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Comparison of asymptotic stability for recursive ROS-multirate methods

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I Abstract

Ordinary differential equation systems containing different time scales can be efficiently treated by so-called multirate strategies based on a singlerate integration method. Since the stability character of a singlerate method usually isn't carried over to the corresponding multirate method, it is convenient to investigate the so-called asymptotic stability of multirate methods. Here a recursive multirate procedure using different Rosenbrock methods is investigated.

II Introduction

Many physical phenomena can be described by partial differential equations (PDEs) which lead to a set of N coupled ordinary differential equations (ODEs) in time, when they are discretized in space:

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t) + f(t, \mathbf{y}(t)) := F(t, \mathbf{y}(t)), \quad \mathbf{y}(0) = \mathbf{y}_0 \in \mathbb{R}^N. \tag{1}$$

Each ODE describes the behaviour of one component (one spatial discretization point) that typically includes the coupling with some of the other components.

A normal singlerate time integrator solves all ODEs with the same time step sizes, which are determined by taking all the components into account. This might produce very small time steps that also have to be applied to components with much less activity.

The idea of multirate methods is to use different time step sizes for different components, depending on the individual activity of the solution, which means there will be a differentiation between *active* and *latent* components. The coupling can be managed by interpolation/extrapolation.

We follow the recursive multirate strategy proposed by Savcenco et al. [8], which includes one tentative global time step applied to all components and an abitrary number of subsequent recursive refinement steps for all components which are not accurate enough. For the present stability analysis, only one refinement step will be considered.

In [5], Savcenco investigated the stability by using ROS1 and ROS2 with different interpolation methods and in [6], the fourth order RODAS method with a third order continuous extension was considered.

Here, we will compare the stability of multirate methods using the Rosenbrock methods ROS2, ROS3PL and RODAS [9, 3, 1]. It will be shown how the stability of the interpolation method used influences the stability of the corresponding multirate method. We will see that the multirate-ROS3PL method, equipped with the second order continuous extension of Ostermann [4], has very good stability properties.

III Rosenbrock integration methods

An s-stage Rosenbrock method is a linear-implicit one-step method of the form

$$y_{n+1} = y_n + \sum_{j=1}^{s} b_j k_{nj} \tag{2}$$

$$\left(I - \tau\gamma F_y(t_n, y_n)\right) k_{ni} = \tau F\left(t_n + \alpha_i \tau, y_n + \sum_{j=1}^{i-1} \alpha_{ij} k_{nj}\right) \tag{3}$$

$$+ \gamma_i \tau^2 F_t(t_n, y_n) + \tau F_y(t_n, y_n) \sum_{j=1}^{i-1} \gamma_{ij} k_{nj}, \quad i = 1, \dots, s \tag{4}$$

with parameters γ_{ij}, α_{ij}, b_j and $\gamma_i = \sum_{j=1}^{i} \gamma_{ij}$, $\alpha_i = \sum_{j=1}^{i-1} \alpha_{ij}$. If the Rosenbrock method has order p, an error estimator can be easily obtained by calculating the difference to an embedded solution of order $p-1$.

A Rosenbrock method is called A-stable, if the stability function of the Rosenbrock method

$$R(\tau\lambda) = 1 + \tau\lambda b^T (I_s - \tau\lambda B)^{-1} \mathbb{1}_s \tag{5}$$

with $B = \left(\beta_{ij}\right)_{ij} := \left(\alpha_{ij} + \gamma_{ij}\right)_{ij}$, $\beta_{ii} = \gamma$, satisfies

$$|R(z)| \leq 1 \quad \forall z \in \mathbb{C}^-. \tag{6}$$

The method is called L-stable, if it is A-stable and

$$R(z) \to 0 \quad \text{for} \quad \Re(z) \to -\infty. \tag{7}$$

In this work we want to consider the second order L-stable ROS2-method [9], which has two stages, the 4-stage third-order L-stable ROS3PL-method [3] and the 6-stage fourth-order A-stable RODAS-method [1].

Since the coefficient set of ROS3PL is needed later on, it is given in Table 1 depending on γ with $b_i = \beta_{4i}$.

Table 1: Set of coefficients for ROS3PL depending on γ

$\alpha_{21} = \frac{1}{2}$	$\beta_{21} = 0$
$\alpha_{31} = \frac{1}{2}$	$\beta_{31} = \dfrac{\frac{1}{2} - \frac{10}{3}\gamma + 9\gamma^2 - 8\gamma^3}{\frac{1}{6} - \gamma}$
$\alpha_{32} = \frac{1}{2}$	$\beta_{32} = \dfrac{8\gamma^3 - 8\gamma^2 + \frac{4}{3}\gamma}{\frac{1}{6} - \gamma}$
$\alpha_{41} = \frac{1}{2}$	$\beta_{41} = \frac{1}{6}$
$\alpha_{42} = \frac{1}{2}$	$\beta_{42} = \frac{2}{3}$
$\alpha_{43} = 0$	$\beta_{43} = \frac{1}{6} - \gamma$

