OPTIMAL INITIAL VALUE CONDITIONS FOR LOCAL STRONG SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN EXTERIOR DOMAINS

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Abstract. Let u be a weak solution of the Navier-Stokes equations in an exterior domain $\Omega \subseteq \mathbb{R}^3$ and a time interval $[0,T[\,,\ 0 < T \le \infty$, with initial value u_0 and external force $f = \operatorname{div} F$. Here we address the problem to find the optimal (weakest possible) initial value condition in order to obtain a strong solution $u \in L^s(0,T;L^q(\Omega))$ in some time interval $[0,T[\,,\ 0 < T \le \infty$, where s,q with $3 < q < \infty$ and $\frac{2}{s} + \frac{3}{q} = 1$ are so-called Serrin exponents. Our main result states, for Serrin exponents s,q with $q \in [\frac{24}{7}, 8]$, a smallness condition on $\int_0^T \|e^{-\nu\tau A}u_0\|_q^s \, d\tau$ to imply existence of a strong solution $u \in L^s(0,T;L^q(\Omega))$; here A denotes the Stokes operator. Moreover, for Serrin exponents s,q with $3 < q < \infty$ we will prove the necessity of the condition $\int_0^\infty \|e^{-\nu\tau A}u_0\|_q^s \, d\tau < \infty$ to get a strong solution u on $[0,T[$, $0 < T \le \infty$.

1. Introduction and main results

In this paper, $\Omega \subseteq \mathbb{R}^3$ is an exterior domain, i.e. an open, connected subset having a nonempty, compact complement in $\mathbb{R}^3$, with smooth boundary $\partial\Omega \in C^{2,1}$, and $[0,T[\,, 0 < T \le \infty$, denotes the time interval. In $[0,T[\times\Omega$ we consider the instationary Navier-Stokes equations

$$\begin{aligned} u_t - \nu\Delta u + u\cdot\nabla u + \nabla p = f \quad &\text{in }]0,T[\times\Omega\,, \\ \operatorname{div} u = 0 \quad &\text{in }]0,T[\times\Omega\,, \\ u = 0 \quad &\text{on }]0,T[\times\partial\Omega\,, \\ u = u_0 \quad &\text{for } t = 0\,, \end{aligned} \tag{1.1}$$

with constant viscosity $\nu > 0$, an external force $f = \operatorname{div}F = (\sum_{i=1}^3 \partial_i F_{i,j})_{j=1}^3$ and initial value u_0. First we recall the definition of weak and strong solutions. The space of test functions is defined to be

$$C_0^\infty([0,T[;C_{0,\sigma}^\infty(\Omega)) := \{\, u|_{[0,T[\times\Omega} \;; u \in C_0^\infty(]-1,T[\times\Omega) \;; \operatorname{div} u = 0 \,\}.$$

Definition 1.1. Let $\Omega \subseteq \mathbb{R}^3$ be an arbitrary domain, let $0 < T \le \infty\,, \nu > 0$, let $f = \operatorname{div} F$ with $F \in L^1_{\text{loc}}([0,T[;L^2(\Omega))$, and let $u_0 \in L^2_\sigma(\Omega)$. Then a vector field $u \in LH_T$, where LH_T denotes the *Leray-Hopf class*

$$LH_T := L^\infty_{\text{loc}}([0,T[;L^2_\sigma(\Omega)) \cap L^2_{\text{loc}}([0,T[;W^{1,2}_{0,\sigma}(\Omega))\,, \tag{1.2}$$

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is called *weak solution* (in the sense of *Leray-Hopf*) of the instationary Navier-Stokes system (1.1) with data f, u_0 if the following identity is satisfied for all test functions $w \in C_0^\infty([0,T[; C_{0,\sigma}^\infty(\Omega))$:

$$\int_0^T \Big(-\langle u, w_t\rangle_\Omega + \nu\langle \nabla u, \nabla w\rangle_\Omega + \langle u\cdot\nabla u, w\rangle_\Omega \Big)\, dt = \langle u_0, w(0)\rangle_\Omega - \int_0^T \langle F, \nabla w\rangle_\Omega\, dt. \tag{1.3}$$

Given a weak solution $u \in LH_T$ of (1.1), after a possible redefinition on a set of Lebesgue measure 0, we may assume that $u : [0,T[\to L^2_\sigma(\Omega)$ is weakly continuous and the initial value u_0 is attained in the following sense:

$$\langle u(t), \phi\rangle_\Omega \to \langle u_0, \phi\rangle_\Omega\,, \quad t \searrow 0\,, \quad \forall \phi \in L^2_\sigma(\Omega).$$

Moreover, there exists a distribution p, called an associated pressure, such that the equality

$$u_t - \nu\Delta u + u\cdot\nabla u + \nabla p = f$$

holds in the sense of distributions on $]0,T[\times\Omega$, see [21, V.1.7].

For exponents s, q with $1 < q,\ s < \infty$ we define the *Serrin number* by

$$\mathcal{S}(s,q) := \frac{2}{s} + \frac{3}{q}.$$

We recall that by the embedding $W_0^{1,2}(\Omega) \subset L^6(\Omega)$ and Hölder's inequality $u \in LH_T$ satisfies $u \in L^s(0,T; L^q(\Omega))$ for all $s \geq 2$, $q \geq 2$ with $\mathcal{S}(s,q) = \frac{3}{2}$.

