Asymptotic Profile of a Linearized Navier-Stokes Flow Past a Rotating Body

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Consider a rigid body in a three-dimensional Navier-Stokes liquid moving with a non-zero velocity and rotating with a non-zero angular velocity that are both constant when referred to a frame attached to the body. Linearizing the associated steady-state equations of motion, we obtain the exterior domain Oseen equations in a rotating frame of reference. We analyze the structure of weak solutions to these equations, and identify the leading term in the asymptotic expansion of the corresponding velocity field.

1 Introduction

Consider a rigid body moving in a three-dimensional Navier-Stokes liquid that fills the whole space $\Omega \subset \mathbb{R}^3$ exterior to the body. Assume the motion of the body is such that the velocity $\xi \in \mathbb{R}^3$ of its center of mass and its angular velocity $\omega \in \mathbb{R}^3$ are constant when referred to a frame attached to the body. Further assume that ξ and ω are directed along the same axis, which is taken to be the x_3-axis. Due to a simple transformation, see [7, Section 2], this assumption can be made without loss of generality whenever $\xi \cdot \omega \neq 0$ or, trivially, when $\xi = 0$ or $\omega = 0$. A flow is considered with Reynolds number so small that the inertia of the liquid can be disregarded. In this case, the motion of the liquid is described by the Oseen linearization of the Navier-Stokes equations in the case $\xi \neq 0$, and Stokes linearization in the case $\xi = 0$. If v denotes the Eulerian velocity field of the liquid, and p the corresponding pressure, the steady-state equations of motion written

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in a frame attached to the body then read, in an appropriate non-dimensional form,

$$\begin{cases} -\Delta v + \nabla p - \mathcal{R}\partial_3 v - \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big) = f & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = v_* & \text{on } \partial\Omega, \\ \lim_{|x|\to\infty} v(x) = 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is an exterior domain with $0 \in \mathbb{R}^3 \setminus \overline{\Omega}$, f the external force acting on the liquid, v_* the boundary velocity, and e_3 the unit vector directed along the x_3-axis. Moreover, $\mathcal{T}$ is the magnitude of the dimensionless angular velocity, the so-called Taylor number, and $\mathcal{R}$ a dimensionless constant proportional to $\xi \cdot \omega$ if $\xi \cdot \omega \neq 0$, proportional to ξ if $\omega = 0$, and zero if $\xi = 0$. Without loss of generality, $\mathcal{T}$ and $\mathcal{R}$ are taken to be non-negative.

The above system is the classical steady-state Oseen ($\mathcal{R} > 0$) or Stokes ($\mathcal{R} = 0$) problem with the additional term $\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)$, which stems from the rotating frame of reference. Due to the unbounded coefficient $\mathrm{e}_3 \wedge x$, this term can not be treated at a perturbation to the classical Oseen or Stokes operator.

The purpose of this paper is to study the structure of a solution (v, p) to (1.1) at large distances from the body, that is, for large $|x|$. Precise information hereof is essential for deriving even very basic physical properties of the flow. For example, a characterization of the wake behind the body is only possible when the asymptotic profile of v has been identified. If the body is moving without rotation ($\mathcal{R} > 0, \mathcal{T} = 0$) the existence of a paraboloidal wake region is a classical result. A similar characterization of the wake behind a rotating body is, however, not known till now. Another example concerns the kinetic energy of the system. In order to determine if the kinetic energy is finite, detailed knowledge of the decay properties of v at large distances is required. As in the previous example, the answer to this question is known when the body is not rotating, in which case the kinetic energy of the surrounding liquid is, in general, infinite, see [4, Chapter VII], but the problem is unsolved, till now, in the case of a rotating body.

Existence of a solution to (1.1) in the class of velocity fields with bounded Dirichlet integral can be established by standard methods. In the context of Navier-Stokes equations, this class of solutions first appeared in the pioneering work [8] of Leray, and is therefore sometimes referred to as the class of Leray solutions. It is a natural starting point for further investigation, and will serve as such in this paper.

In the case of a non-rotating body ($\mathcal{T} = 0$) a complete characterization of the asymptotic structure of a solution to (1.1) is well-known. The reader is referred to the paper [1] by Chang & Finn, where, for a solution in a class even larger than the Leray class, the validity of the asymptotic expansions, as $|x| \to \infty$,

$$\mathcal{T} = 0, \mathcal{R} = 0: \qquad v(x) = \Gamma_{\mathrm{Stokes}}(x) \cdot \mathcal{F} + O\big(|x|^{-2}\big), \tag{1.2}$$

$$\mathcal{T} = 0, \mathcal{R} > 0: \qquad v(x) = \Gamma_{\mathrm{O}}^{\mathcal{R}}(x) \cdot \mathcal{F} + O\big(|x|^{-3/2}\big) \tag{1.3}$$

is shown. Here, $\mathcal{F}$ denotes the force exerted by the liquid on the body, $\Gamma_{\mathrm{O}}^{\mathcal{R}}$ the fundamental solution tensor to the Oseen system, and Γ_{Stokes} the fundamental solution tensor to the Stokes system. A closed-form expression for $\Gamma_{\mathrm{O}}^{\mathcal{R}}$ and Γ_{Stokes} can be found in [4], see also [9].

For a rotating body ($\mathcal{T} > 0$) the asymptotic structure of a solution has only been successfully characterized when the body is not translating, that is, when $\mathcal{R} = 0$. This result is due to Farwig & Hishida, who have shown in [3] that a Leray solution to (1.1) satisfies

$$\mathcal{T} > 0, \mathcal{R} = 0 : \qquad v(x) = \Gamma_{\mathrm{Stokes}}(x) \cdot (\mathcal{F} \cdot \mathrm{e}_3)\, \mathrm{e}_3 + O\big(|x|^{-2}\big), \tag{1.4}$$

as $|x| \to \infty$.

