Term-wise estimates for the norm inflation solutions to the Navier-Stokes equations

Okihiro Sawada*

Dedicated to Professor Takaaki Nishida on his 70th birthday.

Abstract

The ill-posedness of the Navier-Stokes equations in the critical space is concerned. It is shown that the equicontinuity is not equipped within the biggest class of mild solutions. The proof is based on the norm inflation argument by Bourgain. The term-wise estimates for the successive approximation of the mild solutions and its convergence or divergence are established.

1 Introduction

1.1 Problem

We consider the nonstationary incompressible viscous flow of the ideal fluid in the whole space $\mathbb{R}^n$ (no boundary); $n \in \mathbb{N}$ and $n \geq 2$. This is mathematically described as the Cauchy problem of the Navier-Stokes equations:

$$\text{(NS)} \qquad \begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}^n. \end{cases}$$

This Cauchy problem is called (NS) in here. We define the notations of derivatives as follows: $u_t := \partial_t u := \partial u/\partial t$, $\partial_j := \partial/\partial x_j$ for $j = 1, \dots, n$, $\nabla := (\partial_1, \dots, \partial_n)$, $\Delta := \sum_{j=1}^n \partial_j^2$. Here, for vectors $a = (a^1, \dots, a^n)$ and $b = (b^1, \dots, b^n)$, $a \cdot b$ or (a, b) denotes $\sum_{j=1}^n a^j b^j$. The velocity $u = (u^1, \dots, u^n) = (u^1(x,t), \dots, u^n(x,t))$ and the pressure $p = p(x,t)$ are unknown functions. The problem is to determine the solution (u, p) to (NS) uniquely from the given initial velocity u_0 in some function space. It is natural to impose the compatibility condition on u_0, that is, $\nabla \cdot u_0 = 0$ holds for all $x \in \mathbb{R}^3$.

The mathematical analysis of mechanics of viscous fluid has a long history. Historically, (NS) is derived from the conscientious observation by Navier and Stokes in the nineteen century. The mathematical studying of (NS) was started by Oseen [62] who established

*Department of Mathematical and Design Engineering, Gifu University, Yanagido 1-1, 501-1193, Japan
E-mail address: okihiro@gifu-u.ac.jp

the time-local existence of a classical solution to (NS) with a regular initial datum. One of the most important results on (NS) is obtained by Leray [52, 53] in 1930's. In [52] Leray showed that for $n = 2$ there exists a unique time-global classical solution, when the initial velocity u_0 is square-integrable with $\nabla \cdot u_0 = 0$ in the distribution sense. He also constructed the time-global weak solutions for $n = 3$. His proof is based on the following two methods, the Galerkin method and the Energy estimate. See [52, 53] or, e.g. [30, 50, 55, 73] for the details. Briefly, the Galerkin method is to use the basis of L^2 and determine the coefficient of Fourier expansion. It is a famous open problem whether one can obtain the uniqueness and smoothness of Leray's weak solutions, that is, (NS) admits a time-global unique solution in $L^2(\mathbb{R}^3)$. In this note our aim is different to this, so we do not penetrate into its detail.

By the Duhamel principle we derive the integral equation from (NS)

$$\text{(INT)} \qquad u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}(u(\tau), \nabla)u(\tau)d\tau.$$

See e.g. Fujita and Kato [20, 38]. This notion was also introduced by Browder in [10] to study on the equations of parabolic type. We call the solution of (INT) a **mild solution**. This derivation is understood via the following abstract equation of value in a Banach space:

$$\text{(ABS)} \qquad u' = \Delta u - \mathbb{P}(u, \nabla)u, \quad u(0) = u_0.$$

Here, we denote the heat semigroup $e^{t\Delta} := G_t *$, the Gauss kernel $G_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$, convolution with respect to spatial variables $f * g(x) := \int_{\mathbb{R}^n} f(x-z)g(z)dz$, the Helmholtz projection $\mathbb{P} := (\delta_{ij} + R_iR_j)_{i,j=1,\dots,n}$, Kronecker's delta $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$, the Riesz transform $R_i := \partial_i(-\Delta)^{-1/2} := \mathcal{F}^{-1}\frac{\sqrt{-1}\xi_i}{|\xi|}\mathcal{F}$. The Fourier transform is defined by

$$\mathcal{F}f(\xi) := \hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\sqrt{-1}x\cdot\xi} f(x)dx,$$

and $\mathcal{F}^{-1}$ is its inverse;

$$\mathcal{F}^{-1}f(x) := \check{f}(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{\sqrt{-1}x\cdot\xi} f(\xi)d\xi.$$

If u is a mild solution, then it is expected that (u, p) satisfies (NS). For instance, (u, p) is expected to be a classical solution, i.e. u is in C^1 in t and C^2 in x. This formal equivalency between (INT) and (NS) can be justified when u has a sufficient regularity, provided if p is under the suitable assumption, for example,

$$p = \sum_{i,j=1}^{n} R_iR_ju^iu^j. \tag{1.1}$$

We rather discuss (INT) and mild solutions than (NS) and classical solutions. Mild solutions are usually constructed by the limit of the successive approximation

$$u_1(t) := e^{t\Delta}u_0 \qquad \text{and} \qquad u_{j+1}(t) := u_1 - \mathcal{B}(u_j) \quad \text{for} \quad j \in \mathbb{N} \tag{1.2}$$

in $C([0,T];X)$ for $u_0 \in X$ with a Banach space X, where

$$\mathcal{B}(u,v) := \int_0^t e^{(t-\tau)\Delta}\mathbb{P}(u(\tau),\nabla)v(\tau)d\tau \qquad \text{and} \qquad \mathcal{B}(u) := \mathcal{B}(u,u). \tag{1.3}$$

In this paper we discuss on the time-local solvability, time-global solvability for small data, uniqueness and ill-posedness of the Navier-Stokes equations in the whole space with initial data in the critical spaces, due to the analysis of mild solutions. We will refer to the definition of function spaces and their properties, in particular, the important facts concerning with the mild solutions. This paper is contributed to understand of the positive results by Koch and Tataru [42], and the negative one by Bourgain and Pavlovic [9]. In [9] they showed that the mild solution does not equip the equicontinuity in $C(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))$. We now state the main results of this paper:

Theorem 1.1. *For $T>0$ there exists a u_0 such that $\|u_j(T)\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)}$ does not converge.*

1.2 Motivation

In this subsection we refer to the motivation of recent works on (NS) in Besov spaces. The definition of function spaces will be denoted in section 2. We, in here, mention several known results on the solvability and uniqueness for the Navier-Stokes equations in several function spaces.

To solve (NS) uniquely and time-globally in 3-dimension, one may consider the following steps: firstly the smooth time-local solution is constructed, and secondly the solution is extended uniquely and time-globally. Along this strategy, Kato and Fujita [38] introduced the notion of mild solutions, and proved that (NS) admits a unique time-local smooth solution, when $u_0 \in H^{\frac{n}{2}-1}(\mathbb{R}^n)$. They actually discussed that the approximation sequence $\{u_j\}$ converges to the mild solution in the class $C([0,T];H^{\frac{n}{2}-1})$. Although they established this results in smooth bounded domains with non-slip boundary conditions originally, the proof can be applied to the whole space problem without any difficulty. The details of the proof are shown in [20].

Since the results of Kato and Fujita are splendid, there are a lot of papers of the applications of their method in many directions. Some researcher wanted to eliminate the smoothness on the initial data, since the smoothness of the solutions is automatically obtained by the usual smoothing effect of solutions to equations of parabolic type. For this purpose Kato [37] (in the whole space) and Giga and Miyakawa [27] (in a bounded domain) studied the properties of the heat semigroup in the Lebesgue spaces (using $L^p - L^q$

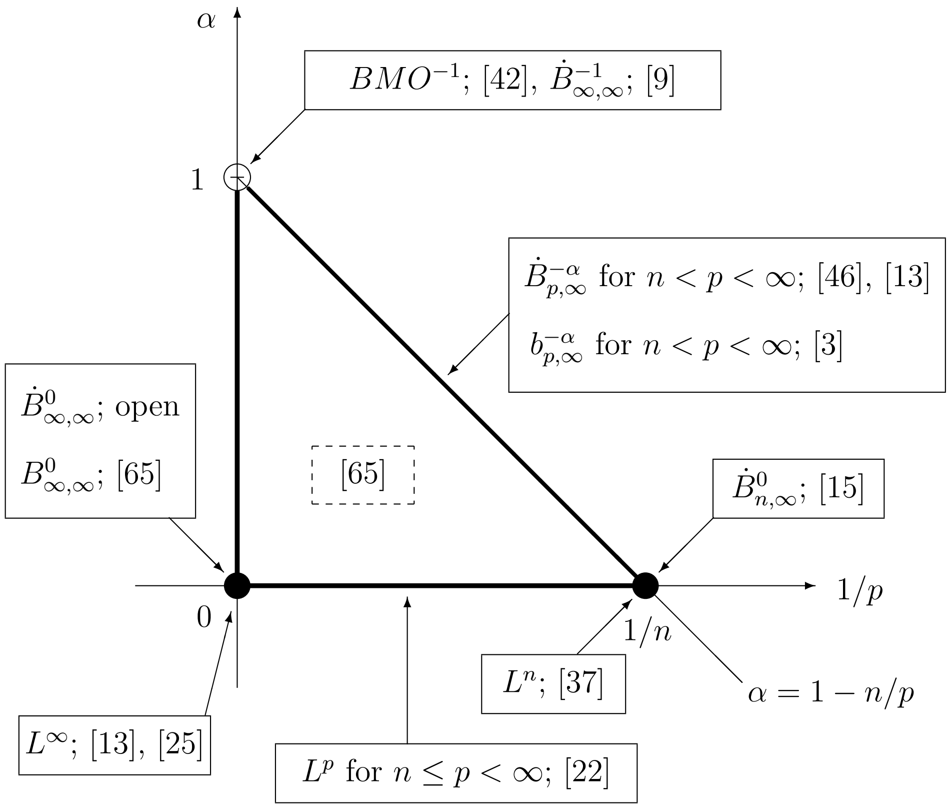

Figure 1:

smoothing estimates), and they proved that (NS) admits a time-local unique smooth solution in $L^n(\mathbb{R}^n)$ for all $n \geq 2$. See Figure 1. Giga [22] also obtained the time-local existence with initial data in $L^p(\mathbb{R}^n)$ for $n \leq p < \infty$. The time-local existence for L^∞ initial data is also constructed by Cannon and Knightly [12], Cannone [13], Giga, Inui and Matsui [25] in general dimension.

We shall explain the scaling invariant space. For $\lambda > 0$

$$u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) := \lambda^2 p(\lambda x, \lambda^2 t).$$

If (u,p) is a solution to (NS), then (u_λ, p_λ) also satisfies (NS), automatically. If $(u_\lambda, p_\lambda) = (u,p)$, then that is called a **self-similar solution**. A study on the self-similar solutions plays an important role for mathematical investigation on partial differential equations. Meyer [57] proposed the notion of the **scaling invariant** spaces with respect to x as follows: we regard X as a scaling invariant space if $\|u\|_X = \|\lambda u(\lambda \cdot + a)\|_X$ for all $\lambda > 0$ and $a \in \mathbb{R}^n$. Concretely, $L^n(\mathbb{R}^n)$ is scaling invariant; in fact, one may check $\|u\|_{L^n_x} = \|\lambda u(\lambda \cdot + a)\|_{L^n_x}$ easily. Once the initial velocity u_0 belongs to a scaling invariant space, and small enough with respect to the norm, there is a chance to get the existence of a time-global smooth unique solution. In 1984 Kato [37] pointed out this fact, he showed it when $u_0 \in L^n(\mathbb{R}^n)$. So, in this paper we call this fact Kato's principle or, time-global well-posedness for small data. This immediately implies that $u = 0$ is a stable stationary solution to (NS) in a small ball of L^n, that is to say, the local stability. We intend to say that, around 1981, Giga and Miyakawa [27, 60, 61] also noticed this fact independently of Kato [37]. Moreover, Giga [22] and von Wahl [76] pointed out that Kato's principle is applicable whence the function space of initial data is scaling invariant. This means that one may not make sense the smallness in not scaling invariant spaces. After [37], there are a lot of contributions on Kato's principle in several scaling invariant spaces. Actually, Kato and Ponce did it in $\dot{H}_2^{\frac{n}{2}-1}$ in [39], Kozono and Yamazaki showed it in $\dot{B}_{p,\infty}^{-1+n/p}$ for $p \in (n, \infty)$ in [46], or Cannone et al. in [13, 14, 15, 63]. In addition, the weak-$L^n(\mathbb{R}^n)$ space (which is equivalent to the Lorentz space $L^{n,\infty}$) is also considered by Kozono and Yamazaki [47]. In 2001 Koch and Tataru proved it by [42] in BMO^{-1}. The function spaces which are concerned are wider and wider:

$$\dot{H}_2^{n/2-1} \subset L^n \subset \dot{B}_{p,\infty}^{-1+n/p} \subset BMO^{-1} = \dot{F}_{\infty,2}^{-1} \subset \dot{F}_{\infty,\infty}^{-1} = \dot{B}_{\infty,\infty}^{-1}$$

for $p \in (n, \infty)$. These embeddings are continuous (in the norms), and $\dot{B}_{\infty,\infty}^{-1}$ is the biggest function space in the scaling invariant spaces. In fact, Meyer showed that all scaling invariant space is a subspace of $\dot{B}_{\infty,\infty}^{-1}$. This implies that all self-similar solution belongs to $\dot{B}_{\infty,\infty}^{-1}$. Therefore, from view point of pure mathematical interests, many researchers tried and still try to investigate (NS) in such function spaces; see e.g. [4, 16, 21, 51].

Very recently, Bourgain and Pavlovic [9] showed the negative results in $\dot{B}_{\infty,\infty}^{-1}$, namely, Kato's principle does not work in the biggest space. Simply saying, (NS) is ill-posed in $\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3)$. The purpose of this paper is to give a rigorous proof of their assertion.

