Asymptotic Structure of a Leray Solution to the Navier-Stokes Flow Around a Rotating Body

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Abstract

Consider a body, $\mathcal{B}$, rotating with constant angular velocity, ω, and fully submerged in a Navier-Stokes liquid that fills the whole space exterior to $\mathcal{B}$. We analyze the flow of the liquid that is steady with respect to a frame attached to $\mathcal{B}$. Our main theorem shows that the velocity field, v, of any weak solution, (v, p), in the sense of Leray, has an asymptotic expansion with a suitable Landau solution as leading term, and a remainder decaying point-wise like $\frac{1}{|x|^{1+\alpha}}$ as $|x| \to \infty$ for any $\alpha \in (0,1)$, provided the magnitude of ω is below a positive constant depending on α. We also furnish analogous expansions for ∇v and for the corresponding pressure field p. These results improve and clarify a recent result of R. Farwig and T. Hishida, Preprint 2591 TU Darmstadt, 2009.

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1 Introduction

Consider a rigid body rotating with prescribed constant angular velocity, $\omega \in \mathbb{R}^3$, in a Navier-Stokes liquid that fills the whole space exterior to the body. We assume that the motion of the liquid with respect to a frame, $\mathcal{S}$, attached to the body is steady. Then, after a suitable non-dimensionalization, the relevant equations for the liquid, in the frame $\mathcal{S}$, become

$$(1.1) \qquad \begin{cases} -\Delta v + v \cdot \nabla v - \omega \wedge x \cdot \nabla v + \omega \wedge v + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = \omega \wedge x & \text{on } \partial\Omega, \\ \lim_{|x| \to \infty} v(x) = 0, & \end{cases}$$

where v is the velocity field, p the corresponding pressure, and $\Omega \subset \mathbb{R}^3$ the region exterior to the body. We assume that Ω is an exterior domain with a C^2-smooth (compact) boundary.

Over the past few years there has been a significant effort devoted to the analysis of the fundamental mathematical properties of solutions to (1.1), including existence, uniqueness, asymptotic behavior, and stability. Without pretending to furnish an exhaustive bibliography, we refer the reader to [1, 13, 10, 5, 6, 18, 17, 19, 20, 21, 23, 4, 11] and to the references cited therein.

One important question that deserves special attention is the behavior of the velocity and pressure fields at large distances. In particular, it is of great relevance to find out the precise asymptotic structure of these fields and, possibly, to single out the corresponding leading terms. Beside its intrinsic mathematical significance, this analysis is also important in several applications, as well as in numerical computations, mainly in the estimation of the error made by approximating the infinite region of flow with a necessarily bounded domain; see, *e.g.*, [3].

The problem of the asymptotic structure of solutions to (1.1) appears to be particularly challenging. In fact, even in the simpler case $\omega = 0$ (and a non-zero right-hand side of compact support in $(1.1)_1$) it has been effectively solved, for small data at least, only lately [22].

Very recently, Farwig and Hishida [8], [9] have investigated the above question for smooth solutions to (1.1), and have furnished a first answer to the problem. More specifically, denoted by $\mathbf{T}(v, p) := -pI + \nabla v + (\nabla v)^T$ (with I the identity tensor) the Cauchy stress tensor, they have shown that the velocity field of any (smooth) solution to (1.1), having norm in a suitable Lorentz space sufficiently small and for which the quantity [1]

$$\left(\int_{\partial\Omega} \mathbf{T}(v, p) \cdot n \, \mathrm{d}S \right) \cdot \frac{\omega}{|\omega|}$$

[1] This quantity represents the force exerted by the liquid on the "body" (the complement of Ω, that is) in the direction of ω.

is also small, can be represented at large distances as

$$v(x)=U(x)+R(x), \tag{1.2}$$

where $U=U(x)$ is the velocity field of a particular Landau solution, that we will recall in a moment, and R is a "remainder" with $R\in L^q(\Omega)$ for some $q\in(3/2,3)$. Since $U(x)$ behaves like $1/|x|$ for large $|x|$, the relation (1.2) indicates that U is the leading term in the Lebesgue summability sense. The Landau solution involved in (1.2) is a field $U\in\mathcal{D}'(\mathbb{R}^3)$ solution to the Navier-Stokes system

$$\begin{cases} -\Delta U+U\cdot\nabla U+\nabla P=\left(\left(\displaystyle\int\limits_{\partial\Omega}\mathbf{T}(v,p)\cdot n\,\mathrm{d}S\right)\cdot\dfrac{\omega}{|\omega|}\right)\dfrac{\omega}{|\omega|}\,\delta,\\ \operatorname{div}U=0,\end{cases} \tag{1.3}$$

with δ denoting the delta distribution supported at $0\in\mathbb{R}^3$; see for example [9] and (3.2) below for an explicit form of (U,P). Here we only note that U is smooth away from the origin, and satisfies $U=O\big(\frac{1}{|x|}\big)$ and $\nabla U=O\big(\frac{1}{|x|^2}\big)$ as $|x|\to\infty$.

Objective of the present paper is to furnish a further contribution to the problem of asymptotic structure of solutions to (1.1) by improving, on the one hand, and clarifying, on the other hand, the results of [8], [9].

We establish our findings in the class of *Leray solutions*. The latter are defined as solutions (v,p) to (1.1) such that

$$\nabla v\in L^2(\Omega)\quad\text{and}\quad v\in L^6(\Omega) \tag{1.4}$$

and satisfying the energy *inequality*

$$2\int\limits_{\Omega}|\mathbf{D}v|^2\,\mathrm{d}x\le\int\limits_{\partial\Omega}\big(\mathbf{T}(v,p)\cdot n\big)\cdot(\omega\wedge x)\,\mathrm{d}S, \tag{1.5}$$

where $\mathbf{D}v:=\frac{1}{2}\big(\nabla v+(\nabla v)^T\big)$ is the stretching tensor of the liquid. As is well known, the class of Leray solutions is not empty for any $\omega\in\mathbb{R}^3$ (see for example [1]), and, moreover, by classical elliptic regularity, one shows that they are also smooth [12].