The main result in this paper concerns a rotating and translating body, that is, the case $\mathcal{T} > 0$ and $\mathcal{R} > 0$. More precisely, for a Leray solution to (1.1) it is shown that

$$\mathcal{T} > 0, \mathcal{R} > 0 : \qquad v(x) = \Gamma_{\mathrm{O}}^{\mathcal{R}}(x) \cdot (\mathcal{F} \cdot \mathrm{e}_3)\, \mathrm{e}_3 + O\big(|x|^{-3/2}\big), \tag{1.5}$$

as $|x| \to \infty$. In view of (1.2)–(1.4), this result completes the characterization of the asymptotic structure of a steady-state, linearized Navier-Stokes flow past a body.

It is well-known that the entries in the Oseen fundamental solution $\Gamma_{\mathrm{O}}^{\mathcal{R}}$ exhibits a paraboloidal wake with respect to the direction of translation, see for example [4, Chapter VII]. Since the asymptotic profile identified in (1.5) is given by a linear combination of these entries, the existence and explicit characterization of a paraboloidal wake behind a body that is *translating and rotating* follows.

An asymptotic expansion of a steady-state velocity field is typically stated as in (1.2)–(1.5). However, the proof presented in this paper to establish (1.5) reveals slightly more details on the asymptotic structure of v, namely

$$v(x) = \Gamma_{\mathrm{O}}^{\mathcal{R}}(x) \cdot \big(\mathcal{F} \cdot \mathrm{e}_3\big)\, \mathrm{e}_3 + O\big(\nabla\Gamma_{\mathrm{O}}^{\mathcal{R}}(x)\big). \tag{1.6}$$

It is well-known that the entries in $\nabla\Gamma_{\mathrm{O}}^{\mathcal{R}}$ are square integrable in a neighborhood of infinity, but the entries in $\Gamma_{\mathrm{O}}^{\mathcal{R}}$ are *not*. Consequently, (1.6) implies that the kinetic energy of the velocity field is infinite unless the coefficient $\mathcal{F} \cdot \mathrm{e}_3$ is zero. In other words, it follows that the kinetic energy of a steady-state, linearized Navier-Stokes flow past a *translating and rotating* body is finite if and only if the component in the direction of rotation of the force exerted by the liquid on the body vanishes.

The leading term in the expansion (1.5) above is itself a solution to (1.1) for large x. This observation explains why the leading term in (1.5) only depends on $\Gamma_{\mathrm{O}}^{\mathcal{R}} \cdot \mathrm{e}_3$, and not on $\Gamma_{\mathrm{O}}^{\mathcal{R}} \cdot \mathrm{e}_1$ or $\Gamma_{\mathrm{O}}^{\mathcal{R}} \cdot \mathrm{e}_2$, as the latter two do *not* satisfy (1.1). In contrast, in the absence of rotation ($\mathcal{T} = 0$) all columns of $\Gamma_{\mathrm{O}}^{\mathcal{R}}$ satisfy (1.1), and the corresponding expansion (1.3) may include all parts of the tensor $\Gamma_{\mathrm{O}}^{\mathcal{R}}$ in the leading term. It is worth noting, however, that the asymptotic profile of a solution in both cases $\mathcal{T} > 0$ and $\mathcal{T} = 0$ is derived from the Oseen fundamental solution $\Gamma_{\mathrm{O}}^{\mathcal{R}}$.

The jump in the coefficient in the leading term from $\big(\mathcal{F} \cdot \mathrm{e}_3\big)\, \mathrm{e}_3$ to $\mathcal{F}$ as $\mathcal{T}$ goes to 0, compare (1.3) and (1.5), can be seen as singular behavior. Clearly, this behavior must also manifest itself in the remainder term. In fact, in the proof of the main result below it is found that

$$v(x) = \Gamma_{\mathrm{O}}^{\mathcal{R}}(x) \cdot \big(\mathcal{F} \cdot \mathrm{e}_3\big)\, \mathrm{e}_3 + R_1(x) + R_2(x) \tag{1.7}$$

with $|R_1(x)| \leq C_1|x|^{-3/2}$, C_1 independent on $\mathcal{T}$, and $|R_2(x)| \leq C_2|x|^{-3}$ with $C_2 = C_2(\mathcal{T}) \to \infty$ as $\mathcal{T} \to 0$. Consequently, the aforementioned singular behavior materializes only in the part of the remainder in $O(|x|^{-3})$. A similar conclusion in the case $\mathcal{R} = 0$ was made in [3, (1.10)].

2 Statement of the main result

Before stating the main theorem, we introduce some notation. For a liquid velocity field $v : \mathbb{R}^3 \to \mathbb{R}^3$ and pressure $p : \mathbb{R}^3 \to \mathbb{R}$ we let

$$\mathrm{T}(v, p) := \nabla v + \nabla v^\top - pI$$

denote the Cauchy stress tensor of the (Newtonian) liquid corresponding to the non-dimensional form (1.1) of the Navier-Stokes equations. We let $(i, j = 1, 2, 3)$

$$\big[\Gamma_{\mathrm{O}}^{\mathcal{R}}(x)\big]_{ij} := (\delta_{ij}\Delta - \partial_i\partial_j)\Phi^{\mathcal{R}}(x), \quad \Phi^{\mathcal{R}}(x) := \frac{1}{4\pi\mathcal{R}}\int_0^{\mathcal{R}(|x|+x_3)/2} \frac{1 - \mathrm{e}^{-\tau}}{\tau}\,\mathrm{d}\tau$$

denote the three-dimensional Oseen fundamental solution, see also [4, Chapter VII.3], and

$$\Gamma_{\mathrm{L}}(x) := \frac{1}{4\pi}\frac{1}{|x|}$$

the fundamental solution to the Laplace equation.

