Pseudodifferential operators on periodic graphs

V. Rabinovich, S. Roch

Abstract

The main aim of the paper is Fredholm properties of a class of bounded linear operators acting on weighted Lebesgue spaces on an infinite metric graph Γ which is periodic with respect to the action of the group $\mathbb{Z}^n$. The operators under consideration are distinguished by their local behavior: they act as (Fourier) pseudodifferential operators in the class OPS^0 on every open edge of the graph, and they can be represented as a matrix Mellin pseudodifferential operator on a neighborhood of every vertex of Γ. We apply these results to study the Fredholm property of a class of singular integral operators and of certain locally compact operators on graphs.

1 Introduction

Schrödinger operators on combinatorial and quantum graphs have attracted a lot of attention in the last time due to their interesting properties and existing and expected applications in nano-structures (see, for instance, [2, 4, 6, 7, 8] and the references cited there). The present paper is devoted to a quite general class of bounded linear operators acting on weighted Lebesgue spaces on an infinite metric graph Γ which is periodic with respect to the action of the group $\mathbb{Z}^n$. Our main emphasis is on Fredholm properties of these operators.

More precisely, the operators under consideration are distinguished by their local behavior: they act as (Fourier) pseudodifferential operators in the class OPS^0 on every open edge of the graph, and they can be represented as a matrix Mellin pseudodifferential operator on a neighborhood of every vertex of Γ. The appearance of Mellin convolution operators in this context is quite natural: near a vertex of the graph, a pseudodifferential operator can be written as a sum of a singular integral operator on a system of rays and a locally compact operator, and every operator of this form corresponds to a matrix Mellin convolution or, more general, a matrix Mellin pseudodifferential operator (see, for instance, Chap. 4 in [16], Chap. 4.6 in [13], [14] and references cited there). Mellin convolution operators were used in [10] to study pseudodifferential operators on *finite* graphs.

When studying Fredholm properties of general (non-periodic) operators, the most challenging part is to understand their local invertibility at infinity. For this goal, we use the limit operators method (see [13] for an overview) which will allow us to reduce the local invertibility at infinity to the invertibility of every single operator in a family of periodic operators, the so-called limit operators. The limit operators method already proved to be a very effective tool for the investigation of essential spectra of operators on combinatorial periodic graphs in [11, 12] and of essential spectra of Schrödinger, Dirac and Klein-Gordon operators on $\mathbb{R}^n$ in [15].

The paper is organized as follows. In Section 2 we recall some auxiliary material concerning Wiener algebras on $\mathbb{Z}^n$, (Fourier) pseudodifferential operators on $\mathbb{R}$, and matrix Mellin pseudodifferential operators on $\mathbb{R}_+ = [0, \infty)$. In Section 3 we introduce a class of pseudodifferential operators on a periodic graph and establish necessary and sufficient conditions for their Fredholmness. Our basic tool to study the Fredholm property is Simonenko's local principle (see [18] and Section 2.5 in [16]). The limit operators method can be used to relate the local invertibility at infinity to the invertibility of periodic operators, to which then a version of Floquet theory can be applied. In Section 4 we consider two applications:

(i) the Fredholm property of singular integral operators $A = aI + b\mathcal{S}_{\Gamma,\phi}$ on a periodic graph $\Gamma \subset \mathbb{R}^2$, where a, b are certain bounded slowly oscillating and piecewise continuous functions with discontinuities only at the vertices of Γ, and where $\mathcal{S}_{\Gamma,\phi}$ is the singular integral operator

$$(\mathcal{S}_{\Gamma,\phi}u)(x) = \frac{1}{\pi i}\int_\Gamma \frac{\phi(x, x-y)u(y)}{y-x}dy, \quad x \in \Gamma,$$

with $\phi \in C^\infty(\Gamma \times \mathbb{R}^2)$ a function the decaying behavior of which will be specified below;

(ii) the Fredholm property of integral operators $A = aI + bT$ on a periodic graph $\Gamma \subset \mathbb{R}^2$ where a, b are bounded, uniformly continuous and slowly oscillating functions on Γ and T is an integral operator of the form

$$(Tu)(x) = \int_\Gamma k(x-y)u(y)dy, \quad x \in \Gamma,$$

where $k : \mathbb{R}^2 \to \mathbb{C}$ is a continuous function with the property that there are positive constants C and ε such that

$$|k(z)| \leq C(1+|z|)^{-2-\varepsilon} \quad \text{for all } z \in \mathbb{R}^2.$$

This work was supported by CONACYT Project 81615 and DFG Grant Ro 1100/8-1.

2 Auxiliary material

2.1 Wiener algebras on $\mathbb{Z}^n$

Given a complex Banach space X, let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators on X and $\mathcal{K}(X)$ the closed ideal of $\mathcal{B}(X)$ of all compact operators. For $A \in \mathcal{B}(X)$, we write $sp_X A$ for the spectrum and $ess\, sp_X A$ for the essential spectrum of A, i.e., for the set of all $\lambda \in \mathbb{C}$ such that the operator $A - \lambda I$ is not Fredholm on X. Further we let $l^p(\mathbb{Z}^n, X)$ stand for the Banach space of all functions $u : \mathbb{Z}^n \to X$ with

$$\|u\|^p_{l^p(\mathbb{Z}^n,\, X)} := \sum_{x \in \mathbb{Z}^n} \|u(x)\|^p_X < \infty$$

if $p \in [1, \infty)$ and

$$\|u\|_{l^\infty(\mathbb{Z}^n,\, X)} := \sup_{x \in \mathbb{Z}^n} \|u(x)\|_X < \infty.$$

On $l^p(\mathbb{Z}^n, X)$, we consider operators of the form

$$A = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha \tau_\alpha \tag{1}$$

where $a_\alpha \in l^\infty(\mathbb{Z}^n, \mathcal{B}(X))$ and τ_α is the operator of shift by $\alpha \in \mathbb{Z}^n$,

$$(\tau_\alpha u)(x) := u(x - \alpha), \quad x \in \mathbb{Z}^n.$$

We say that the operator A in (1) belongs to the Wiener algebra $W(\mathbb{Z}^n, X)$ if

$$\|A\|_{W(\mathbb{Z}^n, X)} := \sum_{\alpha \in \mathbb{Z}^n} \|a_\alpha\|_{\mathcal{B}(X)} < \infty.$$

It is well known that $W(\mathbb{Z}^n, X) \subset \mathcal{B}(l^p(\mathbb{Z}^n, X))$ for every $p \in [1, \infty]$ and that $sp_{l^p(\mathbb{Z}^n, X)} A$ does not depend on p if A belongs to $W(\mathbb{Z}^n, X)$.

Finally, we denote the operator of multiplication by a function f by fI and let $\mathcal{H}$ stand for the set of all sequences $h : \mathbb{N} \to \mathbb{Z}^n$ which tend to infinity in the sense that, for every $R > 0$ there is an m_0 such that $|h(m)| > R$ for all $m \geq m_0$.

Definition 1 *Let $A \in \mathcal{B}(l^p(\mathbb{Z}^n, X))$. An operator $A^h \in \mathcal{B}(l^p(\mathbb{Z}^n, X))$ is called the* limit operator of A defined by the sequence $h \in \mathcal{H}$ *if, for every function φ on $\mathbb{Z}^n$ with finite support,*

$$\lim_{m \to \infty} \|(\tau^{-1}_{h(m)} A \tau_{h(m)} - A^h)\varphi I\|_{\mathcal{B}(l^p(\mathbb{Z}^n,\, X))} = 0$$

and

$$\lim_{m \to \infty} \|\varphi(\tau^{-1}_{h(m)} A \tau_{h(m)} - A^h)\|_{\mathcal{B}(l^p(\mathbb{Z}^n,\, X))} = 0.$$

We denote the set of all limit operators of A by $Lim(A)$.

We say that an operator $A \in \mathcal{B}(l^p(\mathbb{Z}^n, X))$ is rich if every sequence $h \in \mathcal{H}$ has a subsequence g such that the limit operator A^g of A with respect to g exists. For $r > 0$, let χ_r denote the characteristic function of the set $\{x \in \mathbb{Z}^n : |z| > r\}$.

Definition 2 *The operator $A \in \mathcal{B}(l^p(\mathbb{Z}^n, X))$ is called* locally invertible at infinity *if there exist an $r > 0$ and operators $L_r, R_r \in \mathcal{B}(l^p(\mathbb{Z}^n, X))$ such that $L_r A \chi_r I = \chi_r I$ and $\chi_r A R_r = \chi_r I$.*

The following is Theorem 2.5.7 in [13].

Proposition 3 *Let $A \in W(\mathbb{Z}^n, X)$ be a rich operator. Then A is locally invertible at infinity on $l^p(\mathbb{Z}^n, X)$ for some $p \in (1, \infty)$ if and only every limit operator A^h of A is invertible on one of the spaces $l^p(\mathbb{Z}^n, X)$ with $p \in [1, \infty]$.*

2.2 Pseudodifferential operators on $\mathbb{R}$

The theory of pseudodifferential operators is developed in several textbooks, e.g. [17, 19]. Here we only fix the notation and collect some facts for later reference. For proofs of Propositions 4 and 5 see [20] and Chapter 3 of [19], respectively.