γ is about 0.43586652150845, which satisfies the L-stability condition

$$\gamma^4 - 3\gamma^3 + \frac{3}{2}\gamma^2 - \frac{1}{6}\gamma = 0 \tag{8}$$

and the A-stability condition

$$0.22364780 \le \gamma \le 0.57281606. \tag{9}$$

IV Interpolation methods and continuous extensions

For the interpolation of the needed values in the multirate procedure, continuous extensions of the Rosenbrock methods

$$y_I(t_{n-1} + \theta\tau) = y_{n-1} + \sum_{i=1}^{s} \theta\, b_i(\theta) k_{ni}, \quad 0 \le \theta \le 1 \tag{10}$$

can be used, where $b_i(\theta)$ are polynomials with $b_i(1) = b_i$. Derivation of these formulas gives a possibility to approximate the needed values $y_I'(t)$ as well. Ostermann showed that a Rosenbrock method of order p has at least one extension of order $(p+1)/2$ rounded down [4]. The order conditions up to order three are given by

$$\begin{aligned}
\sum_i b_i(\theta) &= 1 && \text{(order 1)}\\
\sum_i b_i(\theta)\beta_i &= \frac{1}{2}\theta - \gamma && \text{(order 2)}\\
\sum_i b_i(\theta)\alpha_i^2 &= \frac{1}{3}\theta^2 && \text{(order 3a)}\\
\sum_{i \ne k} b_i(\theta)\beta_{ik}\beta_k &= \frac{1}{6}\theta^2 - \gamma\theta + \gamma^2 && \text{(order 3b)}
\end{aligned} \tag{11}$$

In general, an interpolation method is called stable with respect to an A-stable Rosenbrock method, if after application to the scalar test equation $y' = \lambda y$, $y(0) = 1$ with $\lambda \in \mathbb{C}^-$ the condition

$$\max_{0\le\theta\le1} |y_I(\theta\tau)| \le c \tag{12}$$

is satisfied for a constant $c \le 1$.

In the following tests, we will use stable second-order continuous extensions for ROS2 [5] and for ROS3PL and a third-order continuous extension for ROS3PL which satisfies equation (12) with a constant $c = 1.0025$. For RODAS we will use a third-order continuous extension, which satisfies equation (12) with $c = 1.04$ [6].

IV.1 A stable second-order continuous extension for ROS3PL

In this subsection we will show the stability of the second-order continuous extension of ROS3PL obtained by he approach of Ostermann [4]. It is given by the polynomials

$$b_1(\theta) = \frac{2}{3}\left(4 - 27\gamma + 81\gamma^2 - 72\gamma^3\right) + \frac{1}{2}\left(-5 + 36\gamma - 108\gamma^2 + 96\gamma^3\right)\cdot\theta \tag{13}$$

$$b_2(\theta) = \frac{4}{3}\left(-1 + 12\gamma - 36\gamma^2 + 36\gamma^3\right) - 2\left(-1 + 8\gamma - 24\gamma^2 + 24\gamma^3\right)\cdot\theta \tag{14}$$

$$b_3(\theta) = \frac{1}{3}\left(-1 + 12\gamma - 36\gamma^2\right) + \frac{1}{2}\left(1 - 10\gamma + 24\gamma^2\right)\cdot\theta \tag{15}$$

$$b_4(\theta) = 2\left(-\gamma + 3\gamma^2\right) + 3\left(\gamma - 2\gamma^2\right)\cdot\theta. \tag{16}$$

We apply one step of ROS3PL to the scalar test equation $y' = \lambda y$, $y(0) = 1$, and get with $z = \lambda\tau$

$$k_{11} = k_{12} = \frac{z}{1 - z\gamma} \tag{17}$$

$$k_{13} = \frac{z}{1 - z\gamma}\left(1 + \frac{\frac{1}{2} - 2\gamma + \gamma^2}{\frac{1}{6} - \gamma}\,\frac{z}{1 - z\gamma}\right) \tag{18}$$

$$k_{14} = \frac{z}{1 - z\gamma}\left(1 + (1-\gamma)\frac{z}{1 - z\gamma} + \left(\frac{1}{2} - 2\gamma + \gamma^2\right)\left(\frac{z}{1 - z\gamma}\right)^2\right). \tag{19}$$

Insertion to formula (10) leads to

$$y_I(\theta\tau) = \frac{1}{-2(1 - z\gamma)^3}\left(-2 + z(6\gamma - 2\theta) + 2z^3\gamma^3(1 - 4\theta + 3\theta^2)\right. \tag{20}$$

$$\left. + z^2((2 - 3\theta)\theta + 12\gamma^3(-1 + \theta)\theta + 6\gamma\theta(-2 + 3\theta) - 6\gamma^3(1 - 6\theta + 6\theta^2))\right) =: R(\theta, z). \tag{21}$$

Defining the denominator of $R(\theta, z)$ by $Q(\theta, z)$ and the numerator by $P(\theta, z)$, condition (12) with $c = 1$ can be transformed for the imaginary axis to

$$P(\theta, ir)\cdot P(\theta, -ir) - Q(\theta, ir)\cdot Q(\theta, -ir) \le 0, \quad \forall r \in \mathbb{R}, \tag{22}$$

which can be shown easily using, for example, Mathematica. Since $R(\theta, z)$ is analytic for $\Re(z) < 0$, the validness of condition (12) with $c = 1$ for all $z \in \mathbb{C}^-$ follows from the maximum principle (see also [1, p.42]). In Fig. 1 $|R(\theta, z)|$ depending on purely imaginary values z for three different values of θ are shown.

