Vorticity, Rotation and Symmetry (II) – Regularity of Fluid Motion

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In the mathematical theory of Navier-Stokes equations different notions of nonstationary solutions have been introduced, e.g., weak and very weak solutions, mild solutions, strong, smooth and regular solutions. Especially very weak solutions have been analyzed during the last years intensively in the context of regularity questions, but the famous problem for instationary solutions in three dimensions is still open: it is unknown whether a weak solution which does exist globally in time is regular or smooth for all times if the initial value and/or the prescribed external force are large. Analogously, it is open whether a regular solution which may be constructed for a sufficiently small time interval does exist globally in time. Related to this problem is the question of uniqueness of weak solutions.

These problems are open since 1934 when J. Leray constructed for the first time global weak solutions of the Navier-Stokes system in the whole space $\mathbb{R}^3$, and the problem has drawn even more attention since 2000 when *Clay Mathematics Institute* of Cambridge, Massachusetts, named this problem one of the seven "Millennium Prize Problems".

To be more precise, given a domain $\Omega \subset \mathbb{R}^3$ and a time interval $(0,T)$, an external force field f on $\Omega \times (0,T)$ and an initial value u_0, we are looking for a velocity field u and a pressure function p solving the Navier-Stokes system

$$\begin{aligned} u_t - \nu\Delta u + u\cdot\nabla u + \nabla p &= f && \text{in } \Omega\times(0,T)\\ \operatorname{div} u &= 0 && \text{in } \Omega\times(0,T)\\ u(0) &= u_0 && \text{at } t=0\\ u &= 0 && \text{on } \partial\Omega\times(0,T). \end{aligned} \tag{1}$$

Under relatively weak assumptions on u_0, f, say,

$$u_0 \in L^2_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_2}, \quad C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega) : \operatorname{div} u = 0\},$$

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and $f \in L^1\big(0,T;L^2(\Omega)\big)$, there exists a weak solution

$$u \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2_{\mathrm{loc}}\big(0,T;H^1_0(\Omega)\big)$$

to (1). But it is an open problem whether u is a strong solution, e.g. in the sense that $u \in L^\infty\big(0,T;H^1_0(\Omega)\big) \cap L^2\big(0,T;H^2(\Omega)\big)$ or even $u \in C^\infty\big(\overline{\Omega} \times (0,T)\big)$ under suitable regularity assumptions on arbitrarily large data u_0 and f. Up to now, this result can be proved only under additional assumptions on u (or p or on other quantities). The classical result of *conditional regularity* is due to J. Serrin (1962/63), requiring that

$$u \in L^s\big(0,T;L^q(\Omega)\big), \ \frac{2}{s} + \frac{3}{q} = 1, \ s > 2, \ q > 3.$$

Since then Serrin's condition has been generalized to the limit cases $s = 2$, $q = \infty$ and, more recently, $s = \infty$, $q = 3$, as well as to related conditions on ∇u, on specific components of either u or ∇u. Other conditions on the given weak solution concern the signs of the eigenvalues of the symmetric matrix of deformation, $\frac{1}{2}\big(\nabla u + (\nabla u)^T\big)$, or the behavior of the pressure which is unique only up to a time-dependent function.

Of special interest is the vorticity $\omega = \operatorname{curl} u$. On the one hand, the nonlinear term $u \cdot \nabla u$ may be written in the form

$$u \cdot \nabla u = \omega \times u + \nabla\big(\frac{1}{2}|u|^2\big) \tag{2}$$

so that $\frac{1}{2}|u|^2$ can be considered as part of the pressure p; then $\frac{1}{2}|u|^2 + p$ defines the so-called total head pressure; however, in most results on the Navier-Stokes system the term $u \cdot \nabla u$ is directly estimated ignoring the special decomposition (2). On the other hand, ω satisfies the *vorticity transport equation*

$$\omega_t - \nu\Delta\omega + u \cdot \nabla\omega - \omega \cdot \nabla u = \operatorname{curl} f. \tag{3}$$

