Finite sections of random Jacobi operators

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Abstract. This article is about a problem in the numerical analysis of random operators. We study a version of the finite section method for the approximate solution of equations $Ax = b$ in infinitely many variables, where A is a random Jacobi operator. In other words, we approximately solve infinite second order difference equations with stochastic coefficients by reducing the infinite volume case to the (large) finite volume case via a particular truncation technique. For most of the paper we consider non-selfadjoint operators A but we also comment on the self-adjoint case when simplifications occur.

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1 Introduction

Let U, V and W be non-empty compact subsets of the complex plane, and put

$$u^* := \max_{u\in U} |u|, \quad v_* := \min_{v\in V} |v|, \quad w^* := \max_{w\in W} |w| \quad \text{and} \quad \delta := v_* - (u^* + w^*). \tag{1}$$

We write $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ for the sets of all positive integer, integer, real and complex numbers.

Infinite matrices. In this paper we study bi- and semi-infinite matrices of the form

$$A = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & v_{-2} & w_{-2} & & & & \\ & u_{-1} & v_{-1} & w_{-1} & & & \\ & & u_0 & \boxed{v_0} & w_0 & & \\ & & & u_1 & v_1 & w_1 & \\ & & & & u_2 & v_2 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad A_+ = \begin{pmatrix} v_1 & w_1 & & & \\ u_2 & v_2 & w_2 & & \\ & u_3 & v_3 & w_3 & \\ & & u_4 & v_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \tag{2}$$

with entries $u_i \in U$, $v_i \in V$ and $w_i \in W$ for all i under consideration, where the box marks the matrix entry of A at $(0, 0)$. As usual, we call $\mathbf{u} := (u_i)$, $\mathbf{v} := (v_i)$, and $\mathbf{w} := (w_i)$ the sub-, main- and superdiagonal of A, resp. A_+. We understand A and A_+ as linear operators, again denoted by A and A_+, acting boundedly, by matrix-vector multiplication, on the standard spaces $\ell^p(\mathbb{Z})$ and $\ell^p(\mathbb{N})$ of bi- and semi-infinite complex sequences with $p \in [1, \infty]$. It is clear that the matrices (2) are in general not self-adjoint. We will study the selfadjoint case, when $w_i = \overline{u}_{i+1}$ for all i, separately in Section 2.6.

The sets of all operators A and A_+ from (2) with entries $u_i \in U$, $v_i \in V$ and $w_i \in W$ for all indices i that occur will be denoted by $M(U,V,W)$ and $M_+(U,V,W)$, respectively. The set of all $n \times n$ matrices with subdiagonal entries in U, main diagonal entries in V and superdiagonal entries in W (and all other entries zero) will be called $M_n(U,V,W)$ for $n \in \mathbb{N}$, and we finally put $M_{\rm fin}(U,V,W) = \cup_{n\in\mathbb{N}} M_n(U,V,W)$.

Recall that a bounded linear operator $B : X \to Y$ between Banach spaces is a *Fredholm operator* if the dimension, α, of its null-space is finite and the codimension, β, of its image in Y is finite. In this case, the image of A is closed in Y and the integer $\operatorname{ind} A := \alpha - \beta$ is called the *index* of A. For a bounded linear operator B on $\ell^p(\mathbb{I})$ with $\mathbb{I} \in \{\mathbb{Z}, \mathbb{N}, \mathbb{Z}\setminus\mathbb{N}\}$, we write $\operatorname{spec}^p B$, $\operatorname{spec}^p_{\rm ess} B$ and $\operatorname{spec}^p_{\rm pt} B$ for the sets of all $\lambda \in \mathbb{C}$ for which $B - \lambda I$ is, respectively, not invertible, not a Fredholm operator or not injective on $\ell^p(\mathbb{I})$. Because A and A_+ in (2) are band matrices, their spectrum and essential spectrum do not depend on the underlying ℓ^p-space [27, 29, 44], so that we will just write $\operatorname{spec} B$ and $\operatorname{spec}_{\rm ess} B$ for operators B in $M(U,V,W)$ and in $M_+(U,V,W)$.

Random alias pseudoergodic operators. Our particular interest is on random operators in $M(U,V,W)$ and $M_+(U,V,W)$. We model randomness by the following concept: Given a metric space $(\mathcal{M}, d)$ and an index set $\mathbb{I} \in \{\mathbb{Z}, \mathbb{N}, \mathbb{Z}\setminus\mathbb{N}\}$, we say that a sequence $a = (a_i)_{i\in\mathbb{I}}$ in $\mathcal{M}$ is *pseudoergodic* if for every $\varepsilon > 0$, all $n \in \mathbb{N}$ and all $b = (b_i)_{i=1}^n \in \mathcal{M}^n$, there is a $k \in \mathbb{I}$ such that $d(a_{k+i}, b_i) < \varepsilon$ for all $i = 1, ..., n$. In particular, if $(\mathcal{M}, d)$ is a discrete space then $a = (a_i)$ is pseudoergodic if and only if every finite vector over $\mathcal{M}$ can be found (as a sequence of consecutive entries) in a. For a finite set $\mathcal{M}$, a pseudoergodic sequence $(a_i)_{i\in\mathbb{N}}$ can be constructed by writing all $\mathcal{M}$-valued sequences of length 1, then 2, then 3, ... in a row. For $\mathcal{M} = \{0,1\}$, this is done by stringing together the binary expansions of all natural numbers.

We will call $(a_i)_{i\in\mathbb{Z}}$: *left-pseudoergodic* if $(a_i)_{i\in\mathbb{Z}\setminus\mathbb{N}}$ is pseudoergodic, *right-pseudoergodic* if $(a_i)_{i\in\mathbb{N}}$ is pseudoergodic, and *bi-pseudoergodic* if both $(a_i)_{i\in\mathbb{Z}\setminus\mathbb{N}}$ and $(a_i)_{i\in\mathbb{N}}$ are pseudoergodic. It is a simple exercise to show that $(a_i)_{i\in\mathbb{Z}}$ is pseudoergodic if and only if it is right- or left-pseudoergodic.

Pseudoergodicity was introduced by Davies [13] to study spectral properties of random operators while eliminating probabilistic arguments. Indeed, if $a = (a_i)_{i\in\mathbb{I}}$, where all entries a_i are independent (or at least not fully correlated) samples from a random variable with values (densely) in $\mathcal{M}$ then, with probability one, a is pseudoergodic in cases $\mathbb{I} \in \{\mathbb{N}, \mathbb{Z}\setminus\mathbb{N}\}$ and bi-pseudoergodic in case $\mathbb{I} = \mathbb{Z}$ (e.g. [30, §5.5.3]).

We call an operator $A \in M(U,V,W)$ *pseudoergodic*, *left-pseudoergodic*, *right-pseudoergodic* or *bi-pseudoergodic* and write $A \in \Psi\mathrm{E}(U,V,W)$, $A \in \Psi\mathrm{E}_{\rm L}(U,V,W)$, $A \in \Psi\mathrm{E}_{\rm R}(U,V,W)$ or $A \in \Psi\mathrm{E}_2(U,V,W)$, respectively, if $a = (a_i)_{i\in\mathbb{Z}}$ with $a_i := (u_i, v_i, w_i) \in \mathcal{M} := U \times V \times W \subset \mathbb{C}^3$ has the corresponding property. So we have

$$\begin{aligned} \Psi\mathrm{E}(U,V,W) &= \Psi\mathrm{E}_{\rm L}(U,V,W) \,\cup\, \Psi\mathrm{E}_{\rm R}(U,V,W), \\ \Psi\mathrm{E}_2(U,V,W) &= \Psi\mathrm{E}_{\rm L}(U,V,W) \,\cap\, \Psi\mathrm{E}_{\rm R}(U,V,W). \end{aligned}$$

If $A \in \Psi\mathrm{E}_{\rm R}(U,V,W)$ then we will write $A_+ \in \Psi\mathrm{E}_+(U,V,W)$ for the corresponding semi-infinite submatrix A_+ of A from (2). We will say a little bit about spectral properties of pseudoergodic operators A and A_+ but will mainly focus on another problem:

The finite section method (FSM). If one wants to solve an equation

$$Ax = b, \qquad \text{i.e.} \qquad \sum_{j\in\mathbb{Z}} a_{ij}\, x(j) = b(i), \quad i \in \mathbb{Z} \tag{3}$$

on $X = \ell^p(\mathbb{Z})$ approximately, where $A : X \to X$ (bounded) and $b \in X$ are given and $x \in X$ is sought for, one often uses a projection method. Therefore, let $P_{l,r} : X \to X$ stand for the operator of multiplication by the characteristic function of the discrete interval $\mathbb{Z} \cap [l, ..., r]$ for $l, r \in \mathbb{Z}$ with $l \le r$ and denote the image of $P_{l,r}$ by $X_{l,r} \cong \mathbb{C}^{r-l+1}$. One then picks sequences of integers $l_1, l_2, ... \to -\infty$ and $r_1, r_2, ... \to +\infty$ and replaces the infinite system (3) by the sequence of finite systems

$$P_{l_n,r_n} A P_{l_n,r_n} x_n \;=\; P_{l_n,r_n} b, \qquad \text{i.e.} \qquad \sum_{l_n \le j \le r_n} a_{ij}\; x_n(j) \;=\; b(i), \quad l_n \le i \le r_n \tag{4}$$

with $n \in \mathbb{N}$. The aim is that, assuming invertibility of A (i.e. unique solvability of (3) for all $b \in X$), also (4) shall be uniquely solvable for all sufficiently large n and the solutions $x_n \in X_{l_n,r_n}$ shall remain bounded in n and converge componentwise[1] to the solution x of (3). If the latter is the case for all right-hand sides $b \in X$ then we say that the *finite section method (short: FSM)* with cut-offs at (l_n) and (r_n) *is applicable* for A.

If (a_{ij}) is a band matrix, i.e. $a_{ij} = 0$ for $|i - j| > d$ with some $d \in \mathbb{N}$ (as is the case for our operators (2)), then the FSM is applicable if and only if A is invertible and the sequence $(A_n) := (P_{l_n,r_n} A P_{l_n,r_n})_{n\in\mathbb{N}}$ is *stable* [46]. By the latter we mean that there exists a $n_0 \in \mathbb{N}$ such that $A_n : X_{l_n,r_n} \to X_{l_n,r_n}$ is invertible for all $n \ge n_0$ and $\sup_{n\ge n_0} \|A_n^{-1}\| < \infty$.