A weak solution of (1.1) is called a *strong solution* if there exist *Serrin exponents* s, q with $\mathcal{S}(s,q) = 1$ such that additionally *Serrin's condition*

$$u \in L^s(0,T; L^q(\Omega)) \tag{1.4}$$

is satisfied. By Serrin's uniqueness Theorem [21, V, Theorem 1.5.1] a weak solution with (1.4) is unique within the class of weak solutions satisfying the energy inequality, i.e. fulfilling

$$\frac{1}{2}\|u(t)\|_2^2 + \nu\int_0^t \|\nabla u\|_2^2\, d\tau \leq \frac{1}{2}\|u_0\|_2^2 - \int_0^t \langle F, \nabla u\rangle_\Omega\, d\tau \tag{1.5}$$

for almost all $t \in [0,T[$. Moreover, such a strong solution satisfies $u\otimes u \in L^2_{\mathrm{loc}}([0,T[; L^2(\Omega))$ and, after a redefinition on a set of vanishing Lebesgue measure, $u : [0,T[\to L^2_\sigma(\Omega)$ is strongly continuous and satisfies the energy identity (1.8) below, cf. [21, V, Theorem 1.4.1]

The existence of weak solutions in smooth exterior domains satisfying (1.5) is well known, see [20, 21]. Up to now, the existence of a strong solution u of (1.1) could only be proven in a sufficiently small interval $[0,T[\,, 0 < T \leq \infty$, and under additional assumptions on Ω, f, and u_0. The first sufficient existence condition in this context seems to be due to [16], yielding a solution class of so called *local strong solutions*. Since then there have been found several sufficient initial value conditions for the existence of local strong solutions, getting weaker step by step, see [2, 9, 10, 13, 15, 17, 19, 20, 21, 22] for bounded and unbounded domains. In [7, 8] the authors considered (1.1) with $3 < q < \infty$ and $\mathcal{S}(s,q) = 1$ in a smooth bounded domain with $f = \mathrm{div}F$ and $F \in L^{\frac{s}{2}}(0,T; L^{\frac{q}{2}}(\Omega))$ and proved that the smallness conditions (1.6), (1.7) below are sufficient for the existence of a strong solution $u \in L^s(0,T; L^q(\Omega))$.

Using L^2-theory of the Stokes operator they also proved in [7] that if Ω is a general domain and $q = 4$ there exists an absolute constant ϵ_*, not depending on Ω, such that the conditions (1.6), (1.7) are sufficient for the existence of a strong solution $u \in L^8(0,T;L^4(\Omega))$. Our first main theorem gives a sufficient criterion on u_0 for the existence of a strong solution $u \in L^s(0,T;L^q(\Omega))$.

Theorem 1.2. *Let* $\Omega \subseteq \mathbb{R}^3$ *be an exterior domain with* $\partial\Omega \in C^{2,1}$, *let exponents* $1 < s, q < \infty$ *be given such that* $\mathcal{S}(s,q) = 1$, *and let* $1 < s_*, q_* < \infty$ *satisfy* $\mathcal{S}(s_*, q_*) = 2$ *where* $\frac{1}{3} + \frac{1}{q} \geq \frac{1}{q_*} \geq \frac{1}{q}$. *Let* $0 < T \leq \infty$, $\nu > 0$, *let* $F \in L^{s_*}(0,T;L^{q_*}(\Omega)) \cap L^2(0,T;L^2(\Omega))$, *and* $u_0 \in L^2_\sigma(\Omega)$.

(1) *If* $q \in [\frac{24}{7}, 8]$ *there exists a constant* $\epsilon_* = \epsilon_*(\Omega, q, q_*) > 0$ *(independent of* T, ν, F, *and* u_0*) with the following property: If the conditions*

$$\left(\int_0^T \|e^{-\nu\tau A} u_0\|_q^s \, d\tau \right)^{\frac{1}{s}} \leq \epsilon_* \nu^{1-\frac{1}{s}}, \tag{1.6}$$

$$\left(\int_0^T \|F(\tau)\|_{q_*}^{s_*} \, d\tau \right)^{\frac{1}{s_*}} \leq \epsilon_* \nu^{1+\frac{3}{2q_*}}, \tag{1.7}$$

are satisfied, then there exists a strong solution $u \in L^s(0,T;L^q(\Omega))$ *of the Navier-Stokes equations (1.1). After a possible redefinition on a set of Lebesgue measure* 0, *we get that* $u : [0,T[\to L^2_\sigma(\Omega)$ *is strongly continuous and satisfies the energy equality*

$$\frac{1}{2}\|u(t)\|_2^2 + \nu \int_0^t \|\nabla u\|_2^2 \, d\tau = \frac{1}{2}\|u_0\|_2^2 - \int_0^t \langle F, \nabla u \rangle_\Omega \, d\tau \tag{1.8}$$

for all $t \in [0,T[$.

(2) *If* $3 < q < \frac{24}{7}$ *or* $8 < q < \infty$, *the conditions (1.6) and (1.7) imply the existence of a strong solution* $u \in L^s(0,T;L^q(\Omega))$ *with the same properties as in (1) under the following additional assumptions on* u_0, F*: There exist* $1 < \gamma, \gamma_*, \rho_* < \infty, \rho \in [\frac{24}{7}, 8]$ *with* $\mathcal{S}(\gamma, \rho) = 1$ *and* $\mathcal{S}(\gamma_*, \rho_*) = 2$ *where* $\frac{1}{3} + \frac{1}{\rho} \geq \frac{1}{\rho_*} \geq \frac{1}{\rho}$ *such that the two conditions*

$$e^{-\nu t A} u_0 \in L^\gamma(0,T;L^\rho(\Omega)), \tag{1.9}$$

$$F \in L^{\gamma_*}(0,T;L^{\rho_*}(\Omega)) \tag{1.10}$$

are satisfied.

For the proof we refer to Section 4. The idea of the proof is to construct u in the form $u = E + \tilde{u}$ where E is the solution of the linear part and $\tilde{u}$ is constructed as a fixed point of a related nonlinear problem, see (2.18). Then $E, \tilde{u}$ satisfy certain representation formulae as in Lemma 3.2 which also helps to get the needed integrability properties of $E, \tilde{u}$. The proof of regularity differs from the case of bounded domains, see [7, 8], where the trivial inclusion $L^q(\Omega) \subseteq L^r(\Omega)$, $q > r$, yielding also better imbedding properties of fractional powers of the Stokes operator, was used several times. This is also the reason why, without additional assumptions of u_0, F, we are able to prove the sufficiency of the condition (1.6), (1.7) only for $q \in [\frac{24}{7}, 8]$.

In Theorem 1.3 below we will formulate a necessary condition for the existence of a strong solution $u \in L^s(0,T;L^q(\Omega))$. If $\Omega \subseteq \mathbb{R}^3$ is a smooth bounded domain the necessity of (1.11) for the existence of a strong solution

$u \in L^s(0,T;L^q(\Omega))$ was proved in [8]; furthermore, for an arbitrary bounded or unbounded domain Ω the condition (1.11) with $s=8, q=4$ is necessary for the existence of a strong solution $u \in L^8(0,T;L^4(\Omega))$.

Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $1 < s, q < \infty$ with $\mathcal{S}(s,q) = 1$ be arbitrary Serrin exponents, let $1 < s_*, q_* < \infty$ with $\mathcal{S}(s_*,q_*) = 2$ where $\frac{1}{3} + \frac{1}{q} \geq \frac{1}{q_*} \geq \frac{1}{q}$. Furthermore, let $0 < T \leq \infty$, $\nu > 0$, assume $F \in L^{s_*}(0,T;L^{q_*}(\Omega)) \cap L^2(0,T;L^2(\Omega))$, and $u_0 \in L^2_\sigma(\Omega)$. Then a necessary condition for the existence of a strong solution $u \in L^s(0,T;L^q(\Omega))$ of the Navier-Stokes equations (1.1) is the condition*

$$\int_0^\infty \|e^{-\nu\tau A}u_0\|_q^s \, d\tau < \infty. \tag{1.11}$$

In the following corollary, which immediately follows from Theorems 1.2 and 1.3, the condition (1.11) on u_0 defines the largest possible class of initial values to get a strong solution $u \in L^s(0,T;L^q(\Omega))$ of (1.1) with $q \in [\frac{24}{7}, 8]$ and the Serrin condition $\mathcal{S}(s,q) = 1$.

Corollary 1.4. *Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $1 < s < \infty, q \in [\frac{24}{7}, 8]$ with $\mathcal{S}(s,q) = 1$. Further, let $1 < s_*, q_* < \infty$ with $\mathcal{S}(s_*,q_*) = 2$ where $\frac{1}{3} + \frac{1}{q} \geq \frac{1}{q_*} \geq \frac{1}{q}$, let $0 < T \leq \infty, \nu > 0$, assume $F \in L^{s_*}(0,T;L^{q_*}(\Omega)) \cap L^2(0,T;L^2(\Omega))$, and $u_0 \in L^2_\sigma(\Omega)$. Then*

$$\int_0^\infty \|e^{-\nu\tau A}u_0\|_q^s \, d\tau < \infty$$

is a necessary and sufficient condition for the existence of a strong solution $u \in L^s(0,T';L^q(\Omega))$ of (1.1) in some interval $[0,T'[$, $0 < T' \leq T$.

After some preliminaries, see Section 2, we discuss the regularity of functions fulfilling a certain class of representation formulae in Section 3. Finally, Section 4 deals with the proofs of Theorems 1.2 and 1.3.

2. Preliminaries

Given $1 \leq q \leq \infty, k \in \mathbb{N}$ we need the usual Lebesgue and Sobolev spaces, $L^q(\Omega)$ and $W^{k,q}(\Omega)$ with norms $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and $\|\cdot\|_{W^{k,q}(\Omega)} = \|\cdot\|_{k,q}$, respectively. For two measurable functions f, g with the property $f \cdot g \in L^1(\Omega)$, where $f \cdot g$ means the usual scalar product of vector or matrix fields, we set

$$\langle f, g\rangle_\Omega := \int_\Omega f(x)\cdot g(x)\, dx.$$

Note that the same symbol $L^q(\Omega)$ etc. will be used for spaces of scalar-, vector- or matrix-valued functions. Let $C^m(\Omega), m = 0, 1, \ldots, \infty$, denote the usual space of functions for which all partial derivatives of order $|\alpha| \leq m$ ($|\alpha| < \infty$ when $m = \infty$) exist and are continuous. As usual, $C_0^m(\Omega)$ is the set of all functions from $C^m(\Omega)$ with compact support in Ω. Further we need the space of smooth solenoidal vector fields

$$C^\infty_{0,\sigma}(\Omega) := \{\, v \in C_0^\infty(\Omega)^3;\ \operatorname{div} v = 0\,\}.$$

We define the spaces $(1 < q < \infty)$

$$L^q_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q},$$
$$W^{1,2}_{0,\sigma}(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}}}.$$

For $1 \le q \le \infty$ let q' be the dual exponent such that $\frac{1}{q} + \frac{1}{q'} = 1$. It is well known that $L^q_\sigma(\Omega)' \cong L^{q'}_\sigma(\Omega)$ using the standard pairing $\langle \cdot, \cdot \rangle_\Omega$.

Given a Banach space X, $1 \le p \le \infty$, and an interval $]0, T[$ we denote by $L^p(0, T; X)$ the space of (equivalence classes of) strongly measurable functions $f :]0, T[\to X$ such that

$$\|f\|_p := \left(\int_0^T \|f(t)\|_X^p \, dt \right)^{\frac{1}{p}} < \infty$$

if $1 \le p < \infty$ and $\|f\|_\infty := \operatorname{ess\,sup}_{t \in]0,T[} \|f(t)\|_X < \infty$ if $p = \infty$. Moreover, we define the set of *locally integrable* functions

$$L^p_{\text{loc}}([0, T[; X) := \{ u : [0, T[\to X;\ u \in L^p(0, T'; X) \ \text{ for all } 0 < T' < T \}.$$

If $X = L^q(\Omega)$, $1 \le q \le \infty$, we denote the norm in $L^p(0, T; L^q(\Omega))$ by $\|f\|_{q,p;T}$.

