A simple proof of L^q-estimates for the steady-state Oseen and Stokes equations in a rotating frame. Part II: Weak solutions

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This is the second of two papers in which simple proofs of L^q-estimates of solutions to the steady-state three-dimensional Oseen and Stokes equations in a rotating frame of reference are given. In this part, estimates are established in terms of data in homogeneous Sobolev spaces of negative order.

1 Introduction

As in [6], we study the system

$$\begin{cases} -\Delta v + \nabla p - \mathcal{R}\partial_3 v - \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big) = f & \text{in } \mathbb{R}^3, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\mathcal{R} \geq 0$ and $\mathcal{T} > 0$ are dimensionless constants. Here, $v : \mathbb{R}^3 \to \mathbb{R}^3$ and $p : \mathbb{R}^3 \to \mathbb{R}$ represents an Eulerian velocity and pressure term, respectively, of a Navier-Stokes liquid in a frame of reference rotating with angular velocity $\mathcal{T}\,\mathrm{e}_3$ relative to some inertial

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frame. The above system is the classical steady-state whole space Oseen ($\mathcal{R} > 0$) or Stokes ($\mathcal{R} = 0$) problem with the extra term $\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)$, which stems from the rotating frame of reference. Due to the unbounded coefficient $\mathrm{e}_3 \wedge x$, this term can not be treated at a perturbation to the Oseen or Stokes operator.

In [6] we gave an elementary proof of L^q-estimates of solutions (v, p) to (1.1) in terms of data $f \in L^q(\mathbb{R}^3)^3$, $1 < q < \infty$. Such estimates had already been shown in [2] and [1], but with very technical and non-trivial proofs based on an appropriate coupling of the Littlewood-Payley decomposition theorem and multiplier theory. In [9] and [8] the approach of [2] and [1] was used to prove L^q-estimates of weak solutions to (1.1) in terms of data f in the homogeneous Sobolev space $D_0^{-1,q}(\mathbb{R}^3)^3$ of negative order. Our aim in this paper is to extend our approach from [6] and give an elementary proof of these estimates of weak solutions.

Our main theorem reads:

Theorem 1.1. *Let $1 < q < \infty$, $\mathcal{R}_0 > 0$, $0 \leq \mathcal{R} < \mathcal{R}_0$, and $\mathcal{T} > 0$. For any $f \in D_0^{-1,q}(\mathbb{R}^3)^3$ there exists a solution $(v, p) \in D^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ to (1.1) that satisfies*

$$\|\nabla v\|_q + \|p\|_q \leq C_1 |f|_{-1,q}, \tag{1.2}$$

with C_1 independent of $\mathcal{R}_0$, $\mathcal{R}$, and $\mathcal{T}$. Moreover,

$$|\mathcal{R}\partial_3 v|_{-1,q} + \big|\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)\big|_{-1,q} \leq C_2\left(1 + \frac{1}{\mathcal{T}^2}\right)|f|_{-1,q}, \tag{1.3}$$

with $C_2 = C_2(\mathcal{R}_0)$. Furthermore, if $(\tilde{v}, \tilde{p}) \in D^{1,r}(\mathbb{R}^3)^3 \times L^r(\mathbb{R}^3)$, $1 < r < \infty$, is another solution to (1.1), then

$$\tilde{v} = v + \alpha\, \mathrm{e}_3 \tag{1.4}$$

for some $\alpha \in \mathbb{R}$.

Remark 1.2. In [8, Theorem 2.1 and Proposition 3.2] it is stated that a solution $(v, p) \in D^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ to (1.1) with $f \in D_0^{-1,q}(\mathbb{R}^3)^3$ satisfies

$$|\mathcal{R}\partial_3 v|_{-1,q} + \big|\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)\big|_{-1,q} \leq C_3 |f|_{-1,q}$$

with C_3 *independent* of $\mathcal{T}$. However, going through the proofs in [8], one finds out that this is not the case, and that the constant C_3 does, in fact, depend on $\mathcal{T}$ in the way shown in (1.3). More specifically, in [8, Appendix 2] the constant in the estimate of the Fourier multiplier clearly depends on $\mathcal{T}$; this estimate is later used in the proof of [8, Proposition 3.2].

