

ON THE ANALYTICITY AND THE ALMOST PERIODICITY OF THE SOLUTION TO THE EULER EQUATIONS WITH NON-DECAYING INITIAL VELOCITY

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ABSTRACT. The Cauchy problem of the Euler equations in the whole space is considered with non-decaying initial velocity in the frame work of $B_{\infty,1}^1$. It is proved that if the initial velocity is real analytic then the solution is also real analytic in spatial variables. Furthermore, a new estimate for the size of the radius of convergence of Taylor's expansion is established. The key of the proof is to derive the suitable estimates for the higher order derivatives of the bilinear terms. It is also shown the propagation of the almost periodicity in spatial variables.

1. INTRODUCTION AND MAIN RESULTS

Let us consider the Euler equations in \mathbb{R}^n with $n \geq 2$, describing the motion of perfect incompressible fluids,

$$(E) \quad \begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases}$$

where the unknown functions $u = u(x, t) = (u^1(x, t), \dots, u^n(x, t))$ and $p = p(x, t)$ denote the velocity field and the pressure of the fluid, respectively, while $u_0 = u_0(x) = (u_0^1(x), \dots, u_0^n(x))$ denotes the given initial velocity field.

The purpose of this paper is to show the propagation properties of the analyticity and the almost periodicity in spatial variables for the solution of (E) with non-decaying initial velocity. For the local-in-time existence and uniqueness of solutions for (E), Kato [5] proved that for the given initial velocity field $u_0 \in H^m(\mathbb{R}^n)^n$ with $m > n/2 + 1$, there exist $T = T(\|u_0\|_{H^m})$ and a unique solution u of (E) in the class $C([0, T]; H^m(\mathbb{R}^n)^n)$. Kato and Ponce [6] extended this result to the fractional-ordered Sobolev spaces $W^{s,p}(\mathbb{R}^n)^n = (1 - \Delta)^{-s/2} L^p(\mathbb{R}^n)^n$ for $s > n/p + 1, 1 < p < \infty$. In order to treat the initial velocity with the minimal regularity, Pak and Park [8] studied in the framework of $B_{\infty,1}^1(\mathbb{R}^n)$ and obtained the following result.

Theorem 1.1 (Pak-Park [8]). *For every $u_0 \in B_{\infty,1}^1(\mathbb{R}^n)^n$ with $\operatorname{div} u_0 = 0$, there exists a $T > 0$ such that (E) possesses a unique solution $u \in C([0, T]; B_{\infty,1}^1(\mathbb{R}^n)^n)$ with the pressure $\nabla p = \sum_{i,j=1}^n \nabla(-\Delta)^{-1} \partial_{x_i} u^j \partial_{x_j} u^i$.*

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The definition of the Besov space $B_{\infty,1}^1(\mathbb{R}^n)$ and its properties will be explained in Section 2. The reader can find the other results concerning the local-in-time existence and uniqueness of solutions to (E) in the reference of [8].

It has already been known that Kato's solution is real analytic in spatial variables if $u_0 \in C^\omega(\mathbb{R}^n)^n$; see Alinhac and Métivier [2], Kukavica and Vicol [7] and the references therein. In this paper, we prove the propagation of analyticity of Pak-Park's solutions. In particular, we give an improvement for the estimate for the size of the radius of convergence of Taylor's expansion.

Before stating our result about the analyticity, we set some notation. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where \mathbb{N} is the set of all positive integers. For $k \in \mathbb{N}_0$, put

$$m_k := c \frac{k!}{(k+1)^2},$$

where c is a positive constant such that one has

$$\begin{aligned} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} m_{|\beta|} m_{|\alpha-\beta|} &\leq m_{|\alpha|}, \quad \alpha \in \mathbb{N}_0^n, \\ \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} m_{|\beta|-1} m_{|\alpha-\beta|+1} &\leq |\alpha| m_{|\alpha|}, \quad \alpha \in \mathbb{N}_0^n \setminus \{0\}^n. \end{aligned}$$

For example, it suffices to take $c \leq 1/16$. For the detail, see Kahane [4] and Alinhac and Métivier [1]. Our result on the propagation of the analyticity now reads:

Theorem 1.2. *Let $u_0 \in B_{\infty,1}^1(\mathbb{R}^n)^n$ with $\operatorname{div} u_0 = 0$, and let $u \in C([0, T]; B_{\infty,1}^1(\mathbb{R}^n)^n)$ be the solution of (E). Suppose that $u_0 \in C^\omega(\mathbb{R}^n)^n$ in the following sense : there exist positive constants K_0 and ρ_0 such that*

$$\|\partial_x^\alpha u_0\|_{B_{\infty,1}^1} \leq K_0 \rho_0^{-|\alpha|} m_{|\alpha|}$$

for all $\alpha \in \mathbb{N}_0^n$. Then, $u(\cdot, t) \in C^\omega(\mathbb{R}^n)^n$ for all $t \in [0, T]$ and satisfies the following estimate : there exist positive constants $K := K(n, K_0)$, $L := L(n, K_0)$ and $\lambda := \lambda(n)$ such that

$$(1.1) \quad \|\partial_x^\alpha u(\cdot, t)\|_{B_{\infty,1}^1} \leq K \left(\frac{\rho_0}{L}\right)^{-|\alpha|} m_{|\alpha|} (1+t)^{\max\{|\alpha|-1, 0\}} \exp \left\{ \lambda |\alpha| \int_0^t \|u(\cdot, \tau)\|_{B_{\infty,1}^1} d\tau \right\}$$

for all $\alpha \in \mathbb{N}_0^n$ and $t \in [0, T]$.