Let $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$. We say that a is a symbol in the class S^m if $a \in C^\infty(\mathbb{R} \times \mathbb{R})$ and

$$|a|_{k,l} := \sup_{(x,\xi) \in \mathbb{R} \times \mathbb{R}} \sum_{\alpha \le k,\, \beta \le l} |\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \langle \xi \rangle^{\alpha - m} < \infty$$

for all $k, l \in \mathbb{N}_0$. To each symbol a, we correspond a (Fourier) pseudodifferential operator (ψdo for short) $Op(a)$ which acts on $u \in C_0^\infty(\mathbb{R})$ by

$$(Op(a)u)(x) := \frac{1}{2\pi} \int_{\mathbb{R}} a(x, \xi) \hat{u}(\xi) e^{ix\xi} d\xi, \quad x \in \mathbb{R},$$

where $\hat{u}$ stands for the Fourier transform of u. By OPS^0 we denote the class of all ψdos with symbol in S^0. We will need the following properties of ψdos.

Proposition 4 *(a) Let $a \in S^0$. Then $Op(a)$ is bounded on $L^p(\mathbb{R})$ for $p \in (1, \infty)$ and*

$$\|Op(a)\|_{\mathcal{B}(L^p(\mathbb{R}))} \le C|a|_{2,2}$$

with a constant C independent of a.
(b) Let $a_1 \in S^{m_1}$ and $a_2 \in S^{m_2}$. Then $Op(a_1)Op(a_2) = Op(c)$ with $c \in S^{m_1+m_2}$ and $a_1 a_2 - c \in S^{m_1+m_2-1}$.

We say that an operator $A \in \mathcal{B}(L^p(\mathbb{R}))$ is *locally Fredholm at $x_0 \in \mathbb{R}$* if there are an open neighborhood U of x_0 and operators $R_{x_0}, L_{x_0} \in \mathcal{B}(L^p(\mathbb{R}))$ such that

$$L_{x_0} A \chi_U I - \chi_U I, \quad \chi_U A R_{x_0} - \chi_U I \in \mathcal{K}(L^p(\mathbb{R})).$$

Proposition 5 *Let $a \in S^0$. Then $Op(a)$ is locally Fredholm at $x_0 \in \mathbb{R}$ if and only if $Op(a)$ is elliptic at x_0, that is if $\liminf_{\xi \to \infty} |a(x_0, \xi)| > 0$.*

2.3 Mellin pseudodifferential operators on $\mathbb{R}_+$

Set $\mathbb{R}_+ := (0, \infty)$. We say that a matrix function $a = (a_{ij})_{i,j=1}^n : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}^{n\times n}$ belongs to $\mathcal{E}(n)$ if every entry a_{ij} is in $C^\infty(\mathbb{R}_+ \times \mathbb{R})$ and

$$|a|_{l_1,l_2} := \max_{1\le i,j\le n} \sup_{(r,\lambda)\in\mathbb{R}_+\times\mathbb{R}} \sum_{\alpha\le l_1,\,\beta\le l_2} |(r\partial_r)^\beta \partial_\lambda^\alpha a_{ij}(r,\,\lambda)|\langle\lambda\rangle^\beta < \infty \tag{2}$$

for all $l_1,\, l_2 \in \mathbb{N}_0$. Let $a \in \mathcal{E}(n)$. Then the operator

$$(op(a)u)(r) := \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}_+} a(r,\,\lambda)\,(r\rho^{-1})^{i\lambda} u(\rho)\rho^{-1} d\rho d\lambda, \quad r \in \mathbb{R}_+, \tag{3}$$

acting on $u \in C_0^\infty(\mathbb{R}_+,\, \mathbb{C}^n)$ is called the *Mellin pseudodifferential operator* (Mellin ψdo for short) with symbol $a \in \mathcal{E}(n)$. The class of all Mellin ψdos with symbol in $\mathcal{E}(n)$ is denoted by $OP\mathcal{E}(n)$.

A function $a \in \mathcal{E}(n)$ is said to be *slowly oscillating at the point* 0 if

$$\lim_{r\to+0} \sup_{\lambda\in\mathbb{R}} |(r\partial_r)^\beta \partial_\lambda^\alpha a_{ij}(r,\,\lambda)|\langle\lambda\rangle^\alpha = 0 \tag{4}$$

for all $\alpha \in \mathbb{N}_0$ and $\beta \in \mathbb{N}$. We let $\mathcal{E}_{sl}(n)$ denote the set of all functions which are slowly oscillating at 0 and write $\mathcal{E}_0(n)$ for the set of all functions $a \in \mathcal{E}(n)$ which satisfy (4) for all $\alpha,\, \beta \in \mathbb{N}_0$. The corresponding classes of Mellin ψdos are denoted by $OP\mathcal{E}_{sl}(n)$ and $OP\mathcal{E}_0(n)$, respectively.

Mellin ψdos are pseudodifferential operators on the multiplicative group $\mathbb{R}_+$ with respect to the invariant measure $d\mu = \frac{dr}{r}$. They can be obtained from (Fourier) ψdos on $\mathbb{R}$ by the change of variables $\mathbb{R} \ni x \mapsto r = e^{-x} \in \mathbb{R}_+$, which transforms the point $+\infty$ to the point 0. Consequently, the main properties of Mellin ψdos follow immediately from the corresponding properties of (Fourier) ψdos on $\mathbb{R}$.

Let $L^p(\mathbb{R}_+,\, d\mu,\, \mathbb{C}^n)$ denote the space of all measurable functions $u : \mathbb{R}_+ \to \mathbb{C}^n$ with

$$\|u\|^p_{L^p(\mathbb{R}_+,\, d\mu,\, \mathbb{C}^n)} := \int_{\mathbb{R}_+} \|u(r)\|^p_{\mathbb{C}^n} d\mu < \infty.$$

The following results can be found in [15].

Proposition 6 *Let $a \in \mathcal{E}(n)$ and $p \in (1,\, \infty)$. Then the Mellin ψdo $op(a)$ is bounded on $L^p(\mathbb{R}_+,\, d\mu,\, \mathbb{C}^n)$ and*

$$\|op(a)\|_{\mathcal{B}(L^p(\mathbb{R}_+,\, d\mu,\, \mathbb{C}^n))} \le C|a|_{2,2} \tag{5}$$

with a constant C independent of a.

Proposition 7 *(a) Let $a, b \in \mathcal{E}(n)$. Then $op(a)op(b) = op(c) \in OP\mathcal{E}(n)$, where*

$$c(r, \lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}_+} a(r, \lambda + \eta) b(r\rho, \lambda) \rho^{-i\eta} d\rho d\eta. \tag{6}$$

(b) Let $a \in \mathcal{E}(n)$ and consider the operator $op(a)$ as acting on $L^p(\mathbb{R}_+, d\mu, \mathbb{C}^n)$. Then $op(a)^* = op(b) \in \mathcal{E}(n)$, where

$$b(r, \lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}_+} a^*(r\rho, \lambda + \eta) \rho^{-i\eta} d\rho d\eta. \tag{7}$$

Here $op(a)^*$ stands for the adjoint operator and a^* for the adjoint matrix function. The integrals in (6) and (7) are understood as oscillatory integrals.

Proposition 8 *(a) Let $a, b \in \mathcal{E}_{sl}(n)$. Then $op(a)op(b) = op(c)$ where $c \in \mathcal{E}_{sl}(n)$ and $c - ab \in \mathcal{E}_0(n)$.*
(b) Let $a \in \mathcal{E}_{sl}(n)$ and consider the operator $op(a)$ as acting on $L^p(\mathbb{R}_+, d\mu, \mathbb{C}^n)$. Then $op(a)^* = op(b)$ where $b \in \mathcal{E}_{sl}(n)$ and $b - a^* \in \mathcal{E}_0(n)$.

In what follows, we will consider Mellin ψdos on the weighted Lebesgue spaces $L^p(\mathbb{R}_+, w, d\mu)$ of all measurable functions with norm

$$\|u\|_{L^p(\mathbb{R}_+, w, d\mu)} := \|wu\|_{L^p(\mathbb{R}_+, d\mu)},$$

where the weight w is of the form $w = \exp \sigma$ with a function σ which satisfies the conditions

$$\lim_{r \to +0} \left(r \frac{d}{dr} \right)^2 \sigma(r) = 0 \quad \text{and} \quad \sup_{r \in \mathbb{R}_+} \left| \left(r \frac{d}{dr} \right)^k \sigma(r) \right| < \infty \tag{8}$$

for every $k \in \mathbb{N}$. Moreover we assume that there is an interval $\mathcal{I} = (c, d)$ which contains 0 such that the function $\varkappa_\sigma(r) := r\sigma'(r)$ satisfies

$$c < \liminf_{r \in \mathbb{R}_+} \varkappa_\sigma(r) \leq \limsup_{r \in \mathbb{R}_+} \varkappa_\sigma(r) < d. \tag{9}$$

We let $\mathcal{R}(\mathcal{I})$ denote the collection of all weights w such that (8) and (9) hold. By $\mathcal{E}(n, \mathcal{I})$ we denote the set of all symbols $a \in \mathcal{E}(n)$ such that the function $a(\cdot, \lambda)$ can be analytically extended with respect to λ into the strip $\Pi := \{\lambda \in \mathbb{C} : \Im(\lambda) \in \mathcal{I}\}$ and this continuation satisfies

$$\sup_{(r, \lambda) \in \mathbb{R}_+ \times \Pi} |(r\partial_r)^\beta \partial_\lambda^\alpha a_{ij}(r, \lambda)| < \infty.$$

The corresponding class of Mellin ψdos with analytical symbol is denoted by $OP\mathcal{E}(n, \mathcal{I})$.