Before giving a proof of Theorem 1.1, we first recall some standard notation. By $L^q(\mathbb{R}^3)$ we denote the usual Lebesgue space with norm $\|\cdot\|_q$. For $m \in \mathbb{N}$ and $1 < q < \infty$, we use $D^{m,q}(\mathbb{R}^3)$ to denote the homogeneous Sobolev space with semi-norm $|\cdot|_{m,q}$, *i.e*,

$$|v|_{m,q} := \left(\sum_{|\alpha|=m} \int_{\mathbb{R}^3} |\partial^\alpha v(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}},\ D^{m,q} := \{v \in L^1_{loc}(\mathbb{R}^3) \mid |v|_{m,q} < \infty\}.$$

We put $D_0^{m,q}(\mathbb{R}^3) := \overline{C_0^\infty(\mathbb{R}^3)}^{|\cdot|_{m,q}}$. We introduce homogeneous Sobolev spaces of negative order as the dual spaces $D_0^{-m,q}(\mathbb{R}^3) := \big(D_0^{m,q'}(\mathbb{R}^3)\big)'$, and denote their norms by $|\cdot|_{-m,q}$. Here, and throughout the paper, $q' := q/(q-1)$ denotes the Hölder conjugate of q. For functions $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, we let $\operatorname{div} u(x,t) := \operatorname{div}_x u(x,t)$, $\Delta u(x,t) := \Delta_x u(x,t)$ etc., that is, unless otherwise indicated, differential operators act in the spatial variable x only. We use $\mathcal{F} f = \widehat{f}$ to denote the Fourier transformation. We put $\mathrm{B}_m := \{x \in \mathbb{R}^3 \mid |x| < m\}$. Finally, note that constants in capital letters in the proofs and theorems are global, while constants in small letters are local to the proof in which they appear.

2 Proof of Main Theorem

As in [6], we make use of an idea going back to [5] and transform solutions to (1.1) into time-periodic solutions to the classical time-dependent Oseen and Stokes problem. For this purpose, we introduce the rotation matrix corresponding to the angular velocity $\mathcal{T}\,\mathrm{e}_3$:

$$Q(t) := \begin{pmatrix} \cos(\mathcal{T}t) & -\sin(\mathcal{T}t) & 0 \\ \sin(\mathcal{T}t) & \cos(\mathcal{T}t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We split the proof into several lemmas. We begin to recall the following result; see [4] or [11].

Lemma 2.1. *Let $\mathcal{R} \geq 0$ and $\mathcal{T} > 0$. For any $h \in C_0^\infty(\mathbb{R}^3)^{3\times 3}$ there is a solution*

$$(v,p) \in D^{1,2}(\mathbb{R}^3)^3 \cap L^6(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3) \tag{2.1}$$

to

$$\begin{cases} -\Delta v + \nabla p - \mathcal{R}\partial_3 v - \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big) = \operatorname{div} h & \text{in } \mathbb{R}^3, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^3 \end{cases} \tag{2.2}$$

that satisfies

$$\|\nabla v\|_2 + \|p\|_2 \leq C_4 \|h\|_2, \tag{2.3}$$

with C_4 independent of $\mathcal{R}$ and $\mathcal{T}$. Moreover

$$(v,p) \in \cap_{m=1}^\infty D^{m+1,2}(\mathbb{R}^3)^3 \times D^{m,2}(\mathbb{R}^3). \tag{2.4}$$

In the next lemma we establish suitable L^q-estimates of the solution introduced above.

Lemma 2.2. *Let $\mathcal{R} \geq 0$ and $\mathcal{T} > 0$. Let $1 < q < \infty$ and $h \in C_0^\infty(\mathbb{R}^3)^{3\times 3}$. The solution (v,p) from Lemma 2.1 satisfies*

$$\|\nabla v\|_q + \|p\|_q \leq C_5 \|h\|_q, \tag{2.5}$$

with C_5 independent of $\mathcal{R}$ and $\mathcal{T}$.