We denote by $L^q(\Omega)$, $1 \leq q \leq \infty$, the usual Lebesgue space. For $m \in \mathbb{N}$ we use $D^{m,q}(\Omega)$ and $W^{m,q}(\Omega)$ to denote the homogeneous and inhomogeneous Sobolev spaces, respectively.

For functions $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ we let $\operatorname{div} u(x,t) := \operatorname{div}_x u(x,t)$, $\Delta u(x,t) := \Delta_x u(x,t)$ etc., that is, unless otherwise indicated, differential operators act in the spatial variable x only.

We put $\mathrm{B}_m := \{x \in \mathbb{R}^3 \mid |x| < m\}$ and $\mathrm{B}^m := \mathbb{R}^3 \setminus \mathrm{B}_m$.

Constants in capital letters in the proofs and theorems are global, while constants in small letters are local to the proof in which they appear.

We now state the main result.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with a C^2-smooth boundary, and $\mathcal{R}, \mathcal{T} > 0$. Moreover, let $f = \operatorname{div} F$ with $F \in C_0^\infty(\overline{\Omega})^{3\times 3}$, and $v_* \in W^{2,1/2}(\partial\Omega)^3$. A solution*

$$(v, p) \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3 \times L^2_{loc}(\Omega) \tag{2.1}$$

to (1.1) satisfies (after possibly adding a constant to p) the asymptotic expansion

$$v(x) = \Gamma_{\mathrm{O}}^{\mathcal{R}}(x) \cdot \big(\mathcal{F} \cdot \mathrm{e}_3\big)\, \mathrm{e}_3 + O\big(|x|^{-3/2}\big), \tag{2.2}$$

$$p(x) = \nabla\Gamma_{\mathrm{L}}(x) \cdot \mathcal{F} - \mathcal{R}\bigg(\int_{\partial\Omega} v \cdot n \,\mathrm{d}S\bigg)\partial_3\Gamma_{\mathrm{L}}(x) + O\big(|x|^{-3}\big), \tag{2.3}$$

as $|x| \to \infty$*, where*

$$\mathcal{F} := \int_{\partial\Omega} \big[\mathrm{T}(v,p) + \mathcal{R} v \otimes \mathrm{e}_3 - \mathcal{T}(\mathrm{e}_3 \wedge x) \otimes v + F\big] \cdot n \,\mathrm{d}S. \tag{2.4}$$

Remark 2.2. Leray solutions belong to the class (2.1). Furthermore, since Ω has a C^2-smooth boundary and $v_* \in W^{2,1/2}(\partial\Omega)^3$, it follows from classical regularity theory for elliptic systems that solutions (v,p) to (1.1) from the class (2.1) satisfy $(v,p) \in W^{2,2}_{loc}(\overline{\Omega})^3 \times W^{1,2}_{loc}(\overline{\Omega})$. Consequently, the boundary integral in the expression for $\mathcal{F}$ in (2.4) is well-defined. In fact, the only reason for the regularity assumptions on $\partial\Omega$ and v_* in Theorem 2.1 is to be able to express the force $\mathcal{F}$ as in (2.4). The theorem continues to hold without the assumptions on $\partial\Omega$ and v_*, but $\mathcal{F}$ must then be expressed in a more indirect way.

Remark 2.3. Note that $\mathcal{F}$ equals the force exerted by the liquid on the body. Thus, the leading term in the expansion (2.2) depends only on the component of this force in the direction of the rotation.

Remark 2.4. Theorem 2.1 holds for arbitrary boundary data v_*. In the case of the so-called "no-slip" boundary condition, that is, $v_* = \mathrm{e}_3 + \mathcal{T}\,\mathrm{e}_3 \wedge x$, the expression for $\mathcal{F}$ reduces to

$$\mathcal{F} = \int_{\partial\Omega} \big[\mathrm{T}(v,p) + F\big] \cdot n \,\mathrm{d}S.$$

Remark 2.5. We have chosen to consider an external force f with very good properties, namely $f = \operatorname{div} F$ with $F \in C_0^\infty(\overline{\Omega})^{3\times 3}$. In principle, more general forces f could be treated. Of course, with more general external forces, the expression for $\mathcal{F}$ must be modified accordingly. Our choice corresponds to the choice made in the related paper [3].

Remark 2.6. Recently, a result in the direction of Theorem 2.1 was obtained in [2, Theorem 5.4]. The leading term in the expansion (2.2) is the fundamental solution corresponding to the Oseen operator $Lv := -\Delta v - \mathcal{R}\partial_3 v$. In [2] an expansion based on the fundamental solution to the full linear operator on the left-hand side of (1.1), that is, the operator $Lv := -\Delta v - \mathcal{R}\partial_3 v - \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)$, was established. If we denote this fundamental solution, which is not of convolution type, by $\mathfrak{Z}(x,y)$, [2, Theorem 5.4] states that

$$\mathcal{T} > 0, \mathcal{R} > 0: \qquad v(x) = \mathfrak{Z}(x,0) \cdot \beta + O\big(|x|^{-3/2}\big) \tag{2.5}$$

for some $\beta \in \mathbb{R}^3$. Since no closed-form expression is available for $\mathfrak{Z}$, it is not possible, however, to derive from (2.5) any physical properties of the flow.

