nabla_x \hat{u})^-\Big)\Big(\int_0^\theta \chi'(\vartheta)(\vartheta - \frac{1-\kappa}{2}\vartheta^2)\, d\vartheta - \frac{1-\kappa}{2}\theta\Big). \end{aligned} \tag{3.17}$$

Since $\chi(1) = 1$, condition (3.13) implies

$$\int_0^1 \chi'(\vartheta)\big(\vartheta - \frac{1-\kappa}{2}\vartheta^2\big)\, d\vartheta = \chi(1)\frac{1+\kappa}{2} - \int_0^1 \chi(\vartheta)\big(1-(1-\kappa)\vartheta\big)d\vartheta = \frac{1-\kappa}{2}.$$

With this equation we obtain from (3.17) for $\theta = 1$ that $\tilde{\psi}(t,\eta,1) = 0$.

To prove (3.14) note first that the integral in (3.16) is non-negative. This follows from the fact that the integrand is non-negative as the product of the derivative $\chi'(\vartheta)$, which is non-negative since χ is increasing, and of a positive definite quadratic form. (3.16) thus implies

$$\tilde{\psi}(t,\eta,\theta) - \hat{\psi}(\theta) \geq -n\cdot[\hat{C}]n\,\theta. \tag{3.18}$$

For brevity we set $p(\vartheta) = \frac{1}{2}\big(\vartheta[\varepsilon(\nabla_x\hat{u})] + \varepsilon(\nabla_x\hat{u})^-\big) : D\big(\vartheta[\varepsilon(\nabla_x\hat{u})] + \varepsilon(\nabla_x\hat{u})^-\big)$. With this notation we obtain from (3.16) that

$$0 = \tilde{\psi}(t,\eta,1) - \hat{\psi}(1) = \tilde{\psi}(t,\eta,\theta) - \hat{\psi}(\theta) - (n\cdot[\hat{C}]n)(1-\theta) + \int_\theta^1 \chi'(\vartheta)p(\vartheta)d\vartheta. \tag{3.19}$$

To estimate $p(\vartheta)$, note that this is a quadratic, non-negative polynomial of second order in ϑ, hence it is a convex function. Since the values of this polynomial at $\vartheta = 0, 1$ are $\frac{1}{2}\varepsilon(\nabla_x\hat{u})^\pm : D\varepsilon(\nabla_x\hat{u})^\pm$ and since (2.11) and (2.12) together imply

$$\begin{aligned}
&\frac{1}{2}|\varepsilon(\nabla_x\hat{u})^+ : D\varepsilon(\nabla_x\hat{u})^+| \leq \frac{1}{1-\kappa}\, n\cdot[\hat{C}]n,\\
&\frac{1}{2}|\varepsilon(\nabla_x\hat{u})^- : D\varepsilon(\nabla_x\hat{u})^-| \leq \frac{1}{1-\kappa}\, n\cdot[\hat{C}]n - \varepsilon(\nabla_x\hat{u})^- : D[\varepsilon(\nabla_x\hat{u})]\\
&\qquad\leq \frac{1}{1-\kappa}\, n\cdot[\hat{C}]n + \frac{1}{(1-\kappa)\kappa}\, n\cdot[\hat{C}]n = \frac{1}{1-\kappa}\big(1+\frac{1}{\kappa}\big)\, n\cdot[\hat{C}]n,
\end{aligned}$$

it follows that

$$p(\vartheta) \leq \frac{1}{1-\kappa}\Big(\vartheta + \frac{\kappa+1}{\kappa}(1-\vartheta)\Big)n\cdot[\hat{C}]n.$$

We use this inequality to estimate $p(\vartheta)$ in (3.19). Since $-n\cdot[\hat{C}]n \leq 0$, by (2.12), we obtain

$$\tilde{\psi}(t,\eta,\theta) - \hat{\psi}(\theta) \geq -\frac{1}{1-\kappa}\int_\theta^1 \chi'(\vartheta)\Big(\vartheta + \frac{\kappa+1}{\kappa}(1-\vartheta)\Big)d\vartheta\,(n\cdot[\hat{C}]n).$$

Combination of this inequality with (3.18) yields (3.14). ■

Example. Condition (3.13) is satisfied if we choose $\chi(\theta) = (1-\kappa)\,\theta^q + \kappa$ with $q = -1/2 + (1/4 + 2/\kappa)^{1/2}$.