Proof. Assume first that $q > 2$. Let $T > 0$. For $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ put

$$u(x,t) := Q(t)v\big(Q(t)^\top x - \mathcal{R}t\,\mathrm{e}_3\big), \quad \mathfrak{p}(x,t) := p\big(Q(t)^\top x - \mathcal{R}t\,\mathrm{e}_3\big),$$
$$H(x,t) := Q(t)h\big(Q(t)^\top x - \mathcal{R}t\,\mathrm{e}_3\big)Q(t)^\top.$$

Then

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} = \operatorname{div} H & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ u(x,0) = v(x) & \text{in } \mathbb{R}^3. \end{cases} \tag{2.6}$$

By well-known theory of the time-dependent Stokes equations, see for example [10, Sec. 5, Theorem 6], the Cauchy problem

$$\begin{cases} \partial_t u_1 - \Delta u_1 = \operatorname{div} H - \nabla \mathfrak{p} & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u_1 = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ \lim_{t\to 0^+} \|u_1(\cdot,t)\|_6 = 0 \end{cases}$$

has a solution with $u_1 \in L^r\big(\mathbb{R}^3 \times (0,T)\big)^3$ for all $1 < r < \infty$, and

$$\|\nabla u_1\|_{L^r(\mathbb{R}^3\times(0,T))} \le c_1 \|H\|_{L^r(\mathbb{R}^3\times(0,T))},$$

with c_1 independent of T. Put

$$u_2(x,t) := (4\pi t)^{-3/2} \int_{\mathbb{R}^3} \mathrm{e}^{-|x-y|^2/4t}\, v(y)\,\mathrm{d}y. \tag{2.7}$$

An elementary calculation shows that $u_2 \in L^6\big(\mathbb{R}^3 \times (0,T)\big)$, $\partial_t u_2, \nabla u_2, \nabla^2 u_2 \in L^6_{loc}\big(\mathbb{R}^3 \times (0,T)\big)$, and that u_2 solves

$$\begin{cases} \partial_t u_2 - \Delta u_2 = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u_2 = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ \lim_{t\to 0^+} \|u_2(\cdot,t) - v(\cdot)\|_6 = 0. \end{cases}$$

Taking derivatives on both sides in (2.7) and applying Young's inequality, we obtain

$$\|\nabla u_2(\cdot,t)\|_{L^q(\mathbb{R}^3)} \le c_2\, t^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{q})}\, \|\nabla v\|_2,$$

with c_2 independent of T. We claim that $u = u_1 + u_2$ in $\mathbb{R}^3 \times (0,T)$. This follows from the fact that $u_1 + u_2$ satisfies (2.6), combined with a uniqueness argument, for example [7, Lemma 3.6]. We can now estimate

$$\begin{aligned} (T-1)\|\nabla v\|_q^q &= \int_1^T \int_{\mathbb{R}^3} |\nabla u(x,t)|^q \,\mathrm{d}x\mathrm{d}t \\ &\le c_3\Big(\|\nabla u_1\|^q_{L^q(\mathbb{R}^3\times(0,T))} + \int_1^T \|\nabla u_2(\cdot,t)\|_q^q \,\mathrm{d}t \Big) \\ &\le c_4\Big(\|H\|^q_{L^q(\mathbb{R}^3\times(0,T))} + \int_1^T t^{-\frac{3q}{2}(\frac{1}{2}-\frac{1}{q})} \|\nabla v\|_2^q \,\mathrm{d}t \Big) \\ &\le c_5\big(T\|h\|_q^q + (T^{-\frac{3q}{2}(\frac{1}{2}-\frac{1}{q})+1} - 1)\|\nabla v\|_2^q \big), \end{aligned}$$

with c_5 independent of T, and also of $\mathcal{R}$ and $\mathcal{T}$. Dividing both sides with T, and subsequently letting $T \to \infty$, we conclude, recall $q > 2$ by assumption, that $\|\nabla v\|_q \leq c_5\|h\|_q$. Finally, we deduce directly from (2.2), applying div on both sides in $(1.1)_1$, that $\Delta p = \operatorname{div}\operatorname{div} h$, which implies that $\|p\|_q \leq c_6\|h\|_q$, with c_6 independent of $\mathcal{R}$ and $\mathcal{T}$. Hence (2.5) follows in the case $q > 2$.