Proposition 9 *Let $a \in \mathcal{E}(n, \mathcal{I})$, $w \in \mathcal{R}(\mathcal{I})$ and $p \in (1, \infty)$. Then the operator $op(a)$ is bounded on $L^p(\mathbb{R}_+, w, d\mu)$, and*

$$\|op(a)\|_{\mathcal{B}(L^p(\mathbb{R}_+, w, d\mu, \mathbb{C}^n))} \le C|a|_{2,4}$$

with a constant C independent of a.

Proposition 10 *Let $a \in \mathcal{E}_{sl}(n, \mathcal{I})$ and $w = \exp \sigma \in \mathcal{R}(\mathcal{I})$. Then $wop(a)w^{-1} = op(b)$ with $b \in \mathcal{E}_{sl}(n)$ and*

$$b(r, \lambda) = a(r, \lambda + i\varkappa_\sigma(r)) + q(r, \lambda) \tag{10}$$

where $q \in \mathcal{E}_0(n)$.

Now we turn to local invertibility properties of Mellin pseudodifferential operators. We say that an operator $A \in \mathcal{B}(L^p(\mathbb{R}_+, w, d\mu, \mathbb{C}^n))$ is *locally invertible at the point* 0, if there are an $r > 0$ and operators L_r and R_r such that

$$L_r A \chi_r I = \chi_r I \quad \text{and} \quad \chi_r A R_r = \chi_r I$$

where χ_r refers to the characteristic function of the interval $[0, r]$.

Proposition 11 *Let $a \in \mathcal{E}_{sl}(n, \mathcal{I})$, $w = \exp \sigma \in \mathcal{R}(\mathcal{I})$, and consider $op(a)$ as acting on $L^p(\mathbb{R}_+, w, d\mu, \mathbb{C}^n)$. Then $op(a)$ is locally invertible at the point 0 if and only if*

$$\liminf_{r \to +0} \inf_{\lambda \in \mathbb{R}} |\det a(r, \lambda + i\varkappa_\sigma(r))| > 0.$$

3 Operators on periodic graphs

3.1 Periodic graphs

By a directed (combinatorial) graph (or digraph for short) we mean a pair $\Gamma_{comb} = (\mathcal{V}, \mathcal{E})$ consisting of a countably infinite set $\mathcal{V}$ of vertices and a set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ of edges. We think of $e = (v, w) \in \mathcal{E}$ as the oriented edge which starts at v and ends at w. We only consider digraphs without loops and without multiple edges, i.e., $\mathcal{E}$ does not contain pairs of the form (v, v), and if $(v, w) \in \mathcal{E}$, then $(w, v) \notin \mathcal{E}$. Given a digraph, there is a related undirected (combinatorial) graph, which arises by ignoring the orientation. Formally, the undirected graph related with Γ_{comb} is the pair $(\mathcal{V}, \mathcal{E}_{ud})$ where $\mathcal{E}_{ud}$ is the set of all subsets of $\mathcal{V}$ consisting of two elements v, w such that $(v, w) \in \mathcal{E}$. We say that $e = \{v, w\}$ connects the vertices v and w and call v and w the endpoints of the edge e. For $v \in \mathcal{V}$, let $\mathcal{E}_v$ denote the set of all edges which have v as one of its endpoints. We assume that the valency $\mathrm{val}(v)$ of any vertex v is finite and positive. In particular, vertices without incident edges are not allowed.

An n-tuple $p = (v_i)_{l=0}^n$ of vertices is called a path in $(\mathcal{V}, \mathcal{E}_{ud})$ if $\{v_i, v_{i+1}\} \in \mathcal{E}_{ud}$ for every $i = 0, \ldots, n-1$. In this case, we call p a path joining v_0 with v_n. An undirected graph is connected if any two of its vertices are connected by a path, and a digraph is connected if the related undirected graph is connected (thus, connectedness is defined independently of orientation).

A function $l : \mathcal{E} \to (0, \infty)$ is called a *length function*, and $l(e)$ is called the length of the edge e. Each triple $\Gamma_{metr} = (\mathcal{V}, \mathcal{E}, l)$ determines a metric graph by identifying the edge e with the line segment $[0, l(e)]$. The formal definition is as follows (the construction we use is quite similar to the definition in [9]). Let

$$\Gamma^{\sim}_{metr} := \{(e, x) \in \mathcal{E} \times [0, \infty) : e \in \mathcal{E},\, x \in [0, l(e)]\}$$

and consider the function $\Pi^{\sim} : \Gamma^{\sim}_{metr} \to \mathcal{V} \cup \mathcal{E}$ defined by

$$\Pi^{\sim}(e, x) = \Pi^{\sim}((v, w), x) := \begin{cases} e & \text{if } x \in (0, l(e)), \\ v & \text{if } x = 0, \\ w & \text{if } x = l(e). \end{cases}$$

Two points $(e, x), (f, y) \in \Gamma^{\sim}_{metr}$ are said to be equivalent if $\Pi^{\sim}(e, x)$ and $\Pi^{\sim}(f, y)$ are in $\mathcal{V}$ and if $\Pi^{\sim}(e, x) = \Pi^{\sim}(f, y)$. This defines an equivalence relation on $\Gamma^{\sim}_{metr}$ which we denote by $\sim$. The equivalence class of (e, x) with respect to $\sim$ is denoted by $(e, x)^{\sim}$. Clearly, the equivalence class of each point (e, x) with $x \neq 0$ and $x \neq l(e)$ is a singleton, whereas points (e, x) with $x = 0$ or $x = l(e)$ are identified if they belong to the "same" vertex of Γ_{comb}.

The set $\Gamma_{metr} := \Gamma^{\sim}_{metr}/\sim$ of all equivalence classes is called the *metric graph* associated with $(\mathcal{V}, \mathcal{E}, l)$. Since $\Pi^{\sim}(e, x)$ only depends on the equivalence class of (e, x), we can define the quotient map

$$\Pi : \Gamma_{metr} \to \mathcal{V} \cup \mathcal{E},\ (e, x)^{\sim} \mapsto \Pi^{\sim}(e, x).$$

The elements of $\Pi^{-1}(\mathcal{V})$ and $\Pi^{-1}(\mathcal{E})$ are called the vertices and the open edges of Γ_{metr}, whereas the unions $\Pi^{-1}(v \cup (v, w) \cup w)$ for an edge $e = (v, w) \in \mathcal{E}$ are called the closed edges of Γ_{metr}. Thus, a closed edge is the union of an open edge with its end points.

There is a natural topology on a metric graph which is defined as follows. Provide $\mathcal{E}$ with the discrete topology and $[0, \infty)$ with the standard (Euclidean) topology and consider on $\Gamma^{\sim}_{metr}$ the restriction of the product topology on $\mathcal{E} \times [0, \infty)$. Then the topology on the metric graph Γ_{metr} is the quotient of the topology on $\Gamma^{\sim}_{metr}$ by the relation $\sim$.

Moreover, this topology is induced by a metric (whence the notion *metric graph*). Given an edge $e = (v, w) \in \mathcal{E}$ and a point $(e, x)^{\sim} \in \Gamma_{metr}$ with $x \neq 0$ and $x \neq l(e)$, we call $[v, x] := \{(e, y) : 0 \leq y \leq x\}$ and $[x, w] := \{(e, y) : x \leq y \leq l(e)\}$ the segments joining (e, x) with the end points of the edge e. With the segments $[v, x]$ and $[x, w]$ we associate the lengths x and $l(e) - x$, respectively.

Let $(e, x)^\sim$, $(f, y)^\sim \in \Gamma_{metr}$. By a path between $(e, x)^\sim$ and $(f, y)^\sim$ we mean an n-tuple $p = (v_i)_{l=0}^n$ of vertices in $\mathcal{V}$ such that v_0 is an endpoint of the edge e and v_n is an endpoint of f and v_0 and v_n are connected by a path in Γ_{comb} as above. The length of this path is defined as the sum of the lengths of the segments joining $(e, x)^\sim$ and $(f, y)^\sim$ with the corresponding endpoints v_0 and v_n, respectively, plus the sum $\sum_{i=0}^{n-1} l(e_i)$ where e_i is the edge $(v_i, v_{i+1}) \in \mathcal{E}$. If $e = f$ we also consider the segment $[x, y] := \{(e, z) : x \leq z \leq y\}$ as a path of length $y - x$ between $(e, x)^\sim$ and $(e, y)^\sim$. The distance of $(e, x)^\sim$ and $(f, y)^\sim$ is then the infimum of the lengths of all paths joining these points. Note that the distance of any two different points is positive since every vertex has finite valency.