IV.2 A third-order continuous extension for ROS3PL

Satisfying the order conditions (11) with second-order polynomials b_i the following third-order continuous extension for ROS3PL is obtained:

$$b_1(\theta) = -1.4523898386593745 + 3.4047796773187495\cdot\theta - 1.7857231719927085\cdot\theta^2 \tag{23}$$

$$b_2(\theta) = 3.2698531182124992 - 4.539706236424999\cdot\theta + 1.9365197848791667\cdot\theta^2 \tag{24}$$

$$b_3(\theta) = 0.227796975781977 - 1.2631935160893306\cdot\theta + 0.7661966854655615\cdot\theta^2 \tag{25}$$

$$b_4(\theta) = -1.0452602553351018 + 2.3981200751955805\cdot\theta - 0.9169932983520198\cdot\theta^2. \tag{26}$$

The stability of this interpolation is numerically checked by applying ROS3PL with the third-order continuous extension to the scalar test equation $y' = \lambda y$, $y(0) = 1$. The result for the value of $\max_{0\le\theta\le1} |y_I(\theta\tau)|$ for different purely imaginary values $z = \tau\lambda$ is given in Fig. 2.

The values $\max_{0\le\theta\le1} |y_I(\theta\tau)|$ are bounded by a value slightly larger than it is demanded in the stability definition. Since this is satisfied for all values of z, no dramatic amplification of the error of the corresponding multirate method is expected.

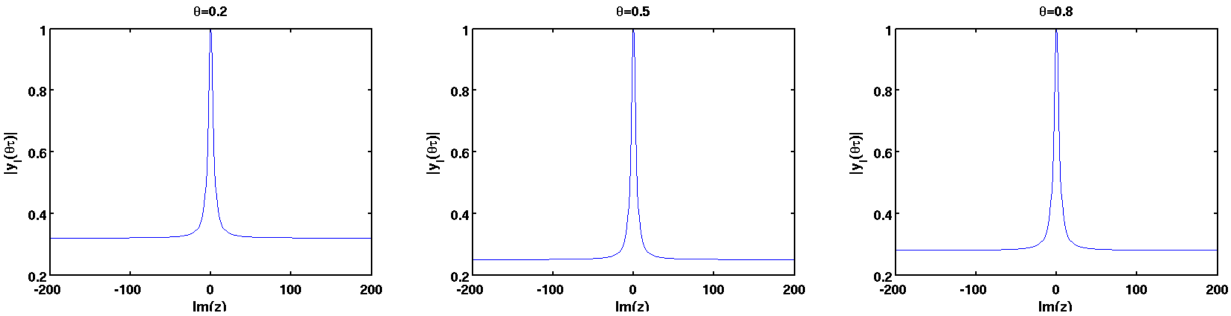

Figure 1: Plot of $|y_I(\theta\tau)|$ for three different θ and a range of purely imaginary values $z = \lambda\tau$.

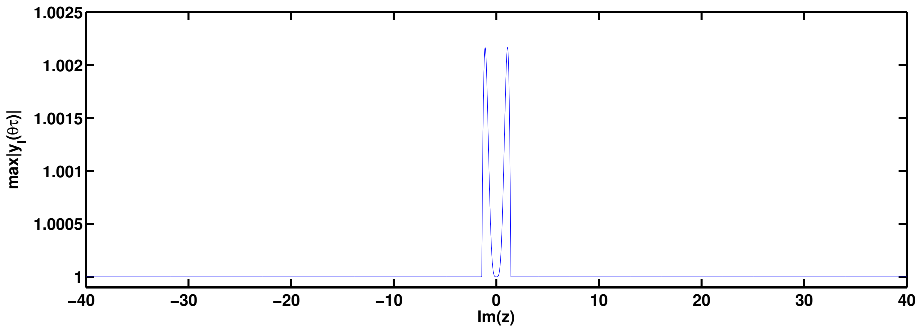

Figure 2: Plot of $\max_{0\le\theta\le 1} |y_I(\theta\tau)|$ for the third-order continuous extension of ROS3PL.

IV.3 Hermite interpolation

Another interpolation we want to consider is the cubic Hermite interpolation, which is defined by

$$y_I(t_{n-1} + \theta\tau) = y_{n-1} + \theta\tau A_{n-1} + (\theta\tau)^2 B_{n-1} + (\theta\tau)^3 C_{n-1}$$

where

$$\begin{aligned} A_{n-1} &= F(t_{n-1}, y_{n-1}) \\ B_{n-1} &= \left(3y_n - 3y_{n-1} - \tau F(t_n, y_n) - 2\tau F(t_{n-1}, y_{n-1})\right)/\tau^2 \\ C_{n-1} &= \left(2y_{n-1} - 2y_n + \tau F(t_{n-1}, y_{n-1}) + \tau F(t_n, y_n)\right)/\tau^3. \end{aligned}$$