Corollary 3.2 *Suppose that the function $(\hat{u},\hat{T})$ and the manifold Γ have the regularity properties stated at the beginning of Section 2. Let $\chi \in C^3([0,1],[0,1])$ be a positive, increasing function satisfying $\chi(0) = \kappa$, $\chi(1) = 1$ and*

$$\int_0^1 \chi(\vartheta)\big(1-(1-\kappa)\vartheta\big)\, d\vartheta = \kappa.$$

Also, assume that $\hat{\psi} \in C^3([0,1])$ *has the following properties:*
1. $\hat{\psi}(0) = \hat{\psi}(1) = 0$.
2. For h *defined in* (3.15) *and* $\theta \in (0,1)$ *we have*

$$\hat{\psi}(\theta) > \min\{\theta, h(\theta)\} \sup_{(t,\eta)\in\Gamma} (n \cdot [\hat{C}]n)(t,\eta).$$

3. There is $c_0 > 0$ *such that*

$$\hat{\psi}'(0) \ge c_0 + \sup_{(t,\eta)\in\Gamma} (n \cdot [\hat{C}]n)(t,\eta), \quad \hat{\psi}'(1) \le -c_0 - \frac{1}{1-\kappa} \sup_{(t,\eta)\in\Gamma} \chi'(1)(n \cdot [\hat{C}]n)(t,\eta).$$

Then $\tilde{\psi}$ *and* $\partial_\theta \tilde{\psi}$ *belong to the space* $C^2(\Gamma \times [0,1])$. *Furthermore, for all* $(t,\eta) \in \Gamma$ *we have* $\tilde{\psi}(t,\eta,0) = \tilde{\psi}(t,\eta,1) = 0$ *and*

$$\partial_\theta \tilde{\psi}(t,\eta,0) \ge c_0, \quad \partial_\theta \tilde{\psi}(t,\eta,1) \le -c_0, \quad \tilde{\psi}(t,\eta,\theta) > 0, \quad \text{for } 0 < \theta < 1. \tag{3.20}$$

This Corollary follows immediately from the representations of $\tilde{\psi}$ and $n \cdot [\hat{C}]n$ in (3.16) and (2.12), respectively, and from Lemma 3.1.

Now we are in a position to state the existence result for the boundary value problem (3.8), (3.11). To this end we consider the initial value problem

$$\varphi_0'(t,\eta,\zeta) = \sqrt{2\tilde{\psi}(t,\eta,\varphi_0(t,\eta,\zeta))}, \qquad \varphi_0(t,\eta,0) = \frac{1}{2}. \tag{3.21}$$

By differentiation of this first order differential equation with respect to ζ we see immediately that a two-times differentiable solution is also a solution of the second order differential equation (3.11). Therefore it suffices to study this initial value problem.

Theorem 3.3 *Suppose that the function* $\tilde{\psi}$ *has all the properties asserted in Corollary 3.2. Then the following assertions hold:*
(i) For all $(t,\eta) \in \Gamma$ *there exist numbers* $-\infty < a = a(t,\eta) < 0 < b = b(t,\eta) < \infty$ *and a unique solution* $\zeta \mapsto \varphi_0(t,\eta,\zeta) : [a,b] \to [0,1]$ *of* (3.21)*, which is strictly increasing and satisfies*

$$\varphi_0(t,\eta,a) = 0, \quad \varphi_0(t,\eta,b) = 1, \quad \varphi_0'(t,\eta,a) = \varphi_0'(t,\eta,b) = 0. \tag{3.22}$$

φ_0 *has continuous derivatives up to second order with respect to all variables on the set* $\Gamma[a,b]$ *defined in* (3.10)*. Hence, these derivatives are bounded. Moreover, the solution satisfies* (3.11) *and*

$$\nabla_\eta \varphi_0(t,\eta,\zeta)\big|_{\zeta=a(t,\eta)} = 0, \quad \nabla_\eta \varphi_0(t,\eta,\zeta)\big|_{\zeta=b(t,\eta)} = 0.$$

(ii) The functions $(t,\eta) \mapsto a(t,\eta)$, $(t,\eta) \mapsto b(t,\eta)$ *are continuously differentiable. All derivatives are bounded.*

This theorem coincides with Theorem 1.1 in [4]. The only difference is that in the theorem above we require a smaller order of differentiability of $\tilde{\psi}$ and consequently obtain a smaller order of differentiability of a, b and φ_0. Since the proof is given in [4], we omit it here.

By Corollary 3.2 and Theorem 3.3 the function φ_0 appearing in (3.2) and in (3.5) can be determined, if the hypotheses of Corollary 3.2 are satisfied. This completes the construction of the function $(u^{(\nu)}, T^{(\nu)}, \varphi^{(\nu)})$.