3 Proof of the main theorem

Lemma 3.1. *Let* $\mathcal{R}, \mathcal{T} > 0$*. There is a constant* $C_3 = C_3(\mathcal{R},\mathcal{T}) > 0$ *such that for all* $k \in \mathbb{Z} \setminus \{0\}$ *and* $x \in \mathbb{R}^3$*:*

$$\mathrm{Re}\Big[-\big(i\mathcal{T}k + (\mathcal{R}/2)^2\big)^{\frac{1}{2}}|x| - (\mathcal{R}/2)x_3\Big] \leq -C_3|k|^{\frac{1}{2}}|x|. \tag{3.1}$$

Proof. First, we compute

$$\begin{aligned}\operatorname{Re}\big[\big(i\mathcal{T}k+(\mathcal{R}/2)^2\big)^{\frac12}\big] &= \big|i\mathcal{T}k+(\mathcal{R}/2)^2\big|^{\frac12}\cos\bigg(\frac12\arctan\bigg(\frac{\mathcal{T}k}{(\mathcal{R}/2)^2}\bigg)\bigg)\\ &= \big(\mathcal{T}^2k^2+(\mathcal{R}/2)^4\big)^{\frac14}\frac{1}{\sqrt2}\bigg(1+\cos\bigg(\arctan\bigg(\frac{\mathcal{T}k}{(\mathcal{R}/2)^2}\bigg)\bigg)\bigg)^{\frac12}\\ &= \big(\mathcal{T}^2k^2+(\mathcal{R}/2)^4\big)^{\frac14}\frac{1}{\sqrt2}\bigg(1+\bigg(1+\frac{\mathcal{T}^2k^2}{(\mathcal{R}/2)^4}\bigg)^{-\frac12}\bigg)^{\frac12}\\ &= \frac{1}{\sqrt2}(\mathcal{R}/2)\bigg(\bigg(1+\frac{\mathcal{T}^2k^2}{(\mathcal{R}/2)^4}\bigg)^{\frac12}+1\bigg)^{\frac12}.\end{aligned}$$

It follows that

$$\forall k\in\mathbb{Z}\setminus\{0\}:\ \operatorname{Re}\big[\big(i\mathcal{T}k+(\mathcal{R}/2)^2\big)^{\frac12}-(\mathcal{R}/2)\big]>0$$

and

$$\lim_{|k|\to\infty}\frac{\operatorname{Re}\big[\big(i\mathcal{T}k+(\mathcal{R}/2)^2\big)^{\frac12}-(\mathcal{R}/2)\big]}{|k|^{\frac12}}=\sqrt{\frac{\mathcal{T}}{2}}.$$

Consequently, there is a constant $C_3=C_3(\mathcal{R},\mathcal{T})>0$ such that

$$\forall k\in\mathbb{Z}\setminus\{0\}:\ \operatorname{Re}\big[\big(i\mathcal{T}k+(\mathcal{R}/2)^2\big)^{\frac12}-(\mathcal{R}/2)\big]\geq C_3|k|^{\frac12},$$

from which we conclude that

$$\begin{aligned}&\operatorname{Re}\big[-\big(i\mathcal{T}k+(\mathcal{R}/2)^2\big)^{\frac12}|x|-(\mathcal{R}/2)x_3\big]\\ &\qquad\leq-\operatorname{Re}\big[\big(i\mathcal{T}k+(\mathcal{R}/2)^2\big)^{\frac12}-(\mathcal{R}/2)\big]|x|\leq-C_3|k|^{\frac12}|x|.\end{aligned}$$

□

Proof of Theorem 2.1. In the first step of the proof, we will reduce (1.1) to a whole space problem. For this purpose, choose $\rho>0$ so large that both $\mathbb{R}^3\setminus\Omega\subset\mathrm{B}_\rho$ and $\operatorname{supp}F\subset\mathrm{B}_\rho$. Let $\psi_\rho\in C^\infty(\mathbb{R}^3)$ be a "cut-off" function with $\psi_\rho=0$ in B_ρ and $\psi_\rho=1$ in $\mathbb{R}^3\setminus\mathrm{B}_{2\rho}$. Since (v,p) solves (1.1), standard regularity theory for elliptic systems implies that $(v,p)\in C^\infty(\Omega\setminus\mathrm{B}_\rho)^3\times C^\infty(\Omega\setminus\mathrm{B}_\rho)$. Now let

$$\sigma(x):=\bigg(-\int_{\partial\Omega}v_*\cdot n\,\mathrm{d}S\bigg)\nabla\Gamma_{\mathrm{L}}(x),\quad \gamma(x):=\mathcal{R}\bigg(-\int_{\partial\Omega}v_*\cdot n\,\mathrm{d}S\bigg)\partial_3\Gamma_{\mathrm{L}}(x).$$

Clearly also $(\sigma,\gamma)\in C^\infty(\Omega\setminus\mathrm{B}_\rho)^3\times C^\infty(\Omega\setminus\mathrm{B}_\rho)$. We can therefore define

$$\begin{aligned}&w:\mathbb{R}^3\to\mathbb{R}^3,\quad w(x):=\psi_\rho(x)\big(v(x)-\sigma(x)\big)-\mathfrak{B}[\nabla\psi_\rho\cdot(v-\sigma)](x),\\ &\mathfrak{q}:\mathbb{R}^3\to\mathbb{R},\quad \mathfrak{q}(x):=\psi_\rho(x)\big(p(x)-\gamma(x)\big),\end{aligned}\tag{3.2}$$

where $\mathfrak{B}$ denotes the so-called "Bogovskiĭ operator", that is, an operator

$$\mathfrak{B} : C_0^\infty(\mathrm{B}_{2\rho}) \to C_0^\infty(\mathrm{B}_{2\rho})^3$$

