On the integer points in a lattice polytope: n-fold Minkowski sum and boundary

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ABSTRACT. In this article we compare the set of integer points in the homothetic copy $n\Pi$ of a lattice polytope $\Pi \subseteq \mathbb{R}^d$ with the set of all sums $x_1 + \cdots + x_n$ with $x_1, ..., x_n \in \Pi \cap \mathbb{Z}^d$ and $n \in \mathbb{N}$. We give conditions on the polytope Π under which these two sets coincide and we discuss two notions of boundary for subsets of $\mathbb{Z}^d$ or, more generally, subsets of a finitely generated discrete group.

1 Introduction

Throughout, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ the natural, integer, rational and real numbers, respectively, and we fix a number $d \in \mathbb{N}$. For arbitrary $n \in \mathbb{N}$, we compare the set of all integer points in the homothetic copy $n\Pi$ of a lattice polytope $\Pi \subseteq \mathbb{R}^d$ (that means Π is the convex hull of a finite number of points in $\mathbb{Z}^d$) with the set of sums $x_1 + \cdots + x_n$ with $x_1, \ldots, x_n \in \Pi \cap \mathbb{Z}^d$. It is easy to see that the latter set is always contained in the first – but in general, they are different. We give conditions on the polytope under which these two sets coincide and we discuss two notions of boundary for subsets of $\mathbb{Z}^d$ and, more generally, of a finitely generated (not necessarily commutative) discrete group.

The motivation for this paper stems from the study of projection methods for the approximate solution of operator equations. Let A be a bounded linear operator acting on a Banach space. To solve the operator equation $Au = f$ numerically, one chooses a sequence (Q_n) of projections (usually assumed to be of finite rank and to converge strongly to the identity operator) and replaces the equation $Au = f$ by the sequence of the linear systems $Q_n A Q_n u_n = Q_n f$. What one expects is that, under suitable conditions, the solutions u_n of these systems converge to the solution u of the original equation. If the Banach space on which A lives consists of functions on a countable set Y (think of a sequence space $l^2(Y)$, for example) then it is convenient to choose an increasing sequence (Y_n) of finite subsets of Y and to specify Q_n as the operator P_{Y_n} which restricts a function on Y to Y_n.

In this paper, we will be concerned with the case when Y is a finitely generated discrete group Γ. In case Γ is the additive group $\mathbb{Z}^d$, a typical (rather geometric) approach [15, 17, 13, 14] to design a projection method is to fix a compact set (for example a lattice polytope) $\Pi \subseteq \mathbb{R}^d$ and to consider the operator P_{Π_n} of restriction to (likewise, the operator of multiplication by the characteristic function of) the set

$$\Pi_n := (n\Pi) \cap \mathbb{Z}^d \tag{1}$$

1

for $n \in \mathbb{N}$. If we assume that the origin is in the interior of Π then it follows that

$$\Pi_n \subseteq \Pi_{n+1} \quad \text{for all } n \in \mathbb{N} \qquad \text{and} \qquad \bigcup_{n \in \mathbb{N}} \Pi_n = \mathbb{Z}^d, \tag{2}$$

whence the sequence $(\Pi_n)_{n \in \mathbb{N}}$ gives rise to an increasing sequence of finite-dimensional projection operators on $l^2(\mathbb{Z}^d)$ that strongly converges to the identity operator as $n \to \infty$.

In the case of a general finitely generated group Γ, this geometric approach is clearly infeasible. A natural idea here is to fix a finite set $\Omega \subseteq \Gamma$ of generators of Γ (that is, we assume that Ω generates Γ as a semi-group) and to consider the set

$$\Omega_n := \{x_1 x_2 \cdots x_n : x_1, x_2, \ldots, x_n \in \Omega\} \tag{3}$$

of all words of length $n \geq 1$ over the alphabet Ω in place of (1). If Ω is symmetric (in the sense that $x^{-1} \in \Omega$ if $x \in \Omega$), contains the identity element e of Γ (in analogy to the above approach in $\mathbb{Z}^d$) and if Γ is equipped with the word metric over Ω then Ω_n is the disk of radius n in Γ around the identity e. Moreover, one gets that also $(\Omega_n)_{n \in \mathbb{N}}$ yields an increasing sequence of finite-dimensional projections with strong limit identity, i.e., (2) holds with Π_m replaced by Ω_m and $\mathbb{Z}^d$ by Γ.