We will prove that, for sufficiently small $|\omega|$, the velocity field v of any Leray solution, (v,p), to (1.1) must obey an asymptotic expansion of the type (1.2), where, unlike [8], [9], $R(x)$ is estimated *point-wise*, with $|R(x)|\le O\big(\frac{1}{|x|^{1+\alpha}}\big)$, for some $\alpha\in(0,1)$. [2] We also show an analogous (improved) point-wise estimate for ∇v, with ∇U as leading term. As far as the pressure field p is concerned, we furnish a similar asymptotic expansion. However, the leading term in this expansion is *not* the pressure P of the Landau solution, but P plus an additional term that depends on the component orthogonal to ω of the force exerted by the liquid on the body. More precisely, we prove the following result:

[2]Notice that, clearly, $R\in L^q$ for large $|x|$, with some $q=q(\alpha)\in(3/2,3)$.

Theorem 1.1 (Main Theorem)**.** *Let $\alpha \in (0,1)$. There is an $\varepsilon = \varepsilon(\alpha) > 0$ so that if $|\omega| < \varepsilon$, then any Leray solution (v,p) to (1.1) obeys the asymptotic expansion*

$$v(x) = U(x) + O\Big(\frac{1}{|x|^{1+\alpha}}\Big) \quad \text{as } |x| \to \infty, \tag{1.6}$$

$$\nabla v(x) = \nabla U(x) + O\Big(\frac{1}{|x|^{2+\alpha}}\Big) \quad \text{as } |x| \to \infty, \tag{1.7}$$

and (after possibly adding a constant to p)

$$p(x) = P(x) + \frac{x}{4\pi|x|^3} \cdot \Big(I - \frac{\omega \otimes \omega}{|\omega|^2}\Big) \cdot \mathcal{F} + O\Big(\frac{1}{|x|^{2+\alpha}}\Big) \quad \text{as } |x| \to \infty, \tag{1.8}$$

where

$$\mathcal{F} := \int_{\partial\Omega} \big(\mathbf{T}(v,p) - v \otimes v\big) \cdot n \,\mathrm{d}S, \tag{1.9}$$

and (U,P) is the Landau solution (U^b, P^b) given by (3.2) corresponding to the parameter $b := \mathcal{F} \cdot \frac{\omega}{|\omega|}\frac{\omega}{|\omega|}$.

Remark 1.2. Note that $\mathcal{F}$ is equal to the (negative) force exerted by the liquid on the body $\mathcal{B}$. We emphasize that the leading term in the expansion (1.6) and (1.7) of v and ∇v, respectively, depends only on the component of $\mathcal{F}$ directed along ω, whereas the leading term in the expansion (1.8) of p also depends on the component of $\mathcal{F}$ orthogonal to ω.

Remark 1.3. It is not known if, in general, one can take $\alpha = 1$ in the above estimates. However, if $\mathbb{R}^3 \setminus \Omega$ possesses suitable rotational symmetry, then $\alpha = 1$ is allowed. However, in such a case, the leading term in the asymptotic expansion is no longer a Landau solution; see [14].

Remark 1.4. As we noticed previously, the formula (1.6) elucidates in a point-wise fashion the result proved in [8], [9] in Lebesgue spaces. However, in [8], [9] no information is provided on the asymptotic structure of ∇v and p. Therefore, (1.7) and (1.8) are completely new achievements.

The proof of Theorem 1.1 relies, basically, on the following two crucial results concerning the *linearized* version of (1.1) in the whole space, this latter being obtained by suppressing the nonlinear term $v \cdot \nabla v$ in (1.1) and by adding a suitable (given) function f, say, on its right-hand side. The first one is the proof of existence of solutions with a suitable decay order, under the assumption that f is of compact support and orthogonal (in the L^2 scalar product) to the direction of ω; see Lemma 2.1. This result can be viewed as a corollary to a very general one proved in [7]. The second one concerns the existence, uniqueness, and corresponding estimates of solutions that converge to zero point-wise, with a specific order of decay, under appropriate decay hypotheses on f; see Lemma

2.2. This result, in turn, is obtained by using the time-dependent transformation and the associated method introduced in [13].

Before discussing some preliminaries in Sect. 2, recalling the definition of Landau solution along with its basic properties in Sect. 3, and presenting the proof of our main results in Sect. 4, we introduce some basic notation. Let $G \subset \mathbb{R}^3$ be any domain, the exterior normal unit vector of which will be denoted by n.

- $\|\cdot\|_{r,G} = \|\cdot\|_r$ is the norm in the Lebesgue space $L^r(G)$, $1 \le r \le \infty$; $\|\cdot\|_{k,r,G}$ is the norm in the usual Sobolev space $W^{k,r}(G)$, $k \in \mathbb{N}$, $1 \le r \le \infty$.
- $D^{1,2}(G) := \{v \in L^1_{loc}(G) \mid |v|_{1,2} < \infty\}$ and $|v|_{1,2} := \big(\int_G |\nabla v|^2 \,\mathrm{d}x\big)^{\frac{1}{2}}$.
- For $\beta \in \mathbb{R}$ define $[\![v]\!]_{\beta,G} := \operatorname{ess\,sup}_{x\in G} |v(x)|(1+|x|)^{\beta}$.
- For $\beta \in \mathbb{R}$, $m \in \mathbb{N} \cup \{0\}$ let $[\![v]\!]_{m,\beta,G} := \sum_{0\le k\le m} [\![\nabla^k v]\!]_{\beta+k,G}$.
- $\mathcal{X}^m_\beta(G) := \{v \in L^1_{loc}(G) \mid [\![v]\!]_{m,\beta,G} < \infty\}$.
- $\mathbb{R}^3_T := \mathbb{R}^3 \times (0,T)$, and $\mathbb{R}^3_\infty := \mathbb{R}^3 \times (0,\infty)$ when $T = \infty$.
- $B_R = \{x \in \mathbb{R}^3 \mid |x| < R\}$ and $B^R = \mathbb{R}^3 \setminus \overline{B}_R$, where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^3$.

For functions $u : \mathbb{R}^3_T \to \mathbb{R}$, $\operatorname{div} u(x,t) := \operatorname{div}_x u(x,t)$, $\Delta u(x,t) := \Delta_x u(x,t)$ etc., that is, unless otherwise indicated, differential operators act in the spatial variables only. Note that constants in capital letters are global, constants in small letters are local.