In the case $l_n = -n,\ r_n = n$ we will speak of the *full FSM* for A. Recently, it has been shown in different situations that (and how) applicability of the FSM can be established by choosing the sequences (l_n) and (r_n) accordingly [32, 41, 42] if the full FSM is not applicable. We will formulate a condition for applicability of the FSM in the case of general sequences (l_n) and (r_n) and then apply this result to the case of pseudoergodic operators (2). For semi-infinite systems on $X = \ell^p(\mathbb{N})$, replace $\mathbb{Z}$ by $\mathbb{N}$ and l_n by 1 in all of the above.

Motivation. A major motivation for the study of random Jacobi operators, their spectra and the solutions of the corresponding operator equations comes from condensed matter physics: Questions about the conductivity of certain (composed, disordered) media, about flux lines in superconductors or about systems of asymmetricly hopping particles have been modeled by random Schrödinger operators (Anderson model [1, 2]), non-selfadjoint versions (Hatano & Nelson [21, 22, 23]) and other non-selfadjoint random Jacobi operators (Feinberg & Zee [16, 17]). Similar models arise in population biology [35]. Besides such discrete models also continuous problems that have been described by a stochastic differential equation in 1D lead, after suitable discretization, to a matrix equation of the kind studied here.

We give some upper and lower bounds on the spectrum of our operators but mainly focus on the approximate solution of operator equations $Ax = b$ via the FSM. The latter can however be useful for spectral studies again: The inverse power method for the computation of the eigenvalue of A that is closest to a given point $z \in \mathbb{C}$ approximates the (in modulus) largest eigenvalue of $(A - zI)^{-1}$ by repeatedly solving equations $(A - zI)x^{(n+1)} = x^{(n)}$, $n = 0, 1, ...$, with a rather arbitrary (non-zero) initial vector $x^{(0)}$.

Historic remarks. The idea of the FSM is so natural that it is difficult to give a historical starting point. First rigorous treatments are from Baxter [3] and Gohberg & Feldman [18] on Wiener-Hopf and convolution operators in dimension $N = 1$ in the early 1960's. For convolution equations in higher dimensions $N \ge 2$, the FSM goes back to Kozak & Simonenko [25, 26], and for general band-dominated operators with scalar [37] and operator-valued [38, 39] coefficients,

[1] For $p \in (1, \infty)$ this is equivalent [40] to convergence of the solutions x_n (extended by zero) to x in $X = \ell^p(\mathbb{Z})$.

most results are due to Rabinovich, Roch & Silbermann. For the state of the art in the scalar case for $p = 2$, see [45]. The quest for stable subsequences if the full FSM itself is instable is getting more attention recently [41, 42, 48, 49, 32]. In [42], the stability theorem for subsequences is used to simplify the criterion in dimension $N = 1$ by removing a uniform boundedness condition. However, we are not aware of a rigorous treatment of random (or pseudoergodic) operators via the finite section method.

2 Main results

2.1 Notations

We first need some geometric notations: For sets $S, T \subseteq \mathbb{C}$ we put $S+T := \{s+t : s \in S,\, t \in T\}$ and we write $s + T := \{s\} + T$ and $sT := \{st : t \in T\}$ if $s \in \mathbb{C}$. By $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$ we denote the unit circle, the unit disk and its closure. So, for example, $S + \varepsilon\overline{\mathbb{D}}$ is the closed ε-neighborhood of $S \subseteq \mathbb{C}$ with $\varepsilon > 0$.

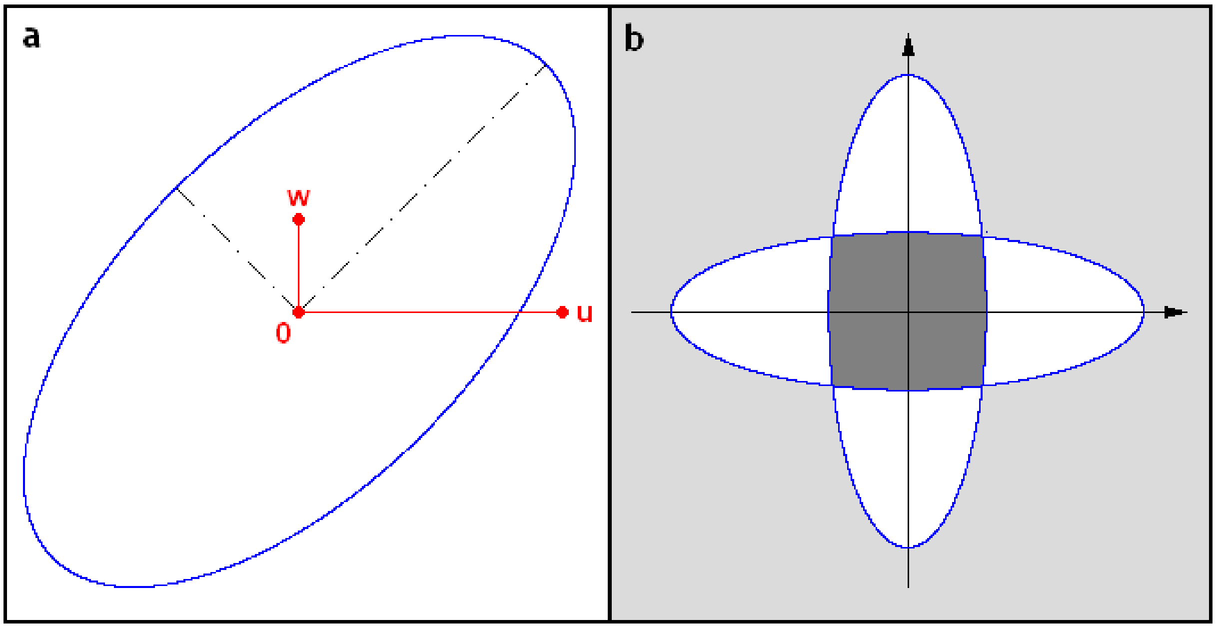

Figure 2.1: a) This is the ellipse $E(u, w)$ with $u = 3$ and $w = \mathrm{i}$. The major axis of the ellipse bisects the angle between u and w at the origin. The half-axes (dotted lines) have length $|3| \pm |\mathrm{i}|$, i.e. 4 and 2.
b) We see $E_+(U, W)$ in dark gray and $E_-(U, W)$ in light gray for $U = \{2\}$ and $W = \{-1, 1\}$. $E(U, W)$ is the union of the two ellipses $E(2, -1)$ and $E(2, 1)$. In Sections 2.2 and 2.3 we show that, for $A \in \Psi\mathrm{E}(U, V, W)$ to be Fredholm (and hence invertible) it is necessary that either $V \subseteq E_+(U, W)$, in which case $\operatorname{ind} A_+ = -1$, or $V \subseteq E_-(U, W)$, in which case $\operatorname{ind} A_+ = 0$. On the other hand, if $V \subset \mathbb{D} \subset E_+(U, W)$ then A is invertible.

For $u, w \in \mathbb{C}$, put

$$E(u, w) \;:=\; \left\{\, v \in \mathbb{C} \;:\; |v + 2\sqrt{uw}| + |v - 2\sqrt{uw}| \;=\; 2(|u| + |w|) \,\right\}, \tag{5}$$

which is the ellipse that is centered at 0, has half-axes of length $|u| + |w|$ and $\big|\,|u| - |w|\,\big|$ and focal points $\pm 2\sqrt{uw}$ (so that the major axis of $E(u, w)$ bisects the angle between u and w at the origin). By $E_+(u, w)$ and $E_-(u, w)$ we denote the bounded (interior) and the unbounded (exterior) component of $\mathbb{C} \setminus E(u,w)$, respectively. Now, for non-empty $U, W \subset \mathbb{C}$, let

$$E(U,W) := \bigcup_{\substack{u \in U \\ w \in W}} E(u,w) \qquad \text{and} \qquad E_\pm(U,W) := \bigcap_{\substack{u \in U \\ w \in W}} E_\pm(u,w).$$

Note that $E(u,w) = -E(u,w)$ for all $u, w \in \mathbb{C}$, so that also $E(U,W) = -E(U,W)$ and $E_\pm(U,W) = -E_\pm(U,W)$ hold (see Figure 2.1 for an example).

2.2 Spectrum and essential spectrum

Let $U, V, W \subset \mathbb{C}$ be non-empty and compact sets, and recall (1). Then we have the following result about spectrum and essential spectrum of our pseudoergodic operators (2):

Theorem 2.1 a) *For $A \in \Psi\mathrm{E}(U,V,W)$ and $A_+ \in \Psi\mathrm{E}_+(U,V,W)$, the following holds:*

$$\begin{aligned}
V + E(U,W) \;&\subseteq \\
\bigcup_{B \in M(U,V,W)} \operatorname{spec} B \;&=\; \bigcup_{B \in M(U,V,W)} \operatorname{spec}_{\mathrm{pt}}^{\infty} B \;=\; \bigcup_{B \in M(U,V,W)} \operatorname{spec}_{\mathrm{ess}} B \;=\; \bigcup_{B_+ \in M_+(U,V,W)} \operatorname{spec}_{\mathrm{ess}} B_+ \\
&=\; \operatorname{spec}_{\mathrm{ess}} A \;=\; \operatorname{spec} A \;=\; \operatorname{spec}_{\mathrm{ess}} A_+ \;\subseteq\; \operatorname{spec} A_+ \\
&\subseteq\; V + (u^* + w^*)\overline{\mathbb{D}}
\end{aligned} \tag{6}$$

b) *The upper bound* $\operatorname{spec} A \subseteq V + (u^* + w^*)\overline{\mathbb{D}}$ *from (6) holds for arbitrary $A \in M(U,V,W)$ (as well as for semi-infinite and finite matrices A). For bi-infinite matrices $A \in M(U,V,W)$ we can improve this upper bound on* $\operatorname{spec} A$ *under one of the following conditions:*

$$\text{if } u_* > w^*, \text{ i.e. } |u| > |w| \ \forall u \in U, w \in W, \text{ then } \operatorname{spec} A \subset \mathbb{C} \setminus \bigcap_{v \in V} \Big(v + (u_* - w^*)\mathbb{D} \Big),$$

$$\text{if } u^* < w_*, \text{ i.e. } |u| < |w| \ \forall u \in U, w \in W, \text{ then } \operatorname{spec} A \subset \mathbb{C} \setminus \bigcap_{v \in V} \Big(v + (w_* - u^*)\mathbb{D} \Big),$$

where, in addition to (1), we define $u_* := \min_{u \in U} |u|$ and $w_* := \min_{w \in W} |w|$.