Given an exterior domain $\Omega \subseteq \mathbb{R}^3$ with $\partial\Omega \in C^{2,1}$ and $1 < q < \infty$, let $P_q : L^q(\Omega)) \to L^q_\sigma(\Omega)$ denote the *Helmholtz projection* with range $\mathcal{R}(P_q) = L^q_\sigma(\Omega)$ and null space $N(P_q) = \{\nabla p \in L^q(\Omega)\,; p \in L^q_{\text{loc}}(\overline{\Omega})\}$. This operator is consistent in the sense that

$$P_q f = P_r f \quad \text{for} \quad f \in L^q(\Omega) \cap L^r(\Omega). \tag{2.1}$$

Furthermore, we get for the dual operator $P'_q \cong P_{q'}$ which means that

$$\langle P_q f, g \rangle_\Omega = \langle f, P_{q'} g \rangle_\Omega \quad \forall f \in L^q(\Omega),\ g \in L^{q'}(\Omega). \tag{2.2}$$

For $1 < q < \infty$ we define the *Stokes operator* by

$$\mathcal{D}(A_q) = L^q_\sigma(\Omega) \cap W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega), \tag{2.3}$$
$$A_q u := -P_q \Delta u\,, \quad u \in \mathcal{D}(A_q). \tag{2.4}$$

The Stokes operator is consistent in the sense that for $1 < q, r < \infty$

$$A_q u = A_r u \quad \forall u \in \mathcal{D}(A_q) \cap \mathcal{D}(A_r). \tag{2.5}$$

Throughout this paper we will write $A = A_2$. It is well known that $-A_q$ generates a uniformly bounded analytic semigroup $\{ e^{-tA_q} : t \ge 0 \}$ on $L^q_\sigma(\Omega)$ satisfying the decay estimate

$$\|A^\alpha_q e^{-tA_q}\|_q \le c\, t^{-\alpha}, \quad t > 0\,, \tag{2.6}$$

where $\alpha \ge 0\,, q > 1$, and $c = c(\Omega, q, \alpha) > 0$.

For $\alpha \in [-1, 1]$ the fractional power $A^\alpha_q : \mathcal{D}(A^\alpha_q) \to L^q_\sigma(\Omega)$ with dense domain $\mathcal{D}(A^\alpha_q) \subseteq L^q_\sigma(\Omega))$ and dense range $\mathcal{R}(A^\alpha_q) \subseteq L^q_\sigma(\Omega))$ is a well defined, injective, closed operator such that

$$(A^\alpha_q)^{-1} = A^{-\alpha}_q\,, \quad \mathcal{R}(A^\alpha_q) = \mathcal{D}(A^{-\alpha}_q), \ \text{ and } \quad (A^\alpha_q)' = A^\alpha_{q'}.$$

In general, $\mathcal{D}(A^\alpha_q)$ will be equipped with the graph norm $\|u\|_{\mathcal{D}(A_q)} := \|u\|_q + \|A^\alpha_q\|_q$ for $u \in \mathcal{D}(A^\alpha_q)$ which makes $\mathcal{D}(A^\alpha_q)$ to a Banach space since A^α_q is closed. In [11, Theorem A] it is proved that A_q has bounded imaginary powers. Consequently, see [11, Theorem B],

$$[L^q_\sigma(\Omega), \mathcal{D}(A_q)]_s = \mathcal{D}(A^s_q) \quad \text{for} \quad 0 \le s \le 1\,, \tag{2.7}$$

where $[L^q_\sigma(\Omega), \mathcal{D}(A_q)]_s$ denotes the complex interpolation space; for the definition of these spaces see e.g. [18, Ch. 2]. It holds $\mathcal{D}(A_q) \subseteq \mathcal{D}(A^\alpha_q) \subseteq \mathcal{D}(A^\beta_q) \subseteq L^q_\sigma(\Omega)$ for $0 \le \beta \le \alpha \le 1$. Furthermore, we have

$$\mathcal{D}(A^{1/2}_q) = W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) \quad \text{for } 1 < q < 3, \tag{2.8}$$

$$\|\nabla u\|_{q,\Omega} \le c\|A^{1/2}_q u\|_{q,\Omega} \quad \text{for } 1 < q < 3 \text{ and } u \in \mathcal{D}(A^{1/2}_q) \tag{2.9}$$

with a constant $c = c(\Omega, q) > 0$. Moreover, for all $u \in \mathcal{D}(A^\alpha_q)$,

$$\|u\|_{\gamma,\Omega} \le c\|A^\alpha_q u\|_{q,\Omega} \quad \text{where } 0 \le \alpha \le \frac{1}{2}\,, 1 < q < 3\,, 2\alpha + \frac{3}{\gamma} = \frac{3}{q}\,, \tag{2.10}$$

with a constant $c = c(\Omega, q, \alpha, \gamma) > 0$. Concerning further information on the Helmholtz projection and the Stokes operator in exterior domains we refer to [3, 4, 11, 12, 14].

Let $2 < q < \infty$ and $u_0 \in L^2_\sigma(\Omega)$ be given. Then from [14, Theorem 1.2 (ii)] we see $e^{-\nu t A} u_0 \in L^q_\sigma(\Omega)$ for all $t > 0$ and

$$\|e^{-\nu t A} u_0\|_q \le c\,(\nu\, t)^{-\frac{3}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \|u_0\|_2 \tag{2.11}$$

with a constant $c = c(\Omega, q) > 0$. If $3 < q < \infty$ with $\mathcal{S}(s, q) = 1$ we get from (2.11) that

$$\int_0^\infty \|e^{-\nu\tau A} u_0\|^s_q \, d\tau < \infty \quad \Longleftrightarrow \quad \int_0^{T_0} \|e^{-\nu\tau A} u_0\|^s_q \, d\tau < \infty \tag{2.12}$$

for any (and consequently for all) $0 < T_0 < \infty$.

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $1 < q\,, s < \infty$, and let $0 < T \le \infty\,, \nu > 0$. Further, let $f \in L^s(0, T; L^q_\sigma(\Omega))$, and let $u_0 \in L^q_\sigma(\Omega)$ such that $\int_0^T \|\nu A_q e^{-\nu t A_q} u_0\|^s_q \, dt < \infty$. Then the instationary Stokes system*

$$\begin{aligned} u_t + \nu A_q u &= f \quad \text{in } (0, T)\,, \\ u(0) &= u_0 \end{aligned} \tag{2.13}$$

has a unique strong solution $u \in L^s_{loc}(0, T; \mathcal{D}(A_q))$ with $u_t \in L^s(0, T; L^q_\sigma(\Omega))$ and $u \in C([0, T[; L^q_\sigma(\Omega))$. Moreover, u satisfies the maximal regularity estimate

$$\|u_t\|_{q,s,\Omega;T} + \|\nu A_q u\|_{q,s,\Omega;T} \le c\left[\left(\int_0^T \|\nu A_q e^{-\nu t A_q} u_0\|^s_{q,\Omega} \, dt\right)^{1/s} + \|f\|_{q,s,\Omega;T}\right] \tag{2.14}$$

with a constant $c = c(\Omega, q, s)$ independent of T, ν and has the representation

$$u(t) = e^{-\nu t A_q} u_0 + \int_0^t e^{-\nu(t-\tau)A_q} f(\tau)\, d\tau \tag{2.15}$$

for all $t \in [0, T[$. In the case $T < \infty$ it even holds $u \in L^s(0, T; \mathcal{D}(A_q))$.

