CONVERGENCE PROPERTIES OF WEAK SOLUTIONS OF THE BOUSSINESQ EQUATIONS IN DOMAINS WITH ROUGH BOUNDARIES

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Abstract. We act on the assumption that the boundary of every 'physical' domain Ω has microscopic asperities which influence the boundary behaviour of weak solutions of the Boussinesq equations. Let $\Omega_n \subseteq \mathbb{R}^3, n \in \mathbb{N}$, be domains with rough boundaries and let Ω_n 'converge to' Ω. Consider a sequence $(u_n, \theta_n)_{n\in\mathbb{N}}$ of weak solutions of the Boussinesq equations with u_n fulfilling the impermeability condition $u_n \cdot N = 0$ on $\partial\Omega_n$ and θ_n fulfilling the Robin boundary condition $\frac{\partial\theta_n}{\partial N} + \alpha(\theta_n - h_0) = 0$ on $\partial\Omega_n$. In this paper the boundary conditions and limit equations of weak limits of (u_n, θ_n) on Ω under certain assumptions on the rugosity of the boundaries will be determined.

1. Introduction and main result

We consider the Boussinesq equations

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \frac{1}{\rho}\nabla p - \beta\theta g &= f_1\,, \\ \operatorname{div} u &= 0\,, \\ \frac{\partial \theta}{\partial t} - \kappa\Delta\theta + (u\cdot\nabla)\theta &= f_2\,, \\ u(x,0) &= u_0(x)\,, \\ \theta(x,0) &= \theta_0(x) \end{aligned} \tag{1.1}$$

for a viscous, incompressible fluid with velocity $u = (u_1, u_2, u_3)$, pressure p, and temperature θ on a domain $\Omega \subseteq \mathbb{R}^3$ and a finite time interval $[0, T[$. In (1.1) we denote by $g = g(x)$ the gravitational vector force. The positive constants appearing in (1.1) have the following meaning: ρ is the density, ν the kinematic viscosity, β the coefficient of thermal expansion, and κ the thermal diffusivity. The equations (1.1) constitute a model of motion of a viscous, incompressible buoyancy-driven fluid flow coupled with heat convection (Boussinesq approximation). For existence and uniqueness of weak or strong solutions with Dirichlet or mixed Dirichlet/Neumann boundary conditions we refer to [4, 10, 11, 12]. We suppose that the boundary is impermeable, i.e.,

$$u \cdot N = 0 \quad \text{on } \partial\Omega\,, \tag{1.2}$$

where N denotes the outer normal unit vector on $\partial\Omega$.

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The present paper is based on the assumption that a 'real domain' Ω is never perfectly smooth, i.e., it contains microscopic asperities influencing the effective boundary conditions for u and θ. In this paper we want to determine the effect on boundary rugosity to weak solutions (u, θ) of the Boussinesq equations (1.1).

The influence of boundary rugosity to weak solutions of the Navier-Stokes equations was observed in several papers. In [5] the authors considered a sequence of weak solutions $u_n, n \in \mathbb{N}$, of the incompressible, instationary Navier-Stokes equations on variable domains D_n converging to a domain D with u_n fulfilling the impermeability condition $u_n \cdot N = 0$ on Γ_n where Γ_n is a periodically oscillating part of the boundary of D_n. They proved that under suitable assumptions on the rugosity of Γ_n that a weak limit u on D of u_n fulfills the no slip condition $u = 0$ on Γ. Here Γ is the part of the boundary of D to which Γ_n converges. In [1] Bucur et al. generalized this criterion by introducing the non-degeneracy condition (see Definition 3.2). The case when the rugosity is degenerate in one direction is considered in [2] for the stationary case and in [3] for the instationary case.

Throughout this work we suppose that the domain Ω has the form

$$\Omega = \{ (x, x_3) \in \mathbb{R}^3;\, x_3 > \psi(x);\, x \in \mathbb{R}^2 \} \tag{1.3}$$

with a Lipschitz continuous function $\psi : \mathbb{R}^2 \to \mathbb{R}^+$. As a model for these microscopic asperities we consider a family of 'domains with rough boundaries'

$$\Omega_n = \{ (x, x_3) \in \mathbb{R}^3;\, x_3 > \psi(x) - \phi_n(x);\, x \in \mathbb{R}^2\} \tag{1.4}$$

with a family of Lipschitz continuous functions $\phi_n : \mathbb{R}^2 \to \mathbb{R}^+$ satisfying the following properties:

(1) $\forall$ compact $K \subseteq \mathbb{R}^2$ it holds $\phi_n(x) \underset{n\to\infty}{\longrightarrow} 0$ uniformly in $x \in K$,

(2) $\dfrac{|\phi_n(x) - \phi_n(y)|}{|x-y|} \le L \quad \forall x, y \in \mathbb{R}^2; x \neq y \quad \forall n \in \mathbb{N}.$

It follows by a well known argument that $\psi \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^2)$, $\phi_n \in W^{1,\infty}(\mathbb{R}^2)$ and $\|\nabla\phi_n\|_\infty \le L$ for all $n \in \mathbb{N}$. Consider a sequence $(u_n, \theta_n)_{n\in\mathbb{N}}$ of weak solutions of the Boussinesq equations (1.1) on Ω_n where u_n satisfies the impermeability condition

$$u_n \cdot N = 0 \quad \text{ on } \partial\Omega_n \tag{1.5}$$

and θ_n satisfies the Robin boundary condition

$$\frac{\partial\theta_n}{\partial N} + \alpha(\theta_n - h_0) = 0 \quad \text{on } \partial\Omega_n \tag{1.6}$$

where h_0 denotes the exterior temperature on $\partial\Omega_n$.

Following the approach in [1] we will introduce a Young measure $\mathcal{R} = (\mathcal{R}_x)_{x\in\mathbb{R}^2}$ associated to a suitable subsequence of $(\nabla\phi_n)_{n\in\mathbb{N}}$ which describes the character of oscillations of $(\nabla\phi_n)_{n\in\mathbb{N}}$. This is motivated by the fact that $\nabla\phi_n$ describes the deviation of the normal vector vector on Ω_n to the normal vector on Ω in the x_1, x_2-plane. Therefore, in our model the rugosity of the boundary is characterized by $\mathcal{R}$.

Assume that u_n converges weakly to u on Ω and that θ_n converges weakly to θ on Ω. We will show that (u, θ) satisfy the Boussinesq equations (1.1) on $[0,T[\times\Omega$. Moreover, it is proved that θ satisfies the Robin boundary condition

$$\frac{\partial\theta}{\partial N} + \alpha\Lambda\cdot(\theta - h_0) = 0 \quad \text{on } \partial\Omega \tag{1.7}$$

with a scalar function $\Lambda(x)\,, x\in\partial\Omega$, which can be explicitly computed using $\mathcal{R}$, see (1.15). Under a non-degeneracy condition on $\mathcal{R}$ we will show that u satisfies the no slip boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.8}$$

This means that the microscopic asperities prevent the fluid from slipping, i.e. u adheres completely to the boundary. By multiplying the Boussinesq system (1.1) with test functions ϕ, w and using integration by parts and (1.7) we obtain the following definition of weak solutions.

Definition 1.1. Let $\Omega\subseteq\mathbb{R}^3$ be a domain with uniformly Lipschitz boundary, let $0<T<\infty$, $g\in L^\infty(0,T;L^\infty(\Omega))$, let $h_0\in L^2(0,T;L^2(\partial\Omega))$, $\Lambda\in L^\infty(0,T;L^\infty(\partial\Omega))$, and let $\nu,\alpha,\beta,\kappa>0$ be constants. Further assume $u_0,\theta_0\in L^2(\Omega)$ and $f_1,f_2\in L^2(0,T;L^2(\Omega))$.

(1) A pair

$$\begin{aligned} &u\in L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;H^1(\Omega))\,,\\ &\theta\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)) \end{aligned} \tag{1.9}$$

is called a weak solution of the Boussinesq equations (1.1) with impermeability condition (1.2) and Robin boundary condition (1.7) if it holds

$$\begin{aligned} &-\int_0^T\langle u,\phi_t\rangle_\Omega\,dt - \int_0^T\langle u\otimes u,\nabla\phi\rangle_\Omega\,dt + \nu\int_0^T\langle\nabla u,\nabla\phi\rangle_\Omega\,dt\\ &= \beta\int_0^T\langle\theta g,\phi\rangle_\Omega\,dt + \int_0^T\langle f_1,\phi\rangle_\Omega\,dt + \langle u_0,\phi(0)\rangle_\Omega \end{aligned} \tag{1.10}$$

for all $\phi\in C_0^\infty([0,T[;C_{0,\sigma}^\infty(\Omega))$ and

$$\begin{aligned} &-\int_0^T\langle\theta,w_t\rangle_\Omega\,dt - \int_0^T\langle u\cdot\nabla w,\theta\rangle_\Omega\,dt + \alpha\kappa\int_0^T\langle\Lambda\cdot(\theta-h_0),w\rangle_{\partial\Omega}\,dt\\ &= -\kappa\int_0^T\langle\nabla\theta,\nabla w\rangle_\Omega\,dt + \int_0^T\langle f_2,w\rangle_\Omega\,dt + \langle\theta_0,w(0)\rangle_\Omega \end{aligned} \tag{1.11}$$

for all $w\in C_0^\infty([0,T[\times\overline{\Omega})$.