We will refer to that many literatures on Kato's principle in several domains are appeared, although that is not the main issue in this paper. In the half space Kozono studied by [44], in the exterior domains that was done by Iwashita [33], on the Riemannian manifold by Taylor [72], in an aperture domain e.g. [17, 28, 48], in a compactly perturbed half space e.g. [49]. It was also studied in the different partial differential equations of parabolic type whether Kato's principle is applicable. For example, the magnetohydrodynamic equations was concerned by e.g. [1, 77], compressible Navier-Stokes equations by [18], moving obstacle in the fluid [41], rotating (and moving) obstacle in the fluid e.g. [29], the Keller-Segel equations e.g. [45].

Furthermore, there are some results on the local existence of mild solutions in the subcritical spaces (not scaling invariant, for example, $B_{p,q}^{-\alpha}$ with $\alpha < 1 - n/p$, below of the critical line $\alpha = 1 - n/p$ in Figure 1); see [3, 27, 40, 54, 60, 65]. In the case of supercritical spaces (upper than the critical line) it seems tough to construct mild solutions by successive approximation, in general. Nevertheless, using L^2-theory by Leray and Hopf, one can obtain the existence of time-global weak solutions when $u_0 \in L^p(\mathbb{R}^n)$ for $p \in (2, n)$; see e.g. [11].

This paper is organized as follows. In section 1 we have stated the problem, main results and our motivation. In section 2 we define the function spaces, Besov spaces and

Triebel-Lizorkin spaces, mainly. Section 3 is devoted to explain the positive results due to Koch and Tataru [42] in BMO^{-1}. We also mention the precise proof of negative results due to Bourgain and Pavlovic [9] in $\dot{B}^{-1}_{\infty,\infty}$ and Theorem 1.1. In this paper we shall argue the convergence of successive approximation, in stead of the method in [9].

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Before closing this section, we will refer to some notation in this paper. Hereafter, we denote the numerical constants by C, which may differ to the others in lines, likely. We do not distinguish scalar valued functions and vector valued, as well as the function spaces, if no confusion occurs. We use Bourgain's notation of an equivalency $A \sim B$, which means that there is a constant C such that $C^{-1}A \le B \le CA$ as well as $\| \cdot \|_A \sim \| \cdot \|_B$ by $C^{-1}\|f\|_A \le \|f\|_B \le C\|f\|_A$ for all f; we use it when we do not have interests in the constant C, particularly.

2 Function spaces

2.1 Sobolev space

We introduce the function spaces in this section. Let $n \in \mathbb{N}$, $s \in \mathbb{R}$ and let $1 \le p, q \le \infty$. The set of test functions is denoted by $\mathcal{D}$ or, $C_c^\infty(\mathbb{R}^n)$. Its topological dual stands for $\mathcal{D}'$, which is the set of distributions. The set of rapidly decreasing functions (in the sense of Schwartz) is written as $\mathcal{S}$; the set of tempered distributions is $\mathcal{S}'$. For $p \in [1, \infty]$, $L^p := L^p(\mathbb{R}^n) := \{f \in L^1_{loc}; \|f\|_p < \infty\}$ is the Lebesgue space of p-th integrable functions whose norm denotes

$$\|f\|_p := \begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} & \text{if} \quad p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |f(x)| & \text{if} \quad p = \infty. \end{cases}$$

We often omit the notation of the domain $(\mathbb{R}^n)$. Note that $\mathcal{S} \subset L^p \subset \mathcal{S}'$, and the first inclusion is dense when $p \in [1, \infty)$. So, we may define the operators ($\mathcal{F}$, $e^{t\Delta}$, R_i, $\mathbb{P}$, etc.) as a tempered distribution.

The solenoidal subspace stands for $L^p_\sigma := \{f \in L^p; \nabla \cdot f = 0\}$, where $\nabla \cdot f = 0$ means in the distribution sense. For $p \in (1, \infty)$ it is well-known that

$$L^p_\sigma = \overline{C^\infty_{c,\sigma}}^{\|\cdot\|_p} := \text{closure of } \{f \in C^\infty_c\, ;\, \nabla \cdot f = 0\} \text{ in } \|\cdot\|_p.$$

For $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$, the Sobolev space $W^{m,p}$ is defined by

$$\begin{aligned} W^{m,p} &:= \Big\{ f \in L^p; \|f\|_{W^{m,p}} < \infty \Big\}, \\ \|f\|_{W^{m,p}} &:= \sum_{|\alpha| \le m} \|\partial^\alpha f\|_p. \end{aligned}$$

Here $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n_0$ is a multi-index; $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Usually, m is called the differentiability exponent, and p is called the integrability exponent. We also use this terminology, throughout of this note. The inhomogeneous Bessel-potential space is defined by

$$H^s_p := (1 - \Delta)^{-s/2} L^p := \{\mathcal{F}^{-1}(1 + |\xi|^2)^{-s/2} \hat{f}\, ;\, f \in L^p\}$$

with $s \in \mathbb{R}$ and $p \in [1, \infty]$. Note that $W^{m,p} = H^m_p$ for $m \in \mathbb{N}_0$.

The homogeneous Sobolev space is defined by

$$\begin{aligned} \dot{W}^{m,p} &:= \Big\{ f \in L^p_{loc}; \|f\|_{\dot{W}^{m,p}} < \infty \Big\}, \\ \|f\|_{\dot{W}^{m,p}} &:= \sum_{|\alpha| = m} \|\partial^\alpha f\|_p. \end{aligned}$$

We denote $\dot{H}^s_p := (-\Delta)^{-s/2} L^p := \{\mathcal{F}^{-1}|\xi|^{-s} \hat{f}\, ;\, f \in L^p\}$ by the homogeneous Bessel-potential space. One can also see that $\dot{W}^{m,p} = \dot{H}^m_p$ for $m \in \mathbb{N}_0$.

Concerning to the fractional order of Sobolev space, we analogously define the Slobodeckij space

$$W^{s,p} := \{f \in W^{[s],p}; \|f\|_{W^{[s],p}} + \sum_{|\alpha| = [s]} \Big(\int\int \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^p}{|x - y|^{n + \{s\}p}} dx dy \Big)^{1/p} < \infty\}$$

for $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $p \in (1, \infty)$. Here we have used the Gauss notation; $s = [s] + \{s\}$ and $[s] \in \mathbb{N}_0$ and $\{s\} \in (0, 1)$. There are many characterization of these function spaces, in particular, using the interpolation theory; see e.g. [5, 74]. However, we omit the details.

2.2 $\mathcal{H}^1$ and BMO

From view points of application of pure mathematical theory, (e.g. the image processing; see [56]) function spaces L^1 and L^∞ are interesting. For example, by numerical simulations it is often used the approximation in phase space (the image of Fourier transform). Fourier transform maps from L^1 to the set of bounded and continuous functions, and one can easily compute such bounded functions. Also, from view point of meteorological observation, it is sometimes better to treat L^∞ data which do not decay at spatial infinity. A pure mathematical motivation for treating L^∞ comes from the fact that L^∞ is Banach algebra with respect to point-wise multiplication, which makes obviously sense to the bilinear terms. However, L^1 and L^∞ are difficult spaces for applying harmonic analysis and singular integral method. Although we must use the Helmholtz projection $\mathbb{P}$, or essentially the Riesz transform R_i, they are not bounded in neither L^1 nor L^∞. In other word, $R_i f \notin L^1$ even if $f \in L^1$ as well as $R_i g \notin L^\infty$ even if $g \in L^\infty$. What is the suitable domain and range? The answer is the Hardy space $\mathcal{H}^1$ corresponding to L^1, and BMO corresponding to L^∞. In fact, we see that

$$\begin{aligned} &R_i : \mathcal{H}^1 \to L^1 \quad \text{and} \quad \mathcal{H}^1 \to \mathcal{H}^1 \quad \text{bounded}, \\ &R_i : L^\infty \to BMO \quad \text{and} \quad BMO \to BMO \quad \text{bounded}. \end{aligned} \tag{2.1}$$

Here we define $\mathcal{H}^1$ by

$$\mathcal{H}^1 := \{f \in L^1; \|f\|_{\mathcal{H}^1} := \|f\|_1 + \sum_{i=1}^n \|R_i f\|_1 < \infty\}.$$

In e.g. the book of Sogge [69] one can find the fact that $\{f \in \mathcal{S}; \int f = 0\}$ is a densely subset of $\mathcal{H}^1$.

Now we consider BMO (Bounded Mean Oscillation) functions:

$$\begin{aligned} BMO &:= \{f \in L^1_{loc}; [[f]]_{BMO} < \infty\}, \\ [[f]]_{BMO} &:= \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy, \\ f_Q &:= \frac{1}{|Q|} \int_Q f(z) dz. \end{aligned}$$

Note that $[[\cdot]]_{BMO}$ is a seminorm, however, not a norm. In fact, $[[f]]_{BMO} = 0$ if and only if f is constant. This comes from the following inequality shown by John and Nirenberg [34]: for all $\varepsilon > 0$ there exists a positive constant C such that

$$\int_{\mathrm{R}^n} \frac{|f(x) - f_Q|}{1 + |x|^{n+\varepsilon}} dx \le C[[f]]_{BMO}, \quad f \in BMO.$$

This estimate also gives us the growth condition at space infinity of BMO functions. Since we choose arbitrarily $\varepsilon > 0$, $f \in BMO$ may grow logarithmically, the growth-rate is less than $|x|^\varepsilon$ for all ε. Actually, one can see that $[x \mapsto \log |x|] \in BMO$; see [34].

The fundamental solution K_n of $-\Delta$ in $\mathbb{R}^n$ is

$$K_n(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & \text{if} \quad n = 2, \\ \frac{1}{\omega_n} \frac{1}{|x|^{n-2}} & \text{if} \quad n \ge 3. \end{cases}$$

Here ω_n is the volume of the unit sphere $\{x \in \mathbb{R}^n; |x| = 1\}$. Once we obtain the precise analysis in the function spaces which contain these K_n, there are a lot of application.

Indeed, Kozono and Yamazaki [46, 47] studied the time-local existence and uniqueness of mild solutions in $\dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$ for $p \in (3, \infty)$ and $L^{3,\infty}(\mathbb{R}^3) = L^3_w$, since they contain $[x \mapsto \frac{1}{|x|}]$ which is nothing else K_3 in 3-dimension, the homogeneous function of degree -1.

Open Problem *Can one get a time-local unique mild solution when $u_0 \in BMO(\mathbb{R}^2)$?*

We should note that $BMO/\mathbb{R}$ (or, $BMO/\mathbb{C}$ if we deal with complex valued functions) is a normed space, then a Banach space. Obviously, $L^\infty \subset BMO \subset \mathcal{S}'$. Notice that $[[f]]_{BMO} \leq 2\|f\|_\infty$. We now introduce the Carleson measure due to Strichartz [71] and this leads us the equivalent norms:

$$[[f]]_{BMO} \sim \sup_{x\in\mathbb{R}^n, R>0} \left(\frac{1}{|B_R(x)|} \int_{B_R(x)} \int_0^{R^2} |e^{t\Delta} f(y)|^2 \frac{dt}{t} dy \right)^{1/2} \tag{2.2}$$

for $f \in BMO/\mathbb{R}$. Here we have used $\sim$ the notation of a norm-equivalency.

At the end of this subsection, we refer to the duality of $\mathcal{H}^1$ and BMO. In the article of Fefferman and Stein [19], the reader find the facts that

$$BMO = (\mathcal{H}^1)' \qquad \text{and} \qquad BMO' \supsetneq \mathcal{H}^1.$$

This relationship is basically similar to that of between L^1 and L^∞. Therefore,

$$\|f\|_\infty \sim \sup_{g\in L^1, \|g\|_1=1} |\langle f, g\rangle| \qquad \text{and} \qquad [[f]]_{BMO} \sim \sup_{g\in \mathcal{H}^1, \|g\|_{\mathcal{H}^1}=1} |\langle f, g\rangle|.$$

2.3 Besov and Triebel-Lizorkin spaces

To define the Besov spaces and Triebel-Lizorkin spaces we now introduce the Paley-Littlewood decomposition. Let us call $\{\phi_j\}_{j=-\infty}^{\infty}$ the Paley-Littlewood decomposition if $\hat{\phi}_0 \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp}\hat{\phi}_0 \subset \{\xi;\, 1/2 \leq |\xi| \leq 2\}$, $\hat{\phi}_j(\xi) = \hat{\phi}_0(2^{-j}\xi)$ and $\sum_{j=-\infty}^{\infty} \hat{\phi}_j(\xi) = 1$ except for $\xi = 0$. And also, let us denote $\psi = \mathcal{F}^{-1}(1 - \sum_{j=1}^{\infty} \hat{\phi}_j)$, so $\{\hat{\psi}, \hat{\phi}_1, \hat{\phi}_2, \ldots\}$ is a dyadic decomposition of the unity in the phase space.

Notice that $\psi, \phi_j \in \mathcal{S}$. We can easily verify by dilation argument that

$$\|\phi_j\|_1 = \|\phi_0\|_1, \qquad j \in \mathbb{Z} \tag{2.3}$$

independently in j. Obviously, $\int \phi_j = 0$ for all $j \in \mathbb{Z}$. Also,

$$\mathcal{F}^{-1}(\hat{\phi}_j \cdot \hat{\phi}_k) = \phi_j * \phi_k = 0 \qquad \text{if} \quad |j-k| \geq 2, \tag{2.4}$$

this fact is called Bony's paraproduct lemma due to [6]. This yields that

$$\phi_j * f = \phi_j * \big(\sum_{k=j-1}^{j+1} \phi_k \big) * f. \tag{2.5}$$

As the same way, $\psi * \phi_j = 0$ for $j \geq 2$. By (2.3) and (2.5) it holds true that for $s \in \mathbb{R}$ there exists a positive constant C such that

$$\begin{aligned} \|(1-\Delta)^{s/2}\phi_j * f\|_p &\leq C2^{sj}\|\phi_j * f\|_p \quad \text{for} \quad j \in \mathbb{N}, \\ \|(-\Delta)^{s/2}\phi_j * f\|_p &\leq C2^{sj}\|\phi_j * f\|_p \quad \text{for} \quad j \in \mathbb{Z}, \end{aligned}$$

which is a sort of Bernstein's inequality; see e.g. [5].