Since the edges of Γ_{comb} and the open (resp. closed) edges of Γ_{metr} are in a one-to-one correspondence, we often use the same notation e both for an edge in $\mathcal{E}$ and for the corresponding open (resp. closed) edge of Γ_{metr}. Moreover, we identify the open (resp. closed) edge of Γ_{metr} which corresponds to $e \in \mathcal{E}$ with the open (resp. closed) interval $(0, l(e))$ (resp. $[0, l(e)]$). Accordingly, we usually identify the point $(e, x)^\sim$ with x and write $x \in e$ in order to indicate that x is considered as a point of the metric graph. Finally, we provide Γ_{metr} by the measure which is induced by the one-dimensional Lebesgue measure on each edge.

In what follows, we let $\Gamma = (\mathcal{V}, \mathcal{E}, l)$ be a metric graph which is periodic with respect to $\mathbb{Z}^n$ (or $\mathbb{Z}^n$-periodic for short) in the following sense: The group $\mathbb{Z}^n$ acts freely on Γ, i.e., there is a mapping

$$\Gamma \times \mathbb{Z}^n \to \Gamma, \ (x, g) \mapsto x + g$$

such that $x + 0 = x$ and $x + (g_1 + g_2) = (x + g_1) + g_2$ for all $g_1, g_2 \in \mathbb{Z}^n$ and $x \in \Gamma$, and if $x = x + g$ for some $x \in \Gamma$ and $g \in \mathbb{Z}^n$, then $g = 0$. Moreover, we assume that every mapping $x \to x + g$ sends vertices to vertices and edges to edges such that $v + g$ and $w + g$ are the endpoints of the image $e + g$ of the edge e with endpoints v and w and that the lengths of e and $e + g$ are equal (thus, the length function l is $\mathbb{Z}^n$-periodic). Then both the valency and the metric on Γ are invariant with respect to the action of $\mathbb{Z}^n$, that is $\mathrm{val}(v + g) = \mathrm{val}(v)$ for every vertex $v \in \mathcal{V}$ and $\rho(x + g, y + g) = \rho(x, y)$ for arbitrary points $x, y \in \Gamma$ and every $g \in \mathbb{Z}^n$. Moreover, we assume that

$$\lim_{\mathbb{Z}^n \ni g \to \infty} \rho(x, y + g) = \infty \quad \text{for } x, y \in \Gamma. \tag{11}$$

If these conditions are satisfied, we call Γ a $\mathbb{Z}^n$-periodic metric graph.

In what follows we also suppose that the fundamental domain $\Gamma_0 := \Gamma/\mathbb{Z}^n$ of Γ with respect to the action of $\mathbb{Z}^n$ is compact in the corresponding quotient topology (thus, the action of $\mathbb{Z}^n$ on Γ is co-compact). Note that this property implies that Γ_0 contains only a finite number of vertices of Γ. For $g \in \mathbb{Z}^n$, we set $\Gamma_g := \{y \in \Gamma : y \in \Gamma_0 + g\}$. Then $\Gamma_{g_1} \cap \Gamma_{g_2} = \emptyset$ if $g_1 \neq g_2$ and $\Gamma = \cup_{g \in \mathbb{Z}^n} \Gamma_g$.

A function $w : \Gamma \to [0, \infty)$ is called a weight if w is continuous on $\Gamma \setminus \mathcal{V}$ and $w^{-1}(\{0, \infty\}) \subset \mathcal{V}$. For $p \in [1, \infty]$, we let $L^p_w(\Gamma)$ denote the space of all measurable functions on Γ such that

$$\|u\|^p_{L^p_w(\Gamma)} := \int_\Gamma |w(x)u(x)|^p dx$$

in case $p < \infty$ and

$$\|u\|_{L^\infty_w(\Gamma)} := \operatorname{esssup}_{x\in\Gamma}|w(x)u(x)|$$

if $p = \infty$ are finite, respectively. We write $L^p(\Gamma)$ instead of $L^p_w(\Gamma)$ if w is identically 1.

In what follows we suppose that the weight is periodic with respect to $\mathbb{Z}^n$, that is $w \circ g = w$ for all $g \in \mathbb{Z}^n$ (where we identify g with the mapping $x \mapsto x+g$). Under this condition, the spaces $L^p_w(\Gamma)$ are invariant with respect to the action of $\mathbb{Z}^n$, i.e., $\|u \circ g\|_{L^p_w(\Gamma)} = \|u\|_{L^p_w(\Gamma)}$ for every $g \in \mathbb{Z}^n$. For $u \in L^p_w(\Gamma)$ and $g \in \mathbb{Z}^n$, set $u_g := u|_{\Gamma_g}$. Then

$$\|u\|^p_{L^p_w(\Gamma)} = \sum_{g\in\mathbb{Z}^n} \|u_g\|^p_{L^p_w(\Gamma_g)}$$

for $p \in [1, \infty)$ and

$$\|u\|_{L^\infty_w(\Gamma)} = \sup_{g\in\mathbb{Z}^n} \|u_g\|_{L^\infty_w(\Gamma_g)}.$$

For $g \in \mathbb{Z}^n$, we let V_g denote the operator of shift by g which acts on functions in $L^p_w(\Gamma)$ as $(V_g u)(x) := u(x - g)$. Since w is $\mathbb{Z}^n$-periodic, the operators V_g are isometries on $L^p_w(\Gamma)$, and $V_g^{-1} = V_{-g}$.

Every weighted Lebesgue space $L^p_w(\Gamma)$ over a $\mathbb{Z}^n$-periodic metric graph is naturally isomorphic to an l^p-space of vector-valued $\mathbb{Z}^n$-sequences as follows. Let $w_0 := w|_{\Gamma_0}$ and $X := L^p_{w_0}(\Gamma_0)$, and consider the operator

$$U : L^p_w(\Gamma) \to l^p(\mathbb{Z}^n, X), \quad (Uu)(g) = (V_g u)|_{\Gamma_0}.$$

The operator U is an isometry with inverse $U^{-1} : l^p(\mathbb{Z}^n, X) \to L^p_w(\Gamma)$ acting as

$$U^{-1} f = \sum_{g\in\mathbb{Z}^n} \chi_g V_g f_g \chi_0 I$$

where χ_0 refers to the characteristic function of Γ_0 and $\chi_g := V_g^{-1}\chi_0$.

Let $A \in \mathcal{B}(L^p_w(\Gamma))$. Then $\tilde{A} := UAU^{-1} \in \mathcal{B}(l^p(\mathbb{Z}^n, X))$ has the matrix representation

$$(\tilde{A}\psi)_\alpha = \sum_{\beta\in\mathbb{Z}^n} \tilde{A}_{\alpha\beta}\tau_\beta\psi_\alpha, \quad \alpha \in \mathbb{Z}^n,$$

where

$$\tilde{A}_{\alpha\beta} = \chi_0 V_{-\alpha} A V_{\alpha-\beta} \chi_0 I = V_{-\alpha}\chi_\alpha A \chi_{\alpha-\beta} V_{\alpha-\beta}.$$

We say that the operator $A \in \mathcal{B}(L^p_w(\Gamma))$ belongs to the Wiener algebra $W_w(\Gamma)$ if

$$\|A\|_{W_w(\Gamma)} := \sup_{\alpha \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{Z}^n} \|\chi_\alpha A \chi_{\alpha-\beta} I\|_{\mathcal{B}(L^p_w(\Gamma))} < \infty.$$

Then the mapping $A \mapsto UAU^{-1}$ is an isometric isomorphism between the Wiener algebras $W_w(\Gamma)$ and $W(\mathbb{Z}^n, X)$. Because operators in $W(\mathbb{Z}^n, X)$ are bounded on $l^p(\mathbb{Z}^n, X)$, operators in $W_w(\Gamma)$ are bounded on $L^p_w(\Gamma)$.

3.2 Simonenko's local principle

We will base our study of pseudodifferential operators on periodic graphs on Simonenko's local principle which we recall here briefly from [18] in a form which is convenient for our purposes; see also Section 2.5 in [16].

We start with some definitions. Let $\dot{\Gamma}$ denote the one-point compactification of the periodic metric graph Γ. For a measurable subset F of $\dot{\Gamma}$, let χ_F denote its characteristic function. An operator $A \in \mathcal{B}(L^p_w(\Gamma))$ is said to be *of local type* on $\dot{\Gamma}$ if, for arbitrary open sets $F_1, F_2 \subset \dot{\Gamma}$ with disjoint closures, the operator $\chi_{F_1} A \chi_{F_2} I$ is compact on $L^p_w(\Gamma)$. Further, the operator $A \in \mathcal{B}(L^p_w(\Gamma))$ is called a *locally Fredholm operator at* $x \in \dot{\Gamma}$ if there are an open neighborhood F of x and operators $L, R \in \mathcal{B}(L^p_w(\Gamma))$ such that

$$LA\chi_F I - \chi_F I, \quad \chi_F AR - \chi_F I \in \mathcal{K}(L^p_w(\Gamma)).$$

Finally, we say that $A \in \mathcal{B}(L^p_w(\Gamma))$ is *locally invertible at infinity* if there are a positive constant r and operators $L, R \in \mathcal{B}(L^p_w(\Gamma))$ such that

$$LA\chi_{B'_r} I = \chi_{B'_r} I \quad \text{and} \quad \chi_{B'_r} AR = \chi_{B'_r} I,$$

where $B'_r := \{x \in \dot{\Gamma} : \rho(x, x_0) > r\}$ for a certain fixed point $x_0 \in \Gamma$ (one easily checks that the property of being locally invertible at infinity does not depend on the choice of x_0). For $p \in (1, \infty)$, it is also easy to see that an operator $A \in \mathcal{B}(L^p_w(\Gamma))$ is locally Fredholm at infinity if and only if it is locally invertible at infinity. The following is one of the main results of [18].