Proof. See [12, Theorem 4.2]. ∎

Finally, we recall the *Hardy-Littlewood inequality*: Let $0 < \alpha < 1, 1 < r < q < \infty$ with $\alpha + \frac{1}{q} = \frac{1}{r}$, and let $f \in L^r(\mathbb{R})$. Then the integral

$$u(t) := \int_{\mathbb{R}} |t-\tau|^{\alpha-1} f(\tau)\, d\tau$$

converges absolutely for almost all $t \in \mathbb{R}$ and it holds

$$\|u\|_{L^q(\mathbb{R})} \leq c\|f\|_{L^r(\mathbb{R})} \tag{2.16}$$

with a constant $c = c(\alpha, q) > 0$; see e.g. [23, Ch. V, 1.2].

The following theorem is central for the construction of the strong solution u in Theorem 1.2.

Theorem 2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an exterior domain with $\partial\Omega \in C^{2,1}$, and let $3 < q < \infty$ with $\mathcal{S}(s,q) = 1$. Let $\alpha := 1 - \frac{1}{s}$. Then there exists a constant $c = c(\Omega, q) > 0$ with the following property: If $0 < T \leq \infty, \nu > 0$, and $E \in L^s(0,T;L^q(\Omega))$ with*

$$\|E\|_{q,s;T} \leq c\,\nu^{\alpha}\,, \tag{2.17}$$

then the nonlinear map

$$(\mathcal{F}(u))(t) := -\int_0^t A_q^{\alpha} e^{-\nu(t-\tau)A_q} A_q^{-\alpha} P_q \mathrm{div}((u+E)\otimes(u+E))(\tau)\, d\tau\,, \tag{2.18}$$

defined for $u \in L^s(0,T;L^q(\Omega))$ and almost all $t \in [0,T[$, has a fixed point in $L^s(0,T;L^q_\sigma(\Omega))$.

Proof. See [6, §5]. □

The background for Theorem 2.2 is the notion of *very weak solutions*, see e.g. [2], [5], [6]. To be more precise, in the setting of Theorem 2.2 a vector field $u \in L^s(0,T;L^q(\Omega))$ is called a very weak solution of the Navier-Stokes system (1.1) with data $f = \operatorname{div} F$, $F \in L^s(0,T;L^r(\Omega))$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$, and initial value u_0 satisfying $\int_0^\infty \|A_q e^{-\nu t A_q} A_q^{-1} P_q u_0\|_q^s\, dt < \infty$ iff

$$\int_0^T \big(-\langle u, w_t\rangle_\Omega - \nu\langle u, \Delta w\rangle_\Omega - \langle u\otimes u, \nabla w\rangle_\Omega\big)\, dt = \langle u_0, w(0)\rangle_\Omega - \int_0^T \langle F, \nabla w\rangle_\Omega\, dt$$

for all test functions $w \in C_0^1([0,T); C^2_{0,\sigma}(\bar\Omega))$ where $C^2_{0,\sigma}(\bar\Omega) = \{v \in C^2(\bar\Omega) :$ div $v = 0$, supp w compact in $\bar\Omega$, $w_{|\partial\Omega} = 0\}$. We note that in our context F lies in $L^{s_*}(0,T;L^{q_*}(\Omega))$ rather than in $L^s(0,T;L^r(\Omega))$; the crucial properties, however, are (4.2), (4.3), (4.4) to be used in (2.17) of Theorem 2.2.

3. Representation Formulae

This section deals with integrability properties of functions satisfying representation formulae as in Lemma 3.2 below. Since in the proof of Theorems 1.2 and 1.3 the terms E_2 and $\tilde u$ satisfy such representation formulae, the results in this section are crucial for the proof of these theorems. We begin with the following technical lemma.

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $p > \frac{3}{2}$, $F \in L^p(\Omega)$. Choose $r, \sigma \geq 0$ with*

$$2\sigma + \frac{3}{r} = \frac{3}{p}\,, \quad 0 \leq \sigma \leq \frac{1}{2}. \tag{3.1}$$

Then there exists a unique element in $L^r_\sigma(\Omega)$ denoted by $A_r^{-1/2-\sigma}P_r\mathrm{div}F \in L^r_\sigma(\Omega)$ with

$$\langle A_r^{-1/2-\sigma}P_r\mathrm{div}F, A_{r'}^{1/2+\sigma}w\rangle_\Omega = -\langle F, \nabla w\rangle_\Omega \tag{3.2}$$

for all $w \in \mathcal{D}(A_{r'}^{1/2+\sigma})$. It holds

$$\|A_r^{-1/2-\sigma}P_r\mathrm{div}F\|_r \le c\|F\|_p \tag{3.3}$$

with a constant $c = c(\Omega, p, r) > 0$.

Proof. We have $2\cdot\frac{1}{2}+\frac{3}{r} \ge \frac{3}{p}$ and consequently $2\cdot\frac{1}{2}+\frac{3}{p'} \ge \frac{3}{r'}$. Then Sobolev's imbedding theorem (see [1, Theorem 4.12]) yields the continuous imbedding

$$W^{1,r'}(\Omega) \hookrightarrow L^{p'}(\Omega). \tag{3.4}$$

From $p > \frac{3}{2}$ it follows $p' < 3$, and (2.8) yields

$$\mathcal{D}(A_{p'}^{1/2}) = W_0^{1,p'}(\Omega) \cap L^{p'}_\sigma(\Omega). \tag{3.5}$$