2 Preliminaries

The proof of our main result relies on two crucial observations concerning the whole space linear problem

$$(2.1)\qquad \begin{cases} -\Delta w - \omega \wedge x \cdot \nabla w + \omega \wedge w + \nabla q = f & \text{in } \mathbb{R}^3, \\ \operatorname{div} w = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

The first observation is due to Farwig and Hishida ([9, Lemma 3.4]), which we state in the following lemma:

Lemma 2.1. *If $f \in C_0^\infty(\mathbb{R}^3)^3$ with*

$$(2.2)\qquad \Bigg(\int\limits_{\mathbb{R}^3} f(x)\,\mathrm{d}x\Bigg) \cdot \omega = 0,$$

then there exists a solution $(w,q) \in \mathcal{X}^1_2(\mathbb{R}^3)^3 \times \mathcal{X}^0_2(\mathbb{R}^3)$ to (2.1).

Proof. We obtain directly from [9, (3.21) and Lemma 3.4] the existence of a solution $(w,q) \in \mathcal{X}_2^0(\mathbb{R}^3)^3 \times \mathcal{X}_2^0(\mathbb{R}^3)$. Moreover, by elliptic regularity theory for the Stokes operator, $w \in C^\infty(\mathbb{R}^3)$. It remains to show $[\![\nabla w]\!]_{3,\mathbb{R}^3} < \infty$. This, however, follows by the same argument as in [9, Lemma 3.7] to prove $|w(x)| \le c_1|x|^{-2}$. This argument relies on the fact that the fundamental solution $\overline{\Gamma}$ to (2.1) (see [9, (3.20)] for an explicit expression) satisfies, after setting, without loss of generality, $\omega = \mathrm{e}_3$, the following expansion for $|y| \le R$ and $|x| \to \infty$:

$$\overline{\Gamma}(x,y) = \Phi(x) + O\bigg(\frac{1}{|x|^2}\bigg), \quad \Phi(x) := \frac{1}{8\pi|x|^3}\begin{pmatrix} 0 & 0 & x_1x_3 \\ 0 & 0 & x_2x_3 \\ 0 & 0 & x_3^2+|x|^2 \end{pmatrix}$$

and

$$w(x) = \int_{\mathbb{R}^3} \overline{\Gamma}(x,y) f(y)\,\mathrm{d}y. \tag{2.3}$$

By analogy to the proof of [7, Proposition 4.1 and 4.2] one can show, for $|y| \le R$ and $|x| \to \infty$:

$$\nabla\overline{\Gamma}(x,y) = \nabla\Phi(x) + O\bigg(\frac{1}{|x|^3}\bigg).$$

Thus, after differentiating in (2.3) and exploiting (2.2) where we have set $\omega = \mathrm{e}_3$, it follows that $|\nabla w(x)| \le c_2|x|^{-3}$, which implies $[\![\nabla w]\!]_{3,\mathbb{R}^3} < \infty$. □

The second observation concerns the solvability of (2.1) in weighted spaces for more general f. We state it as the following lemma:

Lemma 2.2. *Let $\alpha \in (0,1)$. If*[3] $f \in C^\infty(\mathbb{R}^3)^3$ *and* $f = \operatorname{div} F$ *with*[4]

$$[\![F]\!]_{2+\alpha} + [\![\operatorname{div} F]\!]_{3+\alpha} \equiv \sum_{i,j=1}^3 [\![F_{ij}]\!]_{2+\alpha} + \sum_{i=1}^3 [\![\partial_k F_{ki}]\!]_{3+\alpha} < \infty, \tag{2.4}$$

then there exists a unique solution $(w,q) \in \mathcal{X}^1_{1+\alpha}(\mathbb{R}^3)^3 \times \mathcal{X}^0_{2+\alpha}(\mathbb{R}^3)$ to (2.1) that satisfies

$$[\![w]\!]_{1,1+\alpha} + [\![q]\!]_{2+\alpha} \le C_1\,([\![F]\!]_{2+\alpha} + [\![\operatorname{div} F]\!]_{3+\alpha}), \tag{2.5}$$

where $C_1 = C_1(\alpha)$ is independent *of ω.*

Proof. The existence of a weak solution

$$(w,q) \in \big(D^{1,2}(\mathbb{R}^3)^3 \cap L^6(\mathbb{R}^3)^3\big) \times L^2_{loc}(\mathbb{R}^3) \tag{2.6}$$

[3] We take f smooth for simplicity only; this assumption can be substantially weakened.
[4] Throughout this paper, we shall use the summation convention over repeated indexes.

to (2.1) can be shown by a standard Galerkin approximation argument, see for example [26]. We will now prove that this weak solution belongs to the space $\mathcal{X}^1_{1+\alpha}(\mathbb{R}^3)^3 \times \mathcal{X}^0_{2+\alpha}(\mathbb{R}^3)$. To this aim, for $t>0$, we put

$$Q(t) := \exp(\hat{\omega} t), \text{ with } \hat{\omega} := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix},$$

and set

$$\begin{aligned} &u(x,t) := Q(t) w\big(Q^T(t)x\big), \quad \mathfrak{p}(x,t) := q\big(Q^T(t)x\big), \\ &G(x,t) := Q(t) F\big(Q^T(t)x\big); \end{aligned}$$

in particular, $u(\cdot,0)=w$ in the sense that $\lim_{t\to 0}\|u(\cdot,t)-w\|_6 = 0$. Then

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} = \operatorname{div} G & \text{in } \mathbb{R}^3_\infty, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3_\infty, \\ u(\cdot,0) = w & \text{at } t=0, \end{cases} \tag{2.7}$$

and $u \in L^6(\mathbb{R}^3_T)^3$ for all $T>0$.