Proposition 12 *Let $A \in \mathcal{B}(L^p_w(\Gamma))$ be an operator of local type on $\dot{\Gamma}$. Then A is a Fredholm operator if and only if A is a locally Fredholm operator at every point $x \in \dot{\Gamma}$.*

Definition 13 *Let $A \in \mathcal{B}(L^p_w(\Gamma))$ and $h \in \mathcal{H}$. An operator $A^h \in \mathcal{B}(L^p_w(\Gamma))$ is called a* limit operator *of A defined by h if, for every compact subset M of Γ,*

$$\lim_{m\to\infty} \|(V^{-1}_{h(m)} A V_{h(m)} - A^h)\chi_M I\|_{\mathcal{B}(L^p_w(\Gamma))} = 0$$

and

$$\lim_{m\to\infty} \|\chi_M (V^{-1}_{h(m)} A V_{h(m)} - A^h)\|_{\mathcal{B}(L^p_w(\Gamma))} = 0.$$

If $p \in (1, \infty)$, one can show that A^h is the limit operator of A with respect to h if and only if

$$V_{h(m)}^{-1} A V_{h(m)} \to A^h \quad \text{and} \quad V_{h(m)}^{-1} A^* V_{h(m)} \to (A^h)^*$$

strongly as $m \to \infty$. We let $Lim(A)$ denote the set of all limit operators of A, and we call an operator A *rich* if every sequence $h \in \mathcal{H}$ has a subsequence g such that the limit operator A^g of A with respect to g exists.

Proposition 14 *Let $A \in W_w(\Gamma)$ be a rich operator. Then the operator A, considered as an element of $\mathcal{B}(L^p_w(\Gamma))$, is locally invertible at infinity if and only if all limit operators of A are invertible on $L^p_w(\Gamma)$.*

Proof. Let again $X := L^p_{w_0}(\Gamma)$ and consider the operator $\tilde{A} = UAU^{-1} \in W(\mathbb{Z}^n, X)$. It is easy to see that if the limit operator of A with respect to $h \in \mathcal{H}$ exists, then the limit operator of $\tilde{A}$ with respect to h exists, too, and $\tilde{A}^h = UA^hU^{-1}$. In particular, $\tilde{A}^h$ is invertible on $l^p(\mathbb{Z}^n, X)$ if and only if A^h is invertible on $L^p_w(\Gamma)$. Moreover, the operator $A : L^p_w(\Gamma) \to L^p_w(\Gamma)$ is locally invertible at infinity if and only if the operator $\tilde{A} : l^p(\mathbb{Z}^n, X) \to l^p(\mathbb{Z}^n, X)$ has this property. Hence, the assertion follows from Theorem 2.5.7 in [13]. ■

The following theorem is then an immediate consequence of Propositions 12 and 14.

Theorem 15 *Let $A \in W_w(\Gamma)$ be both rich and of local type on $L^p_w(\dot{\Gamma})$. Then A is a Fredholm operator on $L^p_w(\Gamma)$ if and only if A is a locally Fredholm operator at every point $x \in \Gamma$ and if all limit operators of A are invertible on $L^p_w(\Gamma)$.*

3.3 The Fredholm property of pseudodifferential operators on periodic graphs

Let Γ be a $\mathbb{Z}^n$-periodic metric graph and $\mathcal{V}$ and $\mathcal{E}$ the sets of its vertices and edges, respectively. As we agreed above, we identify every edge $e = (v, w)$ with the directed segment $[0, l(e)]$ with endpoints v, w and consider the distance between $x \in e$ and v as the local coordinate of the point x.

Next we describe a class of operators on $L^p_w(\Gamma)$ for which we will derive a Fredholm criterion below. First we have to specify the weight function. Let $F_v \subset \Gamma$ be a sufficiently small neighborhood of a vertex $v \in \mathcal{V}$. Then we can think of this neighborhood as the union $F_v = \cup_{j=1}^{\text{val}(v)} \gamma_j^v$ where the γ_j^v are segments of edges incident to v. We suppose that F_v is such that all segments γ_j^v have the same length, thus they can be identified with a common interval $[0, \varepsilon_v]$, that $w|_{\gamma_j^v} =: w_v$ is independent of j, and that $w_v = e^{\sigma_v} \in \mathcal{R}(\mathcal{I}_v)$ for some open interval $\mathcal{I}_v$. The operators A under consideration are supposed to satisfy the following conditions:

A1 For every open edge e of Γ, there is a symbol $a_e \in S^0(\mathbb{R})$ such that $\varphi A\psi I = \varphi Op(a_e)\psi I$ for all functions $\varphi,\, \psi \in C_0^\infty(e)$.

A2 Let $v \in \mathcal{V}$ and F_v a small neighborhood of v, as specified above. Then there is a symbol $a_v \in \mathcal{E}(\mathrm{val}(v),\, \mathcal{I}_v)$ such that $\varphi A\psi I = \varphi op(a)\psi I$ for all functions $\varphi,\, \psi \in C_0^\infty(F_v)$.

A3 There is a function $f \in l^1(\mathbb{Z}^n)$ such that

$$\|\chi_\alpha A \chi_\beta I\|_{L^p_w(\Gamma)} \le f(\alpha - \beta) \quad \text{for all } \alpha,\, \beta \in \mathbb{Z}^n.$$

A4 The operator A is of local type on $\dot{\Gamma}$.

A5 The operator A is rich.

Assumptions A1 and A2 guarantee that A behaves as a (Fourier) ψdo along every edge and as a Mellin ψdo in a neighborhood of every vertex. Assumption A3 implies that $A \in W_w(\Gamma)$ and, hence, A is bounded on $L^p_w(\Gamma)$.

Theorem 16 *Let A satisfy assumptions* A1-A5. *Then A is a Fredholm operator on $L^p_w(\Gamma)$ if and only if the following conditions hold:*
(a) *for every open edge e of Γ and every $x \in e$,*

$$\liminf_{\xi \to \infty} |a_e(x,\, \xi)| > 0;$$

(b) *for every vertex $v \in \mathcal{V}$,*

$$\liminf_{r \to +0} \inf_{\lambda \in \mathbb{R}} |\det a_v(r,\, \lambda + i\varkappa_{\sigma_v}(r))| > 0;$$

(c) *all limit operators of A are invertible on $L^p_w(\Gamma)$.*

Proof. It follows from Propositions 5 and 11 that condition (a) is necessary and sufficient for the local Fredholmness of A at the every point $x \in \Gamma \setminus \mathcal{V}$, whereas condition (ii) is necessary and sufficient for the local Fredholmness at the every point $v \in \mathcal{V}$, and condition (iii) is necessary and sufficient for the local Fredholmness (= local invertibility) at the point ∞. The assertion follows then from Theorem 15. ■

Corollary 17 *Let A satisfy assumptions* A1-A5 *and conditions* (a) *and* (b) *in Theorem* 16. *Then*

$$ess\, sp_{L^p_w(\Gamma)} A = \cup_{A^g \in Lim(A)} sp_{L^p_w(\Gamma)} A^g.$$

4 Applications

4.1 Singular integral operators

In this section, we let Γ be metric graph which is embedded into $\mathbb{R}^2$. Thus, the vertices of Γ *are* points, and the edges of Γ *are* line segments in the plane. We suppose that Γ is $\mathbb{Z}^n$-periodic with $n \in \{1, 2\}$.

In the following definition of singular integral operators we have to divide by $y - x = (y_1 - x_1, y_2 - x_2) \in \mathbb{R}^2$. This division is understood in the complex sense, i.e., as a (complex) division by $(y_1 - x_1) + i(y_2 - x_2)$. Further we say that a function $\phi : \Gamma \times \mathbb{R}^2 \to \mathbb{C}$ belongs to $C^\infty(\Gamma \times \mathbb{R}^2)$ if, for every $\alpha \in \mathbb{N}_0$ and every multi-index $\beta \in \mathbb{N}_0^2$, the partial derivatives $\partial_x^\alpha \partial_z^\beta (\phi|_{e\times\mathbb{R}^2})$ exist on every set $e \times \mathbb{R}^2$ where e is an open edge of Γ, and if these partial derivatives can be continuously extended to the closure $\bar{e} \times \mathbb{R}^2$ of $e \times \mathbb{R}^2$.