Let $w \in \mathcal{D}(A_{r'})$ be fixed. Using (3.4) and (3.5) we see that $w \in \mathcal{D}(A_{p'}^{1/2})$. We get with the consistency of the Stokes operator, see (2.5), $A_{p'}^{1/2}w = A_{r'}^{1/2}w$ and $A_{r'}^{1/2}w \in \mathcal{D}(A_{r'}^{1/2})$. Now it follows from (2.10) that

$$\begin{aligned}|\langle F, \nabla w\rangle_\Omega| &\le \|F\|_p\|\nabla w\|_{p'}\\ &\le c\|F\|_p\|A_{p'}^{1/2}w\|_{p'} = c\|F\|_p\|A_{r'}^{1/2}w\|_{p'}\\ &\le c\|F\|_p\|A_{r'}^{\sigma}(A_{r'}^{1/2}w)\|_{r'} = c\|F\|_p\|A_{r'}^{1/2+\sigma}w\|_{r'}\end{aligned} \tag{3.6}$$

with a constant $c = c(\Omega, p, r) > 0$. The identity (3.6) is true for all $w \in \mathcal{D}(A_{r'})$. From (2.7) with $s = \frac{1}{2}+\sigma$ we know that $\mathcal{D}(A_{r'})$ is dense in $\mathcal{D}(A_{r'}^{1/2+\sigma})$ with respect to the graph norm $\|\cdot\|_{r'} + \|A_{r'}^{1/2+\sigma}\cdot\|_{r'}$. Hence, by density, we get from (3.6) that

$$|\langle F, \nabla w\rangle_\Omega| \le c\|F\|_p\|A_{r'}^{1/2+\sigma}w\|_{r'} \quad \text{for all} \quad w \in \mathcal{D}(A_{r'}^{1/2+\sigma}) \tag{3.7}$$

with $c = c(\Omega, p, r) > 0$. By [5, Lemma 2.1] applied to (3.7) we obtain a unique element, denoted by $A_r^{-1/2-\sigma}P_r\mathrm{div}F$, in $L^r_\sigma(\Omega)$ fulfilling (3.2). □

The following lemma is a generalization of [5, Lemma 3.2].

Lemma 3.2. *Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $0 < T \le \infty$, $\nu > 0$, and let $s > 1$, $p > \frac{3}{2}$, $F \in L^s(0,T;L^p(\Omega))$. Choose $r, \sigma \ge 0$ with*

$$2\sigma + \frac{3}{r} = \frac{3}{p}, \quad 0 \le \sigma < \frac{1}{2}, \tag{3.8}$$

(i.e. $\sigma = \frac{3}{2}(\frac{1}{p}-\frac{1}{r})$) and let $\beta := \frac{1}{2}+\sigma$. Then

$$\Phi_r(t) := \int_0^t A_r^\beta e^{-\nu(t-\tau)A_r}A_r^{-\beta}P_r\mathrm{div}F(\tau)\,d\tau \tag{3.9}$$

is well defined as an element of $L^r_\sigma(\Omega)$ for almost all $t \in [0,T[$. The following statements are satisfied.

(1) *It holds for almost all* $t \in [0, T[$

$$\int_0^t A_r^\beta e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \mathrm{div} F(\tau)\, d\tau = A_r^\beta \int_0^t e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \mathrm{div} F(\tau)\, d\tau. \tag{3.10}$$

(2) *For* $i \in \{1, 2\}$ *choose* $r_i, \sigma_i \geq 0$ *satisfying (3.8) with* r, σ *replaced by* r_i, σ_i. *Then*

$$\Phi_{r_1}(t) = \Phi_{r_2}(t) \quad \text{for almost all } t \in [0, T[. \tag{3.11}$$

Therefore, we can denote the term in (3.9), which is independent of $r, \sigma \geq 0$ *satisfying (3.8), by* Φ.

(3) *Assume that* s, p *with* $\frac{3}{2} < p < \infty$ *satisfy* $\mathcal{S}(s, p) = 2$. *Let* $r, \sigma \geq 0$ *with* $r > \frac{3}{2}$ *satisfy*

$$2\sigma + \frac{3}{r} = \frac{3}{p}, \quad 0 \leq \sigma \leq \frac{1}{2}, \tag{3.12}$$

(*i.e.* $\sigma = \frac{1}{2}$ *is allowed) and let* $\beta = \frac{1}{2} + \sigma$. *Finally let* $2 < \gamma < \infty$ *be defined by* $\mathcal{S}(\gamma, r) = 1$. *Then*

$$\|\Phi\|_{r,\gamma;T} \leq c\nu^{-\beta} \|F\|_{p,s;T} \tag{3.13}$$

with $c = c(\Omega, p, r) > 0$.

Proof. (1), (2) See [5, Lemma 3.2].

(3) First let $0 \leq \sigma < \frac{1}{2}$ in (3.12). From (3.3) we know $\|A_r^{-\beta} P_r \mathrm{div} F\|_r \leq c\|F\|_p$ with $c = c(\Omega, p, r) > 0$. We use (3.11), (2.6) to get

$$\begin{aligned} \|\Phi(t)\|_r &= \|\Phi_r(t)\|_r \\ &\leq c\nu^{-\beta} \int_0^t |t - \tau|^{-\beta} \|A_r^{-\beta} P_r \mathrm{div} F(\tau)\|_r \, d\tau \\ &\leq c\nu^{-\beta} \int_0^T |t - \tau|^{-\beta} \|F(\tau)\|_p \, d\tau \end{aligned} \tag{3.14}$$

for almost all $t \in [0, T[$ with a constant $c = c(\Omega, p, r) > 0$. Since $(1-\beta) + \frac{1}{\gamma} = \frac{1}{s}$ with $\mathcal{S}(\gamma, r) = 1$, the Hardy-Littlewood inequality (2.16) yields

$$\|\Phi\|_{r,\gamma;T} \leq c\nu^{-\beta} \|F\|_{p,s;T}$$

with $c = c(\Omega, p, r) > 0$. Next, let $\sigma = \frac{1}{2}$ in (3.12) and consequently $2 \cdot \frac{1}{2} + \frac{3}{r} = \frac{3}{p}$, $\gamma = s$. We use (3.10), (2.8) and (3.11) to get

$$\begin{aligned} \|\Phi(t)\|_r &= \|\Phi_p(t)\|_r \\ &= \big\| A_p^{1/2} \int_0^t e^{-\nu(t-\tau)A_p} A_p^{-1/2} P_p \mathrm{div} F(\tau)\, d\tau \big\|_r \\ &\leq c \big\| A_p \int_0^t e^{-\nu(t-\tau)A_p} A_p^{-1/2} P_p \mathrm{div} F(\tau)\, d\tau \big\|_p \end{aligned} \tag{3.15}$$