It can be shown that the Hermite interpolation is not stable for ROS2, ROS3PL and RODAS. For example, the application to the scalar test equation for ROS3PL leads to

$$y_I(\theta\tau) = \frac{1}{-2(1-z\gamma)^3}\Big(-2 + z(6\gamma - 2\theta) + 2z^4\gamma^3(-1+\theta)^2\theta - z^2(6\gamma^2 - 6\gamma\theta + \theta^2) \tag{27}$$

$$+z^3(-1+\theta)(6\gamma^2(1-2\theta)\theta - \theta^2 + 6\gamma\theta^2 + \gamma^3(-2 - 2\theta + 4\theta^2)))\Big). \tag{28}$$

Since the numerator of y_I is a polynomial of higher degree in z than the denominator, y_I cannot have an upper bound for $\Re(z) \to -\infty$.

Numerical results are given in Fig. 3 where the value of $\max_{0\le\theta\le 1} |y_I(\theta\tau)|$ for different purely imaginary values $z = \tau\lambda$ is checked by applying ROS2, ROS3PL and RODAS with the Hermite interpolation to the scalar test equation $y' = \lambda y$, $y(0) = 1$.

V Asymptotic stability for multirate methods

For singlerate methods, the stability is usually studied by applying the method to the scalar test equation

$$y' = \lambda y, \quad \lambda \in \mathbb{C}^-.$$

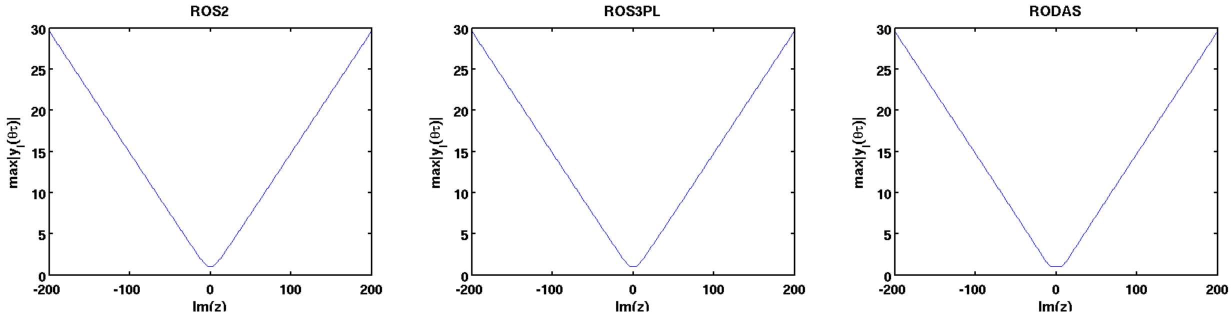

Figure 3: Plot of $\max_{0\leq\theta\leq 1}|y_I(\theta\tau)|$ for the Hermite interpolation with respect to ROS2, ROS3PL and RODAS.

An one step method applied to this equation for a given initial value y_0 can be written as the difference equation

$$y_k = R(\lambda\tau)y_{k-1},$$

where the stability function R is some rational function. Then the method is called *asymptotic stable* if and only if

$$|R(\alpha\tau)| < 1.$$

In this case, we obviously have $\lim_{k\to\infty} y_k = 0$.

Since multirate methods need at least one active component y_A and one latent component y_L, the stability analysis can be done by studying the 2x2 test problem

$$\begin{pmatrix} y_A' \\ y_L' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} y_A \\ y_L \end{pmatrix}$$

where $a_{11}, a_{22} < 0$ and $\gamma = \frac{a_{12}a_{21}}{a_{11}a_{22}} < 1$ ensure that the eigenvalues of the matrix have negative real parts [2]. γ can be considered as a measure for the coupling between the equations and $\kappa = \frac{a_{22}}{a_{11}} \geq 1$ can be introduced as a measure for the stiffness of the system. The application of a multirate method to this test equation with m local time steps in the active component using a one step integration method for integration can be written as

$$\begin{pmatrix} y_{A,m} \\ y_{L,1} \end{pmatrix} = \mathbf{S} \begin{pmatrix} y_{A,0} \\ y_{L,0} \end{pmatrix}.$$

with a so-called *amplification matrix* **S**.

The multirate method is now called *asymptotic stable* if and only if both eigenvalues of **S** are within the unit disk. This guarantees $\|\mathbf{S}^n\| \to 0$ as $n \to \infty$, so that the numerical approximations $\mathbf{S}^n\mathbf{y}_0$ converge to the zero vector for $n \to \infty$.

It can be shown that the eigenvalues of this amplification matrix for the considered multirate methods are only dependent on the three values $\tau a_{11}, \tau a_{22}$ and $\det \tau A$ [5]. That's why the stability regions can be also considered depending on κ and the quantities

$$\xi = \frac{\tau a_{11}}{1-\tau a_{11}} \in (-1,0), \quad \eta = \frac{\gamma}{2-\gamma} \in (-1,1).$$

Numerically calculated results for ROS2, ROS3PL and RODAS using one local refinement in the active component are shown in the following tables. The stability is checked for fixed κ's equal to $1, 10, 100$ and for a ξ/η-grid at the points $(i \cdot 0.01, j \cdot 0.01)$ inside the definition areas with $i, j \in \mathbb{Z}$. The stability results for the Multirate-ROS2- and Multirate-RODAS-method using its continuous extensions were already presented in [5, 6].