To get an integral representation of u, we recall the fundamental solution to the time-dependent Stokes problem, that is, the solution (in the sense of distributions) to $(i,j=1,2,3)$

$$\begin{cases} \partial_t \Gamma_{ij} - \Delta \Gamma_{ij} + \partial_j \gamma_i = \delta_{ij}\delta(t)\delta(x), \\ \partial_k \Gamma_{ik} = 0, \end{cases}$$

where δ_{ij} denotes the Kronecker symbol and $\delta(\cdot)$ the Dirac delta distribution. The fundamental solution takes the form (see [25, §5])

$$\Gamma_{ij} := -\delta_{ij}\Delta\Psi + \partial_i\partial_j\Psi, \quad \gamma_i := \partial_i(\Delta - \partial_t)\Psi,$$

with

$$\Psi(x,t) := \frac{1}{4\pi^{\frac{3}{2}} t^{\frac{1}{2}}} \int_0^1 \mathrm{e}^{-\frac{|x|^2 r^2}{4t}}\,\mathrm{d}r.$$

Using Γ we can write the unique (in the class $L^6(\mathbb{R}^3_T)^3$, $T>0$) solution to (2.7) as (see [16, Section 3])

$$\begin{aligned} u_i(x,t) &= \frac{1}{(4\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \mathrm{e}^{-\frac{|x-y|^2}{4t}}\, w_i(y)\,\mathrm{d}y \\ &\quad - \int_0^t \int_{\mathbb{R}^3} \partial_j \Gamma_{ih}(x-y,t-\tau)\, G_{jh}(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ &=: I_1(x,t) - I_2(x,t). \end{aligned} \tag{2.8}$$

Then Hölder's inequality yields, since $w \in L^6(\mathbb{R}^3)^3$,

$$|I_1\big(Q(t)x,t\big)| = O(t^{-1/4}) \quad \text{as} \;\; t \to \infty, \quad \text{uniformly in } x \in \mathbb{R}^3. \tag{2.9}$$

Moreover, using the estimate on $\int_0^\infty |\nabla\Gamma(x,t)|\,\mathrm{d}t$ from [16, Lemma 3.1], which, as one may easily verify, also holds in the present case of vanishing velocity at infinity (that is, $\mathcal{R}=0$ in [16]), we get

$$|I_2(x,t)| \leq c_1 [\![F]\!]_{2+\alpha} \int\limits_{\mathbb{R}^3} \frac{1}{|x-y|^2(1+|y|)^{2+\alpha}}\,\mathrm{d}y. \tag{2.10}$$

From [12, Lemma II.7.2] we conclude that

$$|I_2(x,t)| \leq [\![F]\!]_{2+\alpha} \frac{c_2}{(1+|x|)^{1+\alpha}}, \quad \text{uniformly in } t>0, \tag{2.11}$$

with $c_2 = c_2(\alpha)$. Since $|w(x)| = |u\big(Q(t)x,t\big)| \leq |I_1\big(Q(t)x,t\big)| + |I_2\big(Q(t)x,t\big)|$ for all $t>0$, from (2.9) and (2.11) we obtain

$$[\![w]\!]_{1+\alpha,\mathbb{R}^3} \leq c_3 [\![F]\!]_{2+\alpha}. \tag{2.12}$$

We now differentiate (2.8) and obtain $\partial_k u(x,t) = \partial_k I_1(x,t) + \partial_k I_2(x,t)$. Then another standard application of Hölder's inequality yields

$$\partial_k I_1(x,t) = O(t^{-3/4}) \quad \text{as} \;\; t\to\infty, \quad \text{uniformly in } x \in \mathbb{R}^3. \tag{2.13}$$

Moreover, we have

$$\partial_k I_2(x,t) = \int\limits_0^t \int\limits_{\mathbb{R}^3} \partial_k \Gamma_{ih}(x-y,t-\tau)\; \partial_j G_{jh}(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau. \tag{2.14}$$

Now fix $0 \neq x \in \mathbb{R}^3$ and let $R = \frac{1}{2}|x|$. Then

$$\begin{aligned} \partial_k I_2(x,t) = &\int\limits_0^t \int\limits_{\mathrm{B}_R} \partial_j\partial_k \Gamma_{ih}(x-y,\tau)\; G_{jh}(y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau \;+ \\ &\int\limits_0^t \int\limits_{\partial\,\mathrm{B}_R} \partial_k \Gamma_{ih}(x-y,\tau)\; G_{jh}(y,t-\tau)\, n_j\,\mathrm{d}S(y)\,\mathrm{d}\tau \;+ \\ &\int\limits_0^t \int\limits_{\mathrm{B}^R} \partial_k \Gamma_{ih}(x-y,\tau)\; \partial_j G_{jh}(y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau \\ =:& \; J_1 + J_2 + J_3. \end{aligned} \tag{2.15}$$

Employing, as above, [16, Lemma 3.1], this time to estimate $\int_0^\infty |\nabla^2\Gamma(x,\tau)|\,\mathrm{d}\tau$, we find

$$
\begin{aligned}
|J_1| &\le c_4 \int\limits_{\mathrm{B}_R} \frac{[\![F]\!]_{2+\alpha}}{|x-y|^3(1+|y|)^{2+\alpha}}\,\mathrm{d}y \\
&\le c_5 \frac{1}{|x|^3} \int\limits_{\mathrm{B}_R} \frac{[\![F]\!]_{2+\alpha}}{(1+|y|)^{2+\alpha}}\,\mathrm{d}y \le [\![F]\!]_{2+\alpha}\big(c_6|x|^{-(2+\alpha)} + c_7|x|^{-3}\big).
\end{aligned}
\tag{2.16}
$$

Furthermore, by [16, Lemma 3.1], we have

$$
|J_2| \le c_8 \int\limits_{\partial\,\mathrm{B}_R} \frac{[\![F]\!]_{2+\alpha}}{|x-y|^2|y|^{2+\alpha}}\mathrm{d}S(y) \le c_9[\![F]\!]_{2+\alpha}|x|^{-(2+\alpha)}.
\tag{2.17}
$$

Finally, using [16, Lemma 3.1] and [12, Lemma II.7.2], we estimate

$$
\begin{aligned}
|J_3| &\le c_{10} \int\limits_{\mathrm{B}^R} \frac{[\![\operatorname{div} F]\!]_{3+\alpha}}{|x-y|^2|y|^{3+\alpha}}\,\mathrm{d}y \\
&\le c_{10} \frac{1}{R} \int\limits_{\mathrm{B}^R} \frac{[\![\operatorname{div} F]\!]_{3+\alpha}}{|x-y|^2|y|^{2+\alpha}}\,\mathrm{d}y \le c_{11}[\![\operatorname{div} F]\!]_{3+\alpha}|x|^{-(2+\alpha)}.
\end{aligned}
\tag{2.18}
$$