The next theorem shows that $\{(u^{(\nu)}, T^{(\nu)}, \varphi^{(\nu)})\}_\nu$ is indeed a family of asymptotic solutions of the hybrid fracture model (1.1) – (1.3):

Theorem 3.4 *Let χ and $\hat{\psi}$ satisfy the hypotheses of Corollary 3.2. Suppose that $(\hat{u}, \hat{T}, \hat{\varphi}, \Gamma)$ is a solution of the sharp interface model (1.8) – (1.12) in the domain Q, and assume that this solution has the regularity properties stated at the beginning of Section 2. For $\nu > 0$ let the function $(u^{(\nu)}, T^{(\nu)}, \varphi^{(\nu)})$ be defined by (3.1) – (3.3) with u_0 and φ_0 satisfying (3.4), (3.8) and (3.11).*

Then $(u^{(\nu)}, T^{(\nu)}, \varphi^{(\nu)})$ belongs to the space $C^2(Q) \times C^1(Q) \times \big(C^1(Q) \cap C^2(\Gamma[\nu])\big)$, the divergence $\mathrm{div}_x T^{(\nu)}$ exists in Q and is continuous, and $(u^{(\nu)}, T^{(\nu)}, \varphi^{(\nu)})$ satisfies (1.2) exactly and (1.1) and (1.3) asymptotically. Precisely, there are constants $K_1 \ldots, K_3 > 0$ such that

$$\big|\mathrm{div}_x T^{(\nu)}(t,x) + \mathsf{b}(t,x)\big| \leq \begin{cases} K_1, & (t,x) \in \Gamma[\nu], \\ K_2\, \nu^{1/2}, & (t,x) \in Q \setminus \Gamma[\nu], \end{cases} \tag{3.23}$$

$$\Big\|\partial_t \varphi^{(\nu)} + f\Big(\psi_\varphi\big(\varepsilon(\nabla_x u^{(\nu)}), \varphi^{(\nu)}\big) - \nu \Delta_x \varphi^{(\nu)}\Big) |\nabla_x \varphi^{(\nu)}|\Big\|_{L^\infty(V)} \leq \begin{cases} K_3, & \textit{for } V = \Gamma[\nu], \\ 0, & \textit{for } V = Q \setminus \Gamma[\nu]. \end{cases} \tag{3.24}$$

Since $\mathrm{meas}(\Gamma[\nu]) \leq K_4 \nu^{1/2}$, this theorem has the following corollary:

Corollary 3.5 *There are constants K_5, K_6 such that*

$$\|\mathrm{div}_x T^{(\nu)} + \mathsf{b}\|_{L^1(Q)} \leq K_5\, \nu^{1/2},$$

$$\Big\|\partial_t \varphi^{(\nu)} + f\Big(\psi_\varphi\big(\varepsilon(\nabla_x u^{(\nu)}), \varphi^{(\nu)}\big) - \nu \Delta_x \varphi^{(\nu)}\Big) |\nabla_x \varphi^{(\nu)}|\Big\|_{L^1(Q)} \leq K_6\, \nu^{1/2}.$$

Sketch of the proof of Theorem 3.4: The proof is very similar to the proof of Theorem 2.9 in [5]. Therefore we only sketch it here. The first step of the proof consists in the construction of an asymptotic expansion for the term $\psi_\varphi\big(\varepsilon(\nabla_x u^{(\nu)}), \varphi^{(\nu)}\big) - \nu \Delta_x \varphi^{(\nu)}$ on the domain $\Gamma[\nu]$. To derive this expansion, we observe first that $\phi = 1$ on $\Gamma[\nu]$ for all sufficiently small $\nu > 0$. Using the definition of $u^{(\nu)}$ in (3.1), (3.4) and the splitting (2.4) of the gradient we thus compute that

$$\begin{aligned} \nabla_x u^{(\nu)} &= \nu^{1/2} \nabla_x u_0 + \nabla_x v = \nu^{1/2} \partial_\xi u_0 \otimes n + \nu^{1/2} \nabla_{\Gamma_\xi} u_0 + \nabla_x v \\ &= (u^* \otimes n)\varphi_0 + \nu^{1/2} \nabla_{\Gamma_\xi} u_0 + (\nabla_x v)^- + \nu^{1/2} (\partial_\xi \nabla_x v)(t, \eta, \xi^*) \frac{\xi}{\nu^{1/2}} \\ &= [\nabla_x \hat{u}]\varphi_0 + (\nabla_x \hat{u})^- + \nu^{1/2} \Big(\nabla_{\Gamma_\xi} u_0 + (\partial_\xi \nabla_x v)(t, \eta, \xi^*) \frac{\xi}{\nu^{1/2}}\Big). \end{aligned}$$