The case $q = 2$ is included in Lemma 2.1. Consider now $1 < q < 2$. In this case we will establish (2.5) by a duality argument. Consider for this purpose $\varphi \in C_0^\infty(\mathbb{R}^3)^{3\times 3}$. For notational purposes, we put

$$Lv := -\Delta v - \mathcal{R}\partial_3 v - \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big), \tag{2.8}$$

$$L^* v := -\Delta v + \mathcal{R}\partial_3 v + \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big). \tag{2.9}$$

As in Lemma 2.1, one can show the existence of a solution (ψ, η), in the class (2.1) and (2.4), to the adjoint problem

$$\begin{cases} L^*\psi + \nabla\eta = \operatorname{div}\varphi & \text{in } \mathbb{R}^3, \\ \operatorname{div}\psi = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.10}$$

By arguments as above, one can also show that

$$\forall r \in (2,\infty): \ \ \|\nabla\psi\|_r + \|\eta\|_r \leq c_7\|\varphi\|_r, \tag{2.11}$$

with c_7 independent of $\mathcal{R}$ and $\mathcal{T}$. The summability properties of (v, p) and (ψ, η), ensured by Lemma 2.1 and supplemented by [6, Theorem 1], enables us to calculate

$$\begin{aligned} |\int_{\mathbb{R}^3} \nabla v : \varphi \,\mathrm{d}x| &= |\int_{\mathbb{R}^3} v \cdot \operatorname{div}\varphi \,\mathrm{d}x| = |\int_{\mathbb{R}^3} v \cdot L^*\psi \,\mathrm{d}x| \\ &= |\int_{\mathbb{R}^3} Lv \cdot \psi \,\mathrm{d}x| = |\int_{\mathbb{R}^3} \operatorname{div} h \cdot \psi \,\mathrm{d}x| = |\int_{\mathbb{R}^3} h : \nabla\psi \,\mathrm{d}x| \\ &\leq \|h\|_q\|\nabla\psi\|_{q'} \leq c_7\|h\|_q\|\varphi\|_{q'}, \end{aligned} \tag{2.12}$$

where the third equality follows by partial integration in the same manner as in [6, Proof of Lemma 2.3], and last estimates from (2.11) since $2 < q' < \infty$. Having established (2.12) for arbitrary φ, we conclude that $\|\nabla v\|_q \leq c_7\|h\|_q$. Finally, the estimate $\|p\|_q \leq c_8\|h\|_q$ follows simply from the fact that $\Delta p = \operatorname{div}\operatorname{div} h$. We have thus established (2.5) also in the case $1 < q \leq 2$. This concludes the lemma. □

In the next lemma we establish estimates of the lower order terms on the left-hand side of (1.1).

Lemma 2.3. *Let $\mathcal{R} > 0$ and $\mathcal{T} > 0$. Let $1 < q < \infty$ and $h \in C_0^\infty(\mathbb{R}^3)^{3\times 3}$. The solution (v, p) from Lemma 2.1 satisfies*

$$|\mathcal{R}\partial_3 v|_{-1,q} + |\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)|_{-1,q} \leq C_6\Big(1 + \frac{1}{\mathcal{T}^2}\Big)\|h\|_q, \tag{2.13}$$

with $C_6 = C_6(\mathcal{R}_0)$.

Proof. Consider first $1 < q \le 2$. For $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ put

$$\begin{aligned} &u(x,t) := Q(t)v\big(Q(t)^\top x\big), \quad \mathfrak{p}(x,t) := p\big(Q(t)^\top x\big),\\ &H(x,t) := Q(t)h\big(Q(t)^\top x\big)Q(t)^\top. \end{aligned}$$

Note that u, $\mathfrak{p}$, and H are smooth and $\frac{2\pi}{\mathcal{T}}$-periodic in the t variable. We can therefore expand these fields in their Fourier-series. More precisely, we have

$$\begin{aligned} &u(x,t) = \sum_{k\in\mathbb{Z}} u_k(x)\,\mathrm{e}^{i\mathcal{T}kt}, \quad \mathfrak{p}(x,t) = \sum_{k\in\mathbb{Z}} \mathfrak{p}_k(x)\,\mathrm{e}^{i\mathcal{T}kt},\\ &H(x,t) = \sum_{k\in\mathbb{Z}} H_k(x)\,\mathrm{e}^{i\mathcal{T}kt}, \end{aligned}$$

with

$$\begin{aligned} &u_k(x) := \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} u(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t, \quad \mathfrak{p}_k(x) := \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} \mathfrak{p}(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t,\\ &H_k(x) := \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} H(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t. \end{aligned}$$