Remark 1.3. (i) Since K, L and λ do not depend on T , (1.1) gives a grow-rate estimate for large time behavior of Pak-Park's solutions provided $\int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau$ is uniformly bounded in time. When $u_0 \in (\dot{B}_{\infty,1}^0 \cap \dot{B}_{\infty,1}^1)(\mathbb{R}^n)^n$, we may obtain the similar estimates of (1.1) replaced $\int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau$ by $\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau$ or $\int_0^t \|\operatorname{rot} u(\tau)\|_{\dot{B}_{\infty,1}^0} d\tau$.

(ii) From (1.1), one can derive the estimate for the size of the uniform analyticity radius of the solutions as follows :

$$\liminf_{|\alpha| \rightarrow \infty} \left(\frac{\|\partial_x^\alpha u(t)\|_{L^\infty}}{\alpha!} \right)^{-\frac{1}{|\alpha|}} \geq \frac{\rho_0}{L} (1+t)^{-1} \exp \left\{ -\lambda \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau \right\}.$$

Recently, Kukavica and Vicol [7] considered the vorticity equations of (E) in $H^s(\mathbb{T}^3)^3$ with $s > 7/2$, and obtained the following estimate for uniform analyticity radius :

$$\liminf_{|\alpha| \rightarrow \infty} \left(\frac{\|\partial_x^\alpha \operatorname{rot} u(t)\|_{L^\infty}}{\alpha!} \right)^{-\frac{1}{|\alpha|}} \geq \rho(1+t^2)^{-1} \exp \left\{ -\lambda \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau \right\}$$

with some $\rho := \rho(r, \operatorname{rot} u_0)$ and $\lambda = \lambda(r)$. Hence our result is an improvement of the previous analyticity-rate in the sense that $(1+t^2)^{-1}$ is replaced by $(1+t)^{-1}$, and clarifies that $\rho = \rho_0/L$.

The proof of Theorem 1.2 is based on the inductive argument with respect to $|\alpha|$. The key of the proof is to derive the suitable estimates for the higher order derivatives of the nonlinear term of (E). To this end, we appeal to the technique due to [4], and use the commutator type estimates, the bilinear estimates (see Lemma 2.2 and Lemma 2.3 below) and the trajectory flow argument.

We next consider the almost periodicity in spatial variables. We recall the definition of the almost periodicity in the sense of Bohr.

Definition 1.4. Let f be a bounded continuous function on \mathbb{R}^n . Put

$$\Sigma_f := \{ \tau_\xi f \mid \xi \in \mathbb{R}^n \} (\subset L^\infty(\mathbb{R}^n)), \quad \tau_\xi f := f(\cdot + \xi).$$

Then, f is called almost periodic in \mathbb{R}^n if Σ_f is relatively compact in $L^\infty(\mathbb{R}^n)$.

We now state the second result of this paper.

Theorem 1.5. Let $u_0 \in B_{\infty,1}^1(\mathbb{R}^n)^n$ with $\operatorname{div} u_0 = 0$ and let $u \in C([0, T]; B_{\infty,1}^1(\mathbb{R}^n)^n)$ be the solution of (E). Suppose that u_0 is almost-periodic in \mathbb{R}^n , then the solution $u(\cdot, t)$ of (E) is almost-periodic in \mathbb{R}^n for all $t \in [0, T]$.

The same assertion is known for the solutions to the Navier-Stokes equations by Giga, Mahalov and Nicolaenko [3]. Recently, Taniuchi, Tashiro and Yoneda [9] proved the almost periodicity of weak solutions to (E) in the whole plane \mathbb{R}^2 when $u_0 \in L^\infty(\mathbb{R}^2)^2$. On the other hand, in the Theorem 1.5, we treat the classical solutions and all space-dimensions $n \geq 2$. The proof of Theorem 1.5 is based on the argument given by [3]. The key of the proof is to use the estimate concerning the continuity with respect to the initial velocities, see Lemma 4.1 below.

This paper is organized as follows. In Section 2, we introduce the notation that will be used throughout the paper, and recall the key lemmas which play important roles in our proof. In Sections 3 and 4, we present the proof of Theorems 1.2 and 1.5, respectively.

2. PRELIMINARIES

In this section, we introduce some notation and the function spaces. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of all rapidly decreasing functions, and let $\mathcal{S}'(\mathbb{R}^n)$ be the space of all tempered distributions. We first recall the definition of the Littlewood-Paley operators. Let $\widehat{\Phi}$ and $\widehat{\varphi}$ be the functions in $\mathcal{S}(\mathbb{R}^n)$ satisfying the following properties :

$$\begin{aligned} \operatorname{supp} \widehat{\Phi} &\subset \{ \xi \in \mathbb{R}^n \mid |\xi| \leq 5/6 \}, \quad \operatorname{supp} \widehat{\varphi} \subset \{ \xi \in \mathbb{R}^n \mid 3/5 \leq |\xi| \leq 5/3 \}, \\ \widehat{\Phi}(\xi) + \sum_{j=0}^{\infty} \widehat{\varphi}_j(\xi) &= 1, \quad \xi \in \mathbb{R}^n, \end{aligned}$$

where $\varphi_j(x) = 2^{jn}\varphi(2^jx)$ and \widehat{f} denotes the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ on \mathbb{R}^n . Given $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote

$$\Delta_j f := \begin{cases} \Phi * f, & j = -1, \\ \varphi_j * f, & j \geq 0, \\ 0, & j \leq -2, \end{cases} \quad S_k f := \sum_{j \leq k} \Delta_j f, \quad k \in \mathbb{Z},$$

where $*$ denotes the convolution operator. Then, we define the Besov spaces $B_{p,q}^s(\mathbb{R}^n)$ by the following definition.