Definition 2.1. *Let $s \in \mathbb{R}$, $p \in [1,\infty]$ and $q \in [1,\infty]$. An inhomogeneous Besov space is defined by*

$$B^s_{p,q} := \big\{ f \in \mathcal{S}'; \|f\|_{B^s_{p,q}} < \infty \big\},$$
$$\|f\|_{B^s_{p,q}} := \begin{cases} \Big[\|\psi * f\|_\infty + \sum_{j=1}^\infty 2^{jsq}\|\phi_j * f\|_p^q \Big]^{1/q} & \text{if} \quad q < \infty, \\ \|\psi * f\|_\infty + \sup_{1 \leq j \leq \infty} 2^{js}\|\phi_j * f\|_p & \text{if} \quad q = \infty. \end{cases}$$

This Besov norm is understood as $\|\cdot\|_{l^q(L^p)}$ in the sense that $\{\|f_j\|_p\}_{j=0}^\infty \in l^q$, where $f \mapsto \{\psi * f, 2^s\phi_1 * f, 2^{2s}\phi_2 * f, \ldots\} =: \{f_j\}_{j=0}^\infty$. Following Johnsen [35], we call s the differentiability-exponent, p the integral-exponent and q the sum-exponent.

Definition 2.2. *An inhomogeneous Triebel-Lizorkin space is defined by*

$$F^s_{p,q} := \big\{ f \in \mathcal{S}'; \|f\|_{F^s_{p,q}} < \infty \big\},$$
$$\|f\|_{F^s_{p,q}} := \begin{cases} \Big\| |\psi * f| + \big(\sum_{j=1}^\infty 2^{jsq}|\phi_j * f|^q\big)^{1/q} \Big\|_p & \text{if} \quad p, q < \infty, \\ \Big\| |\psi * f| + \sup_{1 \leq j \leq \infty} 2^{js}|\phi_j * f| \Big\|_p & \text{if} \quad p < \infty,\ q = \infty, \\ \sup_{k \in \mathbb{N}_0, x \in \mathbb{R}^n} \frac{1}{|B^{2^{-k}}(x)|} \int_{B_{2^{-k}}(x)} \Big(\sum_{j \geq k} 2^{sjq}|\phi_j * f(y)|^q \Big)^{1/q} dy & \text{if} \quad p = \infty,\ q < \infty, \\ \sup_{k \in \mathbb{N}_0, x \in \mathbb{R}^n} \frac{1}{|B^{2^{-k}}(x)|} \int_{B_{2^{-k}}(x)} \sup_{j \geq k} 2^{sj}|\phi_j * f(y)| dy & \text{if} \quad p = q = \infty. \end{cases}$$

Similarly to the Besov norm, this Triebel-Lizorkin norm is understood as $\|\cdot\|_{L^p(l^q)}$ in the sense that $\big\| \|f_j\|_{l^q} \big\|_{L^p}$.

Note. (1) $B^s_{p,q}$ and $F^s_{p,q}$ are Banach spaces. One can easily check that the Cauchy sequence converges. Clearly, $\mathcal{S}$ is a subset of $B^s_{p,q}$ and $F^s_{p,q}$ for all $s \in \mathbb{R}$ and $p, q \in [1,\infty]$; and dense if $p < \infty$ and $q < \infty$.

(2) $B^s_{p,p} = F^s_{p,p}$. Moreover, $B^s_{p,p} = F^s_{p,p} = W^{s,p}$ if $s \in \mathbb{R}_+ \setminus \mathbb{N}$.

(3) The following embeddings hold from Minkowski's inequality ($l^q \subset l^r$ for $q \leq r$):

$$\begin{aligned} &B^s_{p,1} \subset B^s_{p,p} \subset H^s_p \subset B^s_{p,\infty} \quad \text{if} \quad p \leq 2, \\ &B^s_{p,1} \subset H^s_p \subset B^s_{p,p} \subset B^s_{p,\infty} \quad \text{if} \quad p \geq 2, \\ &F^s_{p,1} \subset H^s_p = F^s_{p,2} \subset F^s_{p,\infty} \quad \text{if} \quad p \in (1,\infty). \end{aligned}$$

The last one follows from the fact that $F^0_{p,2} = L^p$ (equivalent norms) and the Mikhlin-Hörmander multiplier theorem.

(4) The embeddings of Sobolev type

$$B^{s_1}_{p_1,q_1} \subset B^{s_2}_{p_2,q_2} \quad \text{and} \quad F^{s_1}_{p_1,q_1} \subset F^{s_2}_{p_2,q_2}$$

hold if either "$s_1 > s_2$ and $p_1 = p_2$" or "$s_1 - n/p_1 = s_2 - n/p_2$, $s_1 > s_2$ and $p_1 < p_2$" without any restriction on the sum-exponents q_1 and q_2.

(5) The equivalency between the Besov space and the Hölder class:

$$B^s_{\infty,\infty} = C^s \quad \text{if} \quad s \in \mathbb{R}_+ \setminus \mathbb{N}$$

holds. For $s \in \mathbb{N}$ the Besov space $B^s_{\infty,\infty}$ is equivalent to the Zygmund class $\mathcal{C}^s$, which is a natural extension for all $s > 0$ of Hölder class; see e.g. the book of Triebel [74].

(6) We easily see that

$$B^0_{\infty,1} \subset BUC \subset L^\infty \subset B^0_{\infty,\infty}.$$

Here BUC stands for the space of bounded and uniformly continuous functions. Only one typographical error in the book of Triebel [74] appears in here: $B^0_{\infty,1}$ seems to be a Banach algebra with respect to the point-wise multiplication. However, that is not true; the concrete explaining is found in the book of Runst and Sickel [64].

(7) For the cases $p \in (0,1)$ or $q \in (0,1)$, then one can analogously define $B^s_{p,q}$ and $F^s_{p,q}$ as quasi-Banach spaces, corresponding quasi-norms, that is, the triangle inequality does not hold in general. We do not penetrate this situation, since we always need the triangle inequality with almost every calculation in this paper, for instance, to construct mild solutions by iteration arguments.

We are now position to define the homogeneous Besov and Triebel-Lizorkin spaces. Let $\mathcal{Z}'$ be the topological dual space of

$$\mathcal{Z} := \Big\{ f \in \mathcal{S}\, ;\, \partial^\alpha \hat{f}(0) = 0,\, \forall \alpha \in \mathbb{N}_0^n \Big\}.$$

Definition 2.3. *For $s \in \mathbb{R}$, $1 \le p \le \infty$ and $1 \le q \le \infty$, we define the homogeneous Besov space by (see e.g. [5, 64, 74, 75]):*

$$\dot{B}^s_{p,q} := \Big\{ f \in \mathcal{Z}'; \|f\|_{\dot{B}^s_{p,q}} < \infty \Big\},$$

$$\|f\|_{\dot{B}^s_{p,q}} := \begin{cases} \Big[\sum_{j=-\infty}^{\infty} 2^{jsq} \|\phi_j * f\|_p^q \Big]^{1/q} & \text{if} \quad q < \infty, \\ \sup_{-\infty \le j \le \infty} 2^{js} \|\phi_j * f\|_p & \text{if} \quad q = \infty. \end{cases}$$

Also, we define the homogeneous Triebel-Lizorkin space by

$$\dot{F}^s_{p,q} := \Big\{ f \in \mathcal{Z}'; \|f\|_{\dot{F}^s_{p,q}} < \infty \Big\},$$

$$\|f\|_{\dot{F}^s_{p,q}} := \Bigg\| \Big[\sum_{j=-\infty}^{\infty} 2^{jsq} |\phi_j * f|^q \Big]^{1/q} \Bigg\|_p \quad \text{if} \quad p, q < \infty,$$

and define it for the cases $p = \infty$ or $q = \infty$ by the same modification of inhomogeneous Triebel-Lizorkin space.

Note. (8) By the definition of ϕ_j it is clear that $\|f\|_{\dot{B}^s_{p,q}} = 0$ if $f \in \mathcal{P} := \{\text{polynomials}\}$. Thus, $\|\cdot\|_{\dot{B}^s_{p,q}}$ and $\|\cdot\|_{\dot{F}^s_{p,q}}$ are seminorms. The quotient spaces divided by polynomials $\dot{B}^s_{p,q}/\mathcal{P}$ and $\dot{F}^s_{p,q}/\mathcal{P}$ are Banach spaces.
(9) Clearly, $\mathcal{Z}$ is a subset of $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$, and dense if $p, q < \infty$.
(10) $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ are subsets of $\mathcal{S}'$ if the exponents satisfy

$$\text{either "}s < n/p\text{" or "}s = n/p \text{ and } q = 1\text{".} \tag{2.6}$$

Under this conditions, the operators $\mathcal{F}$, $e^{t\Delta}$, $\mathbb{P}$, R_i can be defined on the homogeneous spaces as the tempered distribution sense. Also, it is natural to select the representative element such that

$$f = \sum_{j=-\infty}^{\infty} \phi_j * f \quad \text{in} \quad \mathcal{S}'. \tag{2.7}$$

See the details in Bourdaud [7] or Kozono and Yamazaki [46]. Throughout of this note, we basically treat the homogeneous space under the exponents satisfying (2.6) only.
(11) The following equivalences are known:

$$\mathcal{H}^1 = \dot{F}^0_{1,2} \quad \text{and} \quad BMO = \dot{F}^0_{\infty,2}, \tag{2.8}$$

which are equivalent norms. It holds true that $\dot{B}^s_{p,p} = \dot{F}^s_{p,p}$. Also, the homogeneous versions of the embeddings as the same to (3) and (4) hold.
(12) We are mainly interested in the case $p = \infty$, and following continuous embeddings are easily seen:

$$\dot{B}^0_{\infty,1} \subset BUC \subset L^\infty \subset BMO \subset \dot{B}^0_{\infty,\infty}.$$

Typically, thanks to (2.7), we get

$$\|f\|_\infty = \Big\| \sum_{j=-\infty}^{\infty} \phi_j * f \Big\|_\infty \le \sum_{j=-\infty}^{\infty} \|\phi_j * f\|_\infty = \|f\|_{\dot{B}^0_{\infty,1}}.$$

(13) By dilation for any integer j there exists a positive constant C_0 (independent of k and j) such that $\|R_k \phi_j\|_1 \le C_0$. Hence, we see that the Riesz transform is bounded in the homogeneous spaces as subspaces of $\mathcal{S}'$ when the exponents satisfy (2.6).

In this note we mainly deal with the case $p = \infty$. Define $\dot{B}^{-1}_{\infty,\infty}/\mathcal{P}$ by

$$\dot{B}^{-1}_{\infty,\infty}/\mathcal{P} = \Big\{ f \in \mathcal{S}'; \|f\|_{\dot{B}^{-1}_{\infty,\infty}} < \infty \Big\}, \tag{2.9}$$

$$\|f\|_{\dot{B}^{-1}_{\infty,\infty}} \sim \sup_{\rho>0} \sqrt{\rho} \|e^{\rho\Delta} f\|_\infty \quad \text{for} \quad f \in \dot{B}^{-1}_{\infty,\infty}/\mathcal{P}. \tag{2.10}$$

The definition (2.9)-(2.10) and the general definition for $s = -1$ and $p = q = \infty$ are equivalent except for the constant functions; see e.g. [2, 51]. Indeed, for a non-zero constant function $f_c \equiv c \in \mathbb{R}^n \setminus \{0\}$ we see that

$$\sup_{\rho>0} \sqrt{\rho} \|e^{\rho\Delta} f_c\|_\infty = \sup_{\rho>0} \sqrt{\rho} |c| = \infty \ne \sup_{j \in \mathbb{Z}} 2^{-j} \|\phi_j * f_c\|_\infty = 0.$$

The reader should note that the non-zero constant functions f_c do not satisfy (2.7).

3 Local well-posedness in $\dot{f}^{-1}_{\infty,2}$

3.1 Well-posedness in the sense of Hadamard

In this section we explain the results of Koch and Tataru [42]. They constructed time-local unique mild solutions with initial data in vmo^{-1}, and mild solutions can be extended time-globally if BMO^{-1}-norm of the initial velocity is small sufficiently. Before stating their results, we now recall the notion of well-posedness in the sense of Hadamard.

Definition 3.1. *We say that the Cauchy problem is* **(WP)** *well-posed in X if the following three conditions are satisfied:*
(i) *A solution exist.*
(ii) *The solution is unique.*
(iii) *The solution equips the equicontinuity.*

The property (iii) means that the solution depends on the initial data continuously in some reasonable topology e.g. $C([0,\infty);X)$, that is, for all $t>0$ and $\varepsilon>0$ there exists $\delta>0$ such that $\|u_0-\tilde{u}_0\|_X<\delta$ then $\|u(t)-\tilde{u}(t)\|_X<\varepsilon$. Here $u(t)$ and $\tilde{u}(t)$ are solutions at time t with initial data u_0 and $\tilde{u}_0$, respectively. If we only get the time-local existence of unique solution, replacing ∞ by T for some $T\in(0,\infty)$ and $t\in(0,T)$ at (i) and (iii), then it is called (TLWP) time-local well-posed. For the case one can obtain the well-posedness if the initial data are small enough, it is called (GWSD) time-global well-posedness for small data. From view point of the dynamical system, (GWSD) implies the local stability of the zero solution. We call (IP) ill-posed if one of (i) $-$ (iii) is failed. This usual terminology is used throughout this paper.

Leray [52] showed that (NS) is (WP) in $L^2_\sigma(\mathbb{R}^2)$. The famous problem is to show whether (NS) is (WP) in $L^2_\sigma(\mathbb{R}^3)$, or not. Kato [37], Giga and Miyakawa [27] proved that (NS) is (TLWP) and (GWSD) in $L^n_\sigma(\mathbb{R}^n)$.

We will discuss well-posedness of (NS) in Besov or Triebel-Lizorkin spaces closed and related to L^∞, due to the mild solutions. We now focus into the continuity of solutions in time at the initial time. Dealing with L^∞-initial data, we have to take care about the following fact:

Lemma 3.2. *Let $f\in L^\infty$. Then $e^{t\Delta}f\to f$ in L^∞ as $t\to 0$ if and only if $f\in BUC$.*

In other words, $e^{t\Delta}$ is strongly continuous in BUC, but not in L^∞. Or, $e^{t\Delta}$ is (C_0)-semigroup in BUC. Concerning the Heavyside function, $h(x)=1$ for $x\geq 0$ and $h(x)=0$ for $x<0$, it is easy to see that $\|e^{t\Delta}h-h\|_\infty=\frac{1}{2}$ for all $t>0$. The proof of this lemma is found in e.g. [25].