We consider singular integral operators of the form

$$(\mathcal{S}_{\Gamma,\phi} u)(x) = \frac{1}{\pi i} \int_\Gamma \frac{\phi(x, x-y)u(y)}{y-x} dy, \quad x \in \Gamma, \tag{12}$$

where the integral is understood in the sense of the Cauchy principal value,

$$\lim_{\varepsilon\to 0} \int_{\Gamma\cap\{y:|y-x|<\varepsilon\}} \frac{\phi(x, x-y)u(y)}{y-x} dy,$$

and the function $\phi \in C^\infty(\Gamma \times \mathbb{R}^2)$ owns the property that, for all $\alpha, N \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^2$, there is a constant $C_{\alpha\beta N}$ such that

$$|\partial_x^\alpha \partial_z^\beta \phi(x, z)| \le C_{\alpha\beta N}(1 + |z|)^{-N} \quad \text{for all } (x, z) \in \Gamma \times \mathbb{R}^2. \tag{13}$$

In order to show that $\mathcal{S}_{\Gamma,\phi}$ is a pseudodifferential operator on Γ in the sense of our previous definitions, we consider its restriction $\mathcal{S}_{\Gamma,\phi,e}$ to a single edge e of Γ. Identifying this edge with the interval $\{(t, 0) \in \mathbb{R}^2 : t \in (0, l)\}$, we can write this restricted operator in the form

$$(\mathcal{S}_{\Gamma,\phi,e} v)(t) = \frac{1}{\pi i} \int_0^l \frac{\psi(t, t-\tau)v(\tau)}{\tau - t} d\tau, \quad t \in (0, l),$$

where $\psi(t, \tau) := \phi((t, 0), \tau, 0)$ with $t, \tau \in (0, l)$. Hence, $\mathcal{S}_{\Gamma,\phi,e}$ can be identified with the restriction onto the interval $(0, l) \subset \mathbb{R}$ of the operator

$$(S_{\mathbb{R},\phi} f)(x) = \frac{1}{\pi i} \int_\mathbb{R} \frac{\psi(x, x-y)f(y)}{y-x} dy, \quad x \in \mathbb{R},$$

We have to show that this operator is a (Fourier) ψdo in the class OPS^0 with main symbol *sgn*. This fact will follow from Proposition 18 below. We call the function

$$\sigma_{S_{\mathbb{R},\phi}}(x, \xi) := v.p. \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\phi(x, z)}{z} e^{iz\xi} dz$$

on $\mathbb{R} \times \mathbb{R}$ the symbol of the operator $S_{\mathbb{R},\phi}$. The main properties of this function are summarized in the following proposition.

Proposition 18 *The function $\sigma_{S_{\mathbb{R},\phi}}$ satisfies* (13) *in place of ϕ, and*

$$\sigma_{S_{\mathbb{R},\phi}}(x,\, \xi) = \phi(x,\, 0)\operatorname{sgn}\xi + q_\phi(x,\, \xi) \tag{14}$$

where the function q_ϕ is such that $\partial_x^\alpha \partial_\xi^\beta q_\phi(x,\, \xi) = O(|\xi|^{-N})$ for all $\alpha,\, \beta,\, N \in \mathbb{N}_0$ uniformly with respect to $x \in \mathbb{R}$.

Proof. The estimates (13) follow from the identity

$$\frac{\partial \sigma_{S_{\mathbb{R},\phi}}(x,\, \xi)}{\partial \xi} = \frac{1}{\pi}\int_{-\infty}^{\infty} \phi(x,\, z)e^{iz\xi}dz$$

by integrating by parts, whereas (14) is a consequence of the asymptotic behavior of singular integral operators as $\xi \to \infty$; see, for instance, page 112 in [3]. ■

Proposition 19 *Assume that, for every $v \in \mathcal{V}$, the weight w is such that $w_v = \exp\sigma_v \in \mathcal{R}(\mathcal{I})$ where $\mathcal{I} = (-\frac{1}{p},\, 1-\frac{1}{p})$ with $p \in (1,\, \infty)$. Then $\mathcal{S}_{\Gamma,\phi} \in W_w(\Gamma)$ and, consequently, $\mathcal{S}_{\Gamma,\phi}$ is a bounded operator on $L^p_w(\Gamma)$.*

Proof. First we prove that the operator $\mathcal{S}_{\Gamma_0,\phi}$ is bounded on $L^p_w(\Gamma_0)$. Indeed, $\mathcal{S}_{\Gamma_0,\phi} = S_{\Gamma_0} + K_{\Gamma_0,\phi}$, where

$$(S_{\Gamma_0}u)(x) = \frac{\phi(x,\, 0)}{\pi i}\int_{\Gamma_0} \frac{u(y)}{y-x}dy, \quad x \in \Gamma_0,$$

is (a multiple of) the standard singular integral operator and

$$(K_{\Gamma_0,\phi}u)(x) = \int_{\Gamma_0} k(x,\, y)u(y)dy, \quad x \in \Gamma_0,$$

where

$$k(x,\, y) = \frac{1}{\pi i}\frac{(\phi(x,\, x-y) - \phi(x,\, 0))}{y-x}$$

Note that the condition imposed on the weight implies the boundedness of S_{Γ_0} on $L^p_w(\Gamma_0)$ (see, for instance, [1], Section 4.5). Moreover, the hypotheses of the proposition guarantee that $w \in L^p(\Gamma_0)$ and $w^{-1} \in L^q(\Gamma_0)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Since $k \in L^\infty(\Gamma_0 \times \Gamma_0)$, the operator $K_{\Gamma_0,\phi}$ is bounded from $L^1(\Gamma_0)$ to $L^\infty(\Gamma_0)$. Hence, $wK_{\Gamma_0,\phi}w^{-1}I$ is bounded on $L^p(\Gamma_0)$, whence the boundedness of $K_{\Gamma_0,\phi}$ on $L^p_w(\Gamma_0)$.

Next we prove that $\mathcal{S}_{\Gamma,\phi} \in W_w(\Gamma)$. Let $\alpha,\, \beta \in \mathbb{Z}^n$. It is easy to see that the operators defined by

$$\begin{aligned}(\chi_0 V_{-\alpha}\mathcal{S}_{\Gamma,\phi}V_{\alpha-\beta}\chi_0 v)(x) \quad &= \quad \frac{1}{\pi i}\int_{\Gamma_0} \frac{\phi(x,\, x-y+\beta)}{y-x+\beta}v(y)dy \\ &=: \quad (A_\beta v)(x), \quad x \in \Gamma_0,\end{aligned}$$

satisfy the estimate

$$\|A_\beta\|_{\mathcal{L}(L^1(\Gamma_0),L^\infty(\Gamma_0))} \le \sup_{(x,y)\in\Gamma_0\times\Gamma_0} \frac{|\phi(x,\,x-y+\beta)|}{|y-x+\beta|} \tag{15}$$

for all β sufficiently large. Moreover, there is a constant $M > 0$ such that

$$\frac{1}{|y-x+\beta|} \le M$$

for all $x,\, y \in \Gamma_0$ and all sufficiently large $\beta \in \mathbb{Z}^n$. Hence, estimate (15) implies

$$\|A_\beta\|_{\mathcal{L}(L^1(\Gamma_0),L^\infty(\Gamma_0))} \le C_N|\beta|^{-N}$$

for every $N \in \mathbb{N}$ and $\beta \in \mathbb{Z}^n$ large enough, which finally yields

$$\begin{aligned} \|A_\beta\|_{\mathcal{L}(L^p_w(\Gamma_0))} &= \|wA_\beta w^{-1}I\|_{\mathcal{L}(L^p(\Gamma_0))} \\ &\le \|w\|_{L^p(\Gamma_0)}\|A_\beta\|_{\mathcal{L}(L^1(\Gamma_0),L^\infty(\Gamma_0))}\|w^{-1}\|_{L^q(\Gamma_0)} \\ &\le C'_N|\beta|^{-N} \end{aligned}$$

for all sufficiently large β, where we wrote $C'_N := \|w\|_{L^p(\Gamma_0)}\|w^{-1}\|_{L^q(\Gamma_0)}C_N$. Thus, $\mathcal{S}_{\Gamma,\phi} \in W_w(\Gamma)$. ∎

4.2 The Fredholm property of operators $aI + b\mathcal{S}_{\Gamma,\phi}$

We suppose that the coefficients a, b of the operator $A := aI + b\mathcal{S}_{\Gamma,\phi}$ are bounded piecewise continuous functions on Γ which have only discontinuities of the first kind (i.e., jumps) and which are smooth on $\Gamma \setminus \mathcal{V}$. More precisely, for $\omega \in \mathcal{V}$, let F_ω be a sufficiently small neighborhood of ω such that

$$\Gamma \cap F_\omega = \cup_{j=1}^{\mathrm{val}(\omega)} \gamma_j^\omega$$

where the γ_j^ω are rays of the same length incident to the vertex ω. On F_ω, we introduce a local system of coordinates such that

$$\gamma_j^\omega = \{z = \omega + re^{i\theta_j^\omega} : r \in (0,\,\varepsilon)\}$$

where $0 \le \theta_1^\omega < \theta_2^\omega < \ldots < \theta_{\mathrm{val}(\omega)}^\omega < 2\pi$, and we put

$$f_j^\omega(r) := f(\omega + re^{i\theta_j^\omega}), \quad r \in (0, \varepsilon)$$

and $f^\omega := \mathrm{diag}\,(f_1^\omega,\, \ldots,\, f_{\mathrm{val}(\omega)}^\omega)$ for every function f on Γ. We say that a function $f \in L^\infty(\Gamma)$ belongs to the class $PC^\infty(\Gamma)$ if $f \in C^\infty(\Gamma \setminus \mathcal{V})$ and if the one-sided limits

$$\lim_{r\to 0} f_j^\omega(r) =: f_j(\omega)$$

exist for every vertex ω and every $j \in \{1, \ldots, \mathrm{val}(\omega)\}$. In this case we write

$$\tilde{f}(\omega) := \mathrm{diag}\,(f_1(\omega), \ldots, f_{\mathrm{val}(\omega)}). \tag{16}$$