Definition 2.1. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the Besov space $B_{p,q}^s(\mathbb{R}^n)$ is defined to be the set of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the norm

$$\|f\|_{B_{p,q}^s} = \left\| \left\{ 2^{sj} \|\Delta_j f\|_{L^p} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q}$$

is finite.

Note that $B_{p,q}^s(\mathbb{R}^n)$ is a Banach space with its norm $\|\cdot\|_{B_{p,q}^s}$. It is easy to see that

$$\|f\|_{L^\infty} = \left\| \sum_{j \in \mathbb{Z}} \Delta_j f \right\|_{L^\infty} \leq \sum_{j \in \mathbb{Z}} \|\Delta_j f\|_{L^\infty} = \|f\|_{B_{\infty,1}^0}.$$

Therefore $B_{\infty,1}^0(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, and this embedding is continuous. It is also easily obtained that $B_{\infty,1}^0(\mathbb{R}^n) \subset BUC(\mathbb{R}^n)$, where $BUC(\mathbb{R}^n)$ is the space of all bounded uniformly continuous functions on \mathbb{R}^n . Analogously, we can prove that $B_{\infty,1}^1(\mathbb{R}^n) \subset W^{1,\infty}(\mathbb{R}^n)$, which is continuous embedding. Moreover, $B_{\infty,1}^1(\mathbb{R}^n)$ contains some non-decaying functions, for example, $[x \mapsto \sin x]$, $[x \mapsto \cos x]$ and $\left[x \mapsto \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}\right]$. For more details, see Triebel [10].

We now prepare the commutator type estimates and the bilinear estimates for nonlinear terms of (E).

Lemma 2.2 (Pak-Park [8]). *There exists a positive constant $C = C(n)$ such that*

$$\sum_{j \in \mathbb{Z}} 2^j \|(S_{j-2} u \cdot \nabla) \Delta_j f - \Delta_j((u \cdot \nabla) f)\|_{L^\infty} \leq C \|u\|_{B_{\infty,1}^1} \|f\|_{B_{\infty,1}^1}$$

holds for all $(u, f) \in B_{\infty,1}^1(\mathbb{R}^n)^{n+1}$ with $\operatorname{div} u = 0$.

Lemma 2.3. *There exists a positive constant $C = C(n)$ such that*

$$\|fg\|_{B_{\infty,1}^1} \leq C(\|f\|_{L^\infty} \|g\|_{B_{\infty,1}^1} + \|g\|_{L^\infty} \|f\|_{B_{\infty,1}^1})$$

holds for all $f, g \in B_{\infty,1}^1(\mathbb{R}^n)$.

The proof of Lemma 2.3 follows from the characterization by differences of Besov norm, easily; see [10]. Hence we skip the detail of the proof. Next, we give the estimate for the gradient of pressure $\pi = \nabla p$.

Lemma 2.4 (Pak-Park [8]). *There exists a positive constant $C = C(n)$ such that*

$$\|\pi(u, v)\|_{B_{\infty,1}^1} \leq C \|u\|_{B_{\infty,1}^1} \|v\|_{B_{\infty,1}^1}$$

holds for all $u, v \in B_{\infty,1}^1(\mathbb{R}^n)^n$ with $\operatorname{div} u = \operatorname{div} v = 0$, where

$$\pi(u, v) = \sum_{j,k=1}^n \nabla(-\Delta)^{-1} \partial_{x_j} u^k \partial_{x_k} v^j = \nabla(-\Delta)^{-1} \operatorname{div} \{(u \cdot \nabla)v\}.$$

Finally, we recall the Gronwall inequality.

Lemma 2.5 (The Gronwall inequality). *Let $A \geq 0$, and let f, g and h be non-negative, continuous functions on $[0, T]$ satisfying*

$$f(t) \leq A + \int_0^t g(s) ds + \int_0^t h(s) f(s) ds$$

for all $t \in [0, T]$. Then it holds that

$$f(t) \leq A e^{\int_0^t h(\tau) d\tau} + \int_0^t e^{\int_s^t h(\tau) d\tau} g(s) ds$$

for all $t \in [0, T]$.

3. PROOF OF THEOREM 1.2

Proof of Theorem 1.2. We first notice that $u \in C([0, T]; B_{\infty,1}^s(\mathbb{R}^n)^n)$ for all $s \geq 1$, if $u_0 \in B_{\infty,1}^s(\mathbb{R}^n)^n$ for all $s \geq 1$. Hence $u(\cdot, t) \in C^\infty(\mathbb{R}^n)^n$ for all $t \in [0, T]$ and then $u \in C^\infty(\mathbb{R}^n \times [0, T])^n$, if $u_0 \in C^\infty(\mathbb{R}^n)^n$. Moreover, the time-interval in which the solution exists does not depend on s . Indeed, $T \geq C/\|u_0\|_{B_{\infty,1}^1}$ with some constant C depending only on n , and the solution u satisfies

$$(3.1) \quad \sup_{t \in [0, T]} \|u(t)\|_{B_{\infty,1}^1} \leq C_0 \|u_0\|_{B_{\infty,1}^1}$$

with some positive constant C_0 depending only on n .