A natural question before proposing the latter approach for general finitely generated groups Γ is whether or not the geometric approach (1) and the algebraic approach (3) coincide if we have $\Gamma = \mathbb{Z}^d$ and use $\Omega := \Pi \cap \mathbb{Z}^d$ as finite set of generators in (3) with a symmetric lattice polytope $\Pi \subseteq \mathbb{R}^d$ containing the origin in its interior. This question is discussed in Section 2. We will give conditions on the polytope Π under which (1) and (3) coincide – but in general they do not.

In Section 3 we address a further question that arises in the study of projection methods. In [19] it has been pointed out that the "boundaries" $\partial_\Omega \Omega_n$ (if appropriately defined) of the sets Ω_n (or Π_n) hold the key to the answer to whether or not the projection method

$$P_{\Omega_n} A P_{\Omega_n} u_n = P_{\Omega_n} f, \qquad n \in \mathbb{N},$$

yields stable approximations u_n to the solution u of $Au = f$. In Section 3 we give special emphasis to the question whether the "algebraic boundaries" $\partial_\Omega \Omega_n$ coincide with the corresponding "numerical boundaries" $\Omega_n \setminus \Omega_{n-1}$.

2 Lattice polytopes, enlargements and integer points

Now fix $d \in \mathbb{N}$, let $e_1, ..., e_d$ be the standard unit vectors of $\mathbb{R}^d$, and denote the unit simplex $\text{conv}\{0, e_1, ..., e_d\}$ by σ_d. (For standard notions on convex polytopes we recommend [9, 8, 22]; for lattice polytopes see [3, 6, 7, 20, 21].)

Given a non-empty subset S of $\mathbb{R}^d$ and a positive integer n, we write

$$nS := \{ns : s \in S\} \quad \text{and} \quad n * S := \{s_1 + \cdots + s_n : s_1, ..., s_n \in S\} = S + \cdots + S$$

for the ratio-n homothetic copy and n-fold Minkowski sum of S, respectively. For convenience, we also set $0S := \{0\}$ and $0 * S := \{0\}$. It is easy to see that $nS = n * S$ holds for all $n \in \mathbb{N}$ if S is convex. Indeed, the inclusion $nS \subseteq n * S$ is always true, and, by convexity of S, $s_1 + \cdots + s_n$

can be written as ns with $s = (s_1 + \cdots + s_n)/n \in S$ for all $n \in \mathbb{N}$ and $s_1, ..., s_n \in S$. (Note that equality of nS and $n * S$ for all $n \in \mathbb{N}$ does not imply convexity of S, as $S = \mathbb{Q}$ shows.)

For a convex set $\Pi \subseteq \mathbb{R}^d$ containing at least two integer points, it is clear that $(n\Pi) \cap \mathbb{Z}^d \neq n(\Pi \cap \mathbb{Z}^d)$ as soon as $n > 1$. But (as motivated in the introduction) a much more interesting question is whether or not

$$(n\Pi) \cap \mathbb{Z}^d \;=\; n * (\Pi \cap \mathbb{Z}^d) \tag{4}$$

is true for all $n \in \mathbb{N}$. We will study this question for certain polytopes Π.