For $\omega \in \mathcal{V}$, we put $\varepsilon_k := 1$ if ω is the starting point of the oriented edge $e_k^\omega \supset \gamma_k^\omega$ and $\varepsilon_k := -1$ if ω is the terminating point of the edge e_k^ω. Further we define functions $\nu : [0, 2\pi) \times (\mathbb{C} \setminus i\mathbb{Z}) \to \mathbb{C}$ by

$$\nu(\delta, \zeta) := \begin{cases} \coth(\pi\zeta) & \text{if } \delta = 0, \\ \frac{e^{(\pi-\delta)\zeta}}{\sinh(\pi\zeta)} & \text{if } \delta \in (0, 2\pi) \end{cases}$$

and $s_{jk}^\omega : \mathbb{C} \setminus i\mathbb{Z} \to \mathbb{C}$ by

$$s_{jk}^\omega(\zeta) := \varepsilon_k \begin{cases} \nu(2\pi + \theta_j^\omega - \theta_k^\omega, \zeta) & \text{if } j < k, \\ \nu(0, \zeta) & \text{if } j = k, \\ \nu(\theta_j^\omega - \theta_k^\omega, \zeta) & \text{if } j > k \end{cases}$$

and we set, for $(r, \lambda) \in (0, \varepsilon) \times \mathbb{R}$,

$$\left(\hat{\sigma}_\omega(\mathcal{S}_\Gamma)c\right)(r, \lambda) := \left(s_{jk}^\omega\left(\lambda + i\left(\frac{1}{p} + \varkappa_{w_\omega}(r)\right)\right)\right)_{j,k=1}^{\mathrm{val}(\omega)} \tag{17}$$

where $\varkappa_{w_\omega}(r) := r\frac{dv_\omega(r)}{dr}$ for $r \in (0, \varepsilon)$.

We consider the operator $A = aI + b\mathcal{S}_{\Gamma,\phi}$ with coefficients $a, b \in PC^\infty(\Gamma)$. Our goal is to define the symbol of A at every point of Γ and at the infinitely distant point ∞ in such a way that the invertibility of the symbol corresponds to the Fredholm property of A.

If $\omega \in \mathcal{V}$, then the symbol at ω is the function

$$\sigma_A^\omega(r, \lambda) := \tilde{a}(\omega) + \tilde{\phi}(\omega, 0)\tilde{b}(\omega)\hat{\sigma}_\omega(\mathcal{S}_\Gamma)(r, \lambda), \quad (r, \lambda) \in (0, \varepsilon) \times \mathbb{R},$$

where $\tilde{a}(\omega)$, $\tilde{b}(\omega)$ and $\hat{\sigma}_\omega(\mathcal{S}_\Gamma)$ are given by (16) and (17), respectively.

If $x \in \Gamma \setminus \mathcal{V}$, then the symbol at x is given by

$$\sigma_A^x(\xi) := a(x) + b(x)\phi(x, 0)\,\mathrm{sgn}\,\xi, \quad \xi \in \mathbb{R}.$$

The description of the symbol at ∞ is more involved. We will employ the limit operators of A for this goal. First we define the symbol of the operator of multiplication by a function $f \in PC^\infty(\Gamma)$. We write the graph Γ as a countable union $\cup_{j\in\mathbb{N}} e_j$ of edges. It is then a consequence of the Arzela-Ascoli theorem and a standard diagonal argument that every sequence $h \in \mathcal{H}$ has a subsequence g such that

$$(f|_{e_j})(x + g(m)) \to f_j^g(x)$$

uniformly on e_j for every $j \in \mathbb{N}$. With the so-defined family $\{f_j^g\}_{n\in\mathbb{N}}$ of functions on the edges of Γ, we associate a function f^g on all of Γ in the natural way. The function f^g has the property that, for every compact subset K of Γ,

$$\lim_{m\to\infty} \sup_{x\in K} |f(x+g(m)) - f^g(x)| = 0.$$

Hence $f^g I$ is the limit operator of fI with respect to g. Moreover, we obtained that fI is a rich operator.

The limit operators of fI are of a particularly simple form if f is slowly oscillating at infinity. This class of functions is defined as follows. Let $f \in L^\infty(\Gamma)$. We represent f in the form

$$f(x) = f(y+\alpha) =: f_y(\alpha), \quad y \in \Gamma_0,\ \alpha \in \mathbb{Z}^n,$$

with Γ_0 again referring to the fundamental domain of Γ, and we call f *slowly oscillating at infinity* if

$$\lim_{\alpha\to\infty} \sup_{y\in\Gamma_0} |f_y(\beta+\alpha) - f_y(\alpha)| = 0 \quad \text{for every } \beta \in \mathbb{Z}^n. \tag{18}$$

We denote the class of all functions which are slowly oscillating at infinity by $SO(\Gamma)$. If $f \in SO(\Gamma)$, then it follows from (18) that all limit operators $f^h I$ of fI are operators of multiplication by a $\mathbb{Z}^n$-periodic function f^h, that is $V_\beta f^h = f^h$ for every $\beta \in \mathbb{Z}^n$.

We consider the operator $A = aI + b\mathcal{S}_{\Gamma,\phi}$ under the assumptions that $a, b \in PC^\infty(\Gamma) \cap SO(\Gamma)$ and $\phi \in C^\infty(\Gamma \times \mathbb{R}^2)$, and we suppose that the function $(x, z) \mapsto \phi(x, z)$ is slowly oscillating with respect to $x \in \Gamma$ uniformly with respect to z on compacts subsets of $\mathbb{R}^2$. Under these conditions, the operator A is rich, and all limit operators of A are of the form $A^h = a^h I + b^h \mathcal{S}^h_{\Gamma,\phi}$ where

$$(\mathcal{S}^h_{\Gamma,\phi} u)(x) = \frac{1}{\pi i} \int_\Gamma \frac{\phi^h(x,\, x-y) u(y)}{y-x} dy, \quad x \in \Gamma,$$

and ϕ^h is the limit function of ϕ with respect to $h \in \mathcal{H}$ in the sense that

$$\phi^h(x,\, z) = \lim_{m\to\infty} \phi(x + h(m),\, z)$$

uniformly on compact subsets of $\Gamma \times \mathbb{R}^2$. Since ϕ is slowly oscillating with respect to the first variable, the function $\Gamma \times \mathbb{R}^2 \ni (x,\, z) \to \phi^h(x,\, z)$ is periodic with respect to the shifts V_α with $\alpha \in \mathbb{Z}^n$. Consequently, the operator $\mathcal{S}^h_{\Gamma,\phi}$ is invariant with respect to these shifts.

Note that the operator $\tilde{A}^h := UA^hU^{-1}$ is of the form

$$(\tilde{A}^h u)(x,\, \alpha) = a^h(x) I + b^h(x) \sum_{\beta\in\mathbb{Z}^n} \int_{\Gamma_0} k^h(x, x-y+\alpha-\beta) u(y,\, \beta) dy$$

where $(x,\, \alpha) \in \Gamma_0 \times \mathbb{Z}^n$ and

$$k(x,\, z) = -\frac{1}{\pi i}\frac{\phi^h(x,\, z)}{z}, \quad (x,\, z) \in \Gamma_0 \times \mathbb{R}^2.$$

We associate with $\tilde{A}^h$ the operator-valued function

$$\mu(\tilde{A}^h) : \mathbb{T}^2 \to \mathcal{B}(L^p_w(\Gamma_0)), \quad \tau \mapsto a^h I + b^h \mathcal{M}^h(\tau)$$

where

$$\mathcal{M}^h(\tau) := \sum_{\gamma \in \mathbb{Z}^n} M^h_\gamma \tau^\gamma, \quad \tau \in \mathbb{T}^n, \tag{19}$$

and

$$(M^h_\gamma \varphi)(x) = \int_{\Gamma_0} k^h(x,\, x - y + \gamma)\varphi(y)dy, \quad x \in \Gamma_0. \tag{20}$$

Then the conditions

1. $a^h(x) \pm b^h(x)\phi^h(x,\, 0) \neq 0$ for every point $x \in \Gamma_0 \setminus \mathcal{V}$,
2. $\liminf_{r\to 0} \inf_{\lambda\in\mathbb{R}} |\det(\tilde{a}^h(\omega) + \tilde{\phi}^h(\omega,\, 0)\tilde{b}^h(\omega)\hat{\sigma}_\omega(\mathcal{S}_\Gamma)(r,\, \lambda))| > 0$ for every $\omega \in \Gamma_0 \cap \mathcal{V}$,

imply the Fredholm property of all operators $\mu(\tilde{A}^h)(\tau) \in \mathcal{B}(L^p_w(\Gamma_0))$ with $\tau \in \mathbb{T}^n$.