As one may easily verify,

$$\begin{cases} \partial_t u - \Delta u + \nabla\mathfrak{p} - \mathcal{R}\partial_3 u = \operatorname{div} H & \text{in } \mathbb{R}^3\times\mathbb{R},\\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}. \end{cases} \tag{2.14}$$

Replacing in (2.14) u, $\mathfrak{p}$, and H with their respective Fourier series, we find that each Fourier coefficient satisfies

$$\begin{cases} i\mathcal{T}ku_k - \Delta u_k + \nabla\mathfrak{p}_k - \mathcal{R}\partial_3 u_k = \operatorname{div} H_k & \text{in } \mathbb{R}^3,\\ \operatorname{div} u_k = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.15}$$

In the case $k = 0$, (2.15) reduces to the classical Oseen system. By well-known theory, see for example [3, Theorem VII.4.2],

$$\|\nabla u_0\|_q + \mathcal{R}|\partial_3 u_0|_{-1,q} \le c_1\|H_0\|_q \le c_2\|h\|_q, \tag{2.16}$$

with c_2 independent of $\mathcal{R}$ and $\mathcal{T}$. Consider now $k \neq 0$. By Minkowski's integral inequality and Lemma 2.2, we find that

$$\|\nabla u_k\|_q \le \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} \left(\int_{\mathbb{R}^3} |\nabla u(x,t)|^q\,\mathrm{d}x\right)^{1/q}\mathrm{d}t = \|\nabla v\|_q \le C_5\|h\|_q,$$

and similarly $\|\mathfrak{p}_k\|_q \le C_5\|h\|_q$. We can thus conclude from (2.15) that

$$|\mathcal{T}k||u_k|_{-1,q} \le \|\nabla u_k\|_q + \|\mathfrak{p}_k\|_q + \mathcal{R}|\partial_3 u_k|_{-1,q} \le c_3\|h\|_q + \mathcal{R}|\partial_3 u_k|_{-1,q}, \tag{2.17}$$

with c_3 independent of $\mathcal{R}$ and $\mathcal{T}$. A simple interpolation argument yields

$$|\partial_3 u_k|_{-1,q} \leq c_4(\varepsilon |u_k|_{-1,q} + \varepsilon^{-1}\|\nabla u_k\|_q) \tag{2.18}$$

for all $\varepsilon > 0$. We now choose $\varepsilon = |\mathcal{T}k|/(2\mathcal{R}c_4)$ in (2.18) and apply the resulting estimate in (2.17). It follows that

$$|u_k|_{-1,q} \leq c_5 \frac{1}{|\mathcal{T}k|}\Big(1 + \frac{\mathcal{R}^2}{|\mathcal{T}k|}\Big)\|h\|_q \quad (k \neq 0), \tag{2.19}$$

with c_5 independent of $\mathcal{R}$ and $\mathcal{T}$. We observe at this point that $v(x) = u(x,0) = \sum_{k\in\mathbb{Z}} u_k(x)$, and put

$$v_1 := v - u_0. \tag{2.20}$$

We then define

$$U(x,t) := Q(t)v_1\big(Q(t)^\top x\big) = u(x,t) - u_0 = \sum_{k\neq 0} u_k(x)\, \mathrm{e}^{i\mathcal{T}kt}.$$

The first equality above follows from the fact that $Q(t)u_0(Q(t)^\top x) = u_0(x)$ for all $t \in \mathbb{R}$, which one easily verifies directly from the definition of u_0. Now let $\varphi \in C_0^\infty(\mathbb{R}^3)^3$ and put $\Phi(x,t) := Q(t)\varphi(Q(t)^\top x)$. Since Φ is smooth and $2\pi/\mathcal{T}$-periodic in t, we can write Φ in terms of its Fourier-series:

$$\Phi(x,t) = \sum_{k\in\mathbb{Z}} \Phi_k(x)\, \mathrm{e}^{i\mathcal{T}kt}, \quad \Phi_k(x) := \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} \Phi(x,t)\, \mathrm{e}^{-i\mathcal{T}kt}\, \mathrm{d}t.$$