It can be seen that the usage of the instable Hermite interpolation for ROS2, ROS3PL and RODAS in comparison to the usage of the continuous extensions causes much more instability in the resulting multirate method. The stability of the Multirate-ROS3PL method using the continuous extension of second order seems to be not affected by the stiffness and strongness of coupling of the test equation. For most of the other methods, it can be seen that the more the system is stiff and coupled the less stable the methods are.

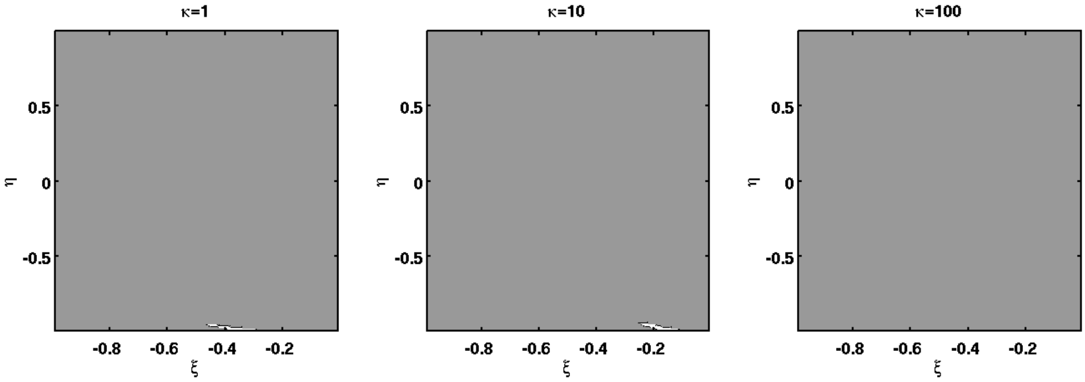

Figure 4: Stability regions (gray areas) for ROS2, 2nd order continuous extension for $\kappa = 1, 10, 100$ (see [5]).

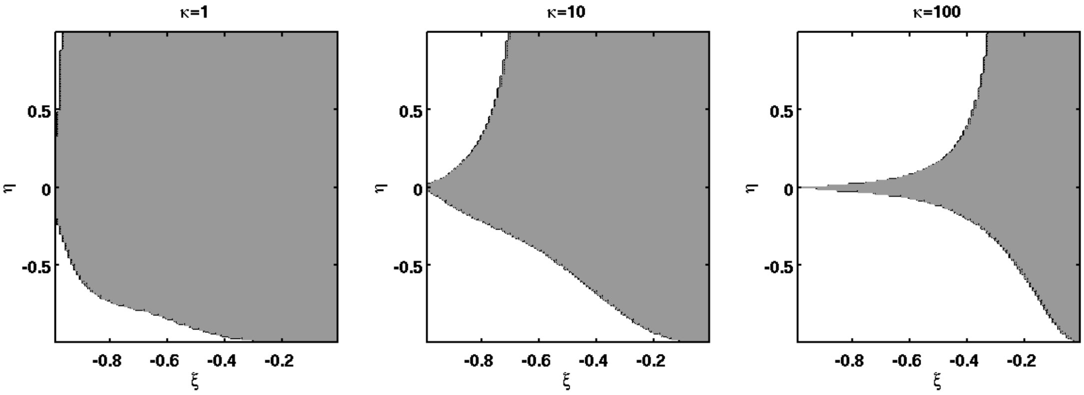

Figure 5: Stability regions (gray areas) for ROS2, Hermite interpolation for $\kappa = 1, 10, 100$.

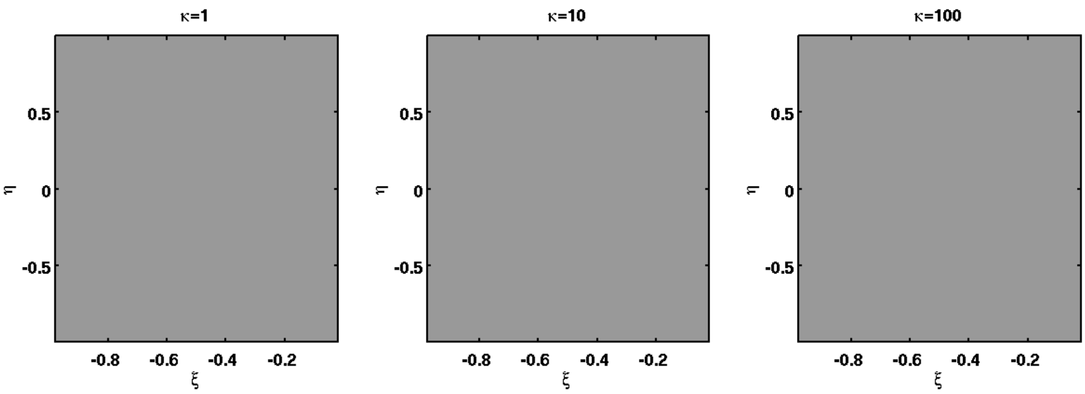

Figure 6: Stability regions (gray areas) for ROS3PL, 2nd order continuous extension for $\kappa = 1, 10, 100$.