Since $|\nabla w(x)| = |\nabla u(Q(t)x,t)| \le |\nabla I_1\big(Q(t)x,t\big)| + |\nabla I_2\big(Q(t)x,t\big)|$, $t>0$, we deduce from (2.13)-(2.18) that

$$
\operatorname*{ess\,sup}_{|x|>1} |\nabla w(x)|(1+|x|)^{2+\alpha} \le c_{12}\big([\![F]\!]_{2+\alpha} + [\![\operatorname{div} F]\!]_{3+\alpha}\big).
\tag{2.19}
$$

To complete the estimate for ∇w, we recall (2.14) and estimate, using [16, Lemma 3.1],

$$
|\partial_k I_2(x,t)| \le c_{13}[\![F]\!]_{3+\alpha} \int\limits_{\mathbb{R}^3} \frac{1}{|x-y|^2(1+|y|)^{3+\alpha}}\,\mathrm{d}y.
$$

It follows that $|\partial_k I_2(x,t)| \le c_{14}[\![F]\!]_{3+\alpha}$ for $|x|\le 1$ and all $t>0$. Combining this estimate with (2.13), we conclude that $\operatorname{ess\,sup}_{|x|\le 1}|\nabla w(x)| \le c_{15}[\![F]\!]_{3+\alpha}$. This, together with (2.19), yields

$$
[\![\nabla w]\!]_{2+\alpha,\mathbb{R}^3} \le c_{16}\big([\![F]\!]_{2+\alpha} + [\![\operatorname{div} F]\!]_{3+\alpha}\big).
\tag{2.20}
$$

We now turn our attention to the pressure term q. Taking div in $(2.1)_1$ we get

$$
\Delta q = -\partial_i\partial_j F_{ij} \quad \text{in } \mathbb{R}^3.
$$

From the fact that $F \in L^{\frac{3}{2}}(\mathbb{R}^3)^{3\times 3}$ it follows, by standard Calderón-Zygmund estimates, that, after possibly modifying q by adding a constant, $q \in L^{\frac{3}{2}}(\mathbb{R}^3)$.

Together with the summability properties of $\operatorname{div} F$, this yields the validity of the representation

$$q(x) = -\int_{\mathbb{R}^3} \partial_j \mathcal{E}(y-x)\, \partial_i F_{ij}(y)\, \mathrm{d}y; \tag{2.21}$$

here $\mathcal{E}$ denotes the fundamental solution to the Laplace equation. Now fix $R = \frac{1}{2}|x| > 0$ and split

$$q(x) = -\int_{\mathrm{B}_R} \partial_i \mathcal{E}(y-x)\, \partial_j F_{ij}(y)\, \mathrm{d}y - \int_{\mathrm{B}^R} \partial_i \mathcal{E}(y-x)\, \partial_j F_{ij}(y)\, \mathrm{d}y =: K_1 + K_2.$$

We can estimate

$$\begin{aligned}
|K_1| &\leq \Big| \int_{\partial \mathrm{B}_R} \partial_i \mathcal{E}(y-x)\, F_{ij}(y)\, n_j\, \mathrm{d}S(y) \Big| + \Big| \int_{\mathrm{B}_R} \partial_j \partial_i \mathcal{E}(y-x)\, F_{ij}(y)\, \mathrm{d}y \Big| \\
&\leq c_{17} \Big(\int_{\partial \mathrm{B}_R} \frac{[\![F]\!]_{2+\alpha}}{|x-y|^2 |y|^{2+\alpha}}\, \mathrm{d}S(y) + \int_{\mathrm{B}_R} \frac{[\![F]\!]_{2+\alpha}}{|x-y|^3 (1+|y|)^{2+\alpha}}\, \mathrm{d}y \Big) \\
&\leq [\![F]\!]_{2+\alpha} \big(c_{18} |x|^{-(2+\alpha)} + c_{19} |x|^{-3} \big).
\end{aligned}$$

Moreover, using again [12, Lemma II.7.2], we obtain

$$\begin{aligned}
|K_2| &\leq \int_{\mathrm{B}^R} \frac{[\![\operatorname{div} F]\!]_{3+\alpha}}{|x-y|^2 |y|^{3+\alpha}}\, \mathrm{d}y \\
&\leq \frac{1}{R} \int_{\mathrm{B}^R} \frac{[\![\operatorname{div} F]\!]_{3+\alpha}}{|x-y|^2 |y|^{2+\alpha}}\, \mathrm{d}y \leq c_{20} [\![\operatorname{div} F]\!]_{3+\alpha} |x|^{-(2+\alpha)}.
\end{aligned}$$

It follows that

$$\operatorname*{ess\,sup}_{|x|>1} |q(x)| (1+|x|)^{2+\alpha} \leq c_{21} \big([\![F]\!]_{2+\alpha} + [\![\operatorname{div} F]\!]_{3+\alpha} \big). \tag{2.22}$$

To complete the estimate for q, we estimate directly from (2.21)

$$|q(x)| \leq c_{22} [\![\operatorname{div} F]\!]_{3+\alpha} \int_{\mathbb{R}^3} \frac{1}{|x-y|^2 (1+|y|)^{3+\alpha}}\, \mathrm{d}y,$$

from which it follows that $\operatorname{ess\,sup}_{|x|\leq 1} |q(x)| \leq c_{23} [\![\operatorname{div} F]\!]_{3+\alpha}$. Combined with (2.22) we thus have

$$[\![q]\!]_{2+\alpha} \leq c_{24} \big([\![F]\!]_{2+\alpha} + [\![\operatorname{div} F]\!]_{3+\alpha} \big). \tag{2.23}$$

Summarizing (2.12), (2.20), and (2.23) we get (2.5) It remains to show uniqueness of the solution in the class $\mathcal{X}^1_{1+\alpha}(\mathbb{R}^3)^3 \times \mathcal{X}^0_{2+\alpha}(\mathbb{R}^3)$. Since (2.1) is a linear problem, we consider only the case $f = 0$ and a solution $(w, q) \in \mathcal{X}^1_{1+\alpha}(\mathbb{R}^3)^3 \times \mathcal{X}^0_{2+\alpha}(\mathbb{R}^3)$. Dot-multiplying the first equation in (2.1) by w, integrating over B_R, and subsequently letting $R \to \infty$, we obtain $\nabla w = 0$. Consequently, $(w, q) = (0, 0)$. □