We see from (6) that the spectrum of A and the essential spectrum of A and A_+ only depend on the sets U, V, W but not on the pseudoergodic operators A and A_+. So all operators in $\Psi\mathrm{E}(U,V,W)$ have the same (essential) spectrum. In particular, in the case of random operators, $\operatorname{spec}_{\mathrm{ess}} A$, $\operatorname{spec} A$ and $\operatorname{spec}_{\mathrm{ess}} A_+$ do not depend on the distributions of the random variables for sub-, main- and superdiagonal – only on their supports U, V, W. (Note that none of the above applies to $\operatorname{spec} A_+$; this set does depend on the concrete operator A_+.)

Example 2.2 a) Anderson model. In [1, 2], the conductivity of 1D disordered media was studied. Here $U = W = \{1\}$ and $V \subset \mathbb{R}$, so that A is a discrete Schrödinger operator with random potential. In this case $E(U,W) = [-2,2]$. So our lower and upper bound from Theorem 2.1 are $V + [-2,2]$ and $V + 2\overline{\mathbb{D}}$. Together with $\operatorname{spec} A \subset \mathbb{R}$, by selfadjointness, we get that the lower bound is also an upper bound, whence all sets in (6) are equal to $V + [-2,2]$.

b) Hatano & Nelson. The so-called non-selfadjoint Anderson model was introduced in [21, 22, 23] for the study of flux lines in type II superconductors under the influence of a tilted external magnetic field. Here $U = \{e^g\}$, $V = [-a,a]$ and $W = \{e^{-g}\}$, where a and g (the strength of the magnetic field) are positive real parameters. Now $E := E(U,W)$ has half-axes of length $e^g \pm e^{-g}$ being part of the real and imaginary axis, and E gets closer to a circle as

$g \to \infty$. Abbreviate $e^g + e^{-g} = 2\cosh g =: c$ and $e^g - e^{-g} = 2\sinh g =: s$. If V is (at least) as long as the major axis of E, i.e. if $a \geq c$, then $V + E$ and $V + c\overline{\mathbb{D}}$, which are the lower and upper bound in (6), only differ by $2e^{-g}$ in Hausdorff distance. It is easy to see that the closed numerical range of A (which is always an upper bound on $\operatorname{spec} A$) is contained in $V + \operatorname{conv}(E)$, which is equal to $V + E$ if and only if $a \geq c$, so that all sets in (6) are equal to $V + E$ in this case (cf. [12]). If $a < c$ then the lower bound $V + E$ has a hole around the origin, so that the best we can say then is $V + E \subseteq \operatorname{spec} A \subseteq V + \operatorname{conv}(E)$. However, if V is shorter than the short axis of E, i.e. if $a < s$, then statement b) of the theorem proves that there is indeed a hole in $\operatorname{spec} A$: Since $u_* = e^g > e^{-g} = w^*$, we have that $\cap_{v \in V}(v + (u_* - w^*)\mathbb{D}) = (-a + s\mathbb{D}) \cap (a + s\mathbb{D}) \neq \varnothing$ is in the resolvent set of A. We summarize these bounds on $\operatorname{spec} A$ in case $a < s$ in Figure 4.1 below. A further study of the shape and size of the hole in $\operatorname{spec} A$ is in [12, 13, 14, 33, 34].

c) Feinberg & Zee. In [16, 17, 24] the case $U = \{1\}$, $V = \{0\}$, $W = \mathbb{T}$ is studied. A simple computation shows that then $E(U, W) = 2\overline{\mathbb{D}} = (u^* + w^*)\overline{\mathbb{D}}$ holds, so that all sets in (6) coincide. In the same papers the much more complicated case with $W = \{\pm 1\}$ is also studied. In this case, $E(U, W) = [-2, 2] \cup [-2\mathrm{i}, 2\mathrm{i}]$ is far away from $(u^* + w^*)\overline{\mathbb{D}} = 2\overline{\mathbb{D}}$ (see [6, 7, 8, 24] for sharper bounds in this case). $\square$

Remark 2.3 The upper and lower bound in (6) might create the impression that spectrum and essential spectrum of A can be written as $V + S(U, W)$ with a set $S(U, W)$ independent of V, in which case it would be sufficient to study the case $V = \{0\}$. To see that this is not true, compare the cases $U \times V \times W = \{0\} \times \{\pm 1\} \times \{1\}$ (see [50, 31]) and $U \times V \times W = \{0\} \times \{0\} \times \{1\}$: In the first case one has $S(U, W) = \overline{\mathbb{D}}$, whereas in the second case, $S(U, W) = \mathbb{T}$. $\square$

Both upper and lower bound in Theorem 2.1 can be improved: The lower bound comes from evaluating $\cup \operatorname{spec} B$ with the union taken over all $B \in M(U, V, W)$ that have constant diagonals. A better lower bound can be derived in concrete examples by also considering operators $B \in M(U, V, W)$ with diagonals of period 2, 3 or more (e.g. [6, 8, 12, 31]). The upper bound can be improved by different approaches such as (higher order) numerical ranges or hulls [14, 15] or by the more recent ideas of [7].

2.3 Fredholmness, invertibility and the full FSM

Here are our results on invertibility, Fredholm property, and applicability of the full FSM.

Theorem 2.4 *If $A \in \Psi\mathrm{E}(U, V, W)$ or $A_+ \in \Psi\mathrm{E}_+(U, V, W)$ is Fredholm then $V \cap E(U, W) = \varnothing$. In fact, either*

(a) $V \subseteq E_-(U, W)$, *or*
(b) $V \subseteq E_+(U, W)$ *and* $u_* > w^*$, *or*
(c) $V \subseteq E_+(U, W)$ *and* $u^* < w_*$,

where, in addition to (1), we define $u_* := \min_{u \in U} |u|$ and $w_* := \min_{w \in W} |w|$.

The three cases correspond to the Fredholm index of A_+ (the so-called plus-index of A): (a) $\iff \operatorname{ind} A_+ = 0$, (b) $\iff \operatorname{ind} A_+ = -1$ and (c) $\iff \operatorname{ind} A_+ = 1$.

Note that, while the index of A_+ can be -1, 0 or 1, the index of A is always zero if A is Fredholm; in fact, A is always invertible if Fredholm (see (6) or the following theorem).

Theorem 2.5 *Let $U, V, W \subset \mathbb{C}$ be non-empty and compact. For $A \in M(U, V, W)$, we look at the following statements:*

(i) *A is a Fredholm operator,*
(ii) *A is invertible,*
(iii) *the full FSM is applicable to A,*
(iv) *all operators in $M(U, V, W)$, $M_+(U, V, W)$ and $M_{\rm fin}(U, V, W)$ are invertible and their inverses are uniformly bounded from above,*
(v) *the full FSM is applicable to all operators in $M(U, V, W)$,*
(vi) *all $B \in M(U, V, W)$ are invertible,*
(vii) *all $B \in M(U, V, W)$ are Fredholm operators.*

a) *For general $A \in M(U, V, W)$, the following implications trivially hold:*

$$\begin{array}{ccccccc} (i) & \Leftarrow & (ii) & \Leftarrow & (iii) & \Leftarrow & \\ \Uparrow & & \Uparrow & & \Uparrow & & (iv) \\ (vii) & \Leftarrow & (vi) & \Leftarrow & (v) & \Leftarrow & \end{array}$$

b) *If $A \in \Psi\mathrm{E}(U, V, W)$ then (i), (ii), (vi) and (vii) are equivalent and (iii) $\Longleftrightarrow$ (v) holds.*

c) *If $A \in \Psi\mathrm{E}(U, V, W)$ and $0 \in U, W$ then (i) − (vii) are all equivalent.*

d) *If $\delta > 0$ in (1) then (i) − (vii) hold, where all the inverses are bounded above by $1/\delta$.*

Remark 2.6 **a)** Statement b) of the theorem shows how 'hard' it is for a pseudoergodic operator to be Fredholm (i.e. invertible) or to even have an applicable full FSM. It also shows that, like the (essential) spectrum, these properties only depend on the sets U, V, W but not on the concrete operator $A \in \Psi\mathrm{E}(U, V, W)$.

b) If U, V, W are discrete sets and $0 \in U, W$ then $A \in \Psi\mathrm{E}(U, V, W)$ decouples into a block diagonal operator $A = \mathrm{Diag}(B_i : i \in \mathbb{Z})$ with every $B \in M_{\rm fin}(U, V, W)$ appearing as one of the blocks B_i, so that some of the above claims in c) become fairly obvious then. However, note that we do not assume U, V, W to be discrete in Theorem 2.5.

c) The condition $\delta > 0$ in d) is equivalent to $V \subset \mathbb{C} \setminus (u^* + w^*)\overline{\mathbb{D}}$ or, to phrase it in the style of (6), to $0 \notin V + (u^* + w^*)\overline{\mathbb{D}}$. □

Since applicability of the full FSM of an operator $A \in \Psi\mathrm{E}(U, V, W)$ is determined by the sets U, V, W only rather than by the operator, it seems advisable to use a version of the more flexible FSM (4) that gives credit to individual features of the concrete pseudoergodic operator A and will work under the sole condition of invertibility of A, where the full FSM might fail.

We say 'might fail' because we do not have an example of sets U, V, W and $A \in \Psi\mathrm{E}(U, V, W)$, where the full FSM fails while the following version applies, unless when $\mathrm{ind}\, A_+ \neq 0$. However, we can prove that our adapted FSM from Section 2.4 generally applies if A is invertible – which we doubt in the case of the full FSM (even if $\mathrm{ind}\, A_+ = 0$).