(2) A pair

$$\begin{aligned} &u\in L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;H^1_0(\Omega))\,,\\ &\theta\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)) \end{aligned} \tag{1.12}$$

is called a weak solution of the Boussinesq equations (1.1) with no slip boundary condition (1.8) and Robin boundary condition (1.7) if the identity (1.10) is satisfied for all $\phi\in C_0^\infty([0,T[;C_{0,\sigma}^\infty(\Omega))$ and (1.11) is satisfied for all $w\in C_0^\infty([0,T[\times\overline{\Omega})$.

Remark. In the 'usual' definition of weak solutions to the Boussinesq equations it is assumed that $u_0 \in L^2_\sigma(\Omega)$. But for the formulation of Theorem 1.2 we only suppose $u_0 \in L^2(\Omega)$ in Definition 1.1. If (u, θ) is a weak solution as in Definition 1.1 then it can be proved that, after a redefinition on a null set, $u : [0, T[\to L^2_\sigma(\Omega)$ is weakly continuous and the initial value is attained in the sense $u(0) = Pu_0$ where P is the usual Helmholtz projection on $L^2(\Omega)$.

Using Galerkin approximation and a Fourier-type Aubin-Lions argument (see [12, 14]) it can be proved that there exists a weak solution (u, θ) of the Boussinesq equations (1.1) with boundary conditions as in (2) of Definition 1.1. Now we are able to formulate our main result. In this theorem a precise formulation of what we have described in the introduction so far will be given.

Theorem 1.2. *Let Ω, Ω_n be as in (1.3), (1.4), let $D \subseteq \mathbb{R}^3$ be open with $\overline{\Omega_n} \subseteq D$ for all $n \in \mathbb{N}$. Let $0 < T < \infty, g \in L^\infty(0, T; L^\infty(D))$, let $h_0 \in L^2(0, T; H^1(D))$, and let $\nu > 0, \alpha > 0, \beta > 0, \kappa > 0$ be constants. Further assume $u_0, \theta_0 \in L^2(D)$ with $\mathrm{div} u_0 = 0$, and $f_1, f_2 \in L^2(0, T; L^2(D))$.*

Consider a sequence of weak solutions $(u_n, \theta_n)_{n\in\mathbb{N}}$ of the Boussinesq equations (1.1) on $[0, T[\times\Omega_n$ (with external forces $f_1, f_2|_{[0,T[\times\Omega_n}$ and initial values $u_0, \theta_0|_{\Omega_n}$) with impermeability condition (1.5) and Robin boundary condition (1.6).

Let $u, \theta \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, and let $(m_k)_{k\in\mathbb{N}}$ be a subsequence such that the following two properties are fulfilled.

(i) *$(u_{m_k})_{k\in\mathbb{N}}$ and $(\theta_{m_k})_{k\in\mathbb{N}}$ are both bounded in $L^\infty(0, T; L^2(\Omega_{m_k})) \cap L^2(0, T; H^1(\Omega_{m_k}))$ and*

$$u_{m_k} \underset{k\to\infty}{\rightharpoonup^*} u \text{ in } L^\infty(0, T; L^2(\Omega)), \quad u_{m_k} \underset{k\to\infty}{\rightharpoonup} u \text{ in } L^2(0, T; H^1(\Omega)), \tag{1.13}$$

$$\theta_{m_k} \underset{k\to\infty}{\rightharpoonup^*} \theta \text{ in } L^\infty(0, T; L^2(\Omega)), \quad \theta_{m_k} \underset{k\to\infty}{\rightharpoonup} \theta \text{ in } L^2(0, T; H^1(\Omega)). \tag{1.14}$$

(ii) *Let $\mathcal{R} = (\mathcal{R}_x)_{x\in\mathbb{R}^2}$ be a Young measure associated to $(\nabla\phi_{m_k})_{k\in\mathbb{N}}$. By Theorem 3.1 choose a compact set $K_\mathcal{R} \subseteq \mathbb{R}^2$ with $\mathrm{supp}(\mathcal{R}_x) \subseteq K_\mathcal{R}$ for almost all $x \in \mathbb{R}^2$.*

Then the following statements are satisfied.

(1) *(u, θ) is a weak solution of the Boussinesq equations (1.1) on $[0, T[\times\Omega$ (with external forces $f_1, f_2|_{[0,T[\times\Omega}$ and initial values $u_0, \theta_0|_\Omega$) with impermeability condition (1.2) and Robin boundary condition (1.7) with the (time independent) 'weight function'*

$$\Lambda(x, \psi(x)) = \frac{1}{\sqrt{1 + |\nabla\psi(x)|^2}} \int_{K_\mathcal{R}} \sqrt{1 + |\nabla\psi(x) - \lambda|^2}\, d\mathcal{R}_x(\lambda) \tag{1.15}$$

for almost all $x \in \mathbb{R}^2$.

(2) *Under the additional assumption that $\mathcal{R}$ is non-degenerate (see Definition 3.2) it holds that u satisfies the no-slip boundary condition (1.8). Therefore (u, θ) is a weak solution of the Boussinesq equations (1.1) with no slip boundary condition (1.8) for u and Robin boundary condition (1.7) with Λ defined by (1.15).*

We remark that this theorem is independent of the concrete boundary conditions that are fulfilled by u. For the existence and properties of Young measures needed for this theorem we refer to Section 3. The proof of this theorem is the content of Section 4. In Section 2 we will discuss some preliminaries, especially we will decompose the pressure in a harmonic and a regular part as introduced in [16].

2. Preliminaries

2.1. **Notation.** Given a Banach space X and an interval $[0,T[\,, 0<T<\infty$, we denote by $L^p(0,T;X)\,, 1\le p\le\infty$ the Banach space of equivalence classes of strongly measurable functions $f:[0,T[\to X$ such that

$$\|f\|_p := \left(\int_0^T \|f(t)\|_X^p\,dt\right)^{\frac{1}{p}} < \infty$$

if $p<\infty$ and

$$\|f\|_\infty := \operatorname{ess\,sup}_{[0,T[}\|f(\cdot)\|_X \ ,$$

if $p=\infty$. In the case that $X=L^q(\Omega)\,, 1\le q\le\infty$, the norm in the space $L^p(0,T;L^q(\Omega))$ is denoted by $\|\cdot\|_{q,p;\Omega;T}$. Let $d\in\mathbb{N}$, let $\Omega\subseteq\mathbb{R}^d$ be an open set, let $1\le p\le\infty$, and $k\in\mathbb{N}$. We denote by $L^p(\Omega), W^{k,p}(\Omega), W_0^{k,p}(\Omega)$ the usual Lebesgue and Sobolev spaces with norm $\|\cdot\|_{L^p(\Omega)}=\|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}(\Omega)}=\|\cdot\|_{k,p}$, respectively. We set $H^k(\Omega):=W^{k,2}(\Omega)$ and $H_0^k(\Omega):=W_0^{k,2}(\Omega)$. Furthermore define $H^0(\Omega):=L^2(\Omega)$ and $H^{-1}(\Omega):=W_0^{1,2}(\Omega)'$. For $s\in\mathbb{R}^+\setminus\mathbb{N}$ let $H^s(\Omega):=W^{s,2}(\Omega)$ denote the usual Sobolev-Slobodeckij space, see [15, Definition II.3.1]. Looking at [15, Satz II.5.3, Satz II.5.4 and Satz II.7.9] we get that for a bounded Lipschitz domain $\Omega\subseteq\mathbb{R}^3$ and $0\le s_2<s_1\le 1$ the embedding

$$H^{s_1}(\Omega)\hookrightarrow H^{s_2}(\Omega) \tag{2.1}$$

is compact. For two measurable functions f,g with the property $f\cdot g\in L^1(\Omega)$ where $f\cdot g$ means the usual scalar product of scalar, vector or matrix fields, we set

$$\langle f,g\rangle_\Omega := \int_\Omega f(x)\cdot g(x)\,dx.$$

Note that (in general) the symbol $L^p(\Omega)$ etc. will be used for spaces of scalar, vector or matrix-valued functions. Let $C^m(\Omega)\,, m=0,1,\dots,\infty$, denote the usual space of functions for which all partial derivatives of finite order $|\alpha|\le m$ exist and are continuous and let $C^m(\overline{\Omega}):=\{\phi|_\Omega;\phi\in C^m(\mathbb{R}^d)\}$. As usual, $C_0^m(\Omega)$ is the set of all functions from $C^m(\Omega)$ with compact support in Ω and let $C_0^\infty(]0,T[\times\Omega)$ denote the space of smooth function with compact support in $]0,T[\times\Omega$. Further we define the following spaces of vector fields

$$\begin{aligned} C_{0,\sigma}^\infty(\Omega) &:= \{\,\phi\in C_0^\infty(\Omega)^d;\ \operatorname{div}\phi=0\,\}\,,\\ L_\sigma^p(\Omega) &:= \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_p}. \end{aligned}$$