Recall the integral equation (INT). It is clear that the second terms of right-hand-side vanish as $t\to 0$ whence it is integrable. So, in order to get the continuity of solutions in time up to initial time, it is naturally required the restriction on $u_0\in X$ satisfying that

$e^{t\Delta}$ is strongly continuous at $t = 0$ in X. For this purpose we now introduce the little Besov space and little Triebel-Lizorkin space.

Definition 3.3. *Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$. Subspaces of $B^s_{p,q}$ and $F^s_{p,q}$ are defined by*

$$b^s_{p,q} := \big\{g \in B^s_{p,q};\ e^{t\Delta}g \to g \ \text{ in } \ B^s_{p,q} \ \text{ as } \ t \to 0\big\},$$
$$f^s_{p,q} := \big\{g \in F^s_{p,q};\ e^{t\Delta}g \to g \ \text{ in } \ F^s_{p,q} \ \text{ as } \ t \to 0\big\}.$$

Assume, in addition, that exponents satisfy (2.6)*, the homogeneous version is defined by*

$$\dot{b}^s_{p,q} := \Big\{g \in \dot{B}^s_{p,q};\ g = \sum_{j=-\infty}^{\infty} \phi_j * g \ \text{ in } \mathcal{S}',\ e^{t\Delta}g \to g \ \text{ in } \ \dot{B}^s_{p,q} \ \text{ as } \ t \to 0\Big\},$$
$$\dot{f}^s_{p,q} := \Big\{g \in \dot{F}^s_{p,q};\ g = \sum_{j=-\infty}^{\infty} \phi_j * g \ \text{ in } \mathcal{S}',\ e^{t\Delta}g \to g \ \text{ in } \ \dot{F}^s_{p,q} \ \text{ as } \ t \to 0\Big\}.$$

They are closed subspace of usual Besov or Triebel-Lizorkin spaces, so Banach spaces. It is easy to check that

$$\overline{C^\infty_c}^{\|\cdot\|_{B^s_{p,q}}} \subset b^s_{p,q} = \overline{B^{s+1}_{p,q}}^{\|\cdot\|_{B^s_{p,q}}} \subset B^s_{p,q}.$$

Also, one may see that $b^s_{p,q} = B^s_{p,q}$ if and only if $q < \infty$. See more details of little Besov spaces in [2, 65].

Next, we refer to the function spaces which are used by Koch and Tataru [42]. Let BMO^{-1} be

$$BMO^{-1} :=\Big\{f \in \mathcal{S}'\,;\ \|f\|_{BMO^{-1}} < \infty\Big\},$$
$$\|f\|_{BMO^{-1}} := \sup_{x\in\mathbb{R}^n, R>0} \left(\frac{1}{|B_R(x)|}\int_0^{R^2}\int_{B_R(x)} |e^{t\Delta}f(y)|^2 dydt\right)^{1/2},$$

where $B_R(x)$ is an open ball radius $R > 0$ centered at $x \in \mathbb{R}^n$. One can easily see that BMO^{-1} is equivalent to the set of first derivatives of BMO functions, and also they coincide the specific homogeneous Triebel-Lizorkin space:

$$BMO^{-1} = \partial BMO = \dot{F}^{-1}_{\infty,2}.$$

Recall (2.2). The reader may find the details of basic properties of BMO or $\dot{F}^s_{p,q}$ in e.g. [42, 64, 70, 71, 79].

One can see that the several interesting functions belong to BMO^{-1} (and then $\dot{B}^{-1}_{\infty,\infty}$), for example, the trigonometric functions e.g. $[x \mapsto \sin x]$ which are not decaying at space infinity, $[x \mapsto \sin x + \sin(\sqrt{2}x)]$ is an almost periodic function, $[x \mapsto e^x \sin(e^x)]$ is a growing and oscillating function, $[x \mapsto p.v.\frac{1}{x}]$ has a singularity.

For $T \in (0, \infty]$ we denote the norm of BMO_T^{-1} by

$$\|f\|_{BMO_T^{-1}} := \sup_{x \in \mathbb{R}^n, R \in (0, \sqrt{T})} \left(\frac{1}{|B_R(x)|} \int_0^{R^2} \int_{B_R(x)} |e^{t\Delta} f(y)|^2 dy dt \right)^{1/2}.$$

Let us now define the bmo^{-1} and vmo^{-1}.

$$\begin{aligned} bmo^{-1} &:= \{f \in \mathcal{S}'; \ \|f\|_{bmo^{-1}} := \|f\|_{BMO_1^{-1}} < \infty\} = F^{-1}_{\infty,2} \supset BMO^{-1} = \dot{F}^{-1}_{\infty,2}, \\ vmo^{-1} &:= \{f \in bmo^{-1}; \ \lim_{T \to 0} \|f\|_{BMO_T^{-1}} = 0\} \\ &= \{f \in bmo^{-1}; \ \lim_{t \to 0} \|e^{t\Delta} f - f\|_{bmo^{-1}} = 0\} = f^{-1}_{\infty,2}. \end{aligned}$$

Here vmo is the localized version of VMO the space of vanishing mean oscillation functions. In the book of Stein [70] VMO functions are required the vanishing in both $\lim_{T\to 0} \|f\|_{BMO_T} = \lim_{T \to \infty} \|f\|_{BMO_T} = 0$, which is slightly different to above.

Let $T \in (0, \infty]$, the function v of x and t we define $\mathcal{E}_T$-norm by

$$\|v\|_{\mathcal{E}_T} := \sup_{0<t<T} \sqrt{t} \|v(t)\|_\infty + \sup_{x \in \mathbb{R}^n, R \in (0, \sqrt{T})} \left(\frac{1}{|B_R(x)|} \int_0^{R^2} \int_{B_R(x)} |v(y,t)|^2 dy dt \right)^{1/2}.$$

This norm is associated to the natural class of solutions of the heat equation as well as the Navier-Stokes equations. Actually, let $v = e^{t\Delta} v_0$ with $v_0 \in BMO^{-1}$, we see that

$$\begin{aligned} \|e^{t\Delta} v_0\|_{\mathcal{E}_T} &= \sup_{0<t<T} \sqrt{t} \|e^{t\Delta} v_0\|_\infty + \sup_{x \in \mathbb{R}^n, R \in (0, \sqrt{T})} \left(\frac{1}{|B_R(x)|} \int_0^{R^2} \int_{B_R(x)} |e^{t\Delta} v_0(y)|^2 dy dt \right)^{1/2} \\ &\le C \|v_0\|_{\dot{B}^{-1}_{\infty,\infty}} + C \|v_0\|_{\dot{F}^{-1}_{\infty,2}} \le C \|v_0\|_{\dot{F}^{-1}_{\infty,2}} < \infty, \end{aligned}$$

the first inequality obviously holds for taking $T = \infty$. The discovering $\mathcal{E}_T$-norm is crucial.

3.2 In $vmo^{-1} = f^{-1}_{\infty,2}$

In this position we give the main results of Koch and Tataru:

Theorem 3.4 (Koch-Tataru [42]). (NS) *is* (TLWP) *in* vmo^{-1}*, i.e.,* $^\forall u_0 \in vmo^{-1}$*,* $^\exists T > 0$ *and mild solution* $^{\exists 1} u \in \mathcal{E}_T \cap C([0,T]; vmo^{-1})$*. Moreover,* (GWSD) *in* BMO^{-1}*, i.e., if we assume, in addition, that* $\|u_0\|_{BMO^{-1}}$ *is small enough, then* $^{\exists 1} u \in \mathcal{E}_\infty \cap C([0,\infty); vmo^{-1})$*.*

Remark 3.5. (i) *When* $u_0 \in bmo^{-1}$*, there is a lack of continuity of mild solutions at* $t = 0$*. Also, uniqueness is not known for large data; see Miura [58].*
(ii) *By definition of* $\mathcal{E}_T$*-norm it is shown that the mild solution* $u(t) \in L^\infty$ *for any small* $t > 0$*. Thus, the pressure giving by* (1.1) *is made sense of value in* BMO *by* (2.1) *for* $t > 0$*. Under this setting* (u, p) *satisfies* (NS) *in the classical sense, and the solution is uniquely determined by* u_0*; see Kato [36].*

(iii) *By smoothing effect for any small* $t_0 > 0$ *the mild solution* $u(t_0) \in W^{1,\infty}(\mathbb{R}^n)$. *So, this* t_0 *can be regarded as a new initial time, and* $u(t_0)$ *as a new bounded and smooth initial velocity. By analysis in* L^∞*-framework by e.g. [13, 25] we may observe the properties of obtained mild solutions, more precisely. Moreover, for the case* $n = 2$, *one can get a time-global unique smooth mild solutions without smallness assumption on the initial velocity. Indeed, one can derive the a priori estimate*

$$\|u(t)\|_\infty \le C_1\|u(t_0)\|_\infty \exp\{C_2\|\omega_0\|_\infty t\} \quad \text{for all} \quad t > t_0 \tag{3.1}$$

with numerical constants C_1, C_2, *and we denote* $\omega_0 := \operatorname{rot} u(t_0)$. *The key of the proof of this a priori estimate is the uniform boundedness of the vorticity (scalar value)* $\omega(t) := \operatorname{rot} u(t) := \partial_1 u^2 - \partial_2 u^1$

$$\|\omega(t)\|_\infty \le \|\omega_0\|_\infty \qquad \text{for all} \quad t > t_0,$$

which is yielded by the maximal principle for the vorticity equation (no stretching terms) due to Oleinik et al. [31]:

$$(\mathrm{VOR})_{2\mathrm{D}} \qquad \omega_t - \Delta\omega + (u,\nabla)\omega = 0 \qquad \text{for} \quad t > t_0, \qquad \omega|_{t=t_0} = \omega_0.$$

See the details in [26, 66].

(iv) *It is well-known that the Serrin's class* $L^s(0,T;L^r)$ *with* $\frac{2}{s} + \frac{n}{r} \le 1$ *satisfying* $s > 2$ *and* $r \in (n,\infty)$ *produces the regularity of solutions to* (NS). *In [68] Serrin proved that* $u(t) \in C^\infty(\mathbb{R}^n)$, *and then* $u \in C^\infty(\mathbb{R}^n \times (0,T))$ *provided* $u \in L^s(0,T;L^r(\mathbb{R}^n))$. *Also, he asked in [68] whether* $u(t) \in C^\omega(\mathbb{R}^n)$, *real analytic with respect to spatial variables under this class, or not? Miura and the author [59] showed the positive answer, using the embedding* $L^s(0,T;L^r) \subset \mathcal{E}_T$. *Indeed, they derived the estimates for higher-order derivatives of mild solutions obtained by Koch and Tataru; there exist positive constants* K_1 *and* K_2

$$\|\partial_x^\beta u(t)\|_\infty \le K_1(K_2|\beta|)^{|\beta|} t^{-|\beta|/2-1/2}$$

for all $t \in (0,T]$ *and* $\beta \in \mathbb{N}_0^n$. *From this estimate, one can deduce the estimate for the size of the radius of convergence of the Taylor's expansion* ($=: \rho(t)$) *from below:*

$$\rho(t) = \lim_{|\beta|\to\infty} \left(\frac{\|\partial_x^\beta u(t)\|_\infty}{\beta!} \right)^{-1/|\beta|} \ge C\sqrt{t} \qquad \text{for} \quad t \in (0,T]$$

with some constant C. *This calculation comes from Stirling's formula and Cauchy's criterion, obviously. The spatial analyticity implies that the propagation speed of* (NS) *in* vmo^{-1} *is infinite as well as the solutions of the heat equation, that is to say, the support of* $u(t)$ *coincides with* $\mathbb{R}^n$ *for any* $t > 0$, *even if the support of* u_0 *is compact.*

We use the iteration scheme (so-called successive approximation or fixed point argument) for the proof of Theorem 3.4. In fact, we successively define $\{u_j\}$ by (1.2) with

(1.3). One can see that the approximation sequence $\{u_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $\mathcal{E}_T \cap C([0,T]; vmo^{-1})$. The Key of the proof is the inequality for estimating to the bilinear terms: there exists a positive constant C such that

$$\|\mathcal{B}(u,v)\|_{\mathcal{E}_T} \leq C\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T} \qquad \text{for} \quad u,v \in \mathcal{E}_T. \tag{3.2}$$

This inequality holds true even for $T=\infty$. One may find the proof of (3.2) due to the point-wise estimates of the heat kernel in [42], and the estimates involving the higher order differentiation in [59].

So far, it is not known the benefit bilinear estimates in neither $C([0,T]; vmo^{-1})$ to which the solutions naturally belong, as long as the author knows. Remark that the function spaces which contain non-decaying functions are usually not Banach algebra with respect to point-wise multiplications, e.g. $\dot{B}^0_{\infty,1}$, vmo^{-1}, BMO^{-1} and $\dot{B}^{-1}_{\infty,\infty}$, except for L^∞. Thus, it seems to be difficult to make sense the bilinear terms in such function spaces, basically. The reader may find some estimates for point-wise multiplications in some function spaces in [13, 35, 64, 65].

4 Ill-posedness in $\dot{f}^{-1}_{\infty,\infty} = \dot{b}^{-1}_{\infty,\infty}$

4.1 Lack of equicontinuity

In this section we will give a rigorous proof of [9] and Theorem 1.1, that is, (NS) is (IP) in $\dot{f}^{-1}_{\infty,\infty} = \dot{b}^{-1}_{\infty,\infty}$ in $\mathbb{R}^3$. Firstly, it is shown a lack of equicontinuity of mild solutions. Also, it seems to be difficult to construct a unique time-local mild solution.