3 Landau Solution

The Landau solution (U^b, P^b), corresponding to a parameter $b \in \mathbb{R}^3$, is a solution in $\mathcal{D}'(\mathbb{R}^3)$ to

$$\begin{cases} -\Delta U + U \cdot \nabla U + \nabla P = b\,\delta, \\ \operatorname{div} U = 0 \end{cases} \tag{3.1}$$

that is axially symmetric about the axis $b\mathbb{R}$ and (-1)-homogeneous. Here δ denotes the delta distribution. The Landau solution can be given explicitly. Assume for simplicity that $b = k\,\mathrm{e}_3$, $k \in \mathbb{R}$, then

$$\begin{aligned} U^b(x) &= \frac{2}{|x|}\left(\frac{c\frac{x_3}{|x|} - 1}{(c - \frac{x_3}{|x|})^2}\frac{x}{|x|} + \frac{1}{c - \frac{x_3}{|x|}}\,\mathrm{e}_3 \right) \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\}, \\ P^b &= \frac{4}{|x|^2}\frac{(c\frac{x_3}{|x|} - 1)}{(c - \frac{x_3}{|x|})^2} \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\}, \end{aligned} \tag{3.2}$$

where

$$k = \frac{8\pi c}{3(c^2-1)}\left(2 + 6c^2 - 3c(c^2-1)\log\frac{c+1}{c-1} \right). \tag{3.3}$$

As one may easily verify, for each $k \in \mathbb{R} \setminus \{0\}$ there exists a unique $c \in \mathbb{R}$ with $|c| > 1$ so that (k, c) satisfies (3.3). Hence, for each $b \in \mathbb{R}^3 \setminus \{0\}$ a Landau solution (U^b, P^b) to (3.1) is given. Moreover, we have $b = k\,\mathrm{e}_3 \to 0$ as $|c| \to \infty$. The Landau solution was originally constructed by Landau [24]. For the explicit calculation of the expressions above, we refer to [2].

An important observation concerning the rotating body case is that

$$b \wedge x \cdot \nabla U^b - b \wedge U^b = 0 \quad \text{in } \mathbb{R}^3 \setminus \{0\},$$

which follows from the fact that U^b is symmetric about $b\mathbb{R}$ (see [9]).

We conclude from the above that (U^b, P^b) is a solution to

$$\begin{cases} -\Delta U^b + U^b \cdot \nabla U^b - b \wedge x \cdot \nabla U^b + b \wedge U^b + \nabla P^b = 0 & \text{in } \mathbb{R}^3 \setminus \{0\}, \\ \operatorname{div} U^b = 0 & \text{in } \mathbb{R}^3 \setminus \{0\} \end{cases} \tag{3.4}$$

that satisfies

$$|U^b(x)| \le \frac{\kappa_1(b)}{|x|} \quad \text{and} \quad |\nabla U^b(x)| \le \frac{\kappa_2(b)}{|x|^2} \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\}, \tag{3.5}$$

with

$$\lim_{b \to 0} \kappa_1(b) = 0 \quad \text{and} \quad \lim_{b \to 0} \kappa_2(b) = 0. \tag{3.6}$$

In fact, the properties (3.4), (3.5), and (3.6) are all we need in order to prove our main theorem.

4 Proof of Main Theorem

We will now prove our main result and, first of all, outline the idea behind the proof.

Let (v, p) be a Leray solution to (1.1) satisfying the energy inequality (1.5). If $|\omega|$ is sufficiently small, it was proved in [15] that

$$\llbracket v \rrbracket_1 + \llbracket \nabla v \rrbracket_2 + \llbracket p \rrbracket_2 < \infty. \tag{4.1}$$

Moreover, by elliptic regularity we conclude that $v,\, p \in C^\infty(\Omega)$. Now let $R > \operatorname{diam}(\mathbb{R}^3 \setminus \Omega)$ and $\chi_R \in C_0^\infty(\mathbb{R}^3)$ be a "cut-off" function with $\chi_R = 0$ in B_R and $\chi_R = 1$ in $\mathbb{R}^3 \setminus \mathrm{B}_{2R}$. Put

$$w := \chi_R v - \mathfrak{B}(\nabla \chi_R \cdot v), \quad q := \chi_R p,$$

where $\mathfrak{B}$ denotes the "Bogovskiĭ operator", *i.e.*, an operator $\mathfrak{B} : C_0^\infty(\mathrm{B}_{2R}) \to C_0^\infty(\mathrm{B}_{2R})^3$ with the property that $\operatorname{div} \mathfrak{B}(f) = f$ whenever $\int_{\mathrm{B}_{2R}} f(x)\,\mathrm{d}x = 0$. We refer to [12, Theorem III.3.2] for details on this operator. Note that in the above case

$$\int\limits_{\mathrm{B}_{2R}} \nabla \chi_R \cdot v \,\mathrm{d}x = \int\limits_{\partial\, \mathrm{B}_{2R}} v \cdot n \,\mathrm{d}S = \int\limits_{\partial\Omega} \omega \wedge x \cdot n \,\mathrm{d}S = 0.$$

Hence (w, q) satisfies

$$\begin{cases} -\Delta w + w \cdot \nabla w - \omega \wedge x \cdot \nabla w + \omega \wedge w + \nabla q = G_v & \text{in } \mathbb{R}^3, \\ \operatorname{div} w = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

with $G_v \in C_0^\infty(\mathbb{R}^3)$, and

$$\llbracket w \rrbracket_1 + \llbracket \nabla w \rrbracket_2 + \llbracket q \rrbracket_2 < \infty. \tag{4.2}$$

Next we introduce the Landau solution (U, P) corresponding to the parameter $b := \mathcal{F} \cdot \frac{\omega}{|\omega|} \frac{\omega}{|\omega|}$, that is, $(U, P) := (U^b, P^b)$. As above we put

$$\widetilde{U} := \chi_R U - \mathfrak{B}(\nabla \chi_R \cdot U), \quad \widetilde{P} = \chi_R P.$$