Moreover, $C_0^\infty(]0,T[;C_{0,\sigma}^\infty(\Omega))$ is the space of smooth, solenoidal vector fields with compact support in $]0,T[\times\Omega$ and

$$\begin{aligned} C_0^\infty([0,T[;C_{0,\sigma}^\infty(\Omega)) &:= \{\,v|_{[0,T[\times\Omega};\ v\in C_0^\infty(]-1,T[;C_{0,\sigma}^\infty(\Omega))\,\}\,,\\ C_0^\infty([0,T[\times\overline{\Omega}) &:= \{\,w|_{[0,T[\times\Omega};\ w\in C_0^\infty(]-1,T[\times\mathbb{R}^d)\,\}. \end{aligned}$$

Let $\Omega \subseteq \mathbb{R}^d, d \geq 2$, be a bounded Lipschitz domain. Let dS denote the surface measure on $\partial\Omega$. The space $L^2(\partial\Omega)$ should denote the usual Lebesgue space on $\partial\Omega$ with scalar product $\langle\cdot,\cdot\rangle_{\partial\Omega}$. For $0 < s < 1$ define $H^s(\partial\Omega) := W^{s,2}(\partial\Omega)$, see [13, I.3.6]. It is well known (see [15, Satz II.8.7]) that for $\frac{1}{2} < s \leq 1$ there exists a continuous, linear trace operator $T : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$ with the property $T\phi = \phi|_{\partial\Omega}$ for $\phi \in C_0^1(\overline{\Omega})$.

Next we will discuss some definitions for Section 3. Let $\mathcal{M}(\mathbb{R}^d)$ denote the Banach space of finite, signed Radon measures on $\mathbb{R}^d$ with the total variation norm $\|\cdot\|_{\mathcal{M}(\mathbb{R}^d)}$. It holds that $C_0(\mathbb{R}^d)' \cong \mathcal{M}(\mathbb{R}^d)$ (see [8, Theorem 1.200]) where $C_0(\mathbb{R}^d) := \{\, \phi \in C(\mathbb{R}^d); \lim_{x\to\infty}\phi(x) = 0 \,\}$. A *Carathéodory function* is a function $G(x,\lambda) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that $G(x,\cdot)$ is a continuous function for all fixed x and $G(\cdot,\lambda)$ is a measurable function for all fixed λ. We call a function $f : \Omega \mapsto \mathcal{M}(\mathbb{R}^d)$ *weakly* measurable* if for all $\phi \in C_0(\mathbb{R}^d)$ the function

$$x \mapsto [f(x),\phi]_{\mathcal{M}(\mathbb{R}^d);C_0(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(x,y)\phi(y)\,dy\,, \quad x \in \Omega,$$

is measurable. Define the space

$$\begin{aligned} L^\infty_{\omega^*}(\Omega;\mathcal{M}(\mathbb{R}^d)) := \{\, & f : \Omega \to \mathcal{M}(\mathbb{R}^d); f \text{ weakly}^* \text{ measurable};\\ & x \mapsto \|f(x)\|_{\mathcal{M}(\mathbb{R}^d)} \text{ measurable and essentially bounded} \,\}. \end{aligned}$$

For two domains Ω_1, Ω_2 we write $\Omega_1 \subseteq\subseteq \Omega_2$ if $\overline{\Omega_1} \subseteq \Omega_2$. Further if $(X_n)_{n\in\mathbb{N}}$ is a sequence of Banach spaces we will write that a sequence $v_n \in X_n, n \in \mathbb{N}$, is bounded in $(X_n)_{n\in\mathbb{N}}$ if there is a constant $M > 0$ such that $\|v_n\|_{X_n} \leq M$ for all $n \in \mathbb{N}$.

2.2. **Decomposition of the pressure.** Let $G \subseteq \mathbb{R}^d, d \geq 2$, be a bounded Lipschitz domain. For $u \in L^2(\Omega)$ we write $\mathrm{div} u = 0$ or $\Delta u = 0$ if these identities are satisfied in the sense of distributions on Ω. If $u \in L^2(\Omega)$ satisfies $\Delta u = 0$, then we can apply Weyl's Lemma to get (after a redefinition on a null set) that u is smooth, i.e. $u \in C^\infty(\Omega)$. Define

$$\begin{aligned} \Delta W_0^{2,2}(G) &:= \{\, \Delta p; p \in W_0^{2,2}(G) \,\}\,,\\ L_0^2(G) &:= \{\, p \in L^2(G); \int_G p\,dx = 0 \,\}. \end{aligned}$$

A major point in the proof of the main theorem is to prove identity (4.3) below. Since the standard Aubin-Lions argument cannot be used in our situation, we formulate and prove the following variant of Theorem 2.6 in [16].

Theorem 2.1. *Let $G \subseteq \mathbb{R}^d, d \geq 2$, be a bounded C^2-domain, let $1 < r < \infty$, $0 < T < \infty$, let $u_0 \in L^2(G)$ with $\mathrm{div} u_0 = 0$, let $Q_1 \in L^r(0,T;L^2(G)^{d^2})$, and let $Q_2 \in L^r(0,T;L^2(G)^d)$. Consider $u \in L^\infty(0,T;L^2(G)^d)$ with $\mathrm{div} u(t) = 0$ for a.a. $t \in]0,T[$ and*

$$-\int_0^T \langle u, \partial_t\phi\rangle_G\,dt + \int_0^T \langle Q_1, \nabla\phi\rangle_G\,dt + \int_0^T \langle Q_2, \phi\rangle_G\,dt - \langle u_0, \phi(0)\rangle_G = 0 \tag{2.2}$$

for all $\phi \in C_0^\infty([0,T[;C_{0,\sigma}^\infty(G))$. Then there exist unique functions $p_r \in L^r(0,T;L_0^2(G))$, $p_h \in L^\infty(0,T;L_0^2(G))$ with $p_r(t) \in \Delta W_0^{2,2}(G)\,, \Delta_x p_h(t) = 0$

for a.a. $t \in]0,T[$ *such that*

$$\begin{aligned}&-\int_0^T \langle u+\nabla_x p_h, \partial_t\phi\rangle_G\,dt + \int_0^T \langle Q_1,\nabla\phi\rangle_G\,dt + \int_0^T \langle Q_2,\phi\rangle_G\,dt\\ &\quad= \langle u_0,\phi(0)\rangle_G + \int_0^T \langle p_r, \mathrm{div}\phi\rangle_G\,dt\end{aligned} \tag{2.3}$$

for all $\phi \in C_0^\infty([0,T[;C_0^\infty(G))$. *The following estimates are satisfied*

$$\|p_r\|_{2,r;G;T} \le c(\|Q_1\|_{2,r;G;T} + \|Q_2\|_{2,r;G;T})\,, \tag{2.4}$$

$$\|p_h\|_{2,\infty;G;T} \le c(\|u\|_{2,\infty;G;T} + \|Q_1\|_{2,r;G;T} + \|Q_2\|_{2,r;G;T}) \tag{2.5}$$

with a constant $c = c(G,r,T) > 0$.