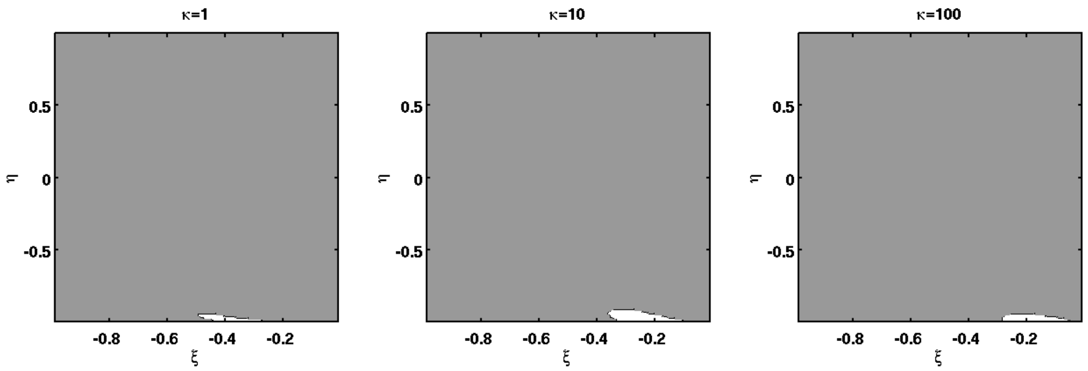

Figure 7: Stability regions (gray areas) for ROS3PL, 3rd order continuous extension for $\kappa = 1, 10, 100$.

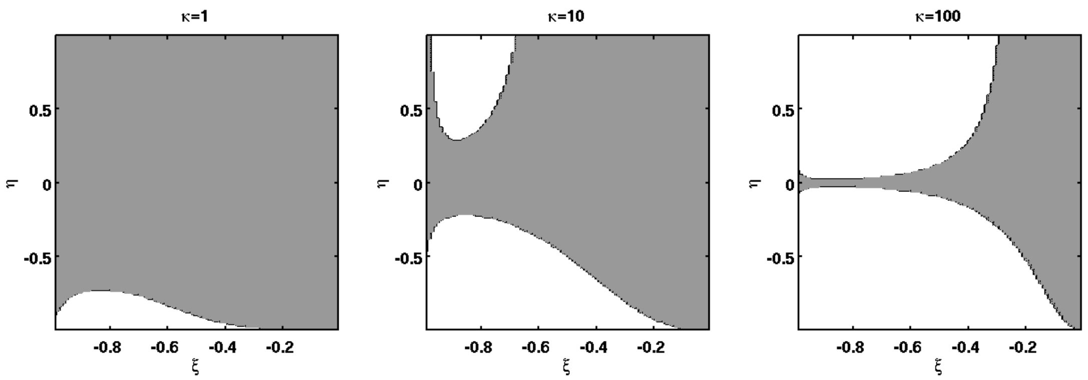

Figure 8: Stability regions (gray areas) for ROS3PL, Hermite interpolation for $\kappa = 1, 10, 100$.

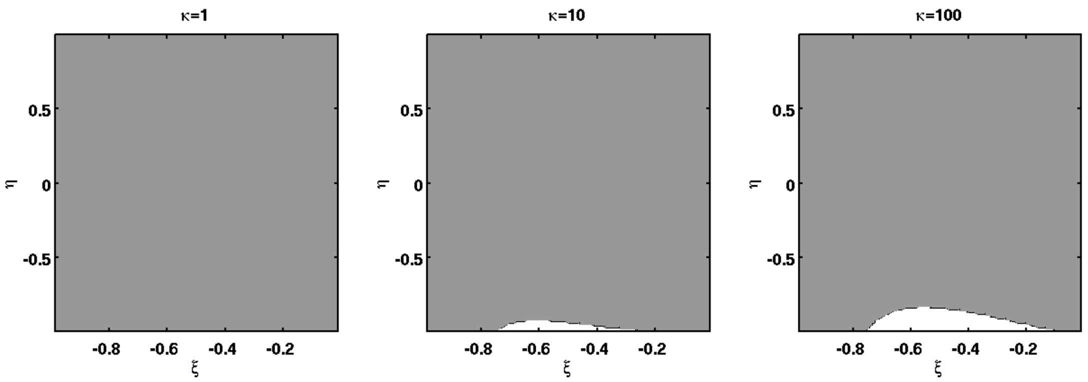

Figure 9: Stability regions (gray areas) for RODAS, 3rd order continuous extension for $\kappa = 1, 10, 100$ (see [7]).