Theorem 20 *Let $a,\, b \in PC^\infty(\Gamma) \cap SO(\Gamma)$. Then the operator $A = aI + b\mathcal{S}_{\Gamma,\phi} : L^p_w(\Gamma) \to L^p_w(\Gamma)$ is a Fredholm operator if and only if the following conditions are satisfied:*

(a) *$a(x) \pm b(x)\phi(x,\, 0) \neq 0$ for every $x \in \Gamma \setminus \mathcal{V}$;*

(b) *$\liminf_{r\to 0} \inf_{\lambda\in\mathbb{R}} |\det(\tilde{a}(\omega) + \tilde{b}(\omega)\tilde{\phi}(x,\, 0)\hat{\sigma}_\omega(\mathcal{S}_\Gamma)(r,\, \lambda))| > 0$ for every $\omega \in \mathcal{V}$;*

(c) *all limit operators A^h of A are invertible.*

Note that conditions (a) - (c) are equivalent to the invertibility of the symbol functions σ^x_A, σ^ω_A and $\mu(\tilde{A}^h)$, respectively, at every point of their domain of definition.

Proof. First we remark that A satisfies conditions A1 - A5. Condition (a) is the condition for the ellipticity of the restriction of A on the open edge e at the point $x \in e$. This condition is necessary and sufficient for the local Fredholmness of A at $x \in e$. Condition (b) is necessary and sufficient for the local invertibility (hence, for the local Fredholmness) of A at the vertex ω; see, for instance, [14] and Chapter 4 in [13]. The final condition (c) is equivalent to the local invertibility of A at the point ∞. Hence, the assertion of the theorem follows from Theorem 16. ■

For an example, let $A = aI + b\mathcal{S}_{\Gamma,\phi}$ and assume that $\lim_{\Gamma\ni x\to\infty} b(x) = 0$. Then condition (c) in Theorem 20 is equivalent to the condition

$$\liminf_{\Gamma\ni x\to\infty} |a(x)| > 0. \tag{21}$$

Hence A is a Fredholm operator on $L^p_w(\Gamma)$ if and only if conditions (a) and (b) from Theorem 20 and condition (21) hold.

4.3 Locally compact operators on graphs

Let again Γ be a metric graph which is embedded into $\mathbb{R}^2$ and $\mathbb{Z}^n$-periodic with $n \in \{1, 2\}$. We consider integral operators $A = aI + bT$ where $a, b \in BUC(\Gamma) \cap SO(\Gamma)$, with $BUC(\Gamma)$ the space of the bounded and uniformly continuous functions on Γ, and T is an integral operator of the form

$$(Tu)(x) = \int_\Gamma k(x-y)u(y)dy, \quad x \in \Gamma,$$

where $k : \mathbb{R}^2 \to \mathbb{C}$ is a continuous function with the property that there exist $C > 0$ and $\varepsilon > 0$ such that

$$|k(z)| \leq C(1+|z|)^{-2-\varepsilon} \quad \text{for all } z \in \mathbb{R}^2. \tag{22}$$

This estimate implies that $A \in W(\Gamma)$ and, consequently, A is a bounded operator on $L^p(\Gamma)$ for every $p \in [1, \infty]$. Moreover, the boundedness of k implies the local compactness of T on every $L^p(\Gamma)$ with $1 < p < \infty$, that is, the operators $T\chi_M I$ and $\chi_M T$ are compact for every compact subset M of Γ. Hence, the operator A is of local type on $L^p(\dot{\Gamma})$.

If $a \in BUC(\Gamma)$ then, by the Arzela-Ascoli theorem, every sequence $h \in \mathcal{H}$ has a subsequence g such that the sequence of the functions $a(\cdot + g(m))$ converges to a limit function a^g uniformly on Γ_0. Hence, A is a rich operator, and every limit operator of A is of the form $A^g = a^g I + b^g T$ where a^g, b^g are limit functions of a, b with respect to $g \in \mathcal{H}$. Because of $a, b \in BUC(\Gamma) \cap SO(\Gamma)$, the functions a^g, b^g are periodic with respect to the action of $\mathbb{Z}^n$. Hence the limit operators A^g are invariant with respect to shifts V_α with $\alpha \in \mathbb{Z}^n$.

As above, we associate with A^g the operator-valued symbol

$$\mu_{A^g} : \mathbb{T}^2 \to \mathcal{B}(L^p(\Gamma_0)), \quad \tau \mapsto a^g I + b^g \mathcal{M}(\tau)$$

where

$$(\mathcal{M}(\tau)u)(x) = \sum_{\alpha \in \mathbb{Z}^n} \left(\int_{\Gamma_0} k(x-y+\alpha)u(y)dy \right) \tau^\alpha, \quad x \in \Gamma_0, \, \tau \in \mathbb{T}^n.$$

Theorem 21 *Under the above assumptions for a, b and k, the operator $A = aI + bT$ is a Fredholm operator on $L^p(\Gamma)$ for $1 < p < \infty$ if and only if the following conditions are satisfied:*

(a) $\inf_{x \in \Gamma} |a(x)| > 0$;

(b) *every limit operator A^g of A is invertible on $L^p(\Gamma)$.*

Note that condition (b) is equivalent to the invertibility of the operator $\mu_{A^g}(\tau)$ on $L^p(\Gamma_0)$ for every $\tau \in \mathbb{T}^n$.

Proof. Since T is locally compact, the operator A is locally Fredholm at the point $x \in \Gamma$ if and only if it satisfies condition (a). Condition (b) is necessary and sufficient for the local invertibility of A at the point ∞. Hence Theorem 21 follows from Simonenko's local principle (Proposition 12). ■

References

[1] A. Böttcher, Yu. Karlovich, Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators, Progress in Mathematics, vol. 154, Birkhäuser.

[2] G. Berkolaiko, R. Carlson, S. Fulling, and P. Kuchment (Editors), Quantum Graphs and Their Applications, Contemp. Math. **415**, Amer. Math. Soc., Providence, R.I., 2006.

[3] M. V. Fedoruk, The Saddle Point Method, Nauka, Moskwa 1977 (Russian).

[4] P. Kuchment (Editor), Quantum graphs and their applications, A special issue of Waves in Random Media, **14** (2004), 1.

[5] P. Kuchment, Quantum graphs: I. Some basic structure, Waves Random Media **14** (2004), S107 - S128.

[6] P. Kuchment, Quantum graphs: II Some spectral properties of quantum and combinatorial graphs, J. Phys. A **38** (2005), 22, 4887 - 4900.

[7] P. Kuchment, On the structure of eigenfunctions corresponding to embedded eigenvalues of locally perturbed periodic graph operators, arXiv:math-ph/0511084 v1 28 Nov 2005.

[8] P. Kuchment, Quantum graphs: an introduction and a brief survey, In: Analysis on Graphs and its Applications, Proc. Symp. Pure Math., AMS 2008, 291 - 314.

[9] D. Lenz, C. Schubert, P. Stollmann, Eigenfunction expansions for Schrödinger operators on metric graphs, Integr. Equ. Oper. Theor. **62** (2008), 4, 541 - 533.

[10] P. Ola and L. Päivärinta, Mellin operators and pseudodifferential operators on graphs, Waves Random Media **14** (2004), S129 - S142, PII: S0959-7174(04)69206-8.

[11] V. Rabinovich, S. Roch, The essential spectrum of Schrödinger operators on lattices, J. Phys. A: Math. Gen. **39** (2006), 8377 - 8394, doi:10.1088/0305-4470/39/26/007

[12] V. Rabinovich, S. Roch, Essential spectra of difference operators on $\mathbb{Z}^n$-periodic graphs, J. Phys. A: Math. Theor. **40** (2007), 10109 - 10128, doi:10.1088/1751-8113/40/33/012

[13] V. Rabinovich, S. Roch, B. Silbermann, Limit Operators and Their Applications in Operator Theory, Operator Theory: Adv. Appl. **150**, Birkhäuser, Basel, Boston, Berlin 2004.

[14] V. S. Rabinovich, Algebras of singular integral operators on complicated contours with nodes being of logarithmic whirl points, Izvestia AN Rossii, ser. matem., **60** (1996), No. 6, 169 - 200 (Russian, English transl.: Izvestia Mathematics **60** (1996), 6, 1261 - 1292).

[15] V. Rabinovich, Essential spectrum of perturbed pseudodifferential operators. Applications to the Schrödinger, Klein-Gordon, and Dirac operators, Russian J. Math. Physics **12** (2005), No. 1, 62 - 80.

[16] S. Roch, P. A. Santos, B. Silbermann, Non-commutative Gelfand Theories, A Tool-Kit for Operators Theorists and Numerical Analysis, Universitext, Springer, 2011.

[17] M. A. Shubin, Pseudodifferential Operators and Spectral Theory, Second Edition, Springer, Berlin, Heidelberg, New York 2001.

[18] I. B. Simonenko, Local Method in the Theory of Invariant with Respect to Shifts Operators and Their Envelopes, Rostov State University, Rostov na Donu 2007 (Russian).

[19] M. E. Taylor, Pseudodifferential Operators, Princeton Univ. Press, Princeton, New Jersey, 1981.

[20] M. E. Taylor, Tools for PDE, Pseudodifferential Operators, Paradifferential Operators and Layer Potentials, AMS Math. Surveys Monographs **81**, 2000.

Authors' addresses:

Vladimir S. Rabinovich, Instituto Politécnico Nacional,
ESIME-Zacatenco, Av. IPN, edif. 1, México D.F., 07738, MÉXICO.
e-mail: vladimir.rabinovich@gmail.com

Steffen Roch, Technische Universität Darmstadt,
Schlossgartenstrasse 7, 64289 Darmstadt, Germany.
e-mail: roch@mathematik.tu-darmstadt.de