with the property that $\operatorname{div} \mathfrak{B}(f) = f$ whenever $\int_{\mathrm{B}_{2\rho}} f(x)\,\mathrm{d}x = 0$. We refer to [4, Theorem III.3.2] for details on this operator. Observe that

$$\int_{\mathrm{B}_{2\rho}} \nabla\psi_\rho(x) \cdot \big(v(x) - \sigma(x)\big)\,\mathrm{d}x = 0$$

and

$$\Delta\sigma = \operatorname{div}\sigma = \mathrm{e}_3 \wedge x \cdot \nabla\sigma - \mathrm{e}_3 \wedge \sigma = 0 \quad \text{in } \mathbb{R}^3 \setminus \{0\},$$

whence $(w, \mathfrak{q})$ is a smooth solution in the class (2.1) to the whole space problem

$$\begin{cases} -\Delta w + \nabla\mathfrak{q} - \mathcal{R}\partial_3 w - \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla w - \mathrm{e}_3 \wedge w\big) = g & \text{in } \mathbb{R}^3, \\ \operatorname{div} w = 0 & \text{in } \mathbb{R}^3 \end{cases} \tag{3.3}$$

with $g \in C_0^\infty(\mathbb{R}^3)^3$ and $\operatorname{supp} g \subset \mathrm{B}_{2\rho}$. In fact, from [6, Lemma 4.2] we deduce that $(w, \mathfrak{q})$ enjoys additional summability properties, namely

$$\begin{aligned} \forall q \in (1,2) : w \in L^{\frac{2q}{2-q}}(\mathbb{R}^3),\ \nabla w \in L^{\frac{4q}{4-q}}(\mathbb{R}^3),\ \partial_3 w \in L^q(\mathbb{R}^3), \\ \nabla^2 w \in L^q(\mathbb{R}^3),\ \mathfrak{q} \in L^{\frac{3q}{3-q}}(\mathbb{R}^3),\ \nabla\mathfrak{q} \in L^q(\mathbb{R}^3). \end{aligned} \tag{3.4}$$

Concerning the right-hand side of (3.3), we can calculate

$$\begin{aligned} \int_{\mathbb{R}^3} g\,\mathrm{d}x &= \int_{\mathrm{B}_{2\rho}} \operatorname{div}\big[-\mathrm{T}(w,\mathfrak{q}) - \mathcal{R}w \otimes \mathrm{e}_3 - \mathcal{T}w \otimes (\mathrm{e}_3 \wedge x) + \mathcal{T}(\mathrm{e}_3 \wedge x) \otimes w\big]\,\mathrm{d}x \\ &= \int_{\partial\mathrm{B}_{2\rho}} \big[-\mathrm{T}(v-\sigma, p-\gamma) - \mathcal{R}(v-\sigma) \otimes \mathrm{e}_3 \\ &\qquad - \mathcal{T}(v-\sigma) \otimes (\mathrm{e}_3 \wedge x) + \mathcal{T}(\mathrm{e}_3 \wedge x) \otimes (v-\sigma)\big] \cdot n\,\mathrm{d}S. \end{aligned}$$

Using that $n = \frac{x}{|x|}$ on $\partial\,\mathrm{B}_{2\rho}$, we obtain by a direct calculation that

$$\int_{\partial\mathrm{B}_{2\rho}} \big[\mathrm{T}(\sigma,\gamma) + \mathcal{R}\sigma \otimes \mathrm{e}_3 + \mathcal{T}\sigma \otimes (\mathrm{e}_3 \wedge x) - \mathcal{T}(\mathrm{e}_3 \wedge x) \otimes \sigma\big] \cdot n\,\mathrm{d}S = 0.$$

Combined with the fact that (v, p) solves (1.1), it then follows that

$$\begin{aligned} \int_{\mathbb{R}^3} g\,\mathrm{d}x &= \int_{\partial\Omega} \big[\mathrm{T}(v,p) + \mathcal{R}v \otimes \mathrm{e}_3 + \mathcal{T}v \otimes (\mathrm{e}_3 \wedge x) - \mathcal{T}(\mathrm{e}_3 \wedge x) \otimes v\big] \cdot n\,\mathrm{d}S \\ &\quad + \int_{\Omega\cap\mathrm{B}_{2\rho}} \operatorname{div} F\,\mathrm{d}x \\ &= \int_{\partial\Omega} \big[\mathrm{T}(v,p) + \mathcal{R}v \otimes \mathrm{e}_3 - \mathcal{T}(\mathrm{e}_3 \wedge x) \otimes v + F\big] \cdot n\,\mathrm{d}S, \end{aligned} \tag{3.5}$$

where the last equality is due to

$$\int_{\partial\Omega} v\otimes(\mathrm{e}_3\wedge x)\cdot n\,\mathrm{d}S=\int_{\partial\,\mathrm{B}_\rho} v\otimes(\mathrm{e}_3\wedge x)\cdot n\,\mathrm{d}S=0,$$

which again is a direct consequence of $n=\frac{x}{|x|}$ on $\partial\,\mathrm{B}_\rho$.