In two dimensions, $\omega = (0,0,\omega_3)$ where ω_3 satisfies the scalar equation

$$\partial_t\omega_3 - \nu\Delta\omega_3 + u \cdot \nabla\omega_3 = \partial_1 f_2 - \partial_2 f_1,$$

the maximum principle holds for ω_3 provided data for $\omega_3|_{\partial\Omega}$ and $\omega_3(t=0)$ are available. In three dimensions, the term $\omega \cdot \nabla u$ in (3) prevents the application of a maximum principle and may lead to the phenomena of *vortex stretching*, a local increase of $|\omega|$ when certain geometric conditions on ω and u are satisfied. In the whole space case the identity $\operatorname{curl}\omega = \operatorname{curl}\ \operatorname{curl} u = -\Delta u$ may be used to get *Biot-Savart's law*

$$u = (-\Delta)^{-1}\operatorname{curl}\omega = \frac{1}{4\pi}\int_{\mathbb{R}^3} \omega(y) \times \frac{x-y}{|x-y|^3}\,dy, \tag{4}$$

i.e., u is defined by ω via a weakly singular integral operator. Moreover, we see that (3) is a nonlinear and nonlocal equation in ω. Actually, not the size of the norm $|\omega|$ is the crucial term, but the change of the orientation in space of the vector $\omega(x)$ when x moves.

Additional problems occur in the analysis of viscous fluid flow around rotating obstacles. Assume that a compact obstacle $K \subset \mathbb{R}^3$ is rotating around a fixed axis of rotation $w = (0, 0, w_3)$ with angular velocity $|w| = w_3$. If K is not axially symmetric with respect to w, the domain $\Omega(t)$ occupied by the fluid is changing in time. Then a change to a coordinate system attached to the rotating body yields the Navier-Stokes equation

$$u_t - \nu\Delta u + u \cdot \nabla u - (w \times x) \cdot \nabla u + w \times u + \nabla p = F. \tag{5}$$

The additional term $w \times u$ represents the Coriolis force, whereas the term $(w \times x) \cdot \nabla u$ is increasing as $|(x_1, x_2)| \to \infty$ and is not subordinate to the Laplacian. Actually, this latter term adds a hyperbolic effect to the (parabolic) Navier-Stokes system. With regard to this hyperbolic influence, the semigroup generated by the operator $A_w u := P\big(-\nu\Delta u - (w \times x) \cdot \nabla u + w \times u\big)$ (with the Helmholtz projection P) is no longer analytic, but only strongly continuous on L^q-spaces. Moreover, the spectrum of $-A_w$ contains an infinite set of equidistant half lines in the left complex half plane. It is an open problem whether there exist additional eigenvalues in the left half plane between the half lines mentioned above. More sophisticated problems occur when the body is not fixed to an axis of rotation but can move and tumble around freely in the fluid and e.g. sink down to the bottom of the fluid container. Further difficulties arise when the body is elastic and the Navier-Stokes system is coupled with nonlinear equations of elasticity to be considered together with a free boundary condition.

In geophysics and atmospheric flows, the Navier-Stokes system with initial values of rotational type is considered. Then a coordinate transform yields the Navier-Stokes equation

$$u_t - \nu\Delta u + u \cdot \nabla u + 2w \times u + \nabla p = F \tag{6}$$

with Coriolic force $2w \times u$, cf. (5). It is known that for sufficiently large $|w|$ solutions to (6) are regular. In other words, a large Coriolis force may help to stabilize and regularize the fluid flow. Moreover, also symmetry helps to prove regularity of weak solutions to the Navier-Stokes system e.g. for helical flows in a pipe.

Recent progress on the above-mentioned topics will be discussed during the conference "Vorticité, Rotation et Symétrie (II) – Régularité des Ecoulements" to be held at the *Centre International de Rencontres Mathématiques* (CIRM) in Luminy (Marseille), May 23 to May 27, 2011, following a previous conference entitled "Vorticité, Rotation et Symétrie – Stabilité des Ecoulements" – at CIRM in 2008. Several talks will be related to the open problem of regularity, but also questions of fluid flow around a single or several rotating obstacles will be addressed. Special emphasis is put on the interaction of the fluid with rigid or elastic bodies moving freely with the flow field. Besides the classical Navier-Stokes system also non-Newtonian fluids, compressible fluid flow and inviscid fluids governed by the Euler equations will be considered. Finally, the analysis of fluid flow in unbounded domains and with non-Dirichlet boundary conditions requires special tools such as e.g. the careful choice of function spaces adapted to the problem, weighted estimates or expansions reflecting the asymptotic behavior of solutions at space infinity or when $t \to \infty$.

The organizers of the conference believe that the lectures as well as the discussions will bring more light – in the inspiring atmosphere of Luminy – into the problems mentioned above and will motivate participants to get new ideas and insight into these puzzling questions open for many years.

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