***Proof*. Step 1.** Define $u(0) := u_0$. For $\psi \in C_{0,\sigma}^\infty(G)\,, \eta \in C_0^\infty([0,T[)$ we have

$$\begin{aligned}&-\int_0^T \langle u(t),\psi\rangle_G\,\eta'(t)\,dt - \langle u_0,\psi(0)\rangle_G\,\eta(0)\\ &\quad= \int_0^T -\big(\langle Q_1(t),\nabla\psi\rangle_G + \langle Q_2(t),\psi\rangle_G\big)\eta(t)\,dt.\end{aligned} \tag{2.6}$$

Identity (2.6) implies that there exists a Lebesgue null set $N = N(\psi)$ such that

$$\langle u(t),\psi\rangle_G - \langle u_0,\psi\rangle_G = -\int_0^t (\langle Q_1(t),\nabla\psi\rangle_G + \langle Q_2(t),\psi\rangle_G)\,dt \tag{2.7}$$

for all $t \in [0,T[\setminus N$. Using a separability argument u can be redefined on a Lebesgue null set of $[0,T[$ such that (2.7) holds for all $t \in [0,T[$ and all $\psi \in C_{0,\sigma}^\infty(G)$ since $u \in L^\infty(0,T;L^2(G)^d)$. For $t \in [0,T[$ define $\widetilde{Q_1}(t) := \int_0^t Q_1(s)\,ds$ and $\widetilde{Q_2}(t) := \int_0^t Q_2(s)\,ds$. We employ Fubini's Theorem and [13, Lemma II.2.2.2] to get for each fixed $t \in [0,T[$ a unique $p(t) \in L_0^2(G)$ such that

$$\langle u(t)-u_0,\psi\rangle_G + \langle \widetilde{Q_1}(t),\nabla\psi\rangle_G + \langle \widetilde{Q_2}(t),\psi\rangle_G = \langle p(t),\mathrm{div}\psi\rangle_G \tag{2.8}$$

for all $\psi \in W_0^{1,2}(G)$. The estimate [13, (II.2.2.6)] yields

$$\|p(t)\|_2 \le c(G,r)(\|u(t)-u_0\|_2 + \|\widetilde{Q}_1(t)\|_2 + \|\widetilde{Q}_2(t)\|_2) \tag{2.9}$$

for all $t \in [0,T[$. Using (2.9) we can show that $t \mapsto p(t)\,, t \in [0,T[$, is Bochner measurable (as a function $[0,T[\to L^2(G)$). Furthermore it holds

$$\|p(t)\|_{2,\infty;G;T} \le c(G,r,T)(\|u\|_{2,\infty;G;T} + \|Q_1\|_{2,r;G;T} + \|Q_2\|_{2,r;G;T}). \tag{2.10}$$

Step 2. [16, Corollary 2.5] implies the existence of unique functions $\widetilde{p_r} \in L^\infty(0,T;L_0^2(G))$ and $p_h \in L^\infty(0,T;L_0^2(G))$ with $p_r(t) \in \Delta W_0^{2,2}(G)$ and $\Delta_x p_h(t) = 0$ for a.a. $t \in [0,T[$ such that

$$p(t) = \widetilde{p_r}(t) + p_h(t)\,, \quad \|\widetilde{p_r}(t)\|_2 + \|p_h(t)\|_2 \le c\|p(t)\|_2 \tag{2.11}$$

with a constant $c = c(G, r, T)$ for a.a. $t \in [0, T[$. From (2.9), (2.11) it follows $\widetilde{p_r}(0) = 0$. We integrate (2.8) over $[0, T[$ and use the Gauss theorem to get

$$\begin{aligned} \int_0^T \langle u(t) - u_0, \phi\rangle_G \, dt + \int_0^T \langle \widetilde{Q_1}(t), \nabla\phi\rangle_G \, dt + \int_0^T \langle \widetilde{Q_2}(t), \phi\rangle_G \, dt \\ = \int_0^T \langle \widetilde{p_r}(t), \mathrm{div}\phi\rangle_G \, dt - \int_0^T \langle \nabla_x p_h(t), \phi\rangle_G \, dt \end{aligned} \tag{2.12}$$

for all $\phi \in C_0^\infty([0, T[\times G)$.

Step 3. Fix $\phi \in C_0^\infty(G)$ and consider $\psi := \nabla\phi$ in (2.8). With $\mathrm{div} u = 0$ and $\Delta_x p_h = 0$ it follows

$$\begin{aligned} \int_G \big(\widetilde{p_r}(t+h) - \widetilde{p_r}(t)\big)\Delta\phi \, dx = \int_G \big(\widetilde{Q_1}(t+h) - \widetilde{Q_1}(t)\big) \cdot \nabla^2\phi \, dx \\ + \int_G \big(\widetilde{Q_2}(t+h) - \widetilde{Q_2}(t)\big) \cdot \nabla\phi \, dx \end{aligned} \tag{2.13}$$

for all $t \in]0, T[$ and all $0 < h < T - t$. Since $\widetilde{p_r}(t) \in \Delta W_0^{2,2}(G)$ we obtain from [16, (2.1),(2.2)] that

$$\|\widetilde{p_r}(t+h) - \widetilde{p_r}(t)\|_2 \le c \sum_{i=1}^2 \|\widetilde{Q_i}(t+h) - \widetilde{Q_i}(t)\|_2$$

for all $t \in]0, T[$ and all $0 < h < T - t$ with a constant $c = c(G, r, T)$. Hence

$$\begin{aligned} \int_0^{T-h} \Big\| \frac{\widetilde{p_r}(t+h) - \widetilde{p_r}(t)}{h} \Big\|_2^r dt &\le c \sum_{i=1}^2 \int_0^{T-h} \Big\| \frac{\widetilde{Q_i}(t+h) - \widetilde{Q_i}(t)}{h} \Big\|_2^r dt \\ &\le c \sum_{i=1}^2 \int_0^{T-h} \big(\frac{1}{h}\big)^r \int_t^{t+h} \|Q_i(s)\|_2^r \, ds \, (h)^{r/r'} \, dt \\ &= c \sum_{i=1}^2 \frac{1}{h} \int_0^T \int_0^{T-h} 1_{[t,t+h]}(s) \|Q_i(s)\|_2^r \, dt \, ds \\ &\le c \sum_{i=1}^2 \int_0^T \|Q_i(s)\|_2^r \, ds \end{aligned} \tag{2.14}$$

for all $t \in]0, T[$ and all $0 < h < T - t$ with a constant $c = c(r, G, T) > 0$ independent of t and h. Estimate (2.14) yields $\widetilde{p_r} \in W^{1,r}(0, T; L^2(G))$ and

$$\|\partial_t \widetilde{p_r}\|_{2,r;T} \le c\,(\,\|Q_1\|_{2,r;T} + \|Q_2\|_{2,r;T}\,). \tag{2.15}$$

Step 4. Let $p_r := \partial_t \widetilde{p_r} \in L^r(0, T; L^2(G))$. For arbitrary $\phi \in C_0^\infty([0, T[\times G)$ consider $\partial_t\phi$ instead of ϕ in (2.12) and integrate by parts to get (2.3). In this argument $\phi(T) = 0$ and $\widetilde{p_r}(0) = 0$ were used.

Proof of uniqueness. Let p_r^2 and p_h^2 be an other pair of functions satisfying the conclusions of this lemma. By putting $p_r - p_r^2$ in (2.4) and $p_h - p_h^2$ in (2.5) we get $p_r(t) = p_r^2(t)$ and $p_h(t) = p_h^2(t)$ for almost all $t \in]0, T[$. $\square$

2.3. **Preliminary Lemmas.**

Lemma 2.2. *Let $\Omega\,,\Omega_n$ be as in (1.3), (1.4), let $K \subseteq \mathbb{R}^2$ be compact.*

(1) *Let $(v_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\Omega_n)$. Then*

$$\lim_{n\to\infty}\int_K |v_n(x,\psi(x)-\phi_n(x)) - v_n(x,\psi(x))|\,dx = 0. \tag{2.16}$$

(2) *If $(\theta_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(0,T;H^1(\Omega_n))$ then*

$$\lim_{n\to\infty}\int_{[0,T[\times K} |\theta_n(t,x,\psi(x)-\phi_n(x)) - \theta_n(t,x,\psi(x))|\,d(x,t) = 0. \tag{2.17}$$

Proof. For $v \in C_0^\infty(\overline{\Omega_n})$ one has

$$\int_K |v(x,\psi(x)-\phi_n(x)) - v(x,\psi(x))|\,dx = \int_K \Big|\int_{\psi(x)-\phi_n(x)}^{\psi(x)} \frac{\partial v}{\partial x_3}(x,\tau)\,d\tau\Big|\,dx. \tag{2.18}$$

By a density argument, the identity (2.18) still holds true for $v_n \in H^1(\Omega_n)$. We deduce with (2.18)

$$\begin{aligned}
&\int_K |v_n(x,\psi(x)-\phi_n(x)) - v_n(x,\psi(x))|\,dx \\
&\le \left(\int_{\Omega_n} |\frac{\partial v_n}{\partial x_3}(x',\tau)|^2\,d\tau\right)^{1/2}\left(\int_K\int_{\psi(x)-\phi_n(x)}^{\psi(x)} 1\,d\tau\,dx\right)^{1/2} \qquad (2.19)\\
&\le \|v_n\|_{W^{1,2}(\Omega_n)}\sqrt{|K|}\,\|\phi_n\|_{\infty,K}^{1/2}.
\end{aligned}$$

By using $\phi_n(x) \to 0$ for $n\to\infty$ uniformly in $x \in K$, we get (2.16). The proof of (2.17) is based on a 'time dependent' version of (2.18) and an argumentation similar to (2.19). □