We now compute, using Parseval's identity and (2.19),

$$\begin{aligned}
|\int_{\mathbb{R}^3} v_1(x)\cdot\varphi(x)\,\mathrm{d}x| &= |\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}}\int_{\mathbb{R}^3} U(x,t)\cdot\Phi(x,t)\,\mathrm{d}x\mathrm{d}t| \\
&= |\int_{\mathbb{R}^3}\sum_{k\neq 0} u_k(x)\cdot\Phi_k(x)\,\mathrm{d}x| \\
&\leq \sum_{k\neq 0} |u_k|_{-1,q}\|\nabla\Phi_k\|_{q'} \\
&\leq c_5\Big(1+\frac{\mathcal{R}^2}{\mathcal{T}}\Big)\|h\|_q \sum_{k\neq 0}\frac{1}{|\mathcal{T}k|}\|\nabla\Phi_k\|_{q'} \\
&\leq c_5\Big(1+\frac{\mathcal{R}^2}{\mathcal{T}}\Big)\frac{1}{\mathcal{T}}\|h\|_q \Big(\sum_{k\neq 0}\frac{1}{|k|^q}\Big)^{\frac{1}{q}} \Big(\sum_{k\neq 0}\|\nabla\Phi_k\|_{q'}^{q'}\Big)^{\frac{1}{q'}}.
\end{aligned}$$

Recalling that $1 < q \leq 2$, we employ the Hausdorff-Young inequality to estimate

$$\Big(\sum_{k\neq 0}\|\nabla\Phi_k\|_{q'}^{q'}\Big)^{\frac{1}{q'}} \leq \Big(\int_{\mathbb{R}^3}\Big[\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} |\nabla\Phi(x,t)|^q\,\mathrm{d}t\Big]^{\frac{q'}{q}}\,\mathrm{d}x\Big)^{\frac{1}{q'}}.$$

Applying Minkowski's integral inequality to the right-hand side above, we obtain

$$\Big(\sum_{k\neq 0}\|\nabla\Phi_k\|_{q'}^{q'}\Big)^{\frac{1}{q'}} \leq \Big(\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}}\Big[\int_{\mathbb{R}^3}|\nabla\Phi(x,t)|^{q'}\,\mathrm{d}x\Big]^{\frac{q}{q'}}\,\mathrm{d}t\Big)^{\frac{1}{q}} = \|\nabla\varphi\|_{q'}.$$

We thus conclude that

$$|\int_{\mathbb{R}^3} v_1(x)\cdot\varphi(x)\,\mathrm{d}x| \leq c_6\Big(1+\frac{\mathcal{R}^2}{\mathcal{T}}\Big)\frac{1}{\mathcal{T}}\|h\|_q\|\nabla\varphi\|_{q'},$$

and consequently, since φ is arbitrary,

$$|v_1|_{-1,q} \leq c_7\Big(1+\frac{\mathcal{R}^2}{\mathcal{T}}\Big)\frac{1}{\mathcal{T}}\|h\|_q, \tag{2.21}$$

with c_7 independent of $\mathcal{R}$ and $\mathcal{T}$. By the same interpolation argument as in (2.18), we estimate

$$|\partial_3 v_1|_{-1,q} \leq c_8(|v_1|_{-1,q}+\|\nabla v_1\|_q). \tag{2.22}$$

Combining now (2.22), (2.21), (2.20), (2.16), and (2.5), we obtain

$$\forall q\in(1,2]:\ |\mathcal{R}\partial_3 v|_{-1,q} \leq c_9\Big(1+\frac{1}{\mathcal{T}^2}\Big)\|h\|_q, \tag{2.23}$$

with $c_9 = c_9(\mathcal{R}_0)$.