Theorem 4.1 (Bourgain-Pavlovic [9])**.** *For $\delta \in (0,1)$ and $T \in (0,1)$ there exists an initial velocity $u_0 \in \dot{b}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ such that $\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} < \delta$ with $\nabla \cdot u_0 = 0$, there exists a mild solution u in $C([0,T]; \dot{b}^{-1}_{\infty,\infty})$ and $\|u(T)\|_{\dot{B}^{-1}_{\infty,\infty}} > 1/\delta$.*

Remark 4.2. (i) *This assertion indicates that in the class $C([0,T]; \dot{b}^{-1}_{\infty,\infty})$ to which mild solutions ought to belong, mild solutions do not have the equicontinuity. Thus, this assertion is to be said* **ill-posedness theorem***. Namely,* (NS) *is not* (TLWP) *in $\dot{b}^{-1}_{\infty,\infty}$ and wider spaces, for example, $b^{-1}_{\infty,\infty}$ and the supercrtical spaces $b^{-\alpha}_{\infty,\infty}$ with $\alpha > 1$. Also,* (NS) *is not* (WPSD) *in $\dot{b}^{-1}_{\infty,\infty}$, even though $\dot{b}^{-1}_{\infty,\infty}$ is scaling invariant. Furthermore, to show the uniqueness of mild solutions in this class seems to be difficult.*
(ii) *This assertion is still true for the case $n \geq 4$ by the simple modification of the proof. However, in the case $n=2$ it is not clear whether the same results can be proved, or not.*
(iii) *It is supposed that one can also obtain the same statement in other function spaces. Recently, the author was informed by Yoneda who wrote [78] for ill-posedness in $\dot{F}^{-1}_{\infty,q}$ with $q \in (2,\infty)$, using the same argument of [9]. Moreover, the author thinks that the similar results can be obtained for strong solutions to other equations of parabolic type, particularly, the Keller-Segel equations; see e.g. Iwabuchi [32].*

4.2 Initial datum

Theorem 4.1 follows from the technique of Bourgain [8] for establishing the similar ill-posedness theorem for the KdV equation. His method is so-called “norm inflation”. Before stating the outline of the proof, we now fix the initial velocity, concretely. In what follows, the initial velocity is fixed to be of the form

$$\begin{aligned} u_0(x) &:= \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} h_s \big[e_2 \cos(k_s \cdot x) + e_3 \cos(l_s \cdot x) \big] \\ &= \Big(0, \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} h_s \cos(h_s x_1), \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} h_s \cos(h_s x_1 - x_2) \Big) \end{aligned} \tag{4.1}$$

with parameters $Q > 0$ and large $r \in \mathbb{N}$; other notations are as follows:

$$\begin{array}{rlll} e_2 := & \overrightarrow{e}_2 := (0,1,0) & & (= v_s \text{ is the notation in [9]}), \\ e_3 := & \overrightarrow{e}_3 := (0,0,1) & & (= v_s' \text{ in [9]}), \\ h_s := & h(s) := 2^{s(s-1)/2}\gamma^{s-1}\eta & & \text{for } s \in \mathbb{N}, \\ k_s := & (h_s, 0, 0), & & \\ l_s := & (h_s, -1, 0) & & (= k_s' \text{ in [9]}). \end{array}$$

Here $\gamma, \eta \in \mathbb{N}$ are also parameters. The specific time T when the inflation occurs can be regarded as a parameter, replacing the time variable $[t \mapsto \lambda t]$ with some $\lambda > 0$. Using this scaling argument, we can relax the restriction $T < 1$. However, for the sake of simplicity of the proof, and for the readers' convenience, T remains as a given small number in this paper.

It is clear by definition that $u_0(x) = (0, u_0^2(x_1), u_0^3(x_1, x_2))$ and $u_0 \in \dot{B}^{-1}_{\infty,\infty}$ by (4.4) below. Moreover, u_0 is a uniformly continuous function, so $u_0 \in \dot{b}^{-1}_{\infty,\infty}$; see [65]. It should be emphasized that we are able to fix the directions of $v_s = e_2$ and $v_s' = e_3$ without loss of generality, since (NS) is invariant under the Galilee transformation. In addition, it should be more emphasized that the selections of v_s and v_s' are slightly different to those of [9]; that is a crucial point noticed by Yoneda.

The proof of the theorem is realized by the suitable selection of the parameters (Q, r, γ, η) for each $\delta, T \in (0,1)$. Since

$$h_{s+1}/h_s = 2^{(2s+1)/2}\gamma, \tag{4.2}$$

it follows that $h_s << h_{s+1}$ for large s or γ; this property is so-called ‘lacunary’. For the sake of simplicity, $h(z) := 2^{z(z-1)/2}\gamma^{z-1}\eta$ denotes the function of $z > 0$. The compatibility condition $\nabla \cdot u_0 = 0$ is satisfied by $e_2 \cdot k_s = 0$ and $e_3 \cdot l_s = 0$, obviously. It is clear that u_0 is a smooth periodic function (thus bounded) with the period $2\pi/h$ in x_1 and 2π in x_2. This implies that the mild solution is also periodic with the period 2π, regarded as a function on the torus $(2\pi\mathbb{T})^3$, as long as the mild solution exists. So, the kinematic energy is bounded by the initial energy $\frac{1}{2}\|u_0\|^2_{L^2((2\pi\mathbb{T})^3)}$; this is huge but finite. The periodicity of solutions is useful for the estimate of the Besov norm; see Proposition 4.3 below. And also, $\hat{u}_0$ is a sum of Dirac's delta functions, therefore, $u_0 \in FM_0$;

$$FM_0 := \{\mathcal{F}^{-1}v \in \mathcal{S}';\ v = \text{sum of finite Radon measures},\ v(0) = 0\}.$$

We refer to the detail of FM_0 in [23, 24].

Let u_1 be the first approximation of iteration scheme[1], that is, the solution to the heat equation with initial datum given by (4.1):

$$u_1(x,t) := e^{t\Delta}u_0(x) = \frac{Q}{\sqrt{r}}\sum_{s=1}^{r} h_s\big[e_2 e^{-h_s^2 t}\cos(h_s x_1) + e_3 e^{-(h_s^2+1)t}\cos(h_s x_1 - x_2)\big].$$

For $t > 0$ we obtain that $u_1(t) := u_1(\cdot,t) \in L^\infty \cap BMO^{-1}$, even though these norms are large; see (4.28) and (4.29) in subsection 4.7. The function u_1 is of the form

$$u_1 = (0, u_1^2(x_1,t), u_1^3(x_1,x_2,t)).$$

It is well-known that one can construct the unique mild solution with initial velocity given by (4.1) in the L^∞-framework, which was shown by [12, 13, 25]. Moreover, in [25] one can estimate for the possible existence time T_* (until when we may construct a mild solution by iteration scheme in $C([0,T_*];L^\infty)$) bounded from below: $T_* \geq C/\|u_0\|_\infty^2 \sim h_r^{-2}$ with the universal constant $C > 0$. Indeed, by $h_r >> r$ we see that

$$\|u_0\|_\infty \sim \frac{Q}{\sqrt{r}}\sum_{s=1}^{r} h_s \sim h_r >> 1 \qquad \text{if} \quad r >> 1.$$

Therefore, T_* might be very tiny. Also, one may see that

$$\|u_0\|_{BMO^{-1}} \sim Q\sqrt{r} >> 1 \qquad \text{if} \quad r >> 1.$$

However, we observe the Besov norm $\|\cdot\|_{\dot{B}^{-1}_{\infty,\infty}} = \sup_{\rho>0}\sqrt{\rho}\|e^{\rho\Delta}\cdot\|_\infty$ as

$$\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} \sim \frac{Q}{\sqrt{r}} << 1 \qquad \text{if} \quad r >> 1. \tag{4.3}$$

In fact, by the definition of Besov norm from Paley-Littlewood decomposition it holds true that

$$\begin{aligned}\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} &\sim \sup_{j\in\mathbb{Z}}\|\phi_j * \nabla^{-1}u_0\|_\infty \\ &\leq \sqrt{2}\frac{Q}{\sqrt{r}}\sup_j\sup_x|\phi_j * \sum_{s=1}^{r}\cos(k_s\cdot)(x)| \\ &\leq 2\sqrt{2}\sup_j\|\phi_j\|_1\frac{Q}{\sqrt{r}} << 1 \qquad \text{if} \quad r >> 1.\end{aligned}$$

[1]The notation of u_1 differs to that of [9]; they use $u_1 := -\mathcal{B}(e^{t\Delta}u_0)$.

Here we have used the two facts that $\|\phi_j\|_1 = \|\phi_0\|_1$ for $j \in \mathbb{Z}$ and for each $s \in \{1, \dots, r\}$ there are at most 2 indices $j \in \mathbb{Z}$ such that $\phi_j * \cos(k_s \cdot x) \neq 0$. For reader's convenience we now give an elementally proof of (4.3) as follows; we will explain the details later:

$$
\begin{aligned}
\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} &= \sup_{\rho>0} \sqrt{\rho} \|e^{\rho\Delta} u_0\|_\infty \\
&= \sup_{\rho>0} \sqrt{\rho} \frac{Q}{\sqrt{r}} \sup_x \left| \sum_{s=1}^r h_s [v e^{-h_s^2 \rho} \cos(k_s \cdot x) + w e^{-(h_s^2+1)\rho} \cos(l_s \cdot x)] \right| \\
&\leq \sqrt{2} \frac{Q}{\sqrt{r}} \sup_{\rho>0} \sum_{s=1}^r \sqrt{\rho} h_s e^{-h_s^2 \rho} \\
&\leq \sqrt{2} \frac{Q}{\sqrt{r}} \left[\sup_{0<\rho<h_r^{-2}} \sum_{s=1}^r \sqrt{\rho} h_s e^{-h_s^2 \rho} + \sup_{\rho \geq h_r^{-2}} \sum_{s=1}^r \sqrt{\rho} h_s e^{-h_s^2 \rho} \right] \\
&\leq \sqrt{2} \frac{Q}{\sqrt{r}} \left[\sqrt{h_r^{-2}} (2h_r) + \sup_{\rho \geq h_r^{-2}} \left\{ \sum_{s=1}^{s_\rho} + \sum_{s=s_\rho+1}^r \right\} \sqrt{\rho} h_s e^{-h_s^2 \rho} \right] \\
&\leq \sqrt{2} \frac{Q}{\sqrt{r}} \left[2 + \sup_{\rho \geq h_r^{-2}} \int_0^{s_\rho - 1} \sqrt{\rho} h(z+1) e^{-h(z+1)^2 \rho} dz \right. \\
&\qquad \left. + \frac{e^{-1/2}}{\sqrt{2}} + \sup_{\rho \geq h_r^{-2}} \int_{s_\rho}^r \sqrt{\rho} h(z) e^{-h(z)^2 \rho} dz \right] \\
&\leq \sqrt{2} \frac{Q}{\sqrt{\rho}} \left[2 + \frac{e^{-1/2}}{\sqrt{2}} + \sup_{\rho \geq h_r^{-2}} \int_1^r \sqrt{\rho} h(z) e^{-h(z)^2 \rho} dz \right] \\
&\leq C_* \frac{Q}{\sqrt{r}}
\end{aligned}
\tag{4.4}
$$

with the numerical constant C_* independent of parameters. We take $g(\varsigma) := \varsigma e^{-\varsigma^2}$, then g is monotone increasing when $\varsigma < 1/\sqrt{2}$, and monotone decreasing when $\varsigma > 1/\sqrt{2}$. Thus, we choose $s_\rho \in \{1, \dots, r\}$ such that

$$
\sqrt{\rho} h_s \leq \sqrt{\rho} h_{s+1} \quad \text{if} \quad s < s_\rho, \qquad \sqrt{\rho} h_s \geq \sqrt{\rho} h_{s+1} \quad \text{if} \quad s \geq s_\rho.
$$

The maximal value of g is taken as $\max g = g(1/\sqrt{2}) = 1/\sqrt{2e}$. By the monotonicity we derive the estimate replaced from sum by integration. The last inequality follows from the fact that the derivation of $\int_1^r \cdots$ with respect to ρ is positive when ρ is small, besides this is negative when ρ is large. In this section we often use the fact that $\sup_{\rho>0} \sum_{s=1}^r \sqrt{\rho} h_s e^{-h_s^2 \rho} \leq C_*$ bounded uniformly in r.

Now we recall the successive approximation and its modification of convergence version. A mild solution u is usually constructed as the limit of function series $\{u_j\}_{j=1}^\infty$ (or, its subsequence if necessary) defined by (1.2). When $u_0 \in BUC$, namely, a bounded uniformly continuous function, $\{u_j\}_{j=1}^\infty$ is a Cauchy sequence in $C([0, T_*]; BUC)$ provided T_* is chosen small enough as h_r^{-2} above, then it has a uniform convergence limit. In order to observe the norm inflation of mild solutions, we always concern at $T > T_*$. Throughout this paper, we use the standard terminology that the bilinear terms denote by (1.3). Let us put the sequence $\{v_k\}_{k=1}^{\infty}$ as

$$v_1(t) := u_1(t) := e^{t\Delta}u_0,$$
$$v_{k+1}(t) := u_{k+1}(t) - u_k(t) = -\mathcal{B}(u_k) + \mathcal{B}(u_{k-1})$$

for $k \in \mathbb{N}$. Therefore, we may rewrite u_j and the mild solution $u = \lim_{j\to\infty} u_j$ as

$$u_j(t) = \sum_{k=1}^{j} v_k(t) \qquad \text{and} \qquad u(t) = \sum_{k=1}^{\infty} v_k(t). \tag{4.5}$$

In what follows, we shall calculate $v_k(t)$ and estimate the Besov norm of them at $t = T$. Moreover, we easily notice that

$$v_k = (0, 0, v_k^3(x_1, x_2, t)) \qquad \text{for} \quad k \geq 2; \tag{4.6}$$

see subsection 4.4 and 4.5.