Then $(\widetilde{U}, \widetilde{P})$ satisfies

$$\begin{cases} -\Delta \widetilde{U} + \widetilde{U} \cdot \nabla \widetilde{U} - \omega \wedge x \cdot \nabla \widetilde{U} + \omega \wedge \widetilde{U} + \nabla \widetilde{P} = G_U & \text{in } \mathbb{R}^3, \\ \operatorname{div} \widetilde{U} = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

with $G_{\widetilde{U}} \in C_0^\infty(\mathbb{R}^3)$, and

$$\llbracket \widetilde{U} \rrbracket_1 + \llbracket \nabla \widetilde{U} \rrbracket_2 + \llbracket \widetilde{P} \rrbracket_2 < \infty, \tag{4.3}$$

which follows from (3.5). A crucial observation at this point is that

$$\int_{\mathbb{R}^3} G_v \,\mathrm{d}x = \int_{\mathrm{B}_{2R}} \operatorname{div}\Big[-\mathbf{T}(w,q) + w\otimes w + w\otimes(\omega\wedge x) - (\omega\wedge x)\otimes w\Big]\,\mathrm{d}x$$
$$= \int_{\partial\,\mathrm{B}_{2R}} \Big[-\mathbf{T}(v,p) + v\otimes v + v\otimes(\omega\wedge x) - (\omega\wedge x)\otimes v\Big]\cdot n\,\mathrm{d}S$$
$$= \int_{\partial\Omega} \Big[\mathbf{T}(v,p) - v\otimes v\Big]\cdot n\,\mathrm{d}S,$$

since $v = \omega\wedge x$ on $\partial\Omega$. Similarly, we have

$$\int_{\mathbb{R}^3} G_U \,\mathrm{d}x = \int_{\mathrm{B}_{2R}} \operatorname{div}\Big[-\mathbf{T}(\widetilde{U},\widetilde{P}) + \widetilde{U}\otimes\widetilde{U} + \widetilde{U}\otimes(\omega\wedge x) - (\omega\wedge x)\otimes\widetilde{U}\Big]\cdot n\,\mathrm{d}S$$
$$= \int_{\partial\,\mathrm{B}_{2R}} \Big[-\mathbf{T}(U,P) + U\otimes U + U\otimes(\omega\wedge x) - (\omega\wedge x)\otimes U\Big]\cdot n\,\mathrm{d}S$$
$$= b,$$

since $(U,P) = (U^b,P^b)$ solves (3.4) with right-hand side $b\delta$. Consequently, by the definition of b,

$$\Big(\int_{\mathbb{R}^3} \big(G_v - G_U\big)\,\mathrm{d}x\Big)\cdot\omega = 0.$$

Thus, by Lemma 2.1, there exists a solution (V_0,P_0) to

$$\text{(4.4)}\qquad \begin{cases} -\Delta V_0 - \omega\wedge x\cdot\nabla V_0 + \omega\wedge V_0 + \nabla P_0 = G_v - G_U & \text{in } \mathbb{R}^3,\\ \operatorname{div} V_0 = 0 & \text{in } \mathbb{R}^3\end{cases}$$

that satisfies

$$\text{(4.5)}\qquad [\![V_0]\!]_2 + [\![\nabla V_0]\!]_3 + [\![P_0]\!]_2 < \infty.$$

Note that, as a consequence of (4.4), $\Delta P_0 = \operatorname{div}(G_v - G_U)$, and hence

$$\text{(4.6)}\qquad P_0(x) = \nabla\mathcal{E}(x)\cdot\int_{\mathbb{R}^3}\big(G_v(y) - G_U(y)\big)\,\mathrm{d}y + O(|x|^{-3}),$$

where $\mathcal{E}$ denotes the fundamental solution to the Laplace equation. Now consider

$$\text{(4.7)}\qquad z := w - \widetilde{U} - V_0 \quad\text{and}\quad \pi := q - \widetilde{P} - P_0.$$

As can easily be verified, (z,π) satisfies the *linear* problem

$$(4.8)\qquad \begin{cases} -\Delta z - \omega\wedge x\cdot\nabla z + \omega\wedge z + z\cdot\nabla w + \widetilde{U}\cdot\nabla z + \nabla\pi = \\ \qquad\qquad\qquad -\operatorname{div}\big[V_0\otimes w + \widetilde{U}\otimes V_0\big] & \text{in } \mathbb{R}^3, \\ \operatorname{div} z = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

with $(z,\pi)\in\mathcal{X}^1_1(\mathbb{R}^3)^3\times\mathcal{X}^0_2(\mathbb{R}^3)$.

Our main result, namely the asymptotic expansions (1.6)-(1.8), now follows if we can show $[\![z]\!]_{1,1+\alpha}+[\![\pi]\!]_{2+\alpha}<\infty$. We show this by first establishing, using Lemma 2.2 in combination with (4.2), (4.3), and (4.5), the existence of a solution to (4.8) with this property, and, secondly, showing uniqueness of solutions to (4.8) in the class $\mathcal{X}^1_1(\mathbb{R}^3)^3\times\mathcal{X}^0_2(\mathbb{R}^3)$.

Lemma 4.1. *Let $\alpha\in(0,1)$. There is an $\varepsilon=\varepsilon(\alpha)>0$ so that if $|\omega|<\varepsilon$ there exists a solution $(z,\pi)\in\mathcal{X}^1_{1+\alpha}(\mathbb{R}^3)^3\times\mathcal{X}^0_{2+\alpha}(\mathbb{R}^3)$ to (4.8).*

Proof. We shall use a perturbation argument in the space

$$X := \{(\mathfrak{z},\mathfrak{p})\in\mathcal{X}^1_{1+\alpha}(\mathbb{R}^3)^3\times\mathcal{X}^0_{2+\alpha}(\mathbb{R}^3)\mid \operatorname{div}\mathfrak{z}=0\},$$
$$\|(\mathfrak{z},\mathfrak{p})\|_X := [\![\mathfrak{z}]\!]_{1,1+\alpha}+[\![\mathfrak{p}]\!]_{2+\alpha}.$$

Clearly, $(X,\|\cdot\|_X)$ is a Banach space. Let $(\mathfrak{z},\mathfrak{p})\in X$. Consider the system

$$(4.9)\qquad \begin{cases} -\Delta z - \omega\wedge x\cdot\nabla z + \omega\wedge z + \nabla\pi = \\ \qquad -\mathfrak{z}\cdot\nabla w - \widetilde{U}\cdot\nabla\mathfrak{z} - \operatorname{div}\big[V_0\otimes w + \widetilde{U}\otimes V_0\big] & \text{in } \mathbb{R}^3, \\ \operatorname{div} z = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