Let $v_1, ..., v_k$ be points of $\mathbb{Z}^d$ with their affine hull equal to $\mathbb{R}^d$ and put $\Pi = \mathrm{conv}\{v_1, ..., v_k\}$. Π is a so-called *lattice polytope* as all its vertices are in $\mathbb{Z}^d$. We will suppose that there is no proper subset I of $\{1, ..., k\}$ with $\Pi = \mathrm{conv}\{v_i : i \in I\}$, so that $v_1, ..., v_k$ are the vertices of Π. If $\Pi \cap \mathbb{Z}^d$ only consists of the vertices of Π then Π is called an *elementary polytope* [10, 20] (or a *lattice-(point-)free polytope* [2, 11]). The following lemma is fairly standard:

Lemma 2.1 *Let $A \in \mathbb{Z}^{d \times d}$ be a matrix and $a_1, ..., a_d \in \mathbb{Z}^d$ its columns.*

   a) *The following conditions are equivalent:*

     (i) $A(\mathbb{Z}^d) = \mathbb{Z}^d$.
    (ii) A^{-1} *is an integer matrix.*
   (iii) $\det A = \pm 1$.
   (iv) *The parallelotope $A([0,1]^d)$ spanned by $a_1, ..., a_d$ has volume 1.*
    (v) *The parallelotope $A([0,1]^d)$ is elementary.*

   b) *The condition*

    (vi) *The simplex $A(\sigma_d) = \mathrm{conv}\{0, a_1, ..., a_d\}$ is elementary.*

   is necessary for (i)–(v); it is moreover sufficient iff $d \in \{1, 2\}$.

If (i)–(v) hold then A is called an *integer unimodular matrix* [7, 20] and the simplex $A(\sigma_d)$ has volume $1/d!$ and is sometimes called a *primitive* (or *unimodular*) *simplex* (e.g. [10]). So primitive simplices are elementary, and the converse holds iff $d \in \{1, 2\}$.

For the sake of completeness, we give a short sketch of the proof of Lemma 2.1:

Proof. Part a) follows by standard arguments using $\det A^{-1} = 1/\det A$ and $\mathrm{vol}(A([0,1]^d)) = |\det A| \, \mathrm{vol}([0,1]^d)$. The implication $(v) \Rightarrow (vi)$ holds by $\sigma_d \subseteq [0,1]^d$. For $d = 1$, the implication $(vi) \Rightarrow (v)$ is clear by $\sigma_1 = [0,1]^1$. For $d = 2$, if x is an integer non-vertex point in $A([0,1]^2)$, then also $a_1 + a_2 - x$ is an integer non-vertex point in $A([0,1]^2)$. But one of the two points is in $A(\sigma_2)$, so that $(vi) \Rightarrow (v)$ holds. For $d \geq 3$, there are elementary but not primitive simplices (see Examples 2.2 a and b below). ∎

Here are two slightly different constructions leading to elementary but not primitive simplices in dimension $d \geq 3$.

Example 2.2 a) Let $d \geq 3$, fix an $m \in \mathbb{N}$, take $a_1 := e_1$, $a_2 := e_2$, ... , $a_{d-1} := e_{d-1} \in \mathbb{Z}^d$ and $a_d := (-1, ..., -1, m)^\top$, and let $A \in \mathbb{Z}^{d \times d}$ be the matrix with columns $a_1, ..., a_d$. Then

$$\Sigma_{d,m} \;:=\; A(\sigma_d) \;=\; \mathrm{conv}\{0, a_1, ..., a_d\}$$

is primitive iff $m = \det A = 1$. On the other hand, the number of integer points in $\Sigma_{d,m}$ apart from its $d+1$ vertices is equal to $k = \lfloor m/d \rfloor$ (integer division), and these k other integer points are $1e_d, ..., ke_d$. (To see this, look at the projection of $\Sigma_{d,m}$ to the hyperplane spanned by $e_1, ..., e_{d-1}$.) So for $m \in \{2, ..., d-1\}$ we have an elementary but not primitive simplex.

b) If we change the last column in the above example from $a_d = (-1, ..., -1, m)^\top$ to $a'_d :=$ $(1, ..., 1, m)^\top$ with $m \in \mathbb{N}$ and call the new matrix A' then, again, the simplex

$$\Sigma'_{d,m} := A'(\sigma_d) = \operatorname{conv}\{0, a_1, ..., a_{d-1}, a'_d\}$$

is not primitive for $m = \det A' \geq 2$ but now it is elementary for all $m \in \mathbb{N}$. So this simplex $\Sigma'_{d,m}$ can have arbitrarily large volume $m/d!$ without containing any integer points other than its vertices. The simplices $\Sigma'_{d,m}$ were first considered (because of this property) by Reeve [16] in case $d = 3$ and have since been termed *Reeve simplices*. There are results that relate the maximal volume of an elementary polytope in $\mathbb{R}^d$ to its surface area [5] or its inradius [2]. $\square$