4.3 Norm inflation

For all $\delta, T \in (0,1)$, we correctly select parameters (Q, r, γ, η) to see that

$$\|v_1(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \leq \|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq C_* \frac{Q}{\sqrt{r}} =: S < \delta, \tag{4.7}$$

$$v_2 = M_2 + R_2, \qquad M_2 := e_3 \frac{Q^2}{4} e^{-t} \sin x_2,$$

$$\|v_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq \|M_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} = C_\flat Q^2 =: L \geq \frac{2}{\delta} \tag{4.8}$$

in subsection 4.4. Here $A \simeq B$ means the almost equal, that is, $A = B + R$ such that $|R| < \frac{1}{3}|B|$ for the scalar valued, and $\|R\|_{\dot{B}^{-1}_{\infty,\infty}} < \frac{1}{3}\|B\|_{\dot{B}^{-1}_{\infty,\infty}}$ for functions; $C_\flat > 0$ is a numerical constant. We will see that M_2 is the major term of v_2 at $t \simeq T$ in the next subsection. Reversely, R_2 is the collection of the remainder terms of v_2 at $t \simeq T$. It is remarkable that $M_k(t)$ no longer might be the leading term if we take neither a different norm nor $t << T$. We further prove that $v_3 = M_3 + R_3$ with

$$M_3 := -\frac{Q^3}{8\sqrt{r}} t e^{-t} \sum_{s=1}^{r} h_s e^{-h_s^2 t} \{\cos(h_s x_1 + x_2) + \cos(h_s x_1 - x_2)\} e_3,$$

$$\|v_3(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq \|M_3(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq \frac{Q^3\sqrt{T}}{8\sqrt{2er}} \simeq \frac{Q^2}{4\eta} S \tag{4.9}$$

for $t \simeq T \simeq \eta^{-2}$. Moreover, we see that for v_4

$$v_4(T) = M_4(T) + R_4(T), \qquad M_4(T) = -K M_2(T), \qquad K := \frac{(1 - 3e^{-2})Q^2}{8r\eta^2} > 0,$$

$$\|v_4(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq \|M_4(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq KL. \tag{4.10}$$

By induction one may also show that $v_k(T) = M_k(T) + R_k(T)$ and

$$M_{2k-1}(T) = (-K)^{k-2} M_3(T) \qquad \text{and} \qquad M_{2k}(T) = (-K)^{k-1} M_2(T) \tag{4.11}$$

for $k \geq 2$ with $t \simeq T \simeq \eta^{-2}$. For the proof of Theorem 4.1, taking parameters such that $K < 1/12$, we may appeal to rough estimates for the remainder terms in the following way. Since the number of terms of v_k is 2^k, and the biggest term in the Besov norm of the components of v_k is that of M_k, it is allowed to compute

$$\|v_{2k-1}(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \leq (\sharp\,\text{terms}) \cdot \|M_{2k-1}(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \leq (4K)^{k-2} S, \tag{4.12}$$

$$\|v_{2k}(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \leq (\sharp\,\text{terms}) \cdot \|M_{2k}(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \leq (4K)^{k-1} L \tag{4.13}$$

for $k \geq 3$. Once we obtain (4.7)-(4.13), it follows from (4.5):

$$\|u(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq \|\sum_{k=1}^{\infty} v_{2k}(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \geq \|v_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} - \sum_{k=2}^{\infty} (4K)^{k-1} \|v_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \geq \frac{L}{2},$$

if $K < 1/12$. We simply discard the sum of odd numbers above, since S is very small compared with L. Finally, the choice of parameters yields that $S \simeq \delta$ and $L \simeq \frac{2}{\delta}$, this completes the proof of Theorem 4.1. Note that the gradient of pressure terms are always annihilated; $\nabla p = 0$ due to (1.1).

The choice of the parameter Q is essential, that is to say, the parameter Q plays an important role for the behavior of the mild solution. Consider the following four cases:

1. It is possible to show that $\|u_j(T)\|_{\dot{B}^{-1}_{\infty,\infty}}$ does not converge as $j \to \infty$ when Q is large so that $K > 4$. This implies that there is no hope to proceed the iteration scheme (1.2) to construct the mild solution up to time T from the initial datum u_0 given by (4.1), even though there exists a unique mild solution at least up to $T_* \sim h_r^{-2}$ in the class $C([0, T_*]; L^\infty)$. We will see the details in subsection 4.6.

2. If Q is large, but not so large compared with r and η such that $K < 1/12$, then the norm inflation occurs, likely. The author guesses that the norm inflation solution can be extended time-global one with exponential decay as $t \to \infty$, since M_k is always the major part of v_k and the estimates $(4.7) - (4.13)$ are valid for all $t > T$.

3. On the other hand, if Q is small such that $C_\sharp Q < 1$, then the norm inflation does not occur, although we can easily derive the estimate for $y := u - u_2$; see subsection 4.7.

4. One can prove that there exists a unique time-global mild solution in the certain class e.g. $C([0, \infty); L^\infty)$ if $Q << 1$, since the initial velocity is periodic; see [24].

We will see the proofs of (4.7) – (4.10) in below. The induction argument yields (4.11) and the estimates for remainder terms (4.12) and (4.13) as the similar way as (4.9) and (4.10), so we omit them in this paper.

Once we get (4.8) with large L, it seems to be difficult to apply the fixed point argument, directly. More precisely, the mapping from the initial data to the mild solutions seems to be not of class C^2; Germain intended to show it in [21]. The proof of Theorem 4.1 in this paper is slightly different to that of [9]. They actually intended to show

$$\|y(T)\|_{\dot{B}^{-1}_{\infty,\infty}} << 1, \tag{4.14}$$

where $y := u - u_2 := u - u_1 + \mathcal{B}(u_1) = \sum_{k=3}^{\infty} v_k$. For the details on y, we will mention in subsection 4.7. Bourgain and Pavlovic [9] proceeded on the investigation to get (4.14). However, it is not clear to the author how to choose the parameters such that (4.14) and (4.8) are satisfied along their strategy, simultaneously. Although they mentioned the way-out by new techniques (slicing the time-interval into many parts) due to Koch and Tzvetkov e.g. [43], it is unlikely to get some advantage by their method in the situation $u, y \in C([0,T]; \dot{b}^{-1}_{\infty,\infty})$. Remark that it seems to be hard to find the associate norm like $\|\cdot\|_{\mathcal{E}_T}$ for Koch-Tataru's solution. As seen in (4.10), it is difficult to show (4.14) without smallness of Q directly, even if y is relatively smaller than v_2. Besides, if Q is small, then the norm inflation does not occur.

Choice of parameters We now refer to the selection of the parameters (Q, r, γ, η) for the proof of Theorem 4.1. Firstly, we always fix $\gamma := 3$. We impose that $\eta \in \mathbb{N}$ with $\eta \geq 2$ large such that $\eta \sim T^{-1/2}$ for $T \in (0,1)$. For any $\delta \in (0,1)$, we fix $Q > 1$ large such that $Q > \sqrt{\frac{3}{C_\flat \delta}}$. Finally, we choose $r \in \mathbb{N}$ large such that $r > 4C_*^2\delta^{-4}$, $T > h_r^{-2}$ and $K < \frac{1}{12}$.

4.4 First and second approximation

In this section we calculate v_1 and v_2, deriving the estimates for BMO^{-1}, $\dot{B}^{-1}_{\infty,\infty}$ and $\mathcal{E}_T$ norms. This section will be devoted to show (4.7) and (4.8) with appropriate selection of parameters.

linear terms For each fixed $t > 0$, if we choose r large enough, then $\dot{B}^{-1}_{\infty,\infty}$ norm and L^∞ norm of $u_1 = v_1$ can be taken arbitrary small. In fact, we see that

$$\begin{aligned}\|v_1(t)\|_{\dot{B}^{-1}_{\infty,\infty}} = \|u_1(t)\|_{\dot{B}^{-1}_{\infty,\infty}} &\leq \sup_{\rho>0} \sqrt{\rho}\|e^{(\rho+t)\Delta}u_0\|_\infty \\ &\leq \sqrt{2}\frac{Q}{\sqrt{r}} \sup_{\rho>0} \sum_{s=1}^{r} \sqrt{\rho} h_s e^{-h_s^2(\rho+t)} \\ &\leq C_* \frac{Q}{\sqrt{r}} \ (= \|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} < \delta)\end{aligned} \tag{4.15}$$

Note that C_* is a numerical constant independent of parameters. Note that (4.15) implies that for each δ, T, Q we may choose r large such that

$$\|v_1(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \le \delta/2. \tag{4.16}$$

We next consider v_2. Divide v_2 into three parts. Let us see

$$
\begin{aligned}
&(u_1(\tau), \nabla) u_1(\tau) \\
&= \sum_{m=1}^{3} \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} h_s [e_2^m e^{-h_s^2 \tau} \cos(k_s \cdot x) + e_3^m e^{-(h_s^2+1)\tau}] \cos(l_s \cdot x)] \\
&\qquad \times \partial_m \left(\frac{Q}{\sqrt{r}} \sum_{q=1}^{r} h_q [e_2 e^{-h_q^2 \tau} \cos(k_q \cdot x) + e_3 e^{-(h_q^2+1)\tau}] \cos(l_q \cdot x)] \right) \\
&= \frac{Q^2}{r} \sum_{s=1}^{r} \sum_{q=1}^{r} h_s h_q e_3 e^{-(h_s^2+h_q^2+1)\tau} \cos(k_s \cdot x) \sin(l_q \cdot x) \\
&= \frac{Q^2}{r} \sum_{s=1}^{r} h_s^2 e_3 e^{-(2h_s^2+1)\tau} \left(-\frac{1}{2}\right) \sin x_2 \\
&\qquad + \frac{Q^2}{r} \sum_{s=1}^{r} h_s^2 e_3 e^{-(2h_s^2+1)\tau} \frac{1}{2} \sin(2h_s x_1 - x_2) \\
&\qquad + \frac{Q^2}{r} \sum_{s,q=1,\, s\neq q}^{r} h_s h_q e_3 e^{-(h_s^2+h_q^2+1)\tau} \cos(k_s \cdot x) \sin(l_q \cdot x) \\
&=: N_1 + N_2 + N_3.
\end{aligned}
$$

For each $\ell = 1, 2, 3$ we set

$$U_\ell := U_\ell(t) := -\int_0^t e^{(t-\tau)\Delta} \mathbb{P} N_\ell(\tau) d\tau.$$

Thus, $v_2 = \sum_{\ell=1}^{3} U_\ell$. In the conclusion U_1 happens "inflation", as the contrast to that U_2 and U_3 are small, when Q and r are large. Notice that v_2 satisfies (4.6). Therefore, $\nabla \cdot N_\ell = 0$ and $\mathbb{P} N_\ell = N_\ell$ as well as $\nabla \cdot v_2 = 0$.

Estimate for U_1 We obtain that

$$
\begin{aligned}
U_1 &= -\int_0^t e^{(t-\tau)\Delta} \mathbb{P} \frac{Q^2}{r} \sum_{s=1}^{r} h_s^2 e^{-(2h_s^2+1)\tau} e_3 \Big(-\frac{1}{2}\Big) \sin x_2 d\tau \\
&= \frac{Q^2}{2r} \sum_{s=1}^{r} h_s^2 \int_0^t e^{-(2h_s^2+1)\tau} e^{(t-\tau)\Delta} e_3 \sin x_2 d\tau \\
&= \frac{Q^2}{2r} \sum_{s=1}^{r} h_s^2 \int_0^t e^{-(2h_s^2+1)\tau} e^{-t+\tau} e_3 \sin x_2 d\tau \\
&= \frac{Q^2}{2r} (e_3 \sin x_2) \sum_{s=1}^{r} h_s^2 e^{-t} \frac{1 - e^{-2h_s^2 t}}{2h_s^2}
\end{aligned}
$$

$$= \frac{Q^2}{4r}(e_3 \sin x_2) e^{-t} \sum_{s=1}^{r} (1 - e^{-2h_s^2 t})$$
$$= M_2 - \frac{Q^2}{4r}(e_3 \sin x_2) e^{-t} \sum_{s=1}^{r} e^{-2h_s^2 t}.$$

We now compute its Besov norm:

$$\begin{aligned}\|M_2(t)\|_{\dot{B}^{-1}_{\infty,\infty}} &= \sup_{\rho>0} \sqrt{\rho} \|e^{\rho\Delta} M_2(t)\|_\infty \\ &= \frac{Q^2}{4} e^{-t} \sup_{\rho>0} \sqrt{\rho} e^{-\rho} \sup_x |e_3 \sin x_2| \geq C_\flat(t) Q^2 \end{aligned} \tag{4.17}$$

with $C_\flat(t) := \frac{1}{4e^t\sqrt{2e}}$. Here, we denote the numerical constant $C_\flat := C_\flat(T) = \frac{1}{4e^T\sqrt{2e}} \in (\frac{1}{4e\sqrt{2e}}, \frac{1}{4\sqrt{2e}})$, when $T \in (0,1)$ is fixed. On the other hand, it is straightforward to show

$$\begin{aligned}\|\frac{Q^2}{4r}(e_3 \sin x_2) e^{-t} \sum_{s=1}^{r} e^{-2h_s^2 t}\|_{\dot{B}^{-1}_{\infty,\infty}} &\sim \frac{Q^2}{r} \sum_{s=1}^{r} \frac{1}{h_s\sqrt{t}} h_s \sqrt{2t} e^{-2h_s^2 t} \\ &\sim \frac{Q^2}{r} \sum_{s=1}^{r} \frac{\eta}{h_s} \sim \frac{Q^2}{r}. \end{aligned} \tag{4.18}$$

These are valid around $t = T$ with $T \simeq \eta^{-2} >> T_*$. In what follows, we always assume that $t \simeq T \simeq \eta^{-2}$. Once we take $Q > \sqrt{3/C_\flat \delta}$, then (4.8) holds if U_2 and U_3 are small as well as (4.18).