Now let u_0 satisfy the assumption of Theorem 1.2. We discuss with the induction argument. In the case $\alpha = 0$, (1.1) follows from (3.1) with $K = C_0 K_0$. Next, we consider the case $|\alpha| \geq 1$. We first introduce some notation. For $l \in \mathbb{N}$ and $\lambda, L > 0$, we put

$$X_l(t) := \max_{|\alpha|=l} \|\partial_x^\alpha u(t)\|_{B_{\infty,1}^1}, \quad t \in [0, T],$$

$$Y_l = Y_l^{\lambda, L} := \max_{1 \leq k \leq l} \sup_{t \in [0, T]} \left\{ \frac{M_k(t)}{m_k} X_k(t) \right\},$$

where

$$M_k(t) = M_k^{\lambda, L}(t) := \rho_0^k L^{-(k-1)} (1+t)^{-(k-1)} e^{-\lambda k \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau}.$$

The similar notation were used in [1] and [2]. In what follows, we shall show that $Y_{|\alpha|} \leq 2K_0$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geq 1$ when λ and L are sufficiently large. We now consider the case $|\alpha| = 1$. Let k be an integer with $1 \leq k \leq n$. Taking the differential operation ∂_{x_k} to the first equation of (E), we have

$$(3.2) \quad \partial_t(\partial_{x_k} u) + (\partial_{x_k} u \cdot \nabla)u + (u \cdot \nabla)\partial_{x_k} u + \partial_{x_k} \pi(u, u) = 0,$$

where

$$\nabla p = \pi(u, u) = \sum_{j,k=1}^n \nabla(-\Delta)^{-1} \partial_{x_j} u^k \partial_{x_k} u^j = \nabla(-\Delta)^{-1} \operatorname{div} \{(u \cdot \nabla)u\}.$$

Applying the Littlewood-Paley operator Δ_j and adding the term $(S_{j-2}u \cdot \nabla)\Delta_j(\partial_{x_k}u)$ to the both sides of (3.2), we have

$$(3.3) \quad \begin{aligned} & \partial_t \Delta_j(\partial_{x_k}u) + (S_{j-2}u \cdot \nabla)\Delta_j(\partial_{x_k}u) \\ &= (S_{j-2}u \cdot \nabla)\Delta_j(\partial_{x_k}u) - \Delta_j((u \cdot \nabla)\partial_{x_k}u) - \Delta_j((\partial_{x_k}u \cdot \nabla)u) - \Delta_j(\partial_{x_k}\pi(u, u)). \end{aligned}$$

Here we consider the family of trajectory flows $\{Z_j(y, t)\}$ defined by the solution of the ordinary differential equations

$$(3.4) \quad \begin{cases} \frac{\partial}{\partial t} Z_j(y, t) = S_{j-2}u(Z_j(y, t), t), \\ Z_j(y, 0) = y. \end{cases}$$

Note that $Z_j \in C^1(\mathbb{R}^n \times [0, T])^n$, and $\operatorname{div} S_{j-2}u = 0$ implies that each $y \mapsto Z_j(y, t)$ is a volume preserving mapping from \mathbb{R}^n onto itself. From (3.3) and (3.4), we see that

$$\partial_t \Delta_j(\partial_{x_k}u) + (S_{j-2}u \cdot \nabla)\Delta_j(\partial_{x_k}u) \Big|_{(x,t)=(Z_j(y,t),t)} = \frac{\partial}{\partial t} \{ \Delta_j(\partial_{x_k}u)(Z_j(y, t), t) \},$$

which yields that

$$(3.5) \quad \begin{aligned} \Delta_j(\partial_{x_k}u)(Z_j(y, t), t) &= \Delta_j(\partial_{x_k}u_0)(y) - \int_0^t \Delta_j((\partial_{x_k}u \cdot \nabla)u)(Z_j(y, s), s) ds \\ &\quad + \int_0^t \{ (S_{j-2}u \cdot \nabla)\Delta_j(\partial_{x_k}u) - \Delta_j((u \cdot \nabla)\partial_{x_k}u) \} (Z_j(y, s), s) ds \\ &\quad - \int_0^t \Delta_j(\partial_{x_k}\pi(u, u))(Z_j(y, s), s) ds. \end{aligned}$$

Since the map $y \mapsto Z_j(y, t)$ is bijective and volume-preserving for all $t \in [0, T]$, by taking the L^∞ -norm with respect to y to both sides of (3.5), we have

$$(3.6) \quad \begin{aligned} \|\Delta_j(\partial_{x_k}u)(t)\|_{L^\infty} &\leq \|\Delta_j(\partial_{x_k}u_0)\|_{L^\infty} + \int_0^t \|\Delta_j((\partial_{x_k}u \cdot \nabla)u)(s)\|_{L^\infty} ds \\ &\quad + \int_0^t \|\{ (S_{j-2}u \cdot \nabla)\Delta_j(\partial_{x_k}u) - \Delta_j((u \cdot \nabla)\partial_{x_k}u) \} (s)\|_{L^\infty} ds \\ &\quad + \int_0^t \|\Delta_j(\partial_{x_k}\pi(u, u))(s)\|_{L^\infty} ds. \end{aligned}$$