Lemma 2.3. *Let $\Omega\,,\Omega_n$ be as in (1.3), (1.4).*

(1) *Let $(v_n)_{n\in\mathbb{N}}$ be bounded in $L^2_\sigma(\Omega_n)$, let $v \in L^2(\Omega)$ with $v_n \rightharpoonup v$ for $n\to\infty$ in $L^2(\Omega)$. Then $v \in L^2_\sigma(\Omega)$.*
(2) *Let $(w_n)_{n\in\mathbb{N}}$ be bounded in $L^2(0,T;L^2_\sigma(\Omega_n))$, let $w \in L^2(0,T;L^2(\Omega))$ with $w_n \rightharpoonup w$ for $n\to\infty$ in $L^2(0,T;L^2(\Omega))$. Then $w(t) \in L^2_\sigma(\Omega)$ for a.a. $t \in [0,T[$.*

Proof. We will only prove the second statement. Let $\phi \in C_0^\infty([0,T[\times\overline{\Omega})$ and let $B \subseteq \mathbb{R}^3$ be a ball with $\mathrm{supp}(\phi(t,\cdot)) \subseteq B$ for all $t \in [0,T[$. Define $B_n := (\Omega_n\setminus\Omega)\cap B$ and consider the identity

$$\int_0^T \langle w_n,\phi\rangle_{\Omega_n}\,dt = \int_0^T \langle w_n,\phi\rangle_{\Omega}\,dt + \int_0^T \langle w_n,\phi\rangle_{B_n}\,dt. \tag{2.20}$$

We get

$$\begin{aligned}
\Big|\int_0^T \langle w_n,\phi\rangle_{B_n}\,dt\Big| &\le \sup_{(t,x)\in[0,T[\times\Omega_n}|\phi(t,x)|\int_{[0,T[\times B_n}|w_n(t,x)|\,d(x,t) \qquad (2.21)\\
&\le c\sqrt{T\,|B_n|}\,\|w_n\|_{L^2(0,T;W^{1,2}(\Omega_n))}
\end{aligned}$$

with a constant $c>0$ independent of $n\in\mathbb{N}$. Since $1_{\Omega_n}\to 1_\Omega$ in $L^1_{\mathrm{loc}}(\mathbb{R}^3)$ for $n\to\infty$, we get $|B_n|\to 0$ for $n\to\infty$ and consequently from (2.20) (2.21), and $w_n \rightharpoonup w$ in $L^2(0,T;L^2(\Omega))$ it follows

$$\lim_{n\to\infty} \int_0^T \langle w_n, \phi\rangle_{\Omega_n}\, dt = \int_0^T \langle w, \phi\rangle_{\Omega}\, dt \tag{2.22}$$

for all $\phi \in C_0^\infty([0,T[\times\overline{\Omega})$. We use $w_n \in L^2(0,T;L^2_\sigma(\Omega_n))$ and (2.22) to get $\int_0^T \langle w, \nabla\phi\rangle_\Omega\, dt = 0$ for all $\phi \in C_0^\infty([0,T[\times\overline{\Omega})$. A cut-off procedure gives us a Lebesgue null set $N \subseteq]0,T[$ such that $\langle w(t), \nabla\psi\rangle_\Omega = 0$ for all $t \in]0,T[\setminus N$ and all $\psi \in C_0^\infty(\overline{\Omega})$. Therefore, combining [9, Lemma III.2.1] with [7, Lemma 2.1(i)] we see that $w(t) \in L^2_\sigma(\Omega)$ for all $t \in]0,T[\setminus N$. $\square$

3. Young measures

The formulation of the main theorem is based on the following existence theorem for Young measures. The main difficulty is that $(\phi_n)_{n\in\mathbb{N}}$ is defined on an unbounded set.

Theorem 3.1. *Let (ϕ_n) be as in (1.4). Then there exists a (not necessarily unique) subsequence $(m_k)_{k\in\mathbb{N}}$ and a unique* Young measure $\mathcal{R} = (\mathcal{R}_x)_{x\in\mathbb{R}^2} \in L^\infty_{\omega^*}(\mathbb{R}^2;\mathcal{M}(\mathbb{R}^2))$ *associated to the subsequence $(\nabla\phi_{m_k})_{k\in\mathbb{N}}$ with the following properties.*

(1) *For a.a. $x \in \mathbb{R}^2$ it holds that $\mathcal{R}_x$ is a (positive) probability measure on $\mathbb{R}^2$ and there exists a compact set $K_\mathcal{R} \subseteq \mathbb{R}^2$ with* $\operatorname{supp}(\mathcal{R}_x) \subseteq K_\mathcal{R}$ *for a.a. $x \in \mathbb{R}^2$.*

(2) *Let $G(x,\lambda) : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$ be a Carathéodory function and define*

$$\overline{G}(x) := \int_{\mathbb{R}^2} G(x,\lambda)\, d\mathcal{R}_x(\lambda)\,, \quad \text{for a.a. } x \in \mathbb{R}^2. \tag{3.1}$$

If the sequence $(G(x,\nabla\phi_{m_k}(x))_{k\in\mathbb{N}}$ is weakly convergent in $L^1(\mathbb{R}^2)$ and $\overline{G} \in L^1(\mathbb{R}^2)$ then it holds

$$G(x,\nabla\phi_{m_k}(x)) \underset{k\to\infty}{\rightharpoonup} \overline{G}(x) \quad \text{in } L^1(\mathbb{R}^2). \tag{3.2}$$

Remark. The subsequence $(m_k)_{k\in\mathbb{N}}$ to which there exists a Young measure associated to $(\nabla\phi_{m_k})_{k\in\mathbb{N}}$ is not unique. But for a fixed subsequence $(m_k)_{k\in\mathbb{N}}$ there is at most one Young measure associated to $(\nabla\phi_{m_k})_{k\in\mathbb{N}}$.

Proof. For each $l \in \mathbb{N}$ there exists a (not necessarily unique) subsequence $(m_k^l)_{k\in\mathbb{N}}$ and a unique Young measure $(\nu_x^l)_{x\in B(0,l)} \in L^\infty_{\omega^*}(B(0,l);\mathcal{M}(\mathbb{R}^2))$ associated to $(\nabla\phi_{m_k^l})_{k\in\mathbb{N}}$. We have employed [8, Proposition 8.4 and Theorem 8.6] which is possible since $B(0,l) := \{x \in \mathbb{R}^2; |x| \le l\}$ is bounded and the sequence $(\nabla\phi_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(B(0,l))$. We remark that $(\nu_x^l)_{x\in B(0,l)} \in L^\infty_{\omega^*}(B(0,l);\mathcal{M}(\mathbb{R}^2))$ is uniquely determined by the convergence property

$$\psi(\nabla\phi_{m_k^l}) \underset{k\to\infty}{\rightharpoonup^*} \int_{\mathbb{R}^2} \psi(\lambda)\, d\nu_x^l(\lambda) \quad \text{in } L^\infty(B(0,l)) \tag{3.3}$$

for all $\psi \in C_0(\mathbb{R}^2)$. Using induction on $l \in \mathbb{N}$ there exists $(m_k^l)_{k\in\mathbb{N}}$ such that $(m_k^{l+1})_{k\in\mathbb{N}}$ is a subsequence of $(m_k^l)_{k\in\mathbb{N}}$ and $(\nu_x^l)_{x\in B(0,l)} \in L^\infty_{\omega^*}(B(0,l);\mathcal{M}(\mathbb{R}^2))$ is the unique Young measure associated to $(\nabla\phi_{m_k^l})_{k\in\mathbb{N}}$. Moreover $\nu_x^{l+1} = \nu_x^l$

for a.a. $x \in B(0,l)$ by (3.3) . Therefore it is possible to obtain a well defined Young measure $(\mathcal{R}_x)_{x\in\mathbb{R}^2} \in L^\infty_{\omega^*}(\mathbb{R}^2;\mathcal{M}(\mathbb{R}^2))$ by requiring

$$(\mathcal{R}_x)_{x\in\mathbb{R}^2}|_{B(0,l)} = (\nu^l_x)_{x\in B(0,l)} \quad \text{for a.a. } x \in B(0,l) \tag{3.4}$$

for all $l \in \mathbb{N}$. Define $m_k := m^k_k, k \in \mathbb{N}$ ('diagonal sequence'). Let $G(x,\lambda) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a Carathéodory function as in (2). Since $\overline{G} \in L^1(\mathbb{R}^2)$ and

$$1_{B(0,l)} G(x, \nabla\phi_{m_k}(x)) \underset{k\to\infty}{\rightharpoonup} 1_{B(0,l)} \int_{\mathbb{R}^2} G(x,\lambda)\, d\mathcal{R}_x(\lambda) \quad \text{in } L^1(\mathbb{R}^2) \tag{3.5}$$

for all $l \in \mathbb{N}$ it follows that (3.2) is true. We choose a compact set $K_\mathcal{R} \subseteq \mathbb{R}^2$ such that $\nabla\phi_{m_k}(x) \in K_\mathcal{R}$ for a.a. $x \in \mathbb{R}^2$ and all $k \in \mathbb{N}$. By considering $\psi \in C_0(\mathbb{R}^2)$ with $\psi(\lambda) = 0$ for all $\lambda \in K_\mathcal{R}$ in (3.3) we finish the proof. □

For the boundary behaviour of u the following definition is needed.