Consider now $2<q<\infty$. Let $\varphi\in C_0^\infty(\mathbb{R}^3)^3$. Recall (2.8) and (2.9). By [6, Lemma 2.1] there is a solution $(\psi,\eta)\in D^{1,2}(\mathbb{R}^3)^3\cap L^6(\mathbb{R}^3)^3\times L^6(\mathbb{R}^3)$ to

$$\begin{cases} L^*\psi+\nabla\eta=\varphi & \text{in } \mathbb{R}^3,\\ \operatorname{div}\psi=0 & \text{in } \mathbb{R}^3\end{cases} \tag{2.24}$$

satisfying (2.4). Moreover, since Δ commutes with L^*, $(\Delta\psi,\Delta\eta)$ satisfies

$$\begin{cases} L^*\Delta\psi+\nabla\Delta\eta=\operatorname{div}\nabla\varphi & \text{in } \mathbb{R}^3,\\ \operatorname{div}\Delta\psi=0 & \text{in } \mathbb{R}^3.\end{cases} \tag{2.25}$$

Repeating the argument from above leading to (2.23), we also obtain

$$\forall r\in(1,2]:\ |\mathcal{R}\partial_3\Delta\psi|_{-1,r} \leq c_{10}\Big(1+\frac{1}{\mathcal{T}^2}\Big)\|\nabla\varphi\|_r, \tag{2.26}$$

with $c_{10}=c_{10}(\mathcal{R}_0)$. As in (2.12), we compute

$$\int_{\mathbb{R}^3}\partial_3 v\cdot\varphi\,\mathrm{d}x = \int_{\mathbb{R}^3}\partial_3 v\cdot L^*\psi\,\mathrm{d}x = -\int_{\mathbb{R}^3} Lv\cdot\partial_3\psi\,\mathrm{d}x = -\int_{\mathbb{R}^3}\operatorname{div}h\cdot\partial_3\psi\,\mathrm{d}x.$$

Put $\Theta_i := \mathcal{F}^{-1}\big[\frac{\xi_j}{|\xi|^2}\widehat{h_{ij}}(\xi)\big]$, $i = 1,2,3$[1]. Then $\Theta \in L^r(\mathbb{R}^3)^3$ for all $r \in (3/2,\infty)$, $\|\nabla\Theta\|_q \leq c_{11}\|h\|_q$, and $\Delta\Theta = \operatorname{div} h$. It follows that

$$|\int_{\mathbb{R}^3} \partial_3 v \cdot \varphi \,\mathrm{d}x| = |\int_{\mathbb{R}^3} \Theta \cdot \partial_3\Delta\psi \,\mathrm{d}x| \leq \|\nabla\Theta\|_q |\partial_3\Delta\psi|_{-1,q'} \leq c_{12}\|h\|_q |\partial_3\Delta\psi|_{-1,q'}.$$

Since $q' \in (1,2)$, we deduce by (2.26) that

$$|\int_{\mathbb{R}^3} \partial_3 v \cdot \varphi \,\mathrm{d}x| \leq c_{13}\Big(1 + \frac{1}{\mathcal{T}^2}\Big)\|h\|_q \|\nabla\varphi\|_{q'}.$$

We conclude $|\mathcal{R}\partial_3 v|_{-1,q} \leq c_{14}(1+\mathcal{T}^{-2})\|h\|_q$, with $c_{14} = c_{14}(\mathcal{R}_0)$.

Since $\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big) = \Delta v - \nabla p + \mathcal{R}\partial_3 v + \operatorname{div} h$, the estimates already obtained in (2.5) together with the estimate for $\mathcal{R}\partial_3 v$ above imply

$$|\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)|_{-1,q} \leq c_{15}\Big(1 + \frac{1}{\mathcal{T}^2}\Big)\|h\|_q,$$

with $c_{15} = c_{15}(\mathcal{R}_0)$. We have thus established (2.13) completely. □

We can now finalize the proof of the main theorem.

Proof of Theorem 1.1. Except for the uniqueness statement, Lemma 2.1–2.3 establish the theorem in the case $f = \operatorname{div} h$ for some $h \in C_0^\infty(\mathbb{R}^3)^{3\times 3}$. It remains to extend to the general case $f \in D_0^{-1,q}(\mathbb{R}^3)^3$. Consider therefore $f \in D_0^{-1,q}(\mathbb{R}^3)^3$. Choose a sequence $\{h_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}^3)^{3\times 3}$ with $\lim_{n\to\infty} \operatorname{div} h_n = f$ in $D_0^{-1,q}(\mathbb{R}^3)^3$. Let (v_n, p_n) be the solution from Lemma 2.1 corresponding to the right-hand side $\operatorname{div} h_n$. Then choose $\kappa_n \in \mathbb{R}^3$ such that $0 = \int_{\mathrm{B}_1} v_n - \kappa_n \,\mathrm{d}x$. From Lemma 2.2 and Poincaré's inequality, it follows that $\{(v_n - \kappa_n, p_n)\}_{n=1}^\infty$ is a Cauchy sequence in the Banach space