2.4 The FSM with adaptive cut-off intervals

If Theorem 2.5 does not yield applicability of the full FSM for $A \in \Psi\mathrm{E}(U, V, W)$ then we propose using the FSM (4) with cut-offs at integer values (l_n) and (r_n) that are adapted to the operator A at hand.

The adaptive FSM in the general case. We start with a statement for general tridiagonal operators $A \in M(U, V, W)$ or, in fact, for even more general operators. For $X = \ell^p(\mathbb{Z})$ with $p \in [1, \infty]$, we write $A \in BO(X)$ and call A a *band operator* if A acts via matrix-vector multiplication by a band matrix. Moreover, we write $A \in BDO(X)$ and call A a *band-dominated operator* if A is the limit (in the operator norm induced by $\|\cdot\|_X$) of a sequence of band operators.

If $A \in BDO(X)$ is given by the matrix $(a_{ij})_{i,j\in\mathbb{Z}}$ and $B \in BDO(X)$ is given by a matrix $(b_{ij})_{i,j\in\mathbb{Z}}$ then we call B a *limit operator* of A if there exists a sequence $h = (h_1, h_2, ...)$ of integers with $|h_n| \to \infty$ and $a_{i+h_n,j+h_n} \to b_{ij}$ as $n \to \infty$ for all $i, j \in \mathbb{Z}$. In this case we write $B =: A_h$. For a given sequence h of integers going to infinity, let

$$\sigma_h^{\mathsf{op}}(A) \;:=\; \{A_g : g \text{ is an infinite subsequence of } h \text{ for which } A_g \text{ exists}\}.$$

By a Bolzano-Weierstrass argument it can be seen that $\sigma_h^{\mathsf{op}}(A)$ is always nonempty. We will also abbreviate

$$\sigma_+^{\mathsf{op}}(A) \;:=\; \sigma_{(1,2,3,...)}^{\mathsf{op}}(A) \qquad \text{and} \qquad \sigma_-^{\mathsf{op}}(A) \;:=\; \sigma_{(-1,-2,-3,...)}^{\mathsf{op}}(A)$$

and put $\sigma^{\mathsf{op}}(A) := \sigma_+^{\mathsf{op}}(A) \cup \sigma_-^{\mathsf{op}}(A)$, so that the latter is the set of all limit operators of A.

In the semi-infinite case $X = \ell^p(\mathbb{N})$, the spaces $BO(X)$ and $BDO(X)$ are defined in the same way. For $A \in BDO(X)$ one then also defines limit operators A_h exactly as above – provided that $h = (h_1, h_2, ...)$ tends to $+\infty$. Note that A_h is in any case bi-infinite, i.e. it acts on $\ell^p(\mathbb{Z})$.

The set of pseudoergodic operators can be equivalently characterized in terms of limit operators (see [28, §3.4.10] or [30, §5.5.3]):

Lemma 2.7 *For an operator $A \in M(U, V, W)$, one has*

$$A \;\in\; \left\{ \begin{array}{ccc} \Psi\mathrm{E_L}(U,V,W) & \Longleftrightarrow & \sigma_-^{\mathsf{op}}(A) \\ \Psi\mathrm{E_R}(U,V,W) & \Longleftrightarrow & \sigma_+^{\mathsf{op}}(A) \\ \Psi\mathrm{E}(U,V,W) & \Longleftrightarrow & \sigma^{\mathsf{op}}(A) \end{array} \right\} \;=\; M(U,V,W).$$

Let $X = \ell^p(\mathbb{Z})$ and $P : X \to X$ denote the operator of multiplication by the characteristic function of $\mathbb{N}$, and let $Q := I - P$ be the complementary projector of P. Given an operator $A \in BDO(X)$ with matrix $(a_{ij})_{i,j\in\mathbb{Z}}$, we write A_+ for the compression $PAP|_{\mathrm{im}\, P}$ of A to $\mathrm{im}\, P \cong \ell^p(\mathbb{N})$; that is, A_+ is the operator of multiplication by the matrix $(a_{ij})_{i,j\in\mathbb{N}}$. Analogously, we write A_- for the compression $QAQ|_{\mathrm{im}\, Q}$ of A to $\mathrm{im}\, Q \cong \ell^p(\mathbb{Z} \setminus \mathbb{N})$; that is, A_- is the operator of multiplication by the matrix $(a_{ij})_{i,j\in\mathbb{Z}\setminus\mathbb{N}}$. When talking about their invertibility, Fredholmness or index, we always understand A_+ and A_- as operators on $\ell^p(\mathbb{N})$, resp. $\ell^p(\mathbb{Z} \setminus \mathbb{N})$. If we are only interested in an operator on $\ell^p(\mathbb{N})$, we usually denote it by A_+ (indicating that it is the compression of an operator A on $\ell^p(\mathbb{Z})$ to the positive half-axis) to remind ourselves of the semi-infinite setting.

The following theorem is a generalization of results from [42, 32] (which can be derived by straightforward changes in the proofs there):

Theorem 2.8 *Let $X = \ell^p(\mathbb{Z})$ with $p \in [1, \infty]$ and fix two sequences $l = (l_n)_{n\in\mathbb{N}}$ and $r = (r_n)_{n\in\mathbb{N}}$ of integers $l_1, l_2, ... \to -\infty$ and $r_1, r_2, ... \to +\infty$. For $A \in BDO(X)$, the finite section method (4) is applicable if and only if the following operators are invertible:*

$$A, \qquad \text{all operators } B_+ \text{ with } B \in \sigma_l^{\mathsf{op}}(A), \qquad \text{all operators } C_- \text{ with } C \in \sigma_r^{\mathsf{op}}(A). \qquad (7)$$

The set of operators (7) is particularly handy if the sequences l and r are such that

$$\text{the sets } \{B_+ : B \in \sigma_l^{\mathsf{op}}(A)\} \text{ and } \{C_- : C \in \sigma_r^{\mathsf{op}}(A)\} \text{ are singletons,} \tag{8}$$

which is equivalent to the existence of the strong limits B_+ and C_- of

$$\begin{pmatrix} v_{l_n} & w_{l_n} & & \\ u_{l_n+1} & v_{l_n+1} & w_{l_n+1} & \\ & u_{l_n+2} & v_{l_n+2} & \ddots \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \ddots & \ddots & & \\ \ddots & v_{r_n-2} & w_{r_n-2} & \\ & u_{r_n-1} & v_{r_n-1} & w_{r_n-1} \\ & & u_{r_n} & v_{r_n} \end{pmatrix}$$

as $n \to \infty$. For (8) it is sufficient (but not necessary) that the limit operators $A_l =: B$ and $A_r =: C$ exist.

Here is the version of Theorem 2.8 for semi-infinite matrices:

Theorem 2.9 *Let $X = \ell^p(\mathbb{N})$ with $p \in [1,\infty]$ and fix a monotonously increasing sequence $r = (r_n)_{n\in\mathbb{N}}$ of positive integers. For $A_+ \in BDO(X)$, the finite section method (4), with $l_n = 1$ for all $n \in \mathbb{N}$, is applicable if and only if the following operators are invertible:*

$$A_+, \qquad \textit{all operators } C_- \textit{ with } C \in \sigma_r^{\mathsf{op}}(A_+). \tag{9}$$

Also here, the set (9) is smallest possible if $\{C_- : C \in \sigma_r^{\mathsf{op}}(A_+)\}$ is a singleton.

Bi-infinite bi-pseudoergodic systems. We demonstrate how, under the sole (and for this purpose minimal – because necessary) assumption of invertibility of A, one can approximately solve operator equations $Ax = b$ on $\ell^p(\mathbb{Z})$ with a bi-pseudoergodic operator A by the finite section method.

Algorithm 2.10 – The ΨE$_2$-FSM. *Suppose $U, V, W \subset \mathbb{C}$ are non-empty and compact sets, $p \in [1,\infty]$, $A \in \Psi\mathrm{E}_2(U,V,W)$ is invertible and $b \in \ell^p(\mathbb{Z})$ is given.*

Step 1. *Pick some arbitrary $u \in U$, $v \in V$ and $w \in W$. Choose integer sequences $l_1, l_2, ...$ monotonically decreasing and $r_1, r_2, ...$ monotonically increasing such that $l_n \le r_n$ and*

$$|u_i - u| + |v_i - v| + |w_i - w| < \frac{1}{n}, \qquad \forall i \in \{l_n, l_n+1, ..., l_n+n\} \cup \{r_n - n, ..., r_n - 1, r_n\}$$

for $n = 1, 2, ...$.

Step 2. *By Theorem 2.4, we know that we are in one of the three cases (a), (b), (c). To find out which of these cases applies, compute*

$$|v + 2\sqrt{uw}| + |v - 2\sqrt{uw}| - 2(|u| + |w|) \begin{cases} > 0 & \Rightarrow \ \textit{case } (a), \\ < 0 : & \textit{compute } |u| - |w| \begin{cases} > 0 & \Rightarrow \ \textit{case } (b), \\ < 0 & \Rightarrow \ \textit{case } (c). \end{cases} \end{cases}$$

If one of the expressions to be computed here is zero or if the outcome of this algorithm depends on the choice of u, v, w then A is not Fredholm, let alone invertible, by Theorem 2.4.