for almost all $t \in [0, T[$. Theorem 2.1 applied to (3.15) implies

$$\|\Phi\|_{r,\gamma;T} \leq c\nu^{-1} \|A_p^{-1/2} P_p \mathrm{div} F\|_{p,\gamma;T} \leq c\nu^{-1} \|F\|_{p,\gamma;T}$$

with $c = c(\Omega, p) > 0$. It holds $\mathcal{S}(\gamma, r) = 1$. □

4. Proof of theorem 1.2 and theorem 1.3

Proof of Theorem 1.2.
Step 1. First, define

$$\begin{aligned} E(t) &:= E_1(t) + E_2(t) \\ &:= e^{-\nu t A} u_0 + \int_0^t A^{1/2} e^{-\nu(t-\tau)A} A^{-1/2} P \mathrm{div} F(\tau)\, d\tau \end{aligned} \tag{4.1}$$

for almost all $t \in [0, T[$. We use [21, IV, Theorems 2.3.1 and 2.4.1] to get that $E_1, E_2 \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; W^{1,2}_{0,\sigma}(\Omega))$ and that E is a weak solution to the Stokes equations with external force $\mathrm{div}F$ and initial value u_0. Assumption (1.6) yields $E_1 \in L^s(0, T; L^q(\Omega))$. We use (2.5) to obtain

$$E_2(t) = \int_0^t A_{q_*}^{1/2} e^{-\nu(t-\tau)A_{q_*}} A_{q_*}^{-1/2} P_{q_*} \mathrm{div} F(\tau)\, d\tau \tag{4.2}$$

for almost all $t \in [0, T[$. From (3.13) it follows $E_2 \in L^s(0, T; L^q(\Omega))$ and

$$\|E_2\|_{q,s;T} \le c\nu^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{q_*} - \frac{1}{q})} \|F\|_{q_*, s_*; T} \tag{4.3}$$

with a constant $c = c(\Omega, q, q_*) > 0$. From now on let $\alpha := \frac{1}{2} + \frac{3}{2q}$.

By Theorem 2.2 there exists a constant $c = c(\Omega, q) > 0$ with the following property: If

$$\|E\|_{q,s;T} < c\nu^\alpha \,, \tag{4.4}$$

then the map $\mathcal{F}$ defined by (2.18) has a fixed point in $L^s(0, T; L^q_\sigma(\Omega))$. Looking at (4.3) we find a constant $\epsilon_* = \epsilon_*(\Omega, q, q_*) > 0$ such that under the conditions (1.6), (1.7) $\mathcal{F}$ has a fixed point $\tilde{u} \in L^s(0, T; L^q_\sigma(\Omega))$.

In the rest of the proof assume that $\tilde{u}$ is a fixed point of $\mathcal{F}$, i.e.

$$\tilde{u}(t) = -\int_0^t A_q^\alpha e^{-\nu(t-\tau)A_q} A_q^{-\alpha} P_q \mathrm{div}((\tilde{u} + E) \otimes (\tilde{u} + E))(\tau)\, d\tau \tag{4.5}$$

for almost all $t \in [0, T[$. Then $u = \tilde{u} + E$ is a very weak solution of (1.1).
Step 2. In this step it will be proved that $\tilde{u} \in L^8(0, T; L^4(\Omega))$.
Case 1. Let $q \in [\frac{24}{7}, 8]$. Then there exists $\sigma \in [0, \frac{1}{2}]$ such that

$$2\sigma + \frac{3}{4} = \frac{3}{\left(\frac{q}{2}\right)}. \tag{4.6}$$

Since $\tilde{u}, E \in L^s(0, T; L^q(\Omega))$ we can use (4.5), (4.6) and (3.13) to get that $\tilde{u} \in L^8(0, T; L^4(\Omega))$. In this case, the additional assumptions (1.9) and (1.10) on u_0, F are not needed.
Case 2. Let $3 < q < \frac{24}{7}$ or $8 < q < \infty$. Define $q_0 := q$ and $s_0 := s$. We choose $k \in \mathbb{N}$, exponents $1 < s_n, q_n < \infty$ with $\mathcal{S}(s_n, q_n) = 1\,, n = 1, \ldots, k$ and $q_k = \rho$ such that there exist $\sigma_n \in [0, \frac{1}{2}[\,, n = 1, \ldots, k$, with

$$2\sigma_n + \frac{3}{q_n} = \frac{3}{\left(\frac{q_{n-1}}{2}\right)}\,, \quad n = 1, \ldots, k. \tag{4.7}$$

Further, as in Case 1, choose $\sigma_{k+1} \in [0, \frac{1}{2}]$ with the property that with $q_{k+1} := 4\,, s_{k+1} := 8$ it holds

$$2\sigma_{k+1} + \frac{3}{q_{k+1}} = \frac{3}{\left(\frac{q_k}{2}\right)}. \tag{4.8}$$

It is easy to see that the sequence $\{q_n\}$ can be chosen strictly decreasing when $8 < q < \infty$ and strictly increasing when $3 < q < \frac{24}{7}$. We already know that $E_1, E_2 \in L^s(0,T;L^q(\Omega))$. The assumption (1.9) yields $E_1 \in L^\gamma(0,T;L^\rho(\Omega))$. Since there exists $\sigma \in [0,\frac{1}{2}]$ with $2\sigma + \frac{3}{\rho} = \frac{3}{\rho_*}$, (1.10) and (3.13) imply $E_2 \in L^\gamma(0,T;L^\rho(\Omega))$. Consequently, by interpolation

$$E_1, E_2 \in L^{s_n}(0,T;L^{q_n}(\Omega)) \tag{4.9}$$

for $n = 0,1,\dots,k$. We use an inductive argument to prove that $\tilde{u} \in L^{s_n}(0,T;L^{q_n}(\Omega))$ for $n = 0,1,\dots,k+1$. For s_0, q_0 this is true since we know $\tilde{u} \in L^s(0,T;L^q(\Omega))$. Moreover, looking at (4.5) we define $\sigma_0 := \frac{3}{2q}$ and see that the representation (4.10) holds with q_0, σ_0. Let $n \geq 1$. We assume that $\tilde{u} \in L^{s_{n-1}}(0,T;L^{q_{n-1}}(\Omega))$ and that