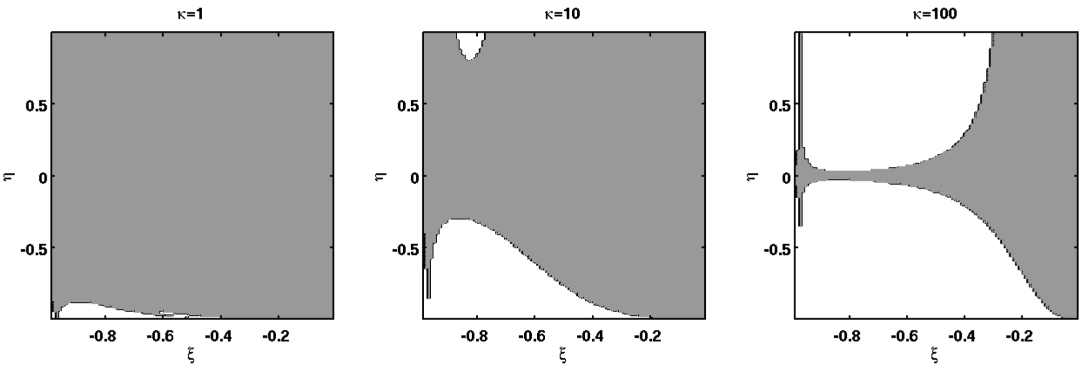

Figure 10: Stability regions (gray areas) for RODAS, Hermite interpolation for $\kappa = 1, 10, 100$.

VI 2d-test problem

To clarify the meaning of the stability concept, we want to consider a specific two dimensional test problem. We choose

$$\begin{pmatrix} y_A' \\ y_L' \end{pmatrix} = \begin{pmatrix} -5 & -1900 \\ 5 & -50 \end{pmatrix} \cdot \begin{pmatrix} y_A \\ y_L \end{pmatrix}, \quad \begin{pmatrix} y_A \\ y_L \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{29}$$

This problem corresponds to $\kappa = 10$, $\eta = -0.95$ and $\xi = -0.2$ for $\tau = 0.05$ and about $\xi = -0.43$ for $\tau = 0.15$.

Considering only the multirate methods using continuous extensions we see from the stability plots in the previous section, that for $\tau = 0.05$ Multirate-ROS2 and Multirate-ROS3PL with third-order continuous extension are instable. Multirate-RODAS and Multirate-ROS3PL with second-order continuous extension are stable. For $\tau = 0.15$ only the Multirate-RODAS is instable and the others are stable.

If we apply the considered multirate methods to this test equation with one local refinement in the first component and equidistant time steps until the end time point $T = 7.5$, we can observe that the instable methods diverge and the stable ones converge to zero, how it is expected. Oscillations in the stable methods can only be avoided when smaller time step sizes are used.

For $\tau = 0.15$ the results are shown in Fig. 11,12,13,14 and for $\tau = 0.05$ in 15,16,17,18.

Therefore it is very advisable that the stability is independent on the time step size and the considered test equation, because if you want to consider much more complex problems, then you don't know anymore where exactly you are in such a stability region diagramm. This means, instability can show up very unpredictable, for example caused by stiffness and coupling. So the multirate-ROS3PL method with the second-order continuous extension has optimal stability properties for the two-dimensional test problems, since the stability is independend on the test problem and on the used time step size.

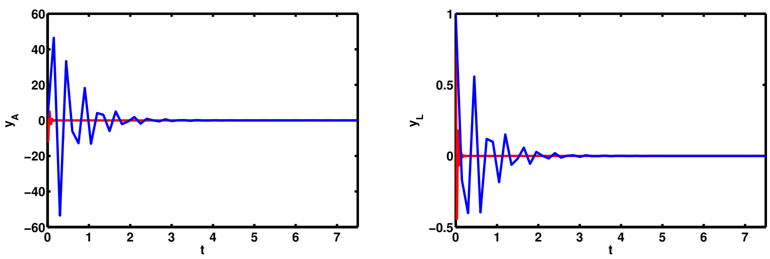

Figure 11: Approximated (blue) vs. reference solution (red) for Multirate-ROS2 + cont. ext. and $\tau = 0.15$.

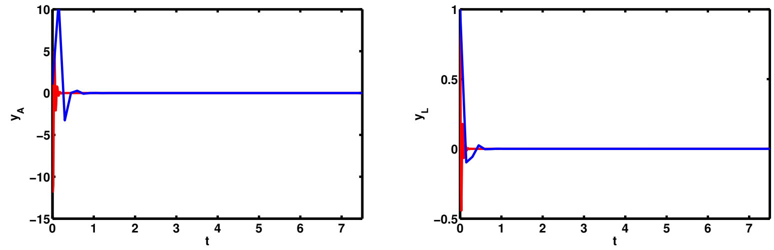

Figure 12: Approximated (blue) vs. reference solution (red) for Multirate-ROS3PL + 2nd order cont. ext. and $\tau = 0.15$.

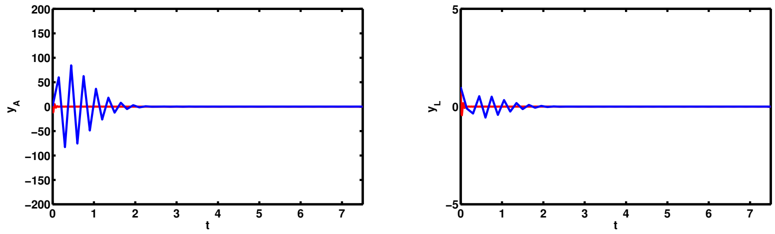

Figure 13: Approximated (blue) vs. reference solution (red) for Multirate-ROS3PL + 3rd order cont. ext. and $\tau = 0.15$.