To get the second equality we applied the mean value theorem to $\nabla_x v$, and to get the last equality we used (2.8) and noted that $(\nabla_x v)^- = (\nabla_x \hat{u})^-$. In the following we write $\zeta = \frac{\xi}{\nu^{1/2}}$. Note that ζ is bounded when (t, η, ξ) varies in $\Gamma[\nu]$. From this equation, from the mean value theorem and from the definition (3.12) of $\tilde{\psi}$ we infer that

$$\begin{aligned} \psi_\varphi\big(\varepsilon(\nabla_x u^{(\nu)}), \varphi^{(\nu)}\big) &= \psi_\varphi\big(\varphi_0\, [\varepsilon(\nabla_x \hat{u})] + \varepsilon(\nabla_x \hat{u})^-, \varphi_0\big) + \nu^{1/2} R_1(\nu, t, \eta, \xi, \zeta) \\ &= \tilde{\psi}_\varphi(t, \eta, \varphi_0) + n \cdot [\hat{C}]n + \nu^{1/2} R_1(\nu, t, \eta, \xi, \zeta). \end{aligned} \tag{3.25}$$

Observe next that

$$\Delta_x \varphi^{(\nu)}(x,t) = \partial_\xi^2\, \varphi^{(\nu)}(t, \eta, \xi) - \kappa(t, \eta, \xi)\, \partial_\xi\, \varphi^{(\nu)}(t, \eta, \xi) + \Delta_{\Gamma_\xi} \varphi^{(\nu)}(t, \eta, \xi), \tag{3.26}$$

where $\kappa(t, \eta, \xi)$ denotes twice the mean curvature of the surface $\Gamma_\xi(t)$ and where Δ_{Γ_ξ} is the surface Laplacian. We insert (3.2) into (3.26) and combine the result with (3.25) to get

$$\psi_\varphi\big(\varepsilon(\nabla_x u^{(\nu)}), \varphi^{(\nu)}\big) - \nu\Delta_x\varphi^{(\nu)} = \tilde{\psi}_\varphi(t, \eta, \varphi_0) - \partial_\zeta^2\varphi_0 + n\cdot[\hat{C}]n + \nu^{1/2}R_2(\nu, t, \eta, \xi, \zeta). \quad (3.27)$$

Since f defined in (1.7) is Lipschitz continuous and since $|\nabla_x\varphi^{(\nu)}| = \nu^{-1/2}\partial_\zeta\varphi_0 + R_3(\nu, t, \eta, \xi, \zeta)$, we conclude from (3.27) that

$$\begin{aligned} & f\big(\psi_\varphi(\varepsilon(\nabla_x u^{(\nu)}), \varphi^{(\nu)}) - \nu\Delta_x\varphi^{(\nu)}\big)|\varphi^{(\nu)}| \\ & \quad = \nu^{-1/2} f\big(\tilde{\psi}_\varphi(t, \eta, \varphi_0) - \partial_\zeta^2\varphi_0 + n\cdot[\hat{C}]n\big)\partial_\zeta\varphi_0 + R_4(\nu, t, \eta, \xi, \zeta). \end{aligned} \quad (3.28)$$

(3.2) implies that

$$\partial_t\varphi^{(\nu)} = \varphi_t^{(\nu)} - \xi\,\partial_t n\cdot\nabla_\eta\varphi^{(\nu)} - s\,\partial_\xi\varphi^{(\nu)} = -\nu^{-1/2}s\,\partial_\zeta\varphi_0 + R_5(\nu, t, \eta, \xi, \zeta),$$

where s is the normal speed of the surface $\Gamma(t)$. This equation and (3.28) yield

$$\begin{aligned} & \partial_t\varphi^{(\nu)} + f\big(\psi_\varphi - \nu\Delta_x\varphi^{(\nu)}\big)|\nabla_x\varphi^{(\nu)}| \\ = \;& \nu^{-1/2}\Big(- s + f\big(\tilde{\psi}_\varphi(t, \eta, \varphi_0) - \partial_\zeta^2\varphi_0 + n\cdot[\hat{C}]n\big)\Big)\partial_\zeta\varphi_0 + R_6(\nu, t, \eta, \xi, \zeta). \end{aligned}$$

From this equation we see that if s satisfies (1.10) and φ_0 fulfills (3.11), then the estimate (3.24) holds on the domain $V = \Gamma[\nu]$. On the set $V = Q\setminus\Gamma[\nu]$ this estimate is obviously satisfied, since $\varphi^{(\nu)}$ is piecewise constant on this set, whence $\partial_t\varphi^{(\nu)} = |\nabla_x\varphi^{(\nu)}| = 0$.

For the proof of the inequality (3.23) we refer to [5]. ■

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