Multiplying both sides of (3.6) by 2^j and then taking the ℓ^1 -norm in j , we obtain that

$$(3.7) \quad \begin{aligned} \|\partial_{x_k}u(t)\|_{B_{\infty,1}^1} &\leq \|\partial_{x_k}u_0\|_{B_{\infty,1}^1} + \int_0^t \|(\partial_{x_k}u \cdot \nabla)u(s)\|_{B_{\infty,1}^1} ds + \int_0^t \|\partial_{x_k}\pi(u, u)(s)\|_{B_{\infty,1}^1} ds \\ &\quad + \int_0^t \sum_{j \in \mathbb{Z}} 2^j \|\{ (S_{j-2}u \cdot \nabla)\Delta_j(\partial_{x_k}u) - \Delta_j((u \cdot \nabla)\partial_{x_k}u) \} (s)\|_{L^\infty} ds \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

It follows from the assumption on u_0 that

$$(3.8) \quad I_1 \leq K_0 \rho_0^{-1} m_1.$$

From Lemma 2.3, we see that

$$(3.9) \quad I_2 \leq C \int_0^t \|\nabla u(s)\|_{L^\infty} \|\nabla u(s)\|_{B_{\infty,1}^1} ds \leq C \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_1(s) ds,$$

where we used the continuous embedding $\|\nabla f\|_{L^\infty} \leq C\|f\|_{B_{\infty,1}^1}$. For the pressure term I_3 , it follows from Lemma 2.4 that

$$(3.10) \quad I_3 \leq 2 \int_0^t \|\pi(\partial_{x_k} u, u)(s)\|_{B_{\infty,1}^1} ds \leq C \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_1(s) ds.$$

For the estimate of I_4 , we have from Lemma 2.2 that

$$(3.11) \quad I_4 \leq C \int_0^t \|u(s)\|_{B_{\infty,1}^1} \|\partial_{x_k} u(s)\|_{B_{\infty,1}^1} ds \leq C \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_1(s) ds.$$

Substituting (3.8), (3.9), (3.10) and (3.11) into (3.7), we have

$$(3.12) \quad \|\partial_{x_k} u(t)\|_{B_{\infty,1}^1} \leq K_0 \rho_0^{-1} m_1 + C_1 \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_1(s) ds$$

with some positive constant C_1 depending only on n . Since $k \in \{1, \dots, n\}$ is arbitrary, it follows from (3.12) that

$$X_1(t) \leq K_0 \rho_0^{-1} m_1 + C_1 \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_1(s) ds,$$

which implies by Lemma 2.5 that

$$(3.13) \quad X_1(t) \leq K_0 \rho_0^{-1} m_1 e^{C_1 \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau}.$$

By choosing $\lambda \geq C_1$, we obtain from (3.13) that

$$\frac{M_1(t)}{m_1} X_1(t) \leq K_0 e^{(C_1 - \lambda) \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau} \leq K_0,$$

which yields that

$$(3.14) \quad Y_1 \leq K_0.$$

Next, we consider the case $|\alpha| \geq 2$. Let α be a multi-index with $|\alpha| \geq 2$. Taking the differential operation ∂_x^α to the first equation of (E), we have

$$(3.15) \quad \partial_t(\partial_x^\alpha u) + \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} (\partial_x^\beta u \cdot \nabla) \partial_x^{\alpha-\beta} u + \partial_x^\alpha \pi(u, u) = 0.$$

Applying the Littlewood-Paley operator Δ_j and adding the term $(S_{j-2} u \cdot \nabla) \Delta_j(\partial_x^\alpha u)$ to the both sides of (3.15), we have

$$(3.16) \quad \begin{aligned} & \partial_t \Delta_j(\partial_x^\alpha u) + (S_{j-2} u \cdot \nabla) \Delta_j(\partial_x^\alpha u) \\ &= (S_{j-2} u \cdot \nabla) \Delta_j(\partial_x^\alpha u) - \Delta_j((u \cdot \nabla) \partial_x^\alpha u) \\ & \quad - \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \Delta_j((\partial_x^\beta u \cdot \nabla) \partial_x^{\alpha-\beta} u) - \Delta_j(\partial_x^\alpha \pi(u, u)) \end{aligned}$$

Similarly to the case of $|\alpha| = 1$, we have from (3.16) that

$$\begin{aligned}
(3.17) \quad & \|\Delta_j(\partial_x^\alpha u)(t)\|_{L^\infty} \leq \|\Delta_j(\partial_x^\alpha u_0)\|_{L^\infty} + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \int_0^t \|\Delta_j((\partial_x^\beta u \cdot \nabla) \partial_x^{\alpha-\beta} u)(s)\|_{L^\infty} ds \\
& + \int_0^t \|\Delta_j(\partial_x^\alpha \pi(u, u))(s)\|_{L^\infty} ds \\
& + \int_0^t \|\{(S_{j-2} u \cdot \nabla) \Delta_j(\partial_x^\alpha u) - \Delta_j((u \cdot \nabla) \partial_x^\alpha u)\}(s)\|_{L^\infty} ds.
\end{aligned}$$