In the next step, following [5], we will transform (3.3) into a time-dependent Oseen problem. For this purpose, we introduce the rotation matrix corresponding to the angular velocity $\mathcal{T}\,\mathrm{e}_3$. More specifically, let $\mathrm{E}_3\in\mathrm{skew}_{3\times3}(\mathbb{R})$ denote the skew-symmetric adjoint of e_3, and put

$$Q(t):=\exp(\mathcal{T}\mathrm{E}_3 t)=\begin{pmatrix}\cos(\mathcal{T}t) & -\sin(\mathcal{T}t) & 0\\ \sin(\mathcal{T}t) & \cos(\mathcal{T}t) & 0\\ 0 & 0 & 1\end{pmatrix}.$$

For $(x,t)\in\mathbb{R}^3\times\mathbb{R}$ we put

$$u(x,t):=Q(t)w\big(Q(t)^\top x\big),\ \mathfrak{p}(x,t):=\mathfrak{q}\big(Q(t)^\top x\big),\ G(x,t):=Q(t)g\big(Q(t)^\top x\big).$$

As one may easily verify,

$$\begin{cases}\partial_t u-\Delta u+\nabla\mathfrak{p}-\mathcal{R}\partial_3 u=G & \text{in } \mathbb{R}^3\times\mathbb{R},\\ \operatorname{div} u=0 & \text{in } \mathbb{R}^3\times\mathbb{R}.\end{cases}\tag{3.6}$$

Note that u, $\mathfrak{p}$, and G are smooth and $\frac{2\pi}{\mathcal{T}}$-periodic in t. We can therefore expand these fields in their Fourier-series with respect to t. More precisely, we have

$$\begin{aligned}&u(x,t)=\sum_{k\in\mathbb{Z}}u_k(x)\,\mathrm{e}^{i\mathcal{T}kt},\quad \mathfrak{p}(x,t)=\sum_{k\in\mathbb{Z}}\mathfrak{p}_k(x)\,\mathrm{e}^{i\mathcal{T}kt},\\ &G(x,t)=\sum_{k\in\mathbb{Z}}G_k(x)\,\mathrm{e}^{i\mathcal{T}kt},\end{aligned}\tag{3.7}$$

with

$$\begin{aligned}&u_k(x):=\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}}u(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t,\quad \mathfrak{p}_k(x):=\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}}\mathfrak{p}(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t,\\ &G_k(x):=\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}}G(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t.\end{aligned}$$

Observe that u_k and $\mathfrak{p}_k$ are smooth and enjoy the same summability properties as w and $\mathfrak{q}$, respectively. In particular, $(u_k,\mathfrak{p}_k)$ lies in the class (3.4). Moreover, note that $G_k\in C_0^\infty(\mathbb{R}^3)^3$. We shall express G_k as

$$G_k(x)=\frac{i}{k\mathcal{T}}\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}}\partial_t G(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t=:\frac{i}{k\mathcal{T}}H_k(x).$$

Clearly, also $H_k \in C_0^\infty(\mathbb{R}^3)^3$. Inserting now the Fourier series from (3.7) into (3.6), we find that each Fourier coefficient satisfies

$$\begin{cases} i\mathcal{T}ku_k - \Delta u_k + \nabla \mathfrak{p}_k - \mathcal{R}\partial_3 u_k = \dfrac{i}{k\mathcal{T}} H_k & \text{in } \mathbb{R}^3, \\ \operatorname{div} u_k = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{3.8}$$

Consequently, $(u_0, \mathfrak{p}_0)$ satisfies the classical Oseen problem, while $(u_k, \mathfrak{p}_k)$ solves an Oseen resolvent-like problem when $k \neq 0$.

We shall now establish an important estimate for u_k in the case $k \neq 0$. We will exploit that $(u_k, \mathfrak{p}_k)$ solves (3.8), and we therefore introduce the fundamental solution to the scalar Oseen resolvent equation:

$$\Gamma^k_{\mathrm{SOR}} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \quad \Gamma^k_{\mathrm{SOR}}(x) := \frac{1}{4\pi} \frac{1}{|x|} \mathrm{e}^{-\left(i\mathcal{T}k + (\mathcal{R}/2)^2\right)^{\frac{1}{2}} |x| - (\mathcal{R}/2)x_3}.$$

More specifically, Γ^k_{SOR} satisfies

$$i\mathcal{T}k\Gamma^k_{\mathrm{SOR}} - \Delta\Gamma^k_{\mathrm{SOR}} - \mathcal{R}\partial_3\Gamma^k_{\mathrm{SOR}} = \delta \quad \text{in } \mathcal{D}'(\mathbb{R}^3).$$

In order to obtain a representation of u_k in terms of the scalar fundamental solution Γ^k_{SOR}, we apply the Helmholtz projection to H_k. More precisely, we put

$$H^s_k(x) := H_k(x) - \nabla\Phi_k, \quad \Phi_k(x) := \Gamma_{\mathrm{L}} * \operatorname{div} H_k. \tag{3.9}$$

Clearly, $\Phi_k \in C^\infty(\mathbb{R}^3)$ and $\operatorname{div} H^s_k = 0$. Furthermore, for $|x| > \operatorname{diam}(\operatorname{supp} H_k) = \operatorname{diam}(\operatorname{supp} g)$ we have

$$\nabla\Phi_k(x) = \nabla^2\Gamma_{\mathrm{L}}(x) \cdot \int_{\mathbb{R}^3} H_k(y)\,\mathrm{d}y - \int_{\mathbb{R}^3} \Big[\nabla^2\Gamma_{\mathrm{L}}(x) - \nabla^2\Gamma_{\mathrm{L}}(x-y)\Big] \cdot H_k(y)\,\mathrm{d}y.$$