Note that $\mathfrak{z}\cdot\nabla w + \widetilde{U}\cdot\nabla\mathfrak{z} = \operatorname{div}\big[\mathfrak{z}\otimes w + \widetilde{U}\otimes\mathfrak{z}\big]$, and put

$$F := \mathfrak{z}\otimes w + \widetilde{U}\otimes\mathfrak{z} + V_0\otimes w + \widetilde{U}\otimes V_0.$$

Since $[\![F]\!]_{2+\alpha}+[\![\operatorname{div} F]\!]_{3+\alpha}<\infty$, there exists, by Lemma 2.2, a unique solution $(z,\pi)\in\mathcal{X}^1_{1+\alpha}(\mathbb{R}^3)^3\times\mathcal{X}^0_{2+\alpha}(\mathbb{R}^3)$ to (4.9). Let us define the map $\mathcal{J}:X\to X$ by $\mathcal{J}(\mathfrak{z},\mathfrak{p}):=(z,\pi)$, and show the existence of a fixed point of $\mathcal{J}$ by the contraction mapping theorem. Therefore, consider $(\mathfrak{z}_1,\mathfrak{p}_1),(\mathfrak{z}_2,\mathfrak{p}_2)\in X$ and put $(z_1,\pi_1):=\mathcal{J}(\mathfrak{z}_1,\mathfrak{p}_1)$ and $(z_2,\pi_2):=\mathcal{J}(\mathfrak{z}_2,\mathfrak{p}_2)$. Clearly, $(z_1-z_2,\pi=\pi_1-\pi_2)$ satisfies

$$(4.10)\qquad \begin{cases} -\Delta(z_1-z_2) - \omega\wedge x\cdot\nabla(z_1-z_2) + \omega\wedge(z_1-z_2) + \nabla\pi = \\ \qquad\qquad -\operatorname{div}\big[(\mathfrak{z}_1-\mathfrak{z}_2)\otimes w + \widetilde{U}\otimes(\mathfrak{z}_1-\mathfrak{z}_2)\big] & \text{in } \mathbb{R}^3, \\ \operatorname{div}(z_1-z_2) = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

Lemma 2.2 implies that

$$[\![z_1-z_2]\!]_{1,1+\alpha}+[\![\pi_1-\pi_2]\!]_{2+\alpha}\le C_1(\alpha)[\![\mathfrak{z}_1-\mathfrak{z}_2]\!]_{1,1+\alpha}\big([\![w]\!]_{1,1}+[\![\widetilde{U}]\!]_{1,1}\big).$$

From [15, Theorem 4.1] we obtain $\lim_{|\omega|\to 0}[\![v]\!]_{1,1,\Omega}=0$. Since $w=\chi_R v - \mathfrak{B}(\nabla\chi_R\cdot v)$, one easily verifies, using well-known L^q-estimates for $\mathfrak{B}$ (see [12, Chapter III.3]) and Sobolev embedding, that $\lim_{|\omega|\to 0}[\![w]\!]_{1,1} = 0$. Moreover, again using [15, Theorem 4.1], we conclude that $\lim_{|\omega|\to 0} b(\omega, v, p) = 0$, which, together with (3.5), (3.6) implies $\lim_{|\omega|\to 0}[\![\widetilde{U}]\!]_{1,1} = 0$. Consequently, for sufficiently small $|\omega|$, $\mathcal{J}$ is a contraction, and, by the contraction mapping theorem, there exists a fixed point $(z, \pi) \in \mathcal{X}^1_{1+\alpha}(\mathbb{R}^3)^3 \times \mathcal{X}^0_{2+\alpha}(\mathbb{R}^3)$ of $\mathcal{J}$. Clearly, by construction of $\mathcal{J}$, this fixed point is a solution to (4.8). ☐

Lemma 4.2. *There is an $\varepsilon > 0$ so that if $|\omega| < \varepsilon$ then a solution (z, π) to (4.8) in $\mathcal{X}^1_1(\mathbb{R}^3)^3 \times \mathcal{X}^0_2(\mathbb{R}^3)$ is unique in this class.*

Proof. Assume that $(z_1, \pi_1), (z_2, \pi_2) \in \mathcal{X}^1_1(\mathbb{R}^3)^3 \times \mathcal{X}^0_2(\mathbb{R}^3)$ both solve (4.8). Then $(z, \pi) := (z_1 - z_2, \pi_1 - \pi_2)$ solves

$$(4.11)\qquad \begin{cases} -\Delta z - \omega \wedge x \cdot \nabla z + \omega \wedge z + \nabla \pi = \\ \qquad\qquad -\operatorname{div}\left[z \otimes w + \widetilde{U} \otimes z\right] & \text{in } \mathbb{R}^3, \\ \operatorname{div} z = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

Testing (4.11) with z, integrating over B_R, subsequently letting $R \to \infty$, and finally applying the Hardy-type inequality

$$\int_{\mathbb{R}^3} \frac{|z|^2}{(1+|x|)^2}\,\mathrm{d}x \le c_1 \int_{\mathbb{R}^3} |\nabla z|^2\,\mathrm{d}x,$$

we obtain $|z|^2_{1,2} \le c_2 |z|^2_{1,2}[\![w]\!]_1$. As in the proof of Lemma 4.1, we use that $\lim_{|\omega|\to 0}[\![w]\!]_1 = 0$, which in this case yields $|z|_{1,2} = 0$ when ω is sufficiently small. Consequently, $(z_1, \pi_1) = (z_2, \pi_2)$. ☐

Combining Lemma 4.1 and Lemma 4.2, we can now prove our main result.

Proof of Theorem 1.1. Since $v(x) - U(x) = w(x) - \widetilde{U}(x)$ for $|x| \ge 2R$, the expansions (1.6) and (1.7) follow if we can show that $[\![w - \widetilde{U}]\!]_{1,1+\alpha} < \infty$. Similarly, since $p(x) - P(x) = q(x) - \widetilde{P}(x)$ for $|x| \ge 2R$, and recalling (4.6), the expansion (1.8) follows if we can show that $[\![q - \widetilde{P} - P_0]\!]_{2+\alpha} < \infty$. Since $[\![V_0]\!]_{1,2} < \infty$, both of these assertions are consequences of the fact that (z, π) defined by (4.7) satisfies $[\![z]\!]_{1,1+\alpha} + [\![\pi]\!]_{2+\alpha} < \infty$, which follows from Lemma 4.1 and Lemma 4.2, provided $|\omega|$ is sufficiently small. ☐

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