Multiplying both sides of (3.17) by 2^j and then taking the ℓ^1 -norm in j , we obtain that

$$\begin{aligned}
(3.18) \quad & \|\partial_x^\alpha u(t)\|_{B_{\infty,1}^1} \leq \|\partial_x^\alpha u_0\|_{B_{\infty,1}^1} + \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \int_0^t \|(\partial_x^\beta u \cdot \nabla) \partial_x^{\alpha-\beta} u(s)\|_{B_{\infty,1}^1} ds \\
& + \int_0^t \|\partial_x^\alpha \pi(u, u)(s)\|_{B_{\infty,1}^1} ds \\
& + \int_0^t \sum_{j \in \mathbb{Z}} 2^j \|\{(S_{j-2} u \cdot \nabla) \Delta_j(\partial_x^\alpha u) - \Delta_j((u \cdot \nabla) \partial_x^\alpha u)\}(s)\|_{L^\infty} ds \\
& =: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

It follows from the assumption on u_0 that

$$(3.19) \quad J_1 \leq K_0 \rho_0^{-|\alpha|} m_{|\alpha|}.$$

For the estimate of J_2 , we have from Lemma 2.3 and the continuous embedding that

$$\begin{aligned}
(3.20) \quad & J_2 \leq C \sum_{0 < \beta \leq \alpha} \binom{\alpha}{\beta} \int_0^t \left(\|\partial_x^\beta u(s)\|_{L^\infty} \|\nabla \partial_x^{\alpha-\beta} u(s)\|_{B_{\infty,1}^1} + \|\nabla \partial_x^{\alpha-\beta} u(s)\|_{L^\infty} \|\partial_x^\beta u(s)\|_{B_{\infty,1}^1} \right) ds \\
& = C \sum_{j=1}^n \binom{\alpha}{e_j} \int_0^t \|\partial_{x_j} u(s)\|_{L^\infty} \|\nabla \partial_x^{\alpha-e_j} u(s)\|_{B_{\infty,1}^1} ds \\
& + C \sum_{\substack{0 < \beta \leq \alpha \\ |\beta| \geq 2}} \binom{\alpha}{\beta} \int_0^t \|\partial_x^\beta u(s)\|_{L^\infty} \|\nabla \partial_x^{\alpha-\beta} u(s)\|_{B_{\infty,1}^1} ds \\
& + C \int_0^t \|\nabla u(s)\|_{L^\infty} \|\partial_x^\alpha u(s)\|_{B_{\infty,1}^1} ds + C \sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} \int_0^t \|\nabla \partial_x^{\alpha-\beta} u(s)\|_{L^\infty} \|\partial_x^\beta u(s)\|_{B_{\infty,1}^1} ds \\
& \leq C |\alpha| \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_{|\alpha|}(s) ds + C \sum_{\substack{0 < \beta \leq \alpha \\ |\beta| \geq 2}} \binom{\alpha}{\beta} \int_0^t X_{|\beta|-1}(s) X_{|\alpha-\beta|+1}(s) ds \\
& + C \sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} \int_0^t X_{|\beta|}(s) X_{|\alpha-\beta|}(s) ds.
\end{aligned}$$

For the pressure term J_3 , from Lemma 2.4, we have

$$\begin{aligned}
J_3 &\leq \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \int_0^t \|\pi(\partial_x^\beta u, \partial_x^{\alpha-\beta} u)(s)\|_{B_{\infty,1}^1} ds \\
&\leq C \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \int_0^t \|\partial_x^\beta u(s)\|_{B_{\infty,1}^1} \|\partial_x^{\alpha-\beta} u(s)\|_{B_{\infty,1}^1} ds \\
(3.21) \quad &\leq C \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_{|\alpha|}(s) ds + C \sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} \int_0^t X_{|\beta|}(s) X_{|\alpha-\beta|}(s) ds.
\end{aligned}$$

For the estimate of J_4 , it follows from Lemma 2.2 that

$$(3.22) \quad J_4 \leq C \int_0^t \|u(s)\|_{B_{\infty,1}^1} \|\partial_x^\alpha u(s)\|_{B_{\infty,1}^1} ds \leq C \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_{|\alpha|}(s) ds.$$

Substituting (3.19), (3.20), (3.21) and (3.22) to (3.18), we have

$$\begin{aligned}
\|\partial_x^\alpha u(t)\|_{B_{\infty,1}^1} &\leq K_0 \rho_0^{-|\alpha|} m_{|\alpha|} + C |\alpha| \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_{|\alpha|}(s) ds \\
(3.23) \quad &\quad + C \sum_{\substack{0 < \beta \leq \alpha \\ |\beta| \geq 2}} \binom{\alpha}{\beta} \int_0^t X_{|\beta|-1}(s) X_{|\alpha-\beta|+1}(s) ds \\
&\quad + C \sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} \int_0^t X_{|\beta|}(s) X_{|\alpha-\beta|}(s) ds.
\end{aligned}$$