Definition 3.2. A Young measure $\mathcal{R} = (\mathcal{R}_x)_{x\in\mathbb{R}^2} \in L^\infty_{\omega^*}(\mathbb{R}^2;\mathcal{M}(\mathbb{R}^2))$ is *non-degenerate* if for a.a. $x \in \mathbb{R}^2$ it holds that $\operatorname{supp}(\mathcal{R}_x)$ contains two linearly independent vectors in $\mathbb{R}^2$.

4. Proof of the main theorem

The crucial point for the proof of the first part of the main theorem is to prove that the Boussinesq system (1.1) has the property that a weak limit of $(u_n,\theta_n)_{n\in\mathbb{N}}$ satisfies the variational identities (1.10), (1.11) with an 'additional weight function' Λ on Ω. The proof of this weak compactness property is the central topic of the lemmas in Section 5.1. To show the second part of the main theorem we need additionally the boundary behaviour of u which will be treated in Theorem 4.4. Without loss of generality we may assume $m_k = k$ for all $k \in \mathbb{N}$.

4.1. Lemmas needed for the proof of statement (1) of Theorem 1.2.

Lemma 4.1. *The weak limit (u,θ) in (1.13), (1.14) satisfies (1.10) on Ω for all $\phi \in C^\infty_0([0,T[;C^\infty_{0,\sigma}(\Omega))$.*

Proof. By interpolation and the continuous imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ we get with (1.13)

$$\int_0^T \|u_n \otimes u_n\|^{4/3}_{2,\Omega}\, dt \le c \int_0^T \|u_n\|^{2/3}_{2,\Omega} \|u_n\|^2_{H^1(\Omega)}\, dt \le c \tag{4.1}$$

with a constant $c > 0$ independent of $n \in \mathbb{N}$. Therefore we find a matrix field in $L^{4/3}(0,T;L^2(\Omega))$, denoted by $\overline{u\otimes u}$, such that (along a not relabeled subsequence)

$$u_n \otimes u_n \underset{n\to\infty}{\rightharpoonup} \overline{u\otimes u} \quad \text{in } L^{4/3}(0,T;L^2(\Omega)). \tag{4.2}$$

The main step in the proof of this lemma is to prove the following assertion.
Assertion. It holds

$$\lim_{n\to\infty} \int_0^T \langle u_n \otimes u_n, \nabla\phi\rangle_\Omega\, dt = \int_0^T \langle u\otimes u, \nabla\phi\rangle_\Omega\, dt. \tag{4.3}$$

for all $\phi \in C^\infty_0([0,T[;C^\infty_{0,\sigma}(\Omega))$.
Proof of (4.3). Fix $\phi \in C^\infty_0([0,T[;C^\infty_{0,\sigma}(\Omega))$. We choose smooth, bounded domains Ω_1,Ω_2 with $\operatorname{supp}(\phi(t,\cdot)) \subseteq\subseteq \Omega_1 \subseteq\subseteq \Omega_2 \subseteq\subseteq \Omega$ for all $t \in [0,T[$.

Since (u_n, θ_n) satisfies (1.10) for all $\psi \in C_0^\infty([0,T[; C_{0,\sigma}^\infty(\Omega_2))$ an application of Theorem 2.1 yields the existence of unique

$$p_{r,n} \in L^{4/3}(0,T; L_0^2(\Omega_2))\,, \quad p_{r,n}(t) \in \Delta W_0^{2,2}(G) \text{ for a.a. } t \in [0,T[\,, \tag{4.4}$$

$$p_{h,n} \in L^\infty(0,T; L_0^2(\Omega_2))\,, \quad \Delta_x p_{h,n}(t) = 0 \text{ for a.a. } t \in [0,T[\tag{4.5}$$

such that

$$\begin{aligned} &-\int_0^T \langle u_n + \nabla_x p_{h,n}, \psi\rangle_{\Omega_2} \eta'(t) \\ &= \int_0^T \big(\langle u_n \otimes u_n, \nabla\psi\rangle_{\Omega_2} - \nu\langle \nabla u_n, \nabla\psi\rangle_{\Omega_2}\big)\eta(t)\, dt \\ &+ \int_0^T \big(\beta\langle \theta_n g, \psi\rangle_{\Omega_2} + \langle f_1, \psi\rangle_{\Omega_2} + \langle p_{r,n}, \mathrm{div}\psi\rangle_{\Omega_2}\big)\eta(t)\, dt \end{aligned} \tag{4.6}$$

for all $\psi \in C_0^\infty(\Omega_2)\,, \eta \in C_0^\infty(]0,T[)$. Consider the estimates (2.4) and (2.5) to see that $(p_{h,n})_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T; L^2(\Omega_2))$ and that $(p_{r,n})_{n\in\mathbb{N}}$ is bounded in $L^{4/3}(0,T; L^2(\Omega_2))$. Hence (along a not relabeled subsequence)

$$p_{h,n} \underset{n\to\infty}{\rightharpoonup^*} p_h \text{ in } L^\infty(0,T; L^2(\Omega_2))\,, \tag{4.7}$$

$$p_{r,n} \underset{n\to\infty}{\rightharpoonup} p_r \quad \text{in } L^{4/3}(0,T; L^2(\Omega_2)). \tag{4.8}$$

By (4.7) we can conclude that $\Delta_x p_h(t) = 0$ for a.a. $t \in [0,T[$. Consequently (4.7) and [6, Theorem 2.2.7] imply that $(p_{h,n})_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T; C^2(\overline{\Omega_1}))$. Therefore

$$p_{h,n} \underset{n\to\infty}{\rightharpoonup} p_h \quad \text{in } L^2(0,T; H^2(\Omega_1)). \tag{4.9}$$

Fix $n \in \mathbb{N}$. With the imbedding $L^2(\Omega_1) \hookrightarrow H^{-1}(\Omega_1)\,, y \mapsto \langle y, \cdot\rangle_{\Omega_1}$, we can write equation (4.6) as (it holds $\Omega_1 \subseteq \Omega_2$)

$$\begin{aligned} \frac{d}{dt}(u_n + \nabla_x p_{h,n}) &= \langle u_n \otimes u_n, \nabla\cdot\rangle_{\Omega_1} \\ &\quad - \nu\langle \nabla u_n\,, \nabla\cdot\rangle_{\Omega_1} + \langle p_{r,n}, \mathrm{div}\cdot\rangle_{\Omega_1} + \beta\theta_n g + f_1 \end{aligned} \tag{4.10}$$

in $L^{4/3}(0,T; H^{-1}(\Omega_1))$. Consider the imbedding scheme

$$H^1(\Omega_1) \underset{\text{compact}}{\hookrightarrow} L^2(\Omega_1) \underset{\text{continuous}}{\hookrightarrow} H^{-1}(\Omega_1). \tag{4.11}$$

Looking at (4.10), (4.8), (1.13) (1.14) we see that $(\frac{d}{dt}(u_n + \nabla_x p_{h,n}))_{n\in\mathbb{N}}$ is bounded in $L^{4/3}(0,T; H^{-1}(\Omega_1))$. Since by (1.13) and (4.9) the sequence $(u_n + \nabla_x p_{h,n})_{n\in\mathbb{N}}$ is bounded in $L^2(0,T; H^1(\Omega_1))$ we get with (4.11) and [14, Theorem 3.2.2] that a subsequence of $(u_n + \nabla_x p_{h,n})_{n\in\mathbb{N}}$ is strongly convergent in $L^2(0,T; L^2(\Omega_1))$. Since $(u_n + \nabla_x p_{h,n})_{n\in\mathbb{N}}$ is also weakly convergent in $L^2(0,T; L^2(\Omega_1))$ to $u + \nabla_x p_h$ we have proven

$$u_n + \nabla_x p_{h,n} \underset{n\to\infty}{\rightarrow} u + \nabla_x p_h \quad \text{strongly in } L^2(0,T; L^2(\Omega_1)). \tag{4.12}$$

It follows (for this fixed ϕ)

$$\begin{aligned}
&\int_0^T \langle \overline{u\otimes u}, \nabla\phi\rangle_{\Omega_1}\,dt\\
&= \lim_{n\to\infty}\int_0^T \langle u_n\otimes u_n, \nabla\phi\rangle_{\Omega_1}\,dt\\
&= \lim_{n\to\infty}\int_0^T \langle (u_n+\nabla_x p_{h,n})\otimes u_n, \nabla\phi\rangle_{\Omega_1}\,dt\\
&- \lim_{n\to\infty}\int_0^T \langle \nabla_x p_{h,n}\otimes (u_n+\nabla_x p_{h,n}), \nabla\phi\rangle_{\Omega_1}\,dt\\
&= \int_0^T \langle (u+\nabla_x p_h)\otimes u, \nabla\phi\rangle_{\Omega_1} - \int_0^T \langle \nabla_x p_h\otimes (u+\nabla_x p_h), \nabla\phi\rangle_{\Omega_1}\,dt\\
&= \int_0^T \langle u\otimes u, \nabla\phi\rangle_{\Omega_1}\,dt.
\end{aligned}$$

In this calculation (1.13), (4.9), (4.12) and the following argument were employed: For $h\in L^2(0,T;H^1(\Omega_1))$ with $\Delta h(t)=0$ for a.a. $t\in]0,T[$ it holds $\int_0^T\langle\nabla_x h\otimes\nabla_x h,\nabla\phi\rangle_{\Omega_1}\,dt = 0$. From (4.2) and the above computation we obtain (4.3).