For $|x| > 2\operatorname{diam}(\operatorname{supp} g)$,

$$\Big|\int_{\mathbb{R}^3} \Big[\nabla^2\Gamma_{\mathrm{L}}(x) - \nabla^2\Gamma_{\mathrm{L}}(x-y)\Big] \cdot H_k(y)\,\mathrm{d}y\Big| \leq c_1|x|^{-4} \int_{\mathbb{R}^3} |y||H_k(y)|\,\mathrm{d}y,$$

with c_1 independent on k. Additionally, we can estimate

$$\begin{aligned} \Big|\int_{\mathbb{R}^3} H_k(y)\,\mathrm{d}y\Big| &\leq \int_{\mathbb{R}^3} \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} |\partial_t G(y,t)|\,\mathrm{d}t\mathrm{d}y \\ &\leq c_2\Big(\int_{\mathbb{R}^3} |g(y)|\,\mathrm{d}y + \int_{\mathbb{R}^3} |y||\nabla g(y)|\,\mathrm{d}y\Big), \end{aligned}$$

with c_2 independent on k, and

$$\int_{\mathbb{R}^3} |y||H_k(y)|\,\mathrm{d}y \leq c_3 \operatorname{diam}(\operatorname{supp} g)\Big(\int_{\mathbb{R}^3} |g(y)|\,\mathrm{d}y + \int_{\mathbb{R}^3} |y||\nabla g(y)|\,\mathrm{d}y\Big),$$

with c_3 independent on k. We thus deduce that

$$\forall |x| > 2\,\mathrm{diam}(\mathrm{supp}\, g): \ |\nabla\Phi_k(x)| \leq c_4 |x|^{-3},$$

with c_4 independent on k. In addition, for $|x| \leq 2\,\mathrm{diam}(\mathrm{supp}\, g)$ we can estimate

$$\begin{aligned}|\nabla\Phi_k(x)| &\leq \Big(\sup_{y\in\mathbb{R}^3} |\mathrm{div}\, H_k(y)|\Big) \cdot \int_{\mathrm{B}_{2\,\mathrm{diam}(\mathrm{supp}\, g)}} |\nabla\Gamma_{\mathrm{L}}(x-y)|\,\mathrm{d}y \\ &\leq c_5 \sup_{y\in\mathbb{R}^3} \big(|\nabla g(y)| + |y||\nabla\,\mathrm{div}\, g(y)|\big) \leq c_6,\end{aligned}$$

with c_6 independent on k. Consequently,

$$\forall x \in \mathbb{R}^3: \ |\nabla\Phi_k(x)| \leq c_7(1+|x|)^{-3}, \tag{3.10}$$

with c_7 independent on k. In a completely similar manner, we prove that

$$\forall x \in \mathbb{R}^3: \ |\Phi_k(x)| \leq c_8(1+|x|)^{-2}, \tag{3.11}$$

with c_8 independent on k. We now return to the task of obtaining a representation of u_k in terms of Γ^k_{SOR}. By Lemma 3.1, we see that $\Gamma^k_{\mathrm{SOR}} \in L^1(\mathbb{R}^3)$. Moreover, from (3.9) and (3.10) we find that $H^s_k \in L^r(\mathbb{R}^3)^3$ for all $1 < r$. We can thus define

$$\tilde{u}_k := \frac{i}{k\mathcal{T}} \Gamma^k_{\mathrm{SOR}} * H^s_k, \tag{3.12}$$

where the convolution is taken component-wise in the components of H^s_k. The difference $(\tilde{u}_k - u_k)$ then satisfies

$$\begin{cases} i\mathcal{T}k(\tilde{u}_k - u_k) - \Delta(\tilde{u}_k - u_k) - \mathcal{R}\partial_3(\tilde{u}_k - u_k) \\ \qquad\qquad\qquad = \dfrac{i}{k\mathcal{T}}(H^s_k - H_k) + \nabla\mathfrak{p}_k & \text{in } \mathbb{R}^3, \\ \mathrm{div}(\tilde{u}_k - u_k) = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{3.13}$$

Taking div in the first equation above, we find that

$$\Delta\Big[\frac{i}{k\mathcal{T}}\Phi_k - \mathfrak{p}_k\Big] = 0 \quad \text{in } \mathbb{R}^3.$$

From (3.4) we know that $\mathfrak{p}_k \in L^{\frac{3r}{3-r}}(\mathbb{R}^3)$ for all $1 < r < 2$. Combined with the decay property (3.11) of Φ_k, we thus deduce that $\mathfrak{p}_k = \frac{i}{k\mathcal{T}}\Phi_k$. Going back to (3.13), we apply the Fourier transformation and deduce

$$(ik\mathcal{T} + |\xi|^2 + i\mathcal{R}\xi_3)\mathcal{F}[\tilde{u}_k - u_k](\xi) = 0 \quad \text{in } \mathbb{R}^3,$$

from which we conclude, recall $k \neq 0$, that $u_k = \tilde{u}_k$. Hence, by (3.12) we obtain the representation

$$u_k(x) = \frac{i}{k\mathcal{T}} \int_{\mathbb{R}^3} \Gamma_{\mathrm{SOR}}(x-y) H^s_k(y)\,\mathrm{d}y. \tag{3.14}$$

We can now establish the estimate we need on u_k. Fix $x \in \mathbb{R}^3$, put $R = \frac{|x|}{2}$, and split

$$\begin{aligned}\int_{\mathbb{R}^3} \Gamma_{\mathrm{SOR}}(x-y) H_k^s(y)\,\mathrm{d}y = \\ \int_{\mathrm{B}_R} \Gamma_{\mathrm{SOR}}(x-y) H_k^s(y)\,\mathrm{d}y + \int_{\mathrm{B}^R} \Gamma_{\mathrm{SOR}}(x-y) H_k^s(y)\,\mathrm{d}y =: I_1 + I_2.\end{aligned} \tag{3.15}$$