Furthermore, for the third term of the right-hand side of (3.23), we see that

$$\begin{aligned}
&\sum_{\substack{0 < \beta \leq \alpha \\ |\beta| \geq 2}} \binom{\alpha}{\beta} \int_0^t X_{|\beta|-1}(s) X_{|\alpha-\beta|+1}(s) ds \\
&= \sum_{\substack{0 < \beta \leq \alpha \\ |\beta| \geq 2}} \binom{\alpha}{\beta} \int_0^t \frac{M_{|\beta|-1}(s)}{m_{|\beta|-1}} X_{|\beta|-1}(s) \frac{M_{|\alpha-\beta|+1}(s)}{m_{|\alpha-\beta|+1}} X_{|\alpha-\beta|+1}(s) \frac{m_{|\beta|-1}}{M_{|\beta|-1}(s)} \frac{m_{|\alpha-\beta|+1}}{M_{|\alpha-\beta|+1}(s)} ds \\
&\leq \sum_{\substack{0 < \beta \leq \alpha \\ |\beta| \geq 2}} \binom{\alpha}{\beta} m_{|\beta|-1} m_{|\alpha-\beta|+1} \rho_0^{-|\alpha|} L^{|\alpha|-2} (Y_{|\alpha|-1})^2 \int_0^t (1+s)^{|\alpha|-2} e^{\lambda|\alpha| \int_0^s \|u(\tau)\|_{B_{\infty,1}^1} d\tau} ds \\
(3.24) \quad &\leq |\alpha| m_{|\alpha|} \rho_0^{-|\alpha|} L^{|\alpha|-2} (Y_{|\alpha|-1})^2 \int_0^t (1+s)^{|\alpha|-2} e^{\lambda|\alpha| \int_0^s \|u(\tau)\|_{B_{\infty,1}^1} d\tau} ds.
\end{aligned}$$

Similarly, for the fourth term of the right hand side of (3.23), we have

$$\begin{aligned}
(3.25) \quad &\sum_{0 < \beta < \alpha} \binom{\alpha}{\beta} \int_0^t X_{|\beta|}(s) X_{|\alpha-\beta|}(s) ds \\
&\leq m_{|\alpha|} \rho_0^{-|\alpha|} L^{|\alpha|-2} (Y_{|\alpha|-1})^2 \int_0^t (1+s)^{|\alpha|-2} e^{\lambda|\alpha| \int_0^s \|u(\tau)\|_{B_{\infty,1}^1} d\tau} ds.
\end{aligned}$$

Substituting (3.24) and (3.25) to (3.23), we have

$$\begin{aligned} \|\partial_x^\alpha u(t)\|_{B_{\infty,1}^1} &\leq K_0 \rho_0^{-|\alpha|} m_{|\alpha|} + C|\alpha| \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_{|\alpha|}(s) ds \\ &\quad + C|\alpha| m_{|\alpha|} \rho_0^{-|\alpha|} L^{|\alpha|-2} (Y_{|\alpha|-1})^2 \int_0^t (1+s)^{|\alpha|-2} e^{\lambda|\alpha| \int_0^s \|u(\tau)\|_{B_{\infty,1}^1} d\tau} ds, \end{aligned}$$

which implies that

$$(3.26) \quad \begin{aligned} X_{|\alpha|}(t) &\leq K_0 \rho_0^{-|\alpha|} m_{|\alpha|} + C|\alpha| \int_0^t \|u(s)\|_{B_{\infty,1}^1} X_{|\alpha|}(s) ds \\ &\quad + C|\alpha| m_{|\alpha|} \rho_0^{-|\alpha|} L^{|\alpha|-2} (Y_{|\alpha|-1})^2 \int_0^t (1+s)^{|\alpha|-2} e^{\lambda|\alpha| \int_0^s \|u(\tau)\|_{B_{\infty,1}^1} d\tau} ds. \end{aligned}$$

By Lemma 2.5, we obtain from (3.26) that

$$\begin{aligned} X_{|\alpha|}(t) &\leq K_0 \rho_0^{-|\alpha|} m_{|\alpha|} e^{C_2|\alpha| \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau} + C_2|\alpha| m_{|\alpha|} \rho_0^{-|\alpha|} L^{|\alpha|-2} (Y_{|\alpha|-1})^2 \\ &\quad \times \int_0^t (1+s)^{|\alpha|-2} e^{C_2|\alpha| \int_s^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau + \lambda|\alpha| \int_0^s \|u(\tau)\|_{B_{\infty,1}^1} d\tau} ds \end{aligned}$$

with some positive constant C_2 depending only on n . By choosing $\lambda \geq C_2$ and $L \geq 1$, we thus have

$$\begin{aligned} \frac{M_{|\alpha|}(t)}{m_{|\alpha|}} X_{|\alpha|}(t) &\leq K_0 L^{-(|\alpha|-1)} (1+t)^{-(|\alpha|-1)} e^{(C_2-\lambda)|\alpha| \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau} \\ &\quad + C_2|\alpha| L^{-1} (1+t)^{-(|\alpha|-1)} (Y_{|\alpha|-1})^2 \int_0^t (1+s)^{|\alpha|-2} e^{(C_2-\lambda)|\alpha| \int_s^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau} ds \\ &\leq K_0 + C_2|\alpha| L^{-1} (1+t)^{-(|\alpha|-1)} (Y_{|\alpha|-1})^2 \int_0^t (1+s)^{|\alpha|-2} ds \\ &\leq K_0 + \frac{2C_2}{L} (Y_{|\alpha|-1})^2. \end{aligned}$$