Now we can finish the proof of this lemma. Let $\phi\in C_0^\infty([0,T[;C_{0,\sigma}^\infty(\Omega))$. Using in the variational identity (1.10) for u_n and θ_n on Ω_n with test function ϕ the convergences (1.13), (1.14) and (4.3) we get that (u,θ) fulfills (1.10). □

Since we want to prove the Robin boundary condition (1.15) it is important that in the following lemma the domain Ω' can share common parts of the boundary with Ω.

Lemma 4.2. *For all bounded Lipschitz domains Ω' with $\Omega'\subseteq\Omega$ and for all $0\le s<1$ it holds*

$$\theta_n \underset{n\to\infty}{\to} \theta \quad \textit{strongly in } L^2(0,T;H^s(\Omega')). \tag{4.13}$$

Proof. Assume by contradiction that there is a bounded Lipschitz domain Ω' with $\Omega'\subseteq\Omega$ and a subsequence $(\theta_{m_k})_{k\in\mathbb{N}}$, an $\epsilon>0$ and $0\le s<1$ with

$$\|\theta_{m_k}-\theta\|_{L^2(0,T;H^s(\Omega'))}\ge\epsilon \quad \forall k\in\mathbb{N}. \tag{4.14}$$

From (1.11) we get for all $\psi\in C_0^\infty(\Omega')$ and $\eta\in C_0^\infty(]0,T[)$

$$\begin{aligned}
&-\int_0^T \langle\theta_n,\psi\rangle_{\Omega'}\eta'(t)\\
&= \int_0^T \big(\langle\theta_n u_n,\nabla\psi\rangle_{\Omega'} - \kappa\langle\nabla\theta_n,\nabla\psi\rangle_{\Omega'} + \langle f_2,\psi\rangle_{\Omega'}\big)\eta(t)\,dt.
\end{aligned} \tag{4.15}$$

In view of the identification $L^2(\Omega')\hookrightarrow H^{-1}(\Omega')\,,y\mapsto\langle y,\cdot\rangle_{\Omega'}$, equation (4.15) means that

$$\frac{d}{dt}\theta_n = \langle\theta_n u_n,\nabla\cdot\rangle_{\Omega'} - \kappa\langle\nabla\theta_n,\nabla\cdot\rangle_{\Omega'} + f_2 \tag{4.16}$$

as identity in $L^{\frac43}(0,T;H^{-1}(\Omega'))$ for every fixed $n\in\mathbb{N}$. By (1.14) we see that the sequence $(\theta_{m_k})_{k\in\mathbb{N}}$ is bounded in $L^2(0,T;H^1(\Omega'))$. From (4.16) with (1.13), (1.14) it follows that $(\frac{d}{dt}\theta_{m_k})_{k\in\mathbb{N}}$ is bounded in $L^{4/3}(0,T;H^{-1}(\Omega'))$. Consider the imbedding scheme

$$H^1(\Omega') \underset{\text{compact}}{\hookrightarrow} H^s(\Omega') \underset{\text{continuous}}{\hookrightarrow} H^{-1}(\Omega'). \tag{4.17}$$

The compactness of the first imbedding follows from (2.1). From (4.17) and [14, Theorem 3.2.2] we get the existence of a subsequence of $(\theta_{m_k})_{k\in\mathbb{N}}$ which is strongly convergent in $L^2(0,T;H^s(\Omega'))$. Looking at (1.14) yields that this strong limit of a subsequence of $(\theta_{m_k})_{k\in\mathbb{N}}$ has to be θ. This contradicts (4.14). □

Lemma 4.3. *(u,θ) satisfies (1.11) on Ω for all $w \in C_0^\infty([0,T[\times\overline{\Omega})$ with Λ defined by (1.15).*

Proof. The crucial point of the lemma is to prove that

$$\lim_{n\to\infty}\int_0^T \langle \theta_n, w\rangle_{\partial\Omega_n}\,dt = \int_0^T \langle \Lambda\cdot\theta, w\rangle_{\partial\Omega}\,dt \tag{4.18}$$

for all $w \in C_0^\infty([0,T[\times\overline{\Omega})$ where Λ is defined by (1.15).
Proof of (4.18). Let $K \subseteq \mathbb{R}^2$ be compact with

$$\operatorname{supp}(w) \subseteq \{\,(t,x,\psi(x)-\phi_n(x));(t,x)\in[0,T[\times K\}$$

for all $n\in\mathbb{N}$. Define

$$s_n(x) := \sqrt{1+|\nabla\psi(x)-\nabla\phi_n(x)|^2}\,,\quad x\in K\,,$$

and $Q := [0,T[\times K$. Then we estimate

$$\begin{aligned}
&\Big|\int_0^T \langle\theta_n,w\rangle_{\partial\Omega_n}\,dt - \int_0^T\langle\Lambda\cdot\theta,w\rangle_{\partial\Omega}\,dt\Big| \\
&\le \int_Q |\,(\theta_n w)(t,x,\psi(x)-\phi_n(x)) - (\theta_n w)(t,x,\psi(x))|\,s_n(x)\,d(x,t) \\
&+ \int_Q |(\theta_n w)(t,x,\psi(x)) - (\theta w)(t,x,\psi(x))|\,s_n(x)\,d(x,t) \\
&+ \Big|\int_Q (\theta w)(t,x,\psi(x))\big(s_n(x) - \int_{K_{\mathcal{R}}}\sqrt{1+|\nabla\psi(x)-\lambda|^2}\,d\mathcal{R}_x(\lambda)\,\big)\,d(x,t)\Big|.
\end{aligned} \tag{4.19}$$

Let $\Omega'\subseteq\Omega$ be a bounded Lipschitz domain such that $\{(x,\psi(x));x\in K\}\subseteq\partial\Omega'$. Fix any $\frac12 < s < 1$. Using the continuous trace operator $H^s(\Omega')\hookrightarrow L^2(\partial\Omega')$ (see section 2.1) it follows from (4.13) that

$$\theta_n \to \theta \quad \text{strongly in } L^2(0,T;L^2(\partial\Omega')). \tag{4.20}$$

A consequence of (2.17) and (4.20) is that the first two terms on the right hand side of (4.19) tend to zero for $n\to\infty$. In this argument we have employed the boundedness of $\nabla\psi\,,\nabla\phi_n$ and the uniform convergence of $w(t,x,\psi(x)-\phi_n(x))$ for $(t,x)\in[0,T[\times K$. To show that the third term in (4.19) converges to zero consider the Carathéodory function

$$H(x,\lambda) := \chi_K(x)\sqrt{1+|\nabla\psi(x)-\lambda|^2}\,,\quad x\in\mathbb{R}^2\,,\lambda\in\mathbb{R}^2. \tag{4.21}$$

Using (3.2) it follows (along a not relabeled subsequence)

$$\chi_K(x)\sqrt{1+|\nabla\psi(x)-\nabla\phi_n(x)|^2} \underset{n\to\infty}{\rightharpoonup} \chi_K(x)\int_{K_{\mathcal{R}}}\sqrt{1+|\nabla\psi(x)-\lambda|^2}\,d\mathcal{R}_x(\lambda) \tag{4.22}$$

in $L^1(\mathbb{R}^2)$ and even in $L^2(\mathbb{R}^2)$ due to the term χ_K. Especially we get for a.a. $t\in[0,T[$

$$\begin{aligned}&\lim_{n\to\infty}\int_K(\theta w)(t,x,\psi(x))s_n(x)\,dx\\&=\int_K(\theta w)(t,x,\psi(x))\int_{K_{\mathcal{R}}}\sqrt{1+|\nabla\psi(x)-\lambda|^2}\,d\mathcal{R}_x(\lambda)\,dx.\end{aligned} \tag{4.23}$$

By (4.23) and the boundedness of $(s_n)_{n\in\mathbb{N}}$ on K we conclude with Lebesgue's theorem on dominated convergence that the third term on the right hand side of (4.19) tends to zero for $n\to\infty$ (along a not relabeled subsequence). Altogether (4.18) holds.