Given a full-dimensional lattice polytope $\Pi \subseteq \mathbb{R}^d$ and a set $\mathcal{T}$ of full-dimensional lattice simplices $S_1, ..., S_m \subseteq \Pi$ with

$$\bigcup_{i=1}^m S_i = \Pi \qquad \text{and} \qquad S_i \cap S_j \text{ is a face of both } S_i \text{ and } S_j, \ \forall i, j,$$

then the set $\mathcal{T} = \{S_1, ..., S_m\}$ is called a *triangulation* of Π. The triangulation $\mathcal{T}$ is called *elementary* or *primitive* if all its elements S_i are, respectively, elementary or primitive simplices.

Here is our main result on the equality (4):

Proposition 2.3 *If a full-dimensional lattice polytope $\Pi \subseteq \mathbb{R}^d$ possesses a primitive triangulation then equality (4) holds for all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$ and $x \in n * (\Pi \cap \mathbb{Z}^d)$. Then $x = p_1 + ... + p_n$ for some $p_1, ..., p_n \in \Pi \cap \mathbb{Z}^d$, so that $x \in \mathbb{Z}^d$ and $x = np$ with $p = (p_1 + ... + p_n)/n$. But $p \in \Pi$ by convexity of Π.

Now let $x \in (n\Pi) \cap \mathbb{Z}^d$, so $x = np$ is an integer point with $p \in \Pi$. Let $\mathcal{T} = \{S_1, ..., S_m\}$ be a primitive triangulation of Π and let $w_0, ..., w_d$ be the vertices of a simplex S_i that contains p. Now there is a unique way to write p as a convex combination of $w_0, ..., w_d$. So there are $\alpha_0, ..., \alpha_d \in [0, 1]$ so that $p = \alpha_0 w_0 + ... + \alpha_d w_d$ and $\alpha_0 + ... + \alpha_d = 1$. Together with $x = np$ this implies

$$
\begin{pmatrix} | & | & & | \\ w_0 & w_1 & \cdots & w_d \\ | & | & & | \\ \hline 1 & 1 & \cdots & 1 \end{pmatrix}
\begin{pmatrix} n\alpha_0 \\ n\alpha_1 \\ \vdots \\ n\alpha_d \end{pmatrix}
= n \begin{pmatrix} | \\ p \\ | \\ \hline 1 \end{pmatrix}
= \begin{pmatrix} | \\ x \\ | \\ \hline n \end{pmatrix}.
\tag{5}
$$

If we refer to the matrix in (5) as M then, after subtracting the first column from all the others and then expanding by the last row,

$$
\det M = \det \begin{pmatrix} | & | & & | \\ w_0 & w_1 - w_0 & \cdots & w_d - w_0 \\ | & | & & | \\ \hline 1 & 0 & \cdots & 0 \end{pmatrix}
$$

$$
= (-1)^{d+1} \det \begin{pmatrix} | & & | \\ w_1 - w_0 & \cdots & w_d - w_0 \\ | & & | \end{pmatrix} \in \{\pm 1\}
$$

by Lemma 2.1 since $w_1 - w_0, \cdots, w_d - w_0$ span the primitive simplex $S_i - w_0$. So M^{-1} exists and is an integer matrix. By (5), it follows that $\beta_0 := n\alpha_0, \cdots, \beta_d := n\alpha_d$ are integers since M^{-1} and x have integer entries. Summarizing, we get that

$$\beta_0 w_0 + \beta_1 w_1 \dots + \beta_d w_d = x, \tag{6}$$

where $\beta_0, ..., \beta_d \in \{0, ..., n\}$ and $\beta_0 + ... + \beta_d = n$, so that (6) is the desired decomposition of x into a sum of n elements from $\Pi \cap \mathbb{Z}^d$. ∎

It is not clear to us whether the existence of a primitive triangulation is necessary for equality (4) to hold for all $n \in \mathbb{N}$. (Is it possible that every $p \in \Pi$ is contained in a primitive simplex $S(p) \subset \Pi$ without the existence of a "global" primitive triangulation of Π?)