The above estimate with (3.14) implies that

$$(3.27) \quad Y_{|\alpha|} \leq K_0 + \frac{2C_2}{L} (Y_{|\alpha|-1})^2$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geq 2$. From (3.14) and (3.27), we obtain by the standard inductive argument that

$$(3.28) \quad Y_{|\alpha|} \leq 2K_0$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geq 1$, provided $\lambda \geq \max\{C_1, C_2\}$ and $L \geq \max\{1, 8C_2K_0\}$. Therefore, it follows from (3.28) that

$$(3.29) \quad \|\partial_x^\alpha u(t)\|_{B_{\infty,1}^1} \leq \frac{2K_0}{L} \left(\frac{\rho_0}{L}\right)^{-|\alpha|} m_{|\alpha|} (1+t)^{|\alpha|-1} e^{\lambda|\alpha| \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau}$$

for all $t \in [0, T]$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geq 1$. From (3.1) and (3.29) with $K = K_0 \max\{C_0, 2/L\}$, we complete the proof of Theorem 1.2. \square

4. PROOF OF THEOREM 1.5

In this section, we present the proof of Theorem 1.5. To this end, we will use the following lemmas.

Lemma 4.1 (Pak-Park [8]). *Let $u_0, v_0 \in B_{\infty,1}^1(\mathbb{R}^n)^n$ with $\operatorname{div} u_0 = \operatorname{div} v_0 = 0$, and let $u, v \in C([0, T]; B_{\infty,1}^1(\mathbb{R}^n)^n)$ be the solutions of (E) with $u(x, 0) = u_0(x)$ and $v(x, 0) = v_0(x)$. Then, there exists a positive constant $C = C(n)$ such that*

$$\|u(t) - v(t)\|_{B_{\infty,1}^0} \leq \|u_0 - v_0\|_{B_{\infty,1}^0} \exp \left\{ C \int_0^t \left(\|u(s)\|_{B_{\infty,1}^1} + \|v(s)\|_{B_{\infty,1}^1} \right) ds \right\}$$

holds for all $t \in [0, T]$.

Lemma 4.2. *Let $f \in B_{\infty,1}^0(\mathbb{R}^n)$. Then, f is almost periodic in \mathbb{R}^n if and only if Σ_f is relatively compact in $B_{\infty,1}^0(\mathbb{R}^n)$.*

Note that $\Sigma_f \subset B_{\infty,1}^0(\mathbb{R}^n)$ if $f \in B_{\infty,1}^0(\mathbb{R}^n)$. We can prove Lemma 4.2 by the similar argument in Giga, Mahalov and Nicolaenko [3], where they proved the case of $\dot{B}_{\infty,1}^0(\mathbb{R}^n)$. Hence we omit the proof.

Proof of Theorem 1.5. Let $\{S(t)\}_{0 \leq t \leq T}$ be the solution maps, that is, $S(t) : B_{\infty,1}^1(\mathbb{R}^n) \rightarrow B_{\infty,1}^1(\mathbb{R}^n)$ is defined by $S(t)u_0 = u(\cdot, t)$. Since (E) is translation invariant with respect to the space variables, it follows from the uniqueness that $S(t)\tau_\eta u_0 = \tau_\eta u(\cdot, t)$. Hence the map $S(t)$ is surjective from Σ_{u_0} onto $\Sigma_{u(\cdot, t)}$.

Let $\{u_j(\cdot, t)\}_{j=1}^\infty$ be an arbitrary sequence in $\Sigma_{u(\cdot, t)}$. Note that u_j can be written as $u_j(\cdot, t) = \tau_{\eta_j} u(\cdot, t)$ with some $\eta_j \in \mathbb{R}^n$. Moreover, it holds that $u_j(\cdot, t) = S(t)\tau_{\eta_j} u_0$ by the surjectivity of $S(t)$. Since $u_0 \in B_{\infty,1}^1(\mathbb{R}^n)$ is almost periodic, by Lemma 4.2, there exists a subsequence of $\{\tau_{\eta_j} u_0\}_{j=1}^\infty$, again denoted by $\{\tau_{\eta_j} u_0\}_{j=1}^\infty$, such that

$$(4.1) \quad \|\tau_{\eta_j} u_0 - \tau_{\eta_k} u_0\|_{B_{\infty,1}^0} \rightarrow 0$$

as $j, k \rightarrow \infty$. We remark that the norm of $B_{\infty,1}^1(\mathbb{R}^n)$ is invariant under the translation. Hence from Lemma 4.1 and (4.1), we obtain that

$$\begin{aligned} \|u_j(t) - u_k(t)\|_{B_{\infty,1}^0} &\leq \|\tau_{\eta_j} u_0 - \tau_{\eta_k} u_0\|_{B_{\infty,1}^0} \exp \left\{ C \int_0^t \left(\|u_j(s)\|_{B_{\infty,1}^1} + \|u_k(s)\|_{B_{\infty,1}^1} \right) ds \right\} \\ &= \|\tau_{\eta_j} u_0 - \tau_{\eta_k} u_0\|_{B_{\infty,1}^0} \exp \left\{ 2C \int_0^t \|u(s)\|_{B_{\infty,1}^1} ds \right\} \rightarrow 0 \end{aligned}$$

as $j, k \rightarrow \infty$, which implies that $u(\cdot, t)$ is almost periodic in \mathbb{R}^n for all $t \in [0, T]$. \square

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