Step 3. *Depending on case (a), (b) or (c), we apply our finite section method (with cut-offs at (l_n) and (r_n) as chosen in step 1) to different equations:*

In case (a), the FSM (4) *is applicable to the equation $Ax = b$, i.e. to*

$$
\left(\begin{array}{ccccccc}
\ddots & \ddots & & & & & \\
\ddots & v_{-2} & w_{-2} & & & & \\
 & u_{-1} & v_{-1} & w_{-1} & & & \\
\hline
 & & u_0 & v_0 & w_0 & & \\
\hline
 & & & u_1 & v_1 & w_1 & \\
 & & & & u_2 & v_2 & \ddots \\
 & & & & & \ddots & \ddots
\end{array}\right)
\begin{pmatrix} \vdots \\ x(-2) \\ x(-1) \\ x(0) \\ x(1) \\ x(2) \\ \vdots \end{pmatrix}
=
\left(\begin{array}{c} \vdots \\ b(-2) \\ b(-1) \\ \hline b(0) \\ \hline b(1) \\ b(2) \\ \vdots \end{array}\right).
$$

In case (b), the FSM (4) *is applicable to the following upward-translated (obviously equivalent) system*

$$
\left(\begin{array}{ccccccc}
\ddots & \ddots & \ddots & & & & \\
 & u_{-1} & v_{-1} & w_{-1} & & & \\
\hline
 & & u_0 & v_0 & w_0 & & \\
\hline
 & & & u_1 & v_1 & w_1 & \\
 & & & & u_2 & v_2 & \ddots \\
 & & & & & u_3 & \ddots \\
 & & & & & & \ddots
\end{array}\right)
\begin{pmatrix} \vdots \\ x(-2) \\ x(-1) \\ x(0) \\ x(1) \\ x(2) \\ \vdots \end{pmatrix}
=
\left(\begin{array}{c} \vdots \\ b(-1) \\ \hline b(0) \\ \hline b(1) \\ b(2) \\ b(3) \\ \vdots \end{array}\right).
$$

Finally, in case (c), the FSM (4) *is applicable to the following downward-translated (obviously equivalent) system*

$$
\left(\begin{array}{ccccccc}
\ddots & & & & & & \\
\ddots & w_{-3} & & & & & \\
\ddots & v_{-2} & w_{-2} & & & & \\
 & u_{-1} & v_{-1} & w_{-1} & & & \\
\hline
 & & u_0 & v_0 & w_0 & & \\
\hline
 & & & u_1 & v_1 & w_1 & \\
 & & & & \ddots & \ddots & \ddots
\end{array}\right)
\begin{pmatrix} \vdots \\ x(-2) \\ x(-1) \\ x(0) \\ x(1) \\ x(2) \\ \vdots \end{pmatrix}
=
\left(\begin{array}{c} \vdots \\ b(-3) \\ b(-2) \\ b(-1) \\ \hline b(0) \\ \hline b(1) \\ \vdots \end{array}\right).
$$

Remark 2.11 – The growth of the intervals $\{l_n, ..., r_n\}$ as $n \to \infty$.

a) If all entries u_i, v_i and w_i of A are independent samples from three uniformly distributed random variables with values (everywhere) in U, V and W then, for all choices u, v, w in step 1, one expects the same exponential growth of $-l_n$ and r_n. For example, if U, V, W are finite with $|U \times V \times W| = m$ then $-l_n$ and r_n are of order m^n. (In the random but not uniformly distributed case, it is certainly advisable to pick some of the more likely u, v, w in step 1 in order to minimize the growth of $-l_n$ and r_n.)

b) The choice of l_n and r_n in step 1 is such that the conditions of Theorem 2.8 and in particular condition (8) are met with operators B_+ and C_- having constant diagonals (containing u, v and w). It is possible to aim at different (non-Toeplitz) operators B_+ and C_- here (as long as they are invertible) via the choice of l_n and r_n, while, possibly, keeping the growth of $-l_n$ and r_n more moderate. Steps 2 (with arbitrarily picked u, v, w) and 3 still remain as shown.

c) One should not be too worried if these finite systems become large very quickly since they can be solved in linear time (as opposed to cubic, for the Gauss algorithm). In case (a) this is done by the so-called Thomas algorithm [11], while in case (b), resp. (c), the solution is calculated successively via backward, resp. forward, substitution.

d) In [41] the finite section method (4) is adapted to the Almost Mathieu operator

$$(Ax)_n \;=\; x_{n-1} \;+\lambda\cos\big(2\pi(n\alpha+\theta)\big)\,x_n \;+\; x_{n+1}, \qquad n\in\mathbb{Z}$$

by putting $-l_n = r_n$ equal to the denominator of the n-th continued fraction approximant of the irrational number $\alpha \in (0,1)$. Note that this means $-l_n = r_n$ also grow exponentially in n. For example, if $\alpha = (\sqrt{5}-1)/2$ is the golden mean then $-l_n = r_n$ is the n-th Fibonacci number. The order of exponential growth is higher if the continued fraction expansion of α contains larger numbers. (For the golden mean, it is $1/(1+1/(1+1/\cdots))$.) $\square$

Semi-infinite pseudoergodic systems. For semi-infinite systems $A_+x = b$ on $\ell^p(\mathbb{N})$, the situation is related but much simpler. Again, we only assume invertibility of the operator.

Algorithm 2.12 – The ΨE_+-FSM. *Suppose $U, V, W \subset \mathbb{C}$ are non-empty and compact sets, $p \in [1,\infty]$, $A_+ \in \Psi E_+(U,V,W)$ is invertible and $b \in \ell^p(\mathbb{N})$ is given.*

Step 1. *Pick some arbitrary $u \in U$, $v \in V$ and $w \in W$. Choose a monotonically increasing sequence $r_1, r_2, ...$ of positive integers such that*

$$|u_i - u| \;+\; |v_i - v| \;+\; |w_i - w| \;<\; \frac{1}{n}, \qquad \forall i \in \{r_n - n, ..., r_n - 1, r_n\}$$

for $n = 1, 2, ...$.

Step 2. *By Theorem 2.4 and the invertibility of A_+, we are automatically in case (a).*

Step 3. *The FSM (4) with $l_n = 1$ for all $n \in \mathbb{N}$ and (r_n) as chosen in step 1 applies to our equation $A_+x = b$, i.e. to*

$$\begin{pmatrix} v_1 & w_1 & & & \\ u_2 & v_2 & w_2 & & \\ & u_3 & v_3 & w_3 & \\ & & u_4 & v_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} x(1)\\ x(2)\\ x(3)\\ x(4)\\ \vdots \end{pmatrix} = \begin{pmatrix} b(1)\\ b(2)\\ b(3)\\ b(4)\\ \vdots \end{pmatrix}.$$

As in Remark 2.11 b), note that one could choose $r = (r_n)$ in step 1 so that C_-, with $C \in \sigma_r^{\mathsf{op}}(A)$, is not of Toeplitz structure but is another (invertible) operator. For example, one could choose r_n such that

$$|u_{r_n-i} - u_{i+2}| \;+\; |v_{r_n-i} - v_{i+1}| \;+\; |w_{r_n-i} - w_i| \;<\; \frac{1}{n}, \qquad \forall i \in \{0, ..., n\}$$

for $n = 1, 2, ...$, so that

$$\begin{pmatrix} \ddots & \ddots & & \\ \ddots & v_{r_n-2} & w_{r_n-2} & \\ & u_{r_n-1} & v_{r_n-1} & w_{r_n-1} \\ & & u_{r_n} & v_{r_n} \end{pmatrix} \to \begin{pmatrix} \ddots & \ddots & & \\ \ddots & v_3 & w_2 & \\ & u_3 & v_2 & w_1 \\ & & u_2 & v_1 \end{pmatrix} =: C_- \tag{10}$$

strongly as $n \to \infty$. But C_- is invertible by our assumption on A_+: We call the matrix C_- in (10) the *reflection* of the operator A_+ from (2) and we will write A_+^R for C_-. Conversely, we also call A_+ the reflection of C_- and denote it by C_-^R. It is easy to see that a semi-infinite matrix is invertible if and only if its reflection is invertible.

2.5 Spectral and pseudospectral approximation

Here we will briefly discuss another feature of the finite section method with adaptive cut-off intervals. We will work exclusively on the Hilbert space $H := \ell^2(\mathbb{Z})$. Let again l and r be sequences of negative and positive integers which converge to $-\infty$ and $+\infty$, respectively. The set $\mathcal{A}_{l,r}$ of all band-dominated operators A on H for which the limit operators A_l and A_r exist is a C^*-algebra, as one easily checks. Note that every band-dominated operator A belongs to an algebra $\mathcal{A}_{l,r}$ with specified sequences l, r.

Proposition 2.13 *Let $A \in \mathcal{A}_{l,r}$. Then the adaptive finite section method (4) with system matrices $P_{l_n,r_n}AP_{l_n,r_n}$ is fractal.*

The notion of a *fractal approximation method* was introduced in [47]. Since already its definition makes heavily use of C^*-algebraic language we will omit all technical details here and refer the interested reader to [47] and [20]. Roughly speaking, an algebra of approximation sequences is fractal if every sequence in the algebra can be reconstructed from each if its (infinite) subsequences modulo a sequence which tends to zero in the norm. A single sequence like $(P_{l_n,r_n}AP_{l_n,r_n})_{n\in\mathbb{N}}$ is called *fractal* if the smallest C^*-algebra which contains this sequence and the sequence $(P_{l_n,r_n})_{n\in\mathbb{N}}$ has the fractal property. The proof of Proposition 2.13 follows easily from Theorem 2.8 above and Theorem 1.69 in [20]. The main point is that the sequence $(P_{l_n,r_n}AP_{l_n,r_n})_{n\in\mathbb{N}}$ is stable by Theorem 2.8 if and only if the operators A, $B_+ = PA_lP$ and $C_- = QA_rQ$ are invertible and that the operators A, B_+ and C_- are strong limits of (shifts of) the sequence $(P_{l_n,r_n}AP_{l_n,r_n})_{n\in\mathbb{N}}$. Since every subsequence has the same strong limits, the result follows from Theorem 1.69 in [20].

Fractal sequences are distinguished by their excellent convergence properties. To mention only a few of them, let $\sigma(A)$ denote the spectrum of an operator A, write $\sigma_2(A)$ for the set of the singular values of A, i.e., $\sigma_2(A)$ is the set of all non-negative square roots of elements in the spectrum of A^*A and finally, for $\varepsilon > 0$, let $\sigma^{(\varepsilon)}(A)$ refer to the ε-pseudospectrum of A, i.e. to the set of all $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is not invertible or $\|(A-\lambda I)^{-1}\| \geq 1/\varepsilon$. Let further

$$d_H(M,\, N) := \max\{\max_{m\in M}\min_{n\in N}|m-n|,\, \max_{n\in N}\min_{m\in M}|m-n|\}$$

denote the Hausdorff distance between the non-empty compact subsets M and N of the complex plane.