Estimate for U_2 We show that $\|U_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}}$ is small. Note that $\mathbb{P}N_2 = N_2$ as the same as U_1. We thus obtain that

$$\begin{aligned} U_2 &= -\int_0^t e^{(t-\tau)\Delta} \mathbb{P} \frac{Q^2}{r} \sum_{s=1}^{r} h_s^2 e^{-(2h_s^2+1)\tau} e_3 \frac{1}{2} \sin(2h_s x_1 - x_2) d\tau \\ &= -\frac{Q^2}{2r} \sum_{s=1}^{r} h_s^2 \int_0^t e^{-(2h_s^2+1)\tau} e^{-(4h_s^2+1)(t-\tau)} e_3 \sin(2h_s x_1 - x_2) d\tau \\ &= -\frac{Q^2}{2r} \sum_{s=1}^{r} h_s^2 e_3 \sin(2h_s x_1 - x_2) e^{-(4h_s^2+1)t} \frac{e^{2h_s^2 t} - 1}{2h_s^2} \\ &= -\frac{Q^2}{4r} \sum_{s=1}^{r} e_3 \sin(2h_s x_1 - x_2) e^{-(4h_s^2+1)t} (e^{2h_s^2 t} - 1). \end{aligned}$$

We now derive the Besov norm of U_2 at t:

$$\begin{aligned}\|U_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} &= \sup_{\rho>0} \sqrt{\rho} \|e^{\rho\Delta} \frac{Q^2}{4r} \sum_{s=1}^{r} e_3 \sin(2h_s x_1 - x_2) e^{-(4h_s^2+1)T} (e^{2h_s^2 t} - 1)\|_\infty \\ &= \frac{Q^2}{4r} \sum_{s=1}^{r} \left(\sup_{\rho>0} \sqrt{\rho} e^{-(4h_s^2+1)\rho} \right) e^{-(4h_s^2+1)T} (e^{2h_s^2 T} - 1) \end{aligned}$$

$$\leq \frac{Q^2}{4r}\sum_{s=1}^{r}\frac{1}{h_s} \simeq \frac{C_\dagger Q^2}{4r\eta}. \tag{4.19}$$

Here $C_\dagger := \sum_{s=1}^{\infty}\frac{\eta}{h_s} \in (1, 3/2)$ is a numerical constant. Therefore, we can take U_2 small as much as we want, provided r is taken large.

<u>**Estimate for U_3**</u> We now see that $\|U_3(T)\|_{\dot{B}^{-1}_{\infty,\infty}}$ is taken small. Since $\mathbb{P}N_3 = N_3$,

$$\begin{aligned}
U_3 &= -\int_0^t e^{(t-\tau)\Delta}\mathbb{P}\frac{Q^2}{r}\sum_{s,q=1,\, s\neq q}^{r} h_s h_q e^{-(h_s^2+h_q^2+1)\tau} e_3 \cos(h_s x_1)\sin(h_q x_1 - x_2)d\tau \\
&= -\frac{Q^2}{r}\int_0^t e^{(t-\tau)\Delta}\sum_{s,q=1,\, s\neq q}^{r} h_s h_q e^{-(h_s^2+h_q^2+1)\tau} e_3 \left\{-\frac{1}{2}\sin(h_s x_1 - h_q x_1 + x_2)\right. \\
&\qquad\qquad \left. +\frac{1}{2}\sin(h_s x_1 + h_q x_1 - x_2)\right\} d\tau \\
&= \frac{Q^2}{2r}e_3\int_0^t \sum_{s\neq q} h_s h_q e^{-(h_s^2+h_q^2+1)\tau}\left\{e^{-[(h_s-h_q)^2+1](t-\tau)}\sin(h_s x_1 - h_q x_1 + x_2)\right. \\
&\qquad\qquad \left. -e^{-[(h_s+h_q)^2+1](t-\tau)}\sin(h_s x_1 + h_q x_1 - x_2)\right\} d\tau \\
&= \frac{Q^2}{4r}e_3\sum_{s\neq q}\Big[\sin(h_s x_1 - h_q x_1 + x_2)(e^{2h_s h_q t} - 1)e^{-(h_s^2 - 2h_s h_q + h_q^2+1)t} \\
&\qquad\qquad - \sin(h_s x_1 + h_q x_1 - x_2)(1 - e^{-2h_s h_q t})e^{-(h_s^2+2h_s h_q + h_q^2+1)t}\Big].
\end{aligned}$$

Notice that the sums over $s < q$ and $s > q$ are symmetric (these values are equivalent). So, we only compute the sum for $s > q$ at $t = T \simeq \eta^{-2}$:

$$\begin{aligned}
\|U_3(T)\|_{\dot{B}^{-1}_{\infty,\infty}} &= \sup_{\rho>0}\sqrt{\rho}\|e^{\rho\Delta}U_3(T)\|_\infty \\
&\leq \frac{Q^2}{2r}\sum_{s=2}^{r}\sum_{q=1}^{s-1}\left(\sup_{\rho>0}\sqrt{\rho}e^{-(h_s^2-2h_s h_q+h_q^2+1)\rho}\right)e^{-(h_s^2+h_q^2+1)T} \\
&\quad + \frac{Q^2}{2r}\sum_{s=2}^{r}\sum_{q=1}^{s-1}\left(\sup_{\rho>0}\sqrt{\rho}e^{-(h_s^2+2h_s h_q+h_q^2+1)\rho}\right)e^{-(h_s^2+2h_s h_q+h_q^2+1)T} \\
&\leq \frac{Q^2}{r}\sum_{s=1}^{r}\frac{2(s-1)}{h_s} \sim \frac{Q^2}{r\eta}.
\end{aligned} \tag{4.20}$$

To deduce the last inequality we have used the property "lacunary" of h_s:

$$h_s^2 - 2h_s h_q + h_q^2 + 1 \geq h_s(h_s - 2h_{s-1}) = h_s^2(1 - 2^{-s+2}\gamma^{-1}) \geq h_s^2/2$$

whence $s \geq 2$, $q < s$ and $\gamma \geq 3$. Here and hereafter, we fix $\gamma = 3$.

Gathering with (4.17), (4.18), (4.19) and (4.20), we get

$$\|v_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq \|M_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq C_\flat Q^2$$

and (4.8) provided if Q and r are large with $T \simeq \eta^{-2}$.

4.5 Calculi for v_3 and v_4

In this subsection we derive the form of v_3 and v_4, concretely. Let us invoke that

$$\begin{aligned} v_3 = u_3 - u_2 &= u_1 - \mathcal{B}(u_2) - \{u_1 - \mathcal{B}(u_1)\} \\ &= -\mathcal{B}(v_1+v_2, v_1+v_2) + \mathcal{B}(v_1, v_1) \\ &= -\mathcal{B}(v_1, v_2) \\ &= -\int_0^t e^{(t-\tau)\Delta} \mathbb{P} e_3 v_1^2(\tau) \partial_2 v_2^3(\tau) d\tau \\ &= -\int_0^t e^{(t-\tau)\Delta} v_1^2(\tau) \partial_2 v_2^3(\tau) d\tau e_3. \end{aligned}$$

Since v_1 and v_2 are functions independent of x_3, the fourth equality holds by $v_2 = (0, 0, v_2^3)$, and the last equality holds by divergence-free of the integrant. Clearly, v_3 satisfies (4.6), that is, $v_3^1 = v_3^2 = 0$. Analogously, we observe that (4.6) is valid and

$$v_{j+1}(t) = -\mathcal{B}(v_1, v_j) = -\int_0^t e^{(t-\tau)\Delta} v_1^2(\tau) \partial_2 v_j^3(\tau) d\tau e_3 \qquad \text{for} \quad j \geq 2. \tag{4.21}$$

We now calculate the concrete expression of v_3^3 at $t \simeq T \simeq \eta^{-2}$:

$$\begin{aligned} v_3^3 = &-\int_0^t e^{(t-\tau)\Delta} \Bigg[\Bigg\{ \frac{Q}{\sqrt{r}} \sum_{s=1}^r h_s e^{-h_s^2 \tau} \cos(h_s x_1) \Bigg\} \cdot \frac{Q^2}{4r} \\ &\cdot \Bigg\{ \sum_{q=1}^r (1 - e^{-2h_q^2 \tau}) e^{-\tau} \cos x_2 + \sum_{q=1}^r (e^{2h_q^2 \tau} - 1) e^{-4(h_q^2+1)\tau} \cos(2h_q x_1 - x_2) \\ &\quad + \sum_{q,p=1, q \neq p}^r (e^{2h_q h_p \tau} - 1) e^{-(h_q^2 + 2h_q h_p + h_p^2 + 1)\tau} \cos(h_q x_1 + h_p x_1 - x_2) \\ &\quad + \sum_{q,p=1, q \neq p}^r (1 - e^{-2h_q h_p \tau}) e^{-(h_q^2 - 2h_q h_p + h_p^2 + 1)\tau} \cos(h_q x_1 - h_p x_1 + x_2) \Bigg\} \Bigg] d\tau \\ = &-\frac{Q^3}{4\sqrt{r}} \int_0^t \sum_{s=1}^r h_s e^{-(h_s^2+1)\tau} e^{(t-\tau)\Delta} \{\cos(h_s x_1) \cos x_2\} d\tau + (remainder) \\ = &-\frac{Q^3}{8\sqrt{r}} t e^{-t} \sum_{s=1}^r h_s e^{-h_s^2 t} \{\cos(h_s x_1 + x_2) + \cos(h_s x_1 - x_2)\} + (remainder) \\ =: &\, M_3(t) + R_3(t). \end{aligned}$$

Here and hereafter, we do not distinguish the vector valued M_k and its third component if no confusion occurs likely, since $M_k = (0, 0, M_k^3)$ for all $k \geq 2$ as well as $R_k = (0, 0, R_k^3)$. It is easy to see that $\|M_3(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \sim Q^3/\sqrt{r} << 1$ and the remainder term R_3 is small compared with the leading term M_3 as the similar to the estimates (4.18), (4.19) and (4.20). Thus, it is straightforward to get that

$$\|v_3(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq \frac{Q^3}{4\sqrt{r}} T e^{-T} \sum_{s=1}^r h_s e^{-h_s^2 T} \sup_{\rho > 0} \sqrt{\rho} e^{-(h_s^2+1)\rho} \sim \frac{Q^3}{\sqrt{r}} << 1 \tag{4.22}$$

with large r and $T \simeq \eta^{-2}$, involving the remainder terms.

Next, we compute v_4. By (4.21) it follows that at $t \simeq T \simeq \eta^{-2}$

$$\begin{aligned} v_4^3 = &- \int_0^t e^{(t-\tau)\Delta} \Bigg[\left\{ \frac{Q}{\sqrt{r}} \sum_{s=1}^r h_s e^{-h_s^2\tau} \cos(h_s x_1) \right\} \cdot \left(-\frac{Q^3}{8\sqrt{r}} \right) \\ &\qquad \cdot \tau e^{-\tau} \sum_{q=1}^r h_q e^{-h_q^2\tau} \left\{ -\sin(h_q x_1 + x_2) + \sin(h_q x_1 - x_2) \right\} \Bigg] d\tau + R_4 \\ = &- \frac{Q^4}{8r} \sum_{s=1}^r h_s^2 e^{-t} \int_0^t \tau e^{-2h_s^2\tau} d\tau \sin x_2 + R_4 \\ = &- \frac{Q^4}{32r} e^{-t} \sin x_2 \left[\sum_{s=1}^r \frac{1}{h_s^2} \left\{ 1 - e^{-2h_s^2 t}(1 + 2h_s^2 t) \right\} \right] + R_4 \\ = &- \frac{(1-3e^{-2})Q^4}{32r\eta^2} e^{-t} \sin x_2 + R_4 = -KM_2 + R_4. \end{aligned}$$

Here we move the summation over $s \geq 2$ to the remainder terms; the remainder term R_4 might differ to the others in lines, likely. Also, it is easy to see that the Besov norm of R_4 is relatively small compared with that of $M_4 = -KM_2$. Then, we have

$$\|v_4(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \simeq KL. \tag{4.23}$$

The same argument indicates (4.11) by (4.21), if $T \simeq \eta^{-2}$. Therefore, the proof of Theorem 4.1 now completes. □

4.6 No convergence of approximation

For the case of huge Q, the successive approximation does not work in $C(0,T;\dot{B}^{-1}_{\infty,\infty})$.

proof of Theorem 1.1. Let us assume $T < 1/4$ without loss of generality. We choose the initial datum u_0 given by (4.1). We will prove that

$$\|R_k(T)\|_{\dot{B}^{-1}_{\infty,\infty}} < \frac{1}{3}\|M_k(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \tag{4.24}$$

for $k \in \mathbb{N}$. Once we get (4.24), one sees

$$\|u_{4j+2}(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \geq \sum_{k=1}^{j} (K/4)^{k-1} \|M_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \to \infty \quad \text{as} \quad j \to \infty \tag{4.25}$$

when $K > 4$. It suffices to show (4.24) under the suitable choice of parameters with $k \geq 3$. Determine $\gamma = 3$ and $r = 2$. Select $\eta \in \mathbb{N}$ as $\eta \geq 2$ and $\eta \simeq T^{-1/2}$. Let Q be taken large such that $K > 4$. If ℓ is odd and $\ell \geq 3$, then we see

$$\frac{\|R_\ell(T)\|_{\dot{B}^{-1}_{\infty,\infty}}}{\|M_\ell(T)\|_{\dot{B}^{-1}_{\infty,\infty}}} \leq 2^\ell \Big(\sum_{s=1}^{2} e^{-h_s^2 T} \Big)^\ell = 2^\ell (e^{-1} + e^{-36})^\ell < \frac{1}{3}.$$

Analogously as the estimates for R_2, in the case for even $\ell \geq 4$ one can prove the similar inequality. This completes the proof of Theorem 1.1. □

The calculation above implies that it seems hard to show the convergence of any subsequence of $\{u_j\}$ in the class $C(0,T;\dot{b}^{-1}_{\infty,\infty})$. The author does not know whether mild solutions exist up to T with the same u_0, or not. Also, it is not clear whether another successive approximation, for example,

$$w_1 := u_1 \qquad \text{and} \qquad w_{j+1} := w_1 - \mathcal{B}(w_j, w_{j+1}),$$

does converge, or not. Although one can easily observe that $\|u_j(T)\|_{BMO^{-1}} \to \infty$ as $j \to \infty$ by the continuous embedding $BMO^{-1} \subset \dot{B}^{-1}_{\infty,\infty}$, it is not clear to the author whether another norms e.g. $\|\cdot\|_{L^3((2\pi\mathbb{T})^3)}$ or $\|\cdot\|_{L^\infty}$ of $\{u_j(T)\}$ diverge as $j \to \infty$, or not. It is obvious that the assertion of Theorem 1.1 does not fit the situation in two-dimension, since the time-global unique solvability in 2D with the initial data given by (4.1) was shown by [26]. Indeed, (3.1) contradicts to Theorem 1.1 in 2D.

4.7 Estimate for y

In the end of this paper we express the remainder y and its property whence Q is relatively small. Before computing the norms, we now establish a proposition of embedding type for periodic functions.