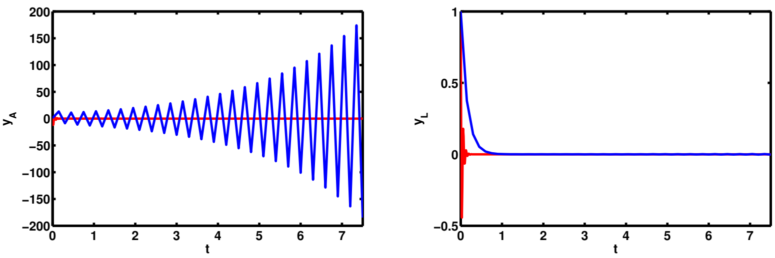

Figure 14: Approximated (blue) vs. reference solution (red) for Multirate-RODAS + cont. ext. and $\tau = 0.15$.

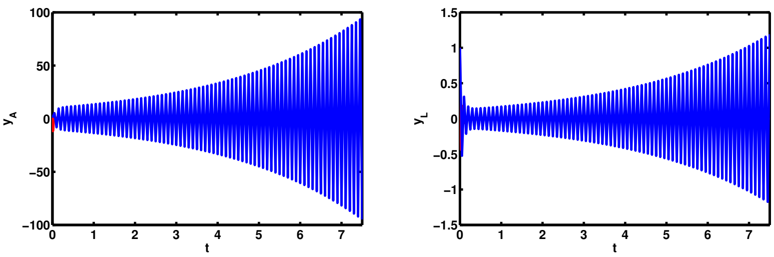

Figure 15: Approximated (blue) vs. reference solution (red) for Multirate-ROS2 + cont. ext. and $\tau = 0.05$.

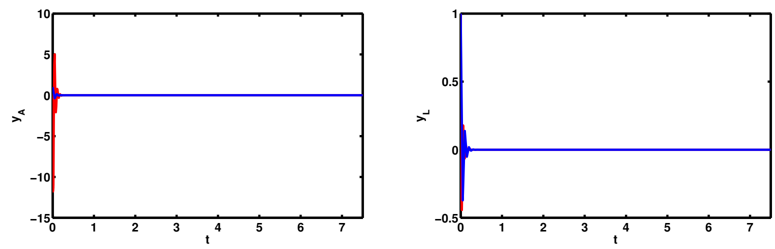

Figure 16: Approximated (blue) vs. reference solution (red) for Multirate-ROS3PL + 2nd order cont. ext. and $\tau = 0.05$.

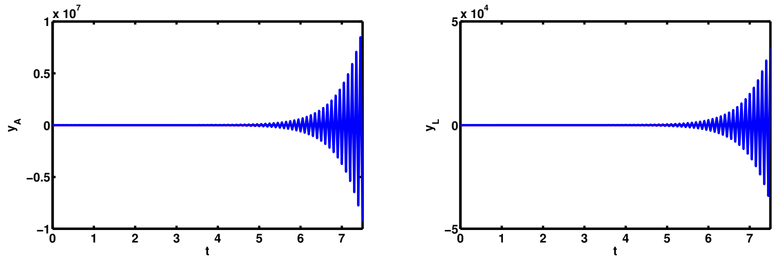

Figure 17: Approximated (blue) vs. reference solution (red) for Multirate-ROS3PL + 3rd order cont. ext. and $\tau = 0.05$.

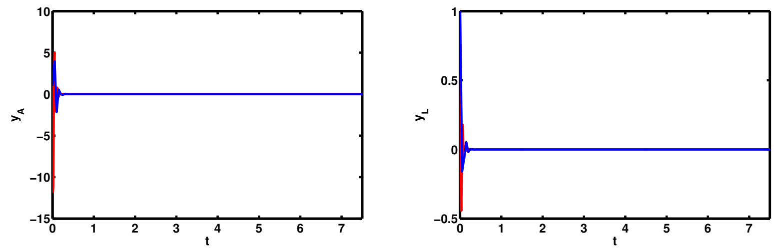

Figure 18: Approximated (blue) vs. reference solution (red) for Multirate-RODAS + cont. ext. and $\tau = 0.05$.

VII Conclusion

The asymptotic stability of a multirate-method using ROS3PL was compared to the methods using ROS2 and RODAS. For interpolation, continuous extensions and the Hermite interpolation were considered and tested for stability. It was shown that the Hermite interpolation is instable and the continuous extensions are stable or at least bounded by a value very near to one.

It turned out that the stability properties for multirate-ROS3PL based on a stable second-order continuous extension was independent on the specific test problem and the used time step size. This was shown numerically and still needs analytical verification. But in general this was in contrast to the other Rosenbrock solvers studied so far, where stability was not achieved when the considered test problem was very stiff and/or coupled. Also it was shown that the stability of the used interpolation method can have strong influence on the stability of the corresponding multirate method. The multirate methods using the Hermite interpolation had much smaller stability regions than the ones using the continuous extensions.

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