$$X_m := \{(v,p) \in L^1_{loc}(\mathbb{R}^3)^3 \times L^1_{loc}(\mathbb{R}^3) \mid \|(v,p)\|_{X_m} < \infty\},$$
$$\|(v,p)\|_{X_m} := \|\nabla v\|_q + \|p\|_q + \|v\|_{L^q(\mathrm{B}_m)}$$

for all $m \in \mathbb{N}$. Consequently, there is an element $(v,p) \in \cap_{m\in\mathbb{N}} X_m$ with the property that $\lim_{n\to\infty}(v_n - \kappa_n, p_n) = (v,p)$ in X_m for all $m \in \mathbb{N}$. Recall (2.8). It follows that $\lim_{n\to\infty}[L(v_n - \kappa_n) + \nabla p_n] = Lv + \nabla p$ in $\mathcal{D}'(\mathbb{R}^3)^3$. By construction, $\lim_{n\to\infty}[Lv_n + \nabla p_n] = f$ in $D_0^{-1,q}(\mathbb{R}^3)^3$. We thus deduce that $\lim_{n\to\infty} L\kappa_n = f - [Lv + \nabla p]$. Consequently, $f - [Lv + \nabla p] = L\kappa$ for some $\kappa \in \mathbb{R}^3$. It follows that $(v+\kappa, p) \in D^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ solves (1.1). Moreover, since (v_n, p_n) satisfies (1.2) and (1.3) for all $n \in \mathbb{N}$, so does $(v+\kappa, p)$. This concludes the first part of the theorem.

To prove the statement of uniqueness, assume that $(\tilde{v}, \tilde{p}) \in D^{1,r}(\mathbb{R}^3)^3 \times L^r(\mathbb{R}^3)$ is another solution to (1.1). Put $w := v - \tilde{v}$ and $\mathfrak{q} := p - \tilde{p}$. It immediately follows that $\Delta\mathfrak{q} = 0$, which, since $\mathfrak{q} \in L^q(\mathbb{R}^3) + L^r(\mathbb{R}^3)$, implies that $\mathfrak{q} = 0$. Now put $U(x,t) :=$

[1] Following the summation convention, we implicitly sum over repeated indices.

$Q(t)w(Q(t)^\top x)$ for $(x,t)\in\mathbb{R}^3\times\mathbb{R}$. Since U is smooth and $2\pi/\mathcal{T}$-periodic in t, we can write U in terms of its Fourier-series

$$U(x,t)=\sum_{k\in\mathbb{Z}}U_k(x)\,\mathrm{e}^{i\mathcal{T}kt},\ U_k(x):=\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}}U(x,t)\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t.$$

As one may easily verify, U_k satisfies $i\mathcal{T}kU_k-\Delta U_k-\mathcal{R}\partial_3 U_k=0$ in $\mathscr{S}'(\mathbb{R}^3)^3$. Thus, Fourier transformation yields $\big(i(\mathcal{T}k-\mathcal{R}\xi_3)+|\xi|^2\big)\widehat{U_k}=0$. It follows that $U_k=0$ for all $k\neq 0$. Moreover, since $\big(-i\mathcal{R}\xi_3+|\xi|^2\big)\widehat{U_0}=0$, it follows that $\operatorname{supp}(\widehat{U_0})\subset\{0\}$. Consequently, since $U_0\in D^{1,q}(\mathbb{R}^3)^3+D^{1,r}(\mathbb{R}^3)^3$, $U_0=b$ for some $b\in\mathbb{R}^3$. It follows that $U(x,t)=b=Q(t)w(Q(t)^\top x)$ for all $t\in\mathbb{R}$ and $x\in\mathbb{R}^3$. Thus, $Q(t)^\top b$ is t-independent, and so $b=\alpha\,\mathrm{e}_3$ for some $\alpha\in\mathbb{R}$. We conclude that $w(x)=U_0(x)=\alpha\,\mathrm{e}_3$. □

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