Theorem 2.14 *Let $A \in \mathcal{A}_{l,r}$ and $A_n := P_{l_n,r_n}AP_{l_n,r_n}$. Then the following sequences converge with respect to the Hausdorff distance as $n \to \infty$:*

(*a*) $\sigma(A_n) \to \sigma(A) \cup \sigma(B_+) \cup \sigma(C_-)$ *if A is self-adjoint;*

(*b*) $\sigma_2(A_n) \to \sigma_2(A) \cup \sigma_2(B_+) \cup \sigma_2(C_-)$*;*

(*c*) $\sigma^{(\varepsilon)}(A_n) \to \sigma^{(\varepsilon)}(A) \cup \sigma^{(\varepsilon)}(B_+) \cup \sigma^{(\varepsilon)}(C_-)$.

The proof follows immediately from the stability criterion in Theorem 2.8, from the fractality of the sequence (A_n) by Proposition 2.13, and from Theorems 3.20, 3.23 and 3.33 in [20]. Let us emphasize that in general one cannot remove the assumption $A = A^*$ in assertion (a), whereas (c) holds without any assumption. This observation is only one reason for the present increasing interest in pseudospectra. For detailed presentations of pseudospectra and their applications as well as of other spectral quantities see the monographs [4, 5, 20, 51] and the references therein.

2.6 The selfadjoint case

We discuss briefly how our results simplify when A is selfadjoint, i.e. $w_i = \overline{u_{i+1}}$ for all i in (2).

In that case, there are only two sets, U and V, of which V is real. Instead of the ellipses $E(u,w)$, one looks at $E(u,\overline{u}) = [-2|u|, 2|u|]$. The set $E(U,W)$ gets replaced by the union of $E(u,\overline{u})$ over all $u \in U$, which is simply $[-2u^*, 2u^*]$.

In Theorem 2.1 a), the lower bound therefore becomes $V + [-2u^*, 2u^*]$. But the upper bound becomes the same (recall Example 2.2 a) since $w^* = u^*$ and since all spectra are real. So

$$\operatorname{spec} A \;=\; \operatorname{spec}_{\mathrm{ess}} A \;=\; \operatorname{spec}_{\mathrm{ess}} A_+ \;=\; \operatorname{spec} A_+ \;=\; V + [-2u^*, 2u^*]. \tag{11}$$

In particular, A is positive definite if and only if $\min_{v \in V} > 2u^*$.

In Theorem 2.4 and anywhere else, case (a) applies (all indices are zero of course). The theorem says that V and $[-2u^*, 2u^*]$ are disjoint if A is Fredholm. From (11) we know that $V \cap [-2u^*, 2u^*] = \varnothing$ is indeed both necessary and sufficient for A to be Fredholm (i.e. invertible).

Concerning the FSM, not much simplification occurs apart from the fact that the full FSM is applicable if A is positive or negative definite.

3 Background theory and proofs

Before we come to the deeper results, let us briefly show how Fredholmness (and index) of A is related to that of its half-axis compressions A_+ and A_-. The following lemma is taken from [36, 43].

Lemma 3.1 *An operator $A \in BO(X)$ is Fredholm if and only if both its compressions A_+ and A_- are Fredholm, i.e.* $\operatorname{spec}_{\mathrm{ess}} A = \operatorname{spec}_{\mathrm{ess}} A_+ \cup \operatorname{spec}_{\mathrm{ess}} A_-$. *Moreover,* $\operatorname{ind} A = \operatorname{ind} A_+ + \operatorname{ind} A_-$.

Proof. Since PAQ and QAP are of finite rank if A is a band operator, one has that $A = PAP + PAQ + QAP + QAQ$ is equivalent, modulo compact operators, to

$$PAP + QAQ \;=\; (PAP + Q)(P + QAQ) \;=\; (P + QAQ)(PAP + Q).$$

So A is Fredholm if and only if $PAP + Q$ and $P + QAQ$, which are the extensions (by identity) of A_+ and A_- to $\ell^p(\mathbb{Z})$, are Fredholm. Moreover,

$$\begin{aligned}
\operatorname{ind} A &= \operatorname{ind}(PAP + PAQ + QAP + QAQ) \;=\; \operatorname{ind}(PAP + QAQ)\\
&= \operatorname{ind}(PAP + Q)(QAQ + P) \;=\; \operatorname{ind}(PAP + Q) + \operatorname{ind}(QAQ + P)\\
&= \operatorname{ind} A_+ + \operatorname{ind} A_-
\end{aligned}$$

holds. ■

One refers to $\operatorname{ind} A_-$ and $\operatorname{ind} A_+$ as the *minus*- and the *plus-index* of A. By Lemma 3.1, the problem of determining Fredholmness (and the index) of A splits into two subproblems. These two subproblems again split into many smaller problems, where the key notion is again that of a limit operator. Besides Lemma 2.7 and Theorems 2.8 and 2.9, limit operators feature in the following characterization of Fredholmness (including the index):

Theorem 3.2 *Let $X = \ell^p(\mathbb{I})$ with $p \in [1,\infty]$ and $\mathbb{I} \in \{\mathbb{Z},\mathbb{N},\mathbb{Z}\setminus\mathbb{N}\}$, and let $A \in BO(X)$.*

a) *The following are equivalent*

(i) *A is Fredholm on X,*

(ii) *all limit operators of A are invertible on $\ell^p(\mathbb{Z})$ [37, 40],*

(iii) *all limit operators of A are injective on $\ell^\infty(\mathbb{Z})$ [9, 10],*

so that, by applying the above to $A - \lambda I$ in place of A,

$$\operatorname{spec}_{\mathrm{ess}} A \;=\; \bigcup_{B\in\sigma^{\mathsf{op}}(A)} \operatorname{spec} B \;=\; \bigcup_{B\in\sigma^{\mathsf{op}}(A)} \operatorname{spec}_{\mathrm{pt}}^{\infty} B. \tag{12}$$

b) *If $A \in BO(\ell^p(\mathbb{Z}))$ is Fredholm then all operators in $\sigma^{\mathsf{op}}_-(A)$ have the same minus-index and all operators in $\sigma^{\mathsf{op}}_+(A)$ have the same plus-index, which also happen to be the minus- and the plus-index of A, respectively [36, 43]. This means* $\operatorname{ind} A = \operatorname{ind} A_- + \operatorname{ind} A_+$, *where*

$$\operatorname{ind} A_- \;=\; \operatorname{ind} B_- \qquad \forall\ B \in \sigma^{\mathsf{op}}_-(A), \tag{13}$$

$$\textit{and} \qquad \operatorname{ind} A_+ \;=\; \operatorname{ind} C_+ \qquad \forall\ C \in \sigma^{\mathsf{op}}_+(A). \tag{14}$$

Let S denote the shift operator $(Sx)(m) := x(m-1)$, $m \in \mathbb{Z}$, on $X = \ell^p(\mathbb{Z})$. If A is pseudoergodic then $\sigma^{\mathsf{op}}(A) = M(U,V,W)$, by Lemma 2.7. Particularly simple elements of $M(U,V,W)$ are operators whose matrix has constant diagonals. So fix $u \in U$, $v \in V$ and $w \in W$, and let $L(u,v,w) := uS + vI + wS^{-1}$ be the single element of $M(\{u\},\{v\},\{w\}) \subseteq M(U,V,W)$, which is a so-called *Laurent operator* (sometimes also called "bi-infinite Toeplitz operator"). It is a standard result [4, 5] that

$$\operatorname{spec} L(u,v,w) \;=\; \{ut^1 + vt^0 + wt^{-1} \,:\, t \in \mathbb{T}\} \;=\; v + E(u,w). \tag{15}$$

Together with (12), the latter proves the lower bound in Theorem 2.1. The rather crude (but still helpful) upper bound in Theorem 2.1 and statement d) in Theorem 2.5 rely on the following simple lemma and its corollary.

Lemma 3.3 *Let A be a finite or (semi- or bi-)infinite matrix with subdiagonal $\mathbf{u} = (u_i)$, main diagonal $\mathbf{v} = (v_i)$ and superdiagonal $\mathbf{w} = (w_i)$. Put*

$$u^* \,:=\, \sup_i |u_i|, \quad v_* \,:=\, \inf_i |v_i|, \quad w^* \,:=\, \sup_i |w_i|, \quad \textit{and} \quad \delta_A \,:=\, v_* - (u^* + w^*). \tag{16}$$

If $\delta_A > 0$ then A is invertible and $\|A^{-1}\| \le 1/\delta_A$.

Proof. Write $A = D + T$ with $D = \operatorname{diag}(v_i)$ and treat A as a perturbation of D. We have $A = D(I + D^{-1}T)$, where $D^{-1}T$ has subdiagonal entries $u_i v_i^{-1}$, superdiagonal entries $w_i v_i^{-1}$ and everything else zero. From $\delta_A > 0$ we get that

$$\|D^{-1}T\| \;\le\; \sup_i |u_i v_i^{-1}| + \sup_i |w_i v_i^{-1}| \;\le\; \frac{u^* + w^*}{v_*} \;<\; 1,$$

so that $I + D^{-1}T$ is invertible by Neumann series. But from $A^{-1} = (I + D^{-1}T)^{-1}D^{-1}$ it follows that also A is invertible and

$$\|A^{-1}\| \;\le\; \|(I + D^{-1}T)^{-1}\|\,\|D^{-1}\| \;\le\; \frac{1}{1 - \|D^{-1}T\|}\,\|D^{-1}\| \;\le\; \frac{1}{1 - \frac{u^*+w^*}{v_*}}\,\frac{1}{v_*} \;=\; \frac{1}{\delta_A},$$

as was claimed. ■

Corollary 3.4 *If $U, V, W \subset \mathbb{C}$ are non-empty and compact,* (1) *holds with $\delta > 0$, and if A is in $M(U, V, W)$ or $M_+(U, V, W)$ or $M_{\rm fin}(U, V, W)$ then A is invertible and $\|A^{-1}\| \le 1/\delta$.*

Proof. Just note that $\delta_A \ge \delta$ for δ_A from (16) and δ from (1) and apply Lemma 3.3. ■

Now we have all the machinery to prove our main results:

Proof of Theorem 2.1. a) All unions in this proof are taken over the set of all $B \in M(U, V, W)$. By (12) and Lemma 2.7,

$$\operatorname{spec}_{\rm ess} A \;=\; \cup \operatorname{spec}_{\rm pt}^{\infty} B \;=\; \cup \operatorname{spec} B \;\supseteq\; \cup \operatorname{spec}_{\rm ess} B \;\supseteq\; \operatorname{spec}_{\rm ess} A,$$

so that equality holds in both "$\supseteq$" signs. Moreover,

$$\operatorname{spec}_{\rm ess} A \;\subseteq\; \operatorname{spec} A \;\subseteq\; \cup \operatorname{spec} B \;=\; \operatorname{spec}_{\rm ess} A$$

holds since A is one of the operators B in this union. So again we have equality everywhere. Equality (12) also holds with A replaced by A_+. Hence, by $\sigma^{\mathsf{op}}(A_+) = M(U, V, W)$,

$$\operatorname{spec} A \;\supseteq\; \operatorname{spec}_{\rm ess} A \;\supseteq\; \operatorname{spec}_{\rm ess} A_+ \;=\; \cup \operatorname{spec} B \;\supseteq\; \operatorname{spec} A$$

holds, which proves the remaining equality in (6). The lower bound $V + E(U, W)$ in (6) now follows by evaluating $\operatorname{spec} B$ from (15) for all Laurent operators $B = L(u, v, w) \in M(U, V, W)$. The upper bound $V + (u^* + w^*)\overline{\mathbb{D}}$ follows from Corollary 3.4 since $A_+ - \lambda I_+$ is invertible if $|v - \lambda| > |u| + |w|$ for all $(u, v, w) \in U \times V \times W$, i.e. if $\operatorname{dist}(\lambda, V) > u^* + w^*$.

b) Let $A \in M(U, V, W)$, $u_* > w^*$ and suppose $\lambda \in \cap_{v \in V}(v + (u_* - w^*)\mathbb{D})$. Then $|v - \lambda| < u_* - w^* \le |u| - |w|$ for all $u \in U$, $v \in V$ and $w \in W$, so that the subdiagonal of $A - \lambda I$ dominates the other two diagonals. By a simple perturbation argument as above (see Lemma 3.3 and Corollary 3.4), $S^{-1}(A - \lambda I) = \operatorname{diag}(u_i)(I + T)$ with $\|T\| < 1$ is invertible, and hence $A - \lambda I$ is invertible. The argument for the case $w_* > u^*$ is completely symmetric. ■

Proof of Theorem 2.4. If A is Fredholm then all its limit operators B, including the Laurent operators $B := L(u, v, w) \in M(U, V, W)$, are invertible. So, for all $(u, v, w) \in U \times V \times W$, we have that $0 \notin \operatorname{spec} L(u, v, w) = v + E(u, w) = v - E(u, w)$, i.e. $v \notin E(u, w)$. The following three cases are possible:

(a) $\operatorname{wind}(E(u, w), v) = 0$, i.e. v is in the exterior of the ellipse $E(u, w)$, or
(b) $\operatorname{wind}(E(u, w), v) = 1$, i.e. v is encircled counter-clockwise by $E(u, w)$, or
(c) $\operatorname{wind}(E(u, w), v) = -1$, i.e. v is encircled clockwise by $E(u, w)$,

where $\operatorname{wind}(C, z)$ denotes the winding number of a closed oriented curve C w.r.t. a point $z \notin C$ and where the ellipse $E(u, w)$ is parametrized (and thereby oriented) by the map $\varphi \mapsto ue^{\mathrm{i}\varphi} + we^{-\mathrm{i}\varphi}$ from $[0, 2\pi)$ to $E(u, w)$. A simple computation shows that $E(u, w)$ is oriented counter-clockwise if $|u| > |w|$ and clockwise if $|u| < |w|$ (while the ellipse degenerates into a line segment if $|u| = |w|$). Let ϱ denote the rotation $z \mapsto \frac{v}{2} - z$ of the complex plane around $\frac{v}{2}$. For the Toeplitz operator B_+, one has (see e.g. [4, 5])

$$\begin{aligned} \operatorname{ind} B_+ &= -\operatorname{wind}(\operatorname{spec} B, 0) = -\operatorname{wind}(v + E(u, w), 0) = -\operatorname{wind}(\varrho(v + E(u, w)), \varrho(0)) \\ &= -\operatorname{wind}(-\tfrac{v}{2} - E(u, w), \tfrac{v}{2}) = -\operatorname{wind}(-E(u, w), v) = -\operatorname{wind}(E(u, w), v), \end{aligned}$$

which is 0 in case (a), -1 in case (b) and 1 in case (c). By (14), we have $\operatorname{ind} B_+ = \operatorname{ind} A_+$ for all $B \in \sigma_+^{\mathsf{op}}(A) = M(U, V, W)$, so that for all choices $(u, v, w) \in U \times V \times W$, the same case, (a), (b) or (c), applies – according to $\operatorname{ind} A_+$. ■

Proof of Theorem 2.5. a) The only implication that is not obvious here is that A is invertible if the full FSM of A is stable. This can be found in [46] (also see [40, 28]).

b) Now let $A \in \Psi\mathrm{E}(U,V,W)$. Properties (i), (ii), (vi) and (vii) are equivalent because (i) implies (vii) by Theorem 3.2 a) and Lemma 2.7. It remains to show that (iii) implies (v). By Theorem 2.8 with $l = (-1,-2,...)$ and $r = (1,2,...)$, property (iii) is equivalent to invertibility of A and all operators B_+ and C_- with $B \in \sigma^{\mathsf{op}}_-(A)$ and $C \in \sigma^{\mathsf{op}}_+(A)$. W.l.o.g suppose A is right-pseudoergodic, so that $\sigma^{\mathsf{op}}_+(A) = M(U,V,W)$ by Lemma 2.7 and hence C_- is invertible for all $C \in M(U,V,W)$. But then, for every $B \in M(U,V,W)$, also B_+ is invertible because its reflection B_+^R is of the form C_- for some $C \in M(U,V,W)$ and is therefore invertible. Finally, since every operator in $M(U,V,W)$ is invertible if A is invertible (see above), we conclude (v), by Theorem 2.8 again.

c) Now let $A \in \Psi\mathrm{E}(U,V,W)$ and $0 \in U,W$. To see that all properties (i)–(vii) are equivalent, it is sufficient to show that (i) implies (iv). So let A be Fredholm. By Theorem 3.2 a) we know that all limit operators B of A, which are all operators in $M(U,V,W)$ by Lemma 2.7, are invertible. Moreover, there is a $c > 0$ (e.g. the norm of a Fredholm regularizer of A, see [37]) such that $\|B^{-1}\| \le c$ for all these operators B. Now we can show (iv): If $D \in M(U,V,W)$ then $B := D \in M(U,V,W)$ is invertible and $\|D^{-1}\| \le c$. If $D_+ \in M_+(U,V,W)$ then $B := \mathrm{Diag}(D_+^R, D_+)$ is in $M(U,V,W)$ since $0 \in U,W$ and hence B is invertible, so that D_+ is invertible and $\|(D_+)^{-1}\| = \|B^{-1}\| \le c$. Finally, if $D \in M_{\mathrm{fin}}(U,V,W)$ then, since $0 \in U,W$, $B := \mathrm{Diag}(\cdots, D, D, D, \cdots) \in M(U,V,W)$ is invertible, so that D is invertible and $\|D^{-1}\| = \|B^{-1}\| \le c$.

d) If $\delta > 0$ then, by Corollary 3.4, property (iv) holds and hence all the others follow. ■

Proof of correctness of Algorithm 2.10. The choice of the sequences $l = (l_n)$ and $r = (r_n)$ in step 1 is such that the sets $\{D_+ : D \in \sigma^{\mathsf{op}}_l(A)\} =: \{B_+\}$ and $\{D_- : D \in \sigma^{\mathsf{op}}_r(A)\} =: \{C_-\}$ are singletons; in fact, B_+ and C_- are Toeplitz operators with diagonals u, v, w. It is possible to choose sequences l and r with these properties because $A \in \Psi\mathrm{E}_2(U,V,W)$.

The test in step 2 exactly follows the geometric definition (5) of the ellipse $E(u,w)$: If $|v+2\sqrt{uw}|+|v-2\sqrt{uw}| > 2(|u|+|w|)$ then $v \in E_-(u,w)$ and we are in case (a) of Theorem 2.4 (also see the proof of Theorem 2.4). If $|v+2\sqrt{uw}|+|v-2\sqrt{uw}| < 2(|u|+|w|)$ then $v \in E_+(u,w)$ and it remains to check the orientation of the ellipse. For $|u| > |w|$, the ellipse is counter-clockwise oriented, so that case (b) applies, and for $|u| < |w|$ the orientation is clockwise and we are in case (c). By Theorem 2.4, the outcome of this test does not depend on the values $u \in U$, $v \in V$, $w \in W$ chosen in step 1 if A is Fredholm. The resulting case corresponds to $\mathrm{ind}\, A_+$.

If we are in case (a) then $\mathrm{ind}\, A_+ = 0$. Because A is invertible, we have $0 = \mathrm{ind}\, A = \mathrm{ind}\, A_+ + \mathrm{ind}\, A_- = \mathrm{ind}\, A_-$. Now let $D \in \sigma^{\mathsf{op}}_r(A)$. By Theorem 3.2 a) and b), D is invertible and $\mathrm{ind}\, D_+ = \mathrm{ind}\, A_+ = 0$, so that $0 = \mathrm{ind}\, D = \mathrm{ind}\, D_+ + \mathrm{ind}\, D_- = \mathrm{ind}\, A_+ + \mathrm{ind}\, C_- = \mathrm{ind}\, C_-$. So C_- is a Toeplitz operator that is Fredholm with index 0. By Coburn's theorem [4, 5], C_- is invertible. By a completely symmetric argument (or simply by noting that $B_+ = C_-^R$) we get that also B_+ is invertible. Since (8) holds, Theorem 2.8 yields the applicability of the FSM (4) with the sequences l and r as chosen in step 1.