Let $w\in C_0^\infty([0,T[\times\overline{\Omega})$. We can consider w as an element in $C_0^\infty([0,T[\times\overline{\Omega_n})$ and therefore (u_n,θ_n) satisfies (1.11) with Λ replaced by 1 and w as test function. In this lemma we have to prove that for n tending to infinity (u,θ) satisfies (1.11) on Ω with Λ defined by (1.15). The desired convergence of the boundary term $\int_0^T\langle\theta_n,w\rangle_{\partial\Omega_n}\,dt$ follows from (4.18). With the same argumentation as in (4.18) (in this case (4.20) is not needed) we show that

$$\lim_{n\to\infty}\int_0^T\langle h_0,w\rangle_{\partial\Omega_n}\,dt=\int_0^T\langle\Lambda\cdot h_0,w\rangle_{\partial\Omega}\,dt.$$

To pass to the limit in the 'other linear terms' we make use of (1.13), (1.14) and (2.22). For the nonlinear term $\int_0^T\langle u_n\cdot\nabla w,\theta_n\rangle_{\Omega_n}$ we additionally need (4.13). □

4.2. **Proof of statement 1 of Theorem 1.2.** Combine Lemma 4.1 and Lemma 4.3 to get that (u,θ) satisfies the identities (1.10) and (1.11) on Ω for all test function ϕ and w as in Definition 1.1. From Lemma 2.3 we obtain $u\in L^\infty(0,T;L^2_\sigma(\Omega))$.

4.3. **Proof of statement 2 of Theorem 1.2.** Since we have already proven statement 1 of the main theorem it is enough to prove the boundary behaviour of u when the Young measure $\mathcal{R}$ is non degenerate. The following theorem describes the boundary behaviour of u under these conditions. The idea of the proof is based on [1]; in our situation we consider a more general (especially unbounded) domain. This theorem is completely independent of the equations that are fulfilled by $(u_n,\theta_n)_{n\in\mathbb{N}}$.

Theorem 4.4. *Let Ω,Ω_n be as in (1.3), (1.4). Let $(u_n)_{n\in\mathbb{N}}$ be bounded in $L^2(0,T;H^1(\Omega_n))$, let $u_n\rightharpoonup u$ for $n\to\infty$ in $L^2(0,T;H^1(\Omega))$ and let $u_n(t)\in L^2_\sigma(\Omega)$ for a.a. $t\in]0,T[$ and for all $n\in\mathbb{N}$. Assume that $\mathcal{R}=(\mathcal{R}_x)_{x\in\mathbb{R}^2}$ is a non-degenerate Young measure associated to $(\nabla\phi_n)_{n\in\mathbb{N}}$. Then $u(t)\in H_0^1(\Omega)$ for a.a. $t\in[0,T[$.*

Proof. It suffices to prove the following assertion.
Whenever $(v_n)_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\Omega_n)\cap L^2_\sigma(\Omega_n)$ with $v_n\rightharpoonup v$

for $n \to \infty$ in $H^1(\Omega)$ then $v \in H^1_0(\Omega)$. Assume this statement is true. For $n \in \mathbb{N}$ and $\delta > 0$ with $\delta < T - \delta$ define

$$u_n^\delta(t) := (\widetilde{u_n} * \rho_\delta)(t) := \int_{\mathbb{R}} \widetilde{u_n}(t-\tau)\rho_\delta(\tau)\, d\tau\,, \quad t \in [\delta, T-\delta]\,, \tag{4.24}$$

where $(\rho_\delta)_{\delta>0}$ is a smooth Dirac sequence with suitable compact support and $\widetilde{u_n}(\tau) := 1_{[0,T]}(\tau)u_n(\tau)$. Then the sequence $(u_n^\delta(t))_{n\in\mathbb{N}}$ with $t \in [\delta, T-\delta]$ has the properties of the sequence $(v_n)_{n\in\mathbb{N}}$ of the statement. Hence $u^\delta(t) \in H^1_0(\Omega)$ for all $t \in [\delta, T-\delta]$. Since $u^\delta(t) \to u(t)$ for $\delta \to 0+$ strongly in $H^1(\Omega)$ for a.a. $t \in [0,T]$ we get $u(t) \in H^1_0(\Omega)$ for a.a. $t \in [0,T[$. It remains to show that the assertion is true.

Let $K_{\mathcal{R}} \subseteq \mathbb{R}^2$ be compact with $\operatorname{supp}(\mathcal{R}_x) \subseteq K_{\mathcal{R}}$ for a.a. $x \in \mathbb{R}^2$. Lemma 2.3 yields $v \in L^2_\sigma(\Omega)$. Fix $D \in C(\mathbb{R}^2)\,, G \in C_0^\infty(\mathbb{R}^2)$. Choose $K \subseteq \mathbb{R}^2$ compact with $\operatorname{supp}(G) \subseteq K$. Define $s_n(x) := (\nabla\psi(x) - \nabla\phi_n(x), -1)$ for $x \in K$. Then

$$\begin{aligned}
0 &= \lim_{n\to\infty} \int_K G(x)D(\nabla\phi_n(x))\big((v_n^1, v_n^2, v_n^3)(x, \psi(x) - \phi_n(x))\big) \cdot s_n(x)\, dx \\
&= \lim_{n\to\infty} \int_K G(x)D(\nabla\phi_n(x))\big((v^1, v^2, v^3)(x, \psi(x))\big) \cdot s_n(x)\, dx \\
&= -\int_K G(x)\big((v^1, v^2)(x, \psi(x))\big) \cdot \int_{K_{\mathcal{R}}} D(\lambda)\lambda\, d\mathcal{R}_x(\lambda)\, dx \\
&\quad + \lim_{n\to\infty} \int_K G(x)D(\nabla\phi_n(x))\big((v^1, v^2, v^3)(x, \psi(x))\big) \cdot (\nabla\psi(x), -1)\, dx \\
&= -\int_K G(x)\big((v^1, v^2)(x, \psi(x))\big) \cdot \int_{K_{\mathcal{R}}} D(\lambda)\lambda\, d\mathcal{R}_x(\lambda)\, dx.
\end{aligned} \tag{4.25}$$

In the first equality of (4.25) we have made use of $v_n \in L^2_\sigma(\Omega_n)$ and the fact that $s_n(x)$ is parallel to the normal vector on $\partial\Omega_n$ for a.a. $x \in K$. Let $\Omega' \subseteq \Omega$ be a bounded Lipschitz domain with $\{(x, \psi(x)); x \in K\} \subseteq \partial\Omega'$. From $v_n \rightharpoonup v$ for $n \to \infty$ in $H^1(\Omega')$ it follows $v_n \to v$ for $n \to \infty$ (strongly) in $L^2(\partial\Omega')$. Further we use (2.16) and an argumentation similar to (4.19) to get the second equality in (4.25). To get the first integral in the third equality of (4.25) consider the Carathéodory function $H(x, \lambda) := G(x)D(\lambda)[v^1, v^2](x, \psi(x)) \cdot \lambda$ for $x, \lambda \in \mathbb{R}^2$ and use (3.2). The argumentation is similar to that in (4.21), (4.22). Moreover we use $v \in L^2_\sigma(\Omega)$ to obtain that the second term in the third equality of (4.25) equals zero for every $n \in \mathbb{N}$.

A separability argument shows that there exists a null set $M \subseteq \mathbb{R}^2$ such that

$$\big((v^1, v^2)(x, \psi(x)\big) \cdot \int_{K_{\mathcal{R}}} D(\lambda)\lambda\, d\mathcal{R}_x(\lambda) = 0 \tag{4.26}$$

for all $D \in C(\mathbb{R}^2)$ and all $x \in \mathbb{R}^2 \setminus M$. Consider $x \in \mathbb{R}^2 \setminus M$ such that $\mathcal{R}_x$ is non-degenerate. Fix $(y_1, z_1), (y_2, z_2) \in \operatorname{supp}(\mathcal{R}_x)$ and $r > 0$ such that w for all $v \in B_r(y_1, z_1)$ and $w \in B_r(y_2, z_2)$ the vectors v and w are linearly independent in $\mathbb{R}^2$. For $i = 1, 2$ choose $D_i \in C(\mathbb{R}^2)$ with $0 \le D_i \le 1\,, D_i(y_i, z_i) = 1$ and $\operatorname{supp}(D_i) \subseteq B_r(y_i, z_i)$. Then the vectors $\int_{K_{\mathcal{R}}} D_i(\lambda)\lambda\, d\mathcal{R}_x(\lambda)\,, i = 1, 2$ are linearly independent in $\mathbb{R}^2$. As consequence of (4.26) it follows $\big((v^1, v^2)(x, \psi(x))\big) = 0$ for all $x \in \mathbb{R}^2 \setminus M$. With $v \cdot N = 0$

on $\partial\Omega$ we get $v \in H_0^1(\Omega)$. This finishes the proof of Theorem 4.4 and consequently the proof of the Main Theorem 1.2. □

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