Proposition 4.3. *Let $\kappa > 0$ and $n \in \mathbb{N}$. Assume that $v \in L^\infty \cap \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^n)$ is periodic with period $2\pi/\kappa$ in x_j for all $j = \{1, \dots, n\}$. Then*

$$\|v\|_{\dot{B}^{-1}_{\infty,\infty}} \leq C\kappa^{-n}\|v\|_\infty \tag{4.26}$$

holds true with constant C depending only on n.

(Proof) Let $n = 1$. By the assumption v has an expansion given by

$$v(x) = \sum_{\ell=1}^{\infty} \alpha_\ell \sin(\kappa\ell x) + \beta_\ell \cos(\kappa\ell x)$$

with some α_ℓ and β_ℓ. Hence, we calculate that

$$\begin{aligned}
\|v\|_{\dot{B}^{-1}_{\infty,\infty}} &\leq \sup_{j\in\mathbb{Z}} \|\phi_j * \nabla^{-1} v\|_\infty \\
&\leq \sup_x \sup_j \left| \sum_{\ell=1}^{\infty} \phi_j * \left(\frac{\alpha_\ell}{\ell\kappa}\sin(\kappa\ell\cdot) + \frac{\beta_\ell}{\ell\kappa}\cos(\kappa\ell\cdot) \right)(x) \right| \\
&\leq 2\|\phi_0\|_1\kappa^{-1}\left(\sup_\ell \frac{|\alpha_\ell|}{\ell} + \sup_\ell \frac{|\beta_\ell|}{\ell} \right) \\
&\leq 4\|\phi_0\|_1\kappa^{-1}\|v\|_\infty.
\end{aligned}$$

Here we have used the fact that for $\ell \in \mathbb{N}$ there are at most 2 indices $j \in \mathbb{Z}$ such that $\phi_j * \cos(\kappa\ell\cdot) \neq 0$. The proof for general $n \in \mathbb{N}$ is the same. $\square$

It is easy to see that for $v \in \mathcal{E}_T$ enjoying the period $2\pi/\kappa$ in $\mathbb{R}^3$

$$\|v(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \leq C\kappa^{-n}\|v(T)\|_\infty = C\kappa^{-3}T^{-\frac{1}{2}}\|T^{\frac{1}{2}}v(T)\|_\infty \leq C\kappa^{-3}T^{-\frac{1}{2}}\|v\|_{\mathcal{E}_T}$$

by (4.26) and the definition of $\mathcal{E}_T$-norm. This technique leads us to show the smallness of Besov norm of functions at T by the smallness of $\mathcal{E}_T$-norm, even though it seems to be tough to compute the Besov norm directly.

Now we estimate y. Let u be a mild solution, and let $y := u - v_1 - v_2$. A formal calculation yields that

$$\begin{aligned} y_t &= u_t - (v_1)_t - (v_2)_t \\ &= u_t - \Delta e^{t\Delta}u_0 - \mathbb{P}(u_1(t), \nabla)u_1(t) - \int_0^t \Delta e^{(t-\tau)\Delta}\mathbb{P}(u_1(\tau), \nabla)u_1(\tau)d\tau. \end{aligned}$$

Subtracting this to $\Delta y = \Delta u - \Delta v_1 - \Delta v_2$, we have

$$\begin{aligned} y_t - \Delta y &= -\mathbb{P}(u, \nabla)u + \mathbb{P}(u_1, \nabla)u_1 \\ &= G_1 + G_2 + G_3 =: G. \end{aligned}$$

Here we set

$$\begin{aligned} G_1 &:= G_1(t) := -\mathbb{P}\big\{(y, \nabla)(u_1 - v_2) + (u_1 - v_2, \nabla)y\big\}, \\ G_2 &:= G_2(t) := -\mathbb{P}(y, \nabla)y, \\ G_3 &:= G_3(t) := -\mathbb{P}(u_1, \nabla)v_2. \end{aligned}$$

Since

$$v_1 = u_1 = (0, u_1^2(x_1, t), u_1^3(x_1, x_2, t)) \quad \text{and} \quad v_2 = (0, 0, v_2^3(x_1, x_2, t)),$$

it is noticed that $(v_2, \nabla)u_1 = 0$ and $(v_2, \nabla)v_2 = 0$, easily. Furthermore, from $\mathcal{B}(u_1)(0) = 0$ it deduces that $y(x, 0) \equiv 0$. By Duhamel's principle y can be regarded as the solution to the following equation of integral form:

$$y(t) = \int_0^t e^{(t-\tau)\Delta}G(\tau)d\tau. \tag{4.27}$$

In terms of $\mathcal{B}$, we rewrite it by

$$y = -\mathcal{B}(y, u_1 - v_2) - \mathcal{B}(u_1 - v_2, y) - \mathcal{B}(y) - \mathcal{B}(u_1, v_2).$$

Moreover, we obviously seek that $y = (0, 0, y^3(x_1, x_2, t))$ as well as (4.6). We now compute $\mathcal{E}_T$-norm of them. For u_1 we see

$$\|u_1(t)\|_\infty \leq \sqrt{2}\frac{Q}{\sqrt{r}}\sum_{s=1}^r h_s e^{-h_s^2 t} \leq C_*\frac{Q}{\sqrt{r}}t^{-\frac{1}{2}} \tag{4.28}$$

and

$$
\begin{aligned}
\|u_1\|_{\mathcal{E}_T} &= \sup_{t\in(0,T)} \sqrt{t}\|u_1(t)\|_\infty + \sup_{x\in\mathbb{R}^n, R\in(0,\sqrt{T})} \left(\frac{1}{|B_R(x)|} \int_0^{R^2} \int_{B_R(x)} |u_1(y,t)|^2 dydt \right)^{1/2} \\
&\le C\frac{Q}{\sqrt{r}} + \sup_{x\in\mathbb{R}^n, R\in(0,\sqrt{T})} \left(\frac{\sqrt{2}}{|B_R(x)|} \int_0^{R^2} \int_{B_R(x)} \Big(\frac{Q}{\sqrt{r}} \sum_{s=1}^{r} h_s e^{-h_s^2 t} \Big)^2 dydt \right)^{1/2} \\
&\le C\frac{Q}{\sqrt{r}} \left[1 + \sup_{R\in(0,\sqrt{T})} \left\{ \int_0^{R^2} \Big(\sum_{s=1}^{r} h_s e^{-h_s^2 t} \Big)^2 dt \right\}^{1/2} \right] \\
&\le C\frac{Q}{\sqrt{r}} \left[1 + \left\{ \sum_{s,q=1}^{r} \frac{h_s h_q}{h_s^2 + h_q^2} \Big(1 - e^{-(h_s^2+h_q^2)T} \Big) \right\}^{1/2} \right] \le CQ.
\end{aligned} \tag{4.29}
$$

Here the constant C does not depend on γ. The last inequality follows from removing the term $e^{-(h_s^2+h_q^2)T}$, simply. Next, we calculate v_2. Recall that $BMO^{-1} \subset \dot{B}^{-1}_{\infty,\infty}$ and its embedding is continuous, then $\|M_2(T)\|_{BMO^{-1}}$ is also large. Moreover, we figure out that

$$
\|M_2\|_{\mathcal{E}_T} \le C\sqrt{T}Q^2 \tag{4.30}
$$

by simple calculation. Here C is a numerical constant independent of parameters. Even if we appeal to Proposition 4.3 for (4.30), it is still kept that $\|M_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \ge 3/\delta$, assuming that Q is sufficiently large. One may also see that

$$
\|U_2\|_{\mathcal{E}_T} \le C\frac{Q^2}{r} \;\to\; 0 \quad \text{as} \quad \frac{Q}{\sqrt{r}} \;\to\; 0.
$$

Although we have shown the smallness of $\|U_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}}$ directly, it is easily obtained by gathering the above and the Proposition 4.3:

$$
\|U_2(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \le \frac{C}{\sqrt{T}}\|U_2\|_{\mathcal{E}_T} \le \frac{CQ^2}{\sqrt{T}r} \to 0 \quad \text{as } r \to \infty.
$$

Besides,

$$
\|U_3\|_{\mathcal{E}_T} \le C\frac{Q^2}{r} \;\to\; 0 \quad \text{as} \quad \frac{Q}{\sqrt{r}} \;\to\; 0.
$$

As the same to estimate for U_2, by Proposition 4.3 it turns out that

$$
\|U_3(T)\|_{\dot{B}^{-1}_{\infty,\infty}} \le \frac{CQ^2}{\sqrt{T}r} \to 0 \quad \text{as } r \to \infty.
$$

We now choose T small such that $C\sqrt{T}Q^2 < 1/4$ as well as r large such that $CQ/\sqrt{r} < 1/4$. Compute (4.27) in $\mathcal{E}_T$-norm by using (4.30), (4.19), (4.20), (3.2) and the triangle-inequality to have

$$
\|y\|_{\mathcal{E}_T} = \|\mathcal{B}(y, u_1 - v_2) + \mathcal{B}(u_1 - v_2, y) + \mathcal{B}(y) + \mathcal{B}(u_1, v_2)\|_{\mathcal{E}_T}
$$

$$\le C\left\{\left(\|u_1\|_{\mathcal{E}_T}+\|v_2\|_{\mathcal{E}_T}+\|y\|_{\mathcal{E}_T}\right)\|y\|_{\mathcal{E}_T}+\|u_1\|_{\mathcal{E}_T}\|v_2\|_{\mathcal{E}_T}\right\}$$
$$\le C\left\{\left(Q+\sqrt{T}Q^2+\|y\|_{\mathcal{E}_T}\right)\|y\|_{\mathcal{E}_T}+\frac{\sqrt{T}Q^3}{\sqrt{r}}\right\}$$
$$\le \left(C_\sharp Q+C\|y\|_{\mathcal{E}_T}\right)\|y\|_{\mathcal{E}_T}+\frac{1}{4C} \tag{4.31}$$

with some positive constant $C_\sharp$. It is not difficult to show that y is small in this way if $C_\sharp Q < 1$. However, y is out of control when Q is large.

In [9] Bourgain and Pavlovic compute $\|y\|_{\mathcal{E}_T}$, dividing the time-interval into many parts. Although the author thinks that it is unnecessary for us to employ their method on the proof of Theorem 4.1, it is supposed that their technique leads us to some new idea and inspiration. So, the author would give an explaining of their method, in what follows. Let $T_0 \in (0,T)$ be fixed, and let T_0 be assumed as a new initial time for the equation (4.27) with initial datum $y(T_0)$. That is to say, for $t > T_0$

$$y(t)=e^{(t-T_0)\Delta}y(T_0)+\int_{T_0}^t e^{(t-\tau)\Delta}G(\tau)d\tau. \tag{4.32}$$

One can rewrite the second terms in the right hand side of (4.32) by

$$\begin{aligned}\int_{T_0}^t e^{(t-\tau)\Delta}G(\tau)d\tau &= \int_0^t e^{(t-\tau)\Delta}G(\tau)\chi_{[T_0,t]}(\tau)d\tau\\ &= -\mathcal{B}(y^\sharp,u_1^\sharp-v_2^\sharp)-\mathcal{B}(u_1^\sharp-v_2^\sharp,y^\sharp)-\mathcal{B}(y^\sharp)-\mathcal{B}(u_1^\sharp,v_2^\sharp)\end{aligned}$$

in terms of $\mathcal{B}$. Here we have denoted $^\sharp$ by

$$y^\sharp := y^\sharp(t) := \begin{cases} 0 & if \quad t<T_0,\\ y(t) & if \quad t\ge T_0.\end{cases}$$

Analogously, we define $u_1^\sharp$ and $v_2^\sharp$. By semigroup property we also rewrite the first terms in the right hand side of (4.32) by

$$\begin{aligned}e^{(t-T_0)\Delta}y(T_0) &= e^{(t-T_0)\Delta}\int_0^{T_0} e^{(T_0-\tau)\Delta}G(\tau)d\tau\\ &= \int_0^t e^{(t-\tau)\Delta}G(\tau)\chi_{[0,T_0]}(\tau)d\tau\\ &= -\mathcal{B}(y^\flat,u_1^\flat-v_2^\flat)-\mathcal{B}(u_1^\flat-v_2^\flat,y^\flat)-\mathcal{B}(y^\flat)-\mathcal{B}(u_1^\flat,v_2^\flat).\end{aligned}$$

Here $^\flat$ denotes

$$y^\flat := y^\flat(t) := \begin{cases} y(t) & if \quad t<T_0,\\ 0 & if \quad t\ge T_0.\end{cases}$$

Analogously, we define $u_1^\flat$ and $v_2^\flat$. When we settle $T_1 \in (T_0,T)$ small again to deduce that $\|y\|_{\mathcal{E}_{T_1}}$ have a better estimate. By (3.2) and so on, we see

$$\|y\|_{\mathcal{E}_{T_1}} \le C(\|u_1^\sharp\|_{\mathcal{E}_{T_1}}+\|v_2^\sharp\|_{\mathcal{E}_{T_1}}+\|y^\sharp\|_{\mathcal{E}_{T_1}})\|y^\sharp\|_{\mathcal{E}_{T_1}}+C\|u_1^\sharp\|_{\mathcal{E}_{T_1}}\|v_2^\sharp\|_{\mathcal{E}_{T_1}}$$

$$
\begin{aligned}
&+ C\left(\|u_1^\flat\|_{\mathcal{E}_{T_1}} + \|v_2^\flat\|_{\mathcal{E}_{T_1}} + \|y^\flat\|_{\mathcal{E}_{\mathcal{E}_1}}\right)\|y^\flat\|_{\mathcal{E}_{T_1}} + C\|u_1^\flat\|_{\mathcal{E}_{T_1}}\|v_2^\flat\|_{\mathcal{E}_{T_1}} \\
\leq\ & C\left(\frac{Q}{\sqrt{r}} + \sqrt{T_0}Q^2 + \sqrt{T_1 - T_0}Q^2 + \|y\|_{\mathcal{E}_{T_1}}\right)\|y\|_{\mathcal{E}_{T_1}} \\
&+ C\frac{\sqrt{T_0} + \sqrt{T_1 - T_0}}{\sqrt{r}}Q^3.
\end{aligned}
$$

One may have some improvements by this method, repeating and repeating.

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