If we are in case (b) or (c) then $k := \mathrm{ind}\, A_+ = \mp 1$ and there are no sequences l and r for which the FSM (4) could be applicable to $Ax = b$. However, the FSM (4) is applicable to the equivalent system $S^k Ax = S^k b$ by the same arguments as in case (a) since $S^k A$ is invertible and since $\mathrm{ind}\,(S^k A)_+ = \mathrm{ind}\, S_+^k A_+ = \mathrm{ind}\, S_+^k + \mathrm{ind}\, A_+ = -k + k = 0$. ■

Proof of correctness of Algorithm 2.12. The choice of the sequence $r = (r_n)$ in step 1 is such that the set $\{D_- : D \in \sigma_r^{\mathsf{op}}(A)\} =: \{C_-\}$ is a singleton; in fact, C_- is a Toeplitz operator with diagonals u, v, w. It is possible to choose such a sequence r because $A_+ \in \Psi\mathrm{E}_+(U, V, W)$.

Since $\operatorname{ind} A_+ = 0$ by invertibility of A_+, we are automatically in case (a) of Theorem 2.4. Let $D \in \sigma_r^{\mathsf{op}}(A)$. By Theorem 3.2 a) and b), D is invertible and $\operatorname{ind} D_+ = \operatorname{ind} A_+ = 0$, so that $0 = \operatorname{ind} D = \operatorname{ind} D_+ + \operatorname{ind} D_- = \operatorname{ind} A_+ + \operatorname{ind} C_- = \operatorname{ind} C_-$. So C_- is a Toeplitz operator that is Fredholm with index 0. By Coburn's theorem [4, 5], C_- is invertible. Now Theorem 2.9 yields the applicability of the FSM (4) with the sequence r as chosen in step 1. ∎

4 A numerical example

We illustrate our results by a numerical computation, for which we come back to the Hatano-Nelson model from Example 2.2 b). So let $U = \{e^g\}$ and $W = \{e^{-g}\}$ with $g > 0$, put $c := e^g + e^{-g} = 2\cosh g$ and $s := e^g - e^{-g} = 2\sinh g$, and let $V = [-a, a]$ with $0 < a < s < c$.

Now let $A \in \Psi\mathrm{E}(U, V, W)$. From Theorem 2.1 and our discussion in Example 2.2 b), we derive the upper and lower bounds on $\operatorname{spec} A$ as shown in Figure 4.1. For further studies of this operator, including the size and shape of the hole in its spectrum, see [12, 13, 14, 33, 34].

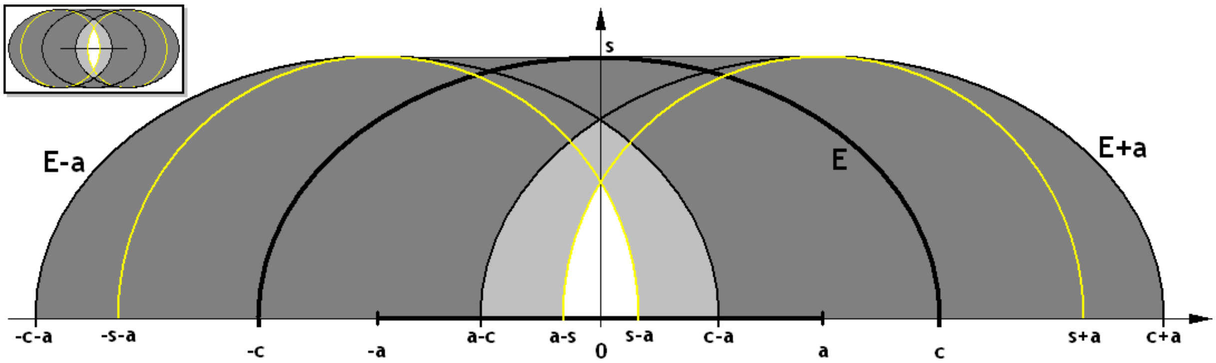

Figure 4.1: Here are our lower (dark gray) and upper (dark+light gray) bound on $\operatorname{spec} A$ from Example 2.2 b) with $g = 1$ and $a = 2$, so that $0 < a < s < c$. The region of uncertainty (light gray) is small if s and c are close to each other (i.e. if g is large). For reasons of symmetry we have only shown the upper half of the complex plane.

While the spectrum of a n-by-n principal submatrix of A is less interesting (each such matrix is similar to a self-adjoint matrix), the spectrum of this finite problem with periodic boundary conditions and its limit as $n \to \infty$ has been described in much detail by Goldsheid and Khoruzhenko [19], who thereby verified numerical observations of Hatano and Nelson [21, 22, 23]. The limiting set as $n \to \infty$ turns out to be the union of certain analytic curves (the so-called 'bubble with wings' [50]) and is entirely different from (although contained in) the spectrum of the infinite matrix A.

We will now apply our adaptive FSM (Algorithm 2.10) to a concrete matrix $A \in M(U, V, W)$, whose main diagonal entries v_i have been chosen independently from $V = [-a, a]$, where the density of our probability distribution on V increases in a certain way towards the endpoints of the interval. Our model has the parameters $g = 1$ (so that $c = 2\cosh 1 \approx 3.0862$ and $s = 2\sinh 1 \approx 2.3504$) and $a = 2 < s$, whence $0 \notin \operatorname{spec} A$.

In step 1 of the algorithm, we choose, as motivated in Remark 2.11 a), $v = 2$, which is one of the values with the highest probability density, besides the obvious choices $u = e^1$ and $w = e^{-1}$.

Then, for $n = 1, 2, ...,$ we look for n consecutive entries of the main diagonal that are within $1/n$ of $v = 2$ to find our cut-off bounds $l_n < l_{n-1}$ and $r_n > r_{n-1}$ (see Figure 4.2) with $l_0 = 0 = r_0$.

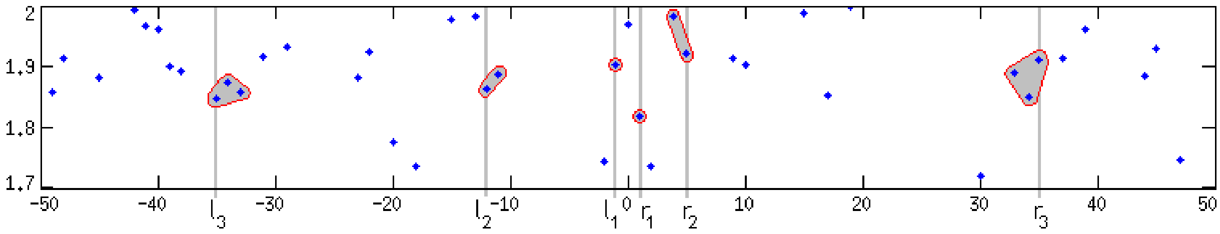

Figure 4.2: These are the main diagonal entries v_{-50} to v_{50} that are close to 2. Encircled are the groups of $n = 1, 2, 3$ consecutive entries that are within $1/n$ of $v = 2$ and therefore lead to the definition of l_n and r_n.

In step 2 of the algorithm, we find that we are in case (b), which says that v lies inside the ellipse $E = E(u, w)$ and it is encircled counter-clockwise w.r.t. the parametrization $\varphi \mapsto ue^{i\varphi} + we^{-i\varphi} = e^{1+i\varphi} + e^{-1-i\varphi}$ of E. In other words: $\operatorname{ind} A_+ = -1$.

This means that, in step 3, we shift our infinite system up by one row before we truncate it according to our sequences $l = (l_n)$ and $r = (r_n)$. The resulting method is applicable if and only if the inverses of the finite matrices $A_n := P_{l_n,r_n} S^{-1} A P_{l_n,r_n}$ remain uniformly bounded as $n \to \infty$. The following table shows the cut-off sequences $l = (l_n)$ and $r = (r_n)$, the size of the matrices A_n and the norms of their inverses for $n = 1, 2, ..., 8$.

n	l_n	r_n	$r_n - l_n + 1$	$\|A_n^{-1}\|$
1	-1	1	3	0.6816
2	-12	5	18	1.0580
3	-35	35	71	1.2698
4	-41	162	204	1.2698
5	-899	537	1437	1.4121
6	-1068	1183	2252	1.5438
7	-20494	21758	42253	1.6135
8	-469241	41570	510811	1.7500

We see the rather irregular exponential growth of the intervals $\{l_n, ..., r_n\}$ (see Remark 2.11) and the moderate growth of the inverses A_n^{-1}. This numerical evidence is not really convincing that the inverses remain uniformly bounded as $n \to \infty$. However, from the theory behind our Theorem 2.8 it follows that $\limsup_n \|A_n^{-1}\|$ is in case $p = 2$ equal (and otherwise at least bounded above by two times) the maximum of the norms of the inverses of the operators in (7). In our case, this means that

$$\limsup_n \|A_n^{-1}\| \;=\; \max(\,\|A^{-1}\|\,,\,\|B_+^{-1}\|\,)$$

if $p = 2$, where B_+ is the Toeplitz operator (note the translation S^{-1} in step 3)

$$B_+ \;=\; \begin{pmatrix} e^1 & 2 & e^{-1} & \\ & e^1 & 2 & \ddots \\ & & e^1 & \ddots \\ & & & \ddots \end{pmatrix}$$

with symbol $a(t) = e^1 + 2t^{-1} + e^{-1}t^{-2}$, $t \in \mathbb{T}$. But from $\|A^{-1}\| \le (s-2)^{-1} \approx 2.8539$ (the argument is as in the proof of Lemma 3.3) and $\|B_+^{-1}\| = (\min_{t\in\mathbb{T}} |a(t)|)^{-1} = (c-2)^{-1} \approx 0.9207$ (note that

B_+ is upper-triangular, whence its inverse is the Toeplitz operator with symbol $a(t)^{-1}$), we get that $\limsup_n \|A_n^{-1}\|$ is bounded above by $(s-2)^{-1} \approx 2.8539$. In general, it takes very large random matrices A_n to see $\|A_n^{-1}\|$ come close to $\sup_n \|A_n^{-1}\|$ because this requires a particular (and usually long) pattern somewhere on the diagonal(s) of A_n. The latter is reminiscent of the finite but very long time it takes a monkey to type the complete works of Shakespeare [52].

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