From (3.9) and (3.10) we see that H_k^s is point-wise bounded independently on k. Utilizing Lemma 3.1, we thus estimate

$$\begin{aligned}|I_1| &\leq \int_{\mathrm{B}_R} (4\pi)^{-1} |x-y|^{-1}\, \mathrm{e}^{-C_3 |k|^{\frac{1}{2}} |x-y|}\, |H_k(y)|\,\mathrm{d}y \\ &\leq c_9 \int_{\mathrm{B}_R} R^{-1}\, \mathrm{e}^{-C_3 |k|^{\frac{1}{2}} R}\,\mathrm{d}y \\ &= c_{10} R^2\, \mathrm{e}^{-C_3 |k|^{\frac{1}{2}} R} = c_{11} |x|^2\, \mathrm{e}^{-\frac{C_3}{2} |k|^{\frac{1}{2}} |x|},\end{aligned} \tag{3.16}$$

with c_{11} independent on k. By (3.9) and (3.10) also $|H_k^s(x)| \leq c_{12}(1+|x|)^{-3}$, whence

$$\begin{aligned}|I_2| &\leq \int_{\mathrm{B}^R} (4\pi)^{-1} |x-y|^{-1}\, \mathrm{e}^{-C_3 |k|^{\frac{1}{2}} |x-y|}\, |y|^{-3}\,\mathrm{d}y \\ &\leq c_{13} R^{-3} \int_{\mathbb{R}^3} |y|^{-1}\, \mathrm{e}^{-C_3 |k|^{\frac{1}{2}} |y|}\,\mathrm{d}y = c_{14} |k|^{-1} |x|^{-3},\end{aligned} \tag{3.17}$$

with c_{14} independent on k. Combining (3.14)–(3.17), we conclude that

$$|u_k(x)| \leq c_{15} \big(|k|^{-2} |x|^{-3} + |k|^{-1} |x|^2\, \mathrm{e}^{-\frac{C_3}{2} |k|^{\frac{1}{2}} |x|} \big), \tag{3.18}$$

with c_{15} independent on k. This concludes the estimate we need on u_k.

We shall now establish the asymptotic expansion of v. For $|x| > 2\rho$ we have

$$v(x) = w(x) + \sigma(x) = u(x,0) + \sigma(x) = \sum_{k \in \mathbb{Z}} u_k(x) + \sigma(x). \tag{3.19}$$

By (3.8), the coefficient u_0 satisfies the classical Oseen problem

$$\begin{cases} -\Delta u_0 + \nabla \mathfrak{p}_0 - \mathcal{R} \partial_3 u_0 = G_0 & \text{in } \mathbb{R}^3, \\ \operatorname{div} u_0 = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{3.20}$$

Since $(u_0, \mathfrak{p}_0)$ lies in the class (3.4), well-known theory, see for example [4, Section VII.3], then yields

$$\begin{aligned}u_0(x) &= \Gamma_{\mathrm{O}}^{\mathcal{R}}(x) \cdot \int_{\mathbb{R}^3} G_0(y)\,\mathrm{d}y + O\big(\nabla \Gamma_{\mathrm{O}}^{\mathcal{R}}(x)\big) \\ &= \Gamma_{\mathrm{O}}^{\mathcal{R}}(x) \cdot \int_{\mathbb{R}^3} G_0(y)\,\mathrm{d}y + O\big(|x|^{-3/2}\big).\end{aligned} \tag{3.21}$$

We compute

$$\begin{aligned}\int_{\mathbb{R}^3} G_0(y)\,\mathrm{d}y &= \int_{\mathbb{R}^3} \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} Q(t)g(Q(t)^\top y)\,\mathrm{d}t\mathrm{d}y \\ &= \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} Q(t)\cdot\Big[\int_{\mathbb{R}^3} g(y)\,\mathrm{d}y\Big]\mathrm{d}t = \Big[\int_{\mathbb{R}^3} g(y)\,\mathrm{d}y\Big]\cdot \mathrm{e}_3\,\mathrm{e}_3\,.\end{aligned} \tag{3.22}$$

From (3.5), (3.21), and (3.22) we conclude that

$$u_0(x) = \Gamma^{\mathcal{R}}_{\mathrm{O}}(x)\cdot(\mathcal{F}\cdot\mathrm{e}_3)\,\mathrm{e}_3 + O\big(|x|^{-3/2}\big). \tag{3.23}$$

Recalling (3.19) and the fact that $\sigma(x) = O\big(|x|^{-2}\big)$, we now see that (2.2) follows if we can show that

$$v_1(x) := \sum_{k\neq 0} u_k(x) = O\big(|x|^{-3/2}\big). \tag{3.24}$$

Using estimate (3.18), we find

$$|v_1(x)| \leq c_{16}\sum_{k=1}^{\infty} k^{-2}|x|^{-3} + k^{-1}|x|^2\,\mathrm{e}^{-\frac{C_3}{2}k^{\frac12}|x|}\,.$$

Since

$$\Big(\frac{C_3}{2}k^{\frac12}|x|\Big)^5\,\mathrm{e}^{-\frac{C_3}{2}k^{\frac12}|x|} \leq c_{17},$$

with c_{17} independent on $|x|$ and k, we can further estimate

$$|v_1(x)| \leq c_{18}\sum_{k=1}^{\infty}(k^{-2}+k^{-7/2})|x|^{-3} \leq c_{19}|x|^{-3} = O\big(|x|^{-3/2}\big).$$

Consequently, (3.24) holds, and (2.2) follows.

Finally, we must show (2.3). We have $p(x) = \mathfrak{q}(x) + \gamma(x)$ for $|x| > 2\rho$. We deduce directly from (3.3) that $\Delta\mathfrak{q} = \operatorname{div} g$, which implies

$$p(x) = \nabla\Gamma_{\mathrm{L}}(x)\cdot\int_{\mathbb{R}^3} g(y)\,\mathrm{d}y + O\big(|x|^{-3}\big) + \gamma(x).$$

Recalling (3.5) and the definition of γ, we conclude (2.3), and thereby the theorem. □

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