$$\tilde{u}(t) = -A_{q_{n-1}}^{1/2+\sigma_{n-1}} \int_0^t e^{-\nu(t-\tau)A_{q_{n-1}}} A_{q_{n-1}}^{-1/2-\sigma_{n-1}} P_{q_{n-1}} \mathrm{div}((\tilde{u}+E)\otimes(\tilde{u}+E))\, d\tau \tag{4.10}$$

for almost all $t \in [0,T[$. If $n < k+1$, we use (4.7), (4.9), (4.10) and (3.13) to get $\tilde{u} \in L^{s_n}(0,T;L^{q_n}(\Omega))$. Further, since $\tilde{u}, E \in L^{s_n}(0,T;L^{q_n}(\Omega))$ we apply (3.11) to (4.10) to obtain that $\tilde{u}$ satisfies the representation formula (4.10) with q_{n-1} replaced by q_n and σ_{n-1} replaced by σ_n. If $n = k+1$ we use (4.8), (4.9), (4.10) and (3.13) to get $\tilde{u} \in L^8(0,T;L^4(\Omega))$. The iterative argument is finished.

Step 3. Let $u := E + \tilde{u}$. In this final step we prove that u is a weak solution of (1.1). We define q_1, $s_1 > 1$ by

$$\frac{1}{2} = \frac{1}{q} + \frac{1}{q_1}, \quad \frac{1}{2} = \frac{1}{s} + \frac{1}{s_1}.$$

Since $\mathcal{S}(s_1,q_1) = \frac{3}{2}$ and $E \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$ we obtain by interpolation $E \in L^{s_1}(0,T;L^{q_1}(\Omega))$. Let $0 < T' \leq T$ with $T' < \infty$. From Step 2 it follows $\tilde{u}\otimes\tilde{u} \in L^2(0,T';L^2(\Omega))$ and we get

$$\begin{aligned}\|u\otimes u\|_{2,2;T'} &\leq \|\tilde{u}\otimes\tilde{u}\|_{2,2;T'} + \|\tilde{u}\otimes E\|_{2,2;T'} + \|E\otimes\tilde{u}\|_{2,2;T'} + \|E\otimes E\|_{2,2;T'} \\ &\leq \|\tilde{u}\otimes\tilde{u}\|_{2,2;T'} + 2\|\tilde{u}\|_{q,s;T}\|E\|_{q_1,s_1;T'} + \|E\|_{q,s;T'}\|E\|_{q_1,s_1;T'} \\ &< \infty.\end{aligned}$$

Now an application of (2.5) to (4.5) yields

$$\tilde{u}(t) = -\int_0^t A^{1/2} e^{-\nu(t-\tau)A} A^{-1/2} P\mathrm{div}(u\otimes u)(\tau)\, d\tau$$

for almost all $t \in [0,T[$. Therefore, $\tilde{u}$ can be considered as a weak solution of the instationary Stokes system with initial value 0 and external force $\mathrm{div}(-u\otimes u)$ where $u\otimes u \in L^2_{\mathrm{loc}}([0,T;L^2(\Omega))$. Then linear theory (see [21, IV, Theorems 2.3.1 and 2.4.1]) implies $\tilde{u} \in LH_T$ (if $T < \infty$ then even $\tilde{u} \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$).

Altogether, $u = \tilde{u} + E \in L^s(0,T;L^q(\Omega))$ is a strong solution of (1.1) with initial value u_0 and external force $\mathrm{div}F$. Consequently, $u : [0,T[\to L^2_\sigma(\Omega)$ is, after a redefinition on a set of Lebesgue measure 0, strongly continuous and satisfies the energy equality (1.8) for all $t \in [0,T[$. □

Proof of Theorem 1.3. We assume that $u \in L^s(0,T;L^q(\Omega))$ is a strong solution of (1.1) with exponents $q > 3$ and $\mathcal{S}(s,q) = 1$. From [21, IV, Theorem 2.4.1] we get the representation

$$
\begin{aligned}
u(t) &:= E_1(t) + E_2(t) + \tilde{u}(t) \\
&:= e^{-\nu t A} u_0 + \int_0^t A^{1/2} e^{-\nu(t-\tau)A} A^{-1/2} P \mathrm{div} F(\tau)\, d\tau \\
&\quad - \int_0^t A^{1/2} e^{-\nu(t-\tau)A} A^{-1/2} P \mathrm{div}(u \otimes u)(\tau)\, d\tau
\end{aligned}
$$

for almost all $t \in [0, T[$. By (2.5) we know that

$$
E_2(t) = \int_0^t A_{q^*}^{1/2} e^{-\nu(t-\tau)A_{q^*}} A_{q^*}^{-1/2} P \mathrm{div} F(\tau)\, d\tau \tag{4.11}
$$

for almost all $t \in [0, T[$. We apply (3.13) to (4.11) and get $E_2 \in L^s(0, T; L^q(\Omega))$. By (2.5), (3.11) it follows with $\beta = \sigma + \frac{1}{2}$, $\sigma = \frac{3}{2q}$ that

$$
\begin{aligned}
\tilde{u}(t) &= -\int_0^t A_{q/2}^{1/2} e^{-\nu(t-\tau)A_{q/2}} A_{q/2}^{-1/2} P_{q/2} \mathrm{div}(u \otimes u)(\tau)\, d\tau \\
&= -\int_0^t A_q^{\beta} e^{-\nu(t-\tau)A_q} A_q^{-\beta} P_q \mathrm{div}(u \otimes u)(\tau)\, d\tau
\end{aligned}
$$

for almost all $t \in [0, T[$. Then (3.13) yields $\tilde{u} \in L^s(0, T; L^q(\Omega))$. Therefore,

$$
e^{-\nu t A} u_0 = (u - E_2 - \tilde{u}) \in L^s(0, T; L^q(\Omega)).
$$

Finally, from (2.12), we get (1.11).

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