It is not hard to see that every lattice polytope Π possesses an elementary triangulation. (Existence of a triangulation can be shown by induction over the number of vertices of Π, and every non-elementary simplex S_i can be further triangulated with respect to its integer non-vertex points.) Existence of a primitive triangulation however is a different question – at least in dimensions $d \geq 3$.

Corollary 2.4 *If Π is a full-dimensional lattice polytope in $\mathbb{R}^d$ with $d \in \{1, 2\}$ then equality (4) holds for all $n \in \mathbb{N}$.*

Proof. Every lattice polytope has an elementary triangulation. In dimensions $d \in \{1, 2\}$, by Lemma 2.1 b), an elementary triangulation is always primitive. Now apply Proposition 2.3. ∎

In dimension $d \geq 3$, it is generally a difficult question whether or not a given lattice polytope Π has a primitive triangulation (see e.g. [4, 10]). The simplices $\Sigma_{d,m}$ in Example 2.2 a) with $m \in \{2, ..., d-1\}$ and $\Sigma'_{d,m}$ in 2.2 b) with $m \in \{2, 3, ...\}$ are examples of lattice polytopes that have no primitive triangulation. They are also examples, where (4) is not valid for general $n \in \mathbb{N}$. For example, $2\Sigma_{3,2}$ contains e_3, which cannot be written as the sum of two integer points from $\Sigma_{3,2}$. It is even possible to give examples Π where, for a given $k \in \mathbb{N}$, (4) starts to fail at $n > k$ while holding true for $n = 1, 2, ..., k$. As an example, take $\Pi = \Sigma_{2k+1, 2}$.

The more specific question posed in the introduction is whether the fact that $\Omega := \Pi \cap \mathbb{Z}^d$ (i) contains the origin, (ii) is symmetric (i.e. $-x \in \Omega$ if $x \in \Omega$), and (iii) generates $\mathbb{Z}^d$, i.e.

$$\bigcup_{n \in \mathbb{N}} n * \Omega = \mathbb{Z}^d, \tag{7}$$

guarantees equality (4) for all $n \in \mathbb{N}$. But also that has to be answered in the negative, as the example $\Pi = \text{conv}(\Sigma_{3,3} \cup -\Sigma_{3,3}) = \text{conv}\{\pm e_1, \pm e_2, \pm(-1, -1, 3)^\top\} \subseteq \mathbb{R}^3$ shows. Indeed, $(-1, -1, 1)^\top$ is in 2Π but not in $2 * \Omega$ with $\Omega = \Pi \cap \mathbb{Z}^3 = \{\pm e_1, \pm e_2, \pm e_3, \pm(-1, -1, 3)^\top, 0\}$.

After all these examples, here are some results on the positive side:

The symmetric hypercube $[-1, 1]^d$ is the union of 2^d shifted copies of $[0, 1]^d$, each of which has a primitive triangulation. The cross-polytope $\text{conv}\{\pm e_1, ..., \pm e_d\}$ triangulates into 2^d unit simplices around the origin. Much more involved, there is the following result by Kempf, Knudsen, Mumford and Saint-Donat [12]:

Lemma 2.5 *For every lattice polytope $\Pi \subseteq \mathbb{R}^d$, there is an integer $k \in \mathbb{N}$ such that $k\Pi$ possesses a primitive triangulation.*

Consequently, for every lattice polytope Π, there is a $k \in \mathbb{N}$ such that $k\Pi$ satisfies (4) in place of Π for all $n \in \mathbb{N}$.

Remark 2.6 The two conditions (7) with $\Omega := \Pi \cap \mathbb{Z}^d$, which says that Ω generates all of $\mathbb{Z}^d$, and $0 \in \text{int}(\Pi)$, which implies that $n * \Omega \subseteq (n+1) * \Omega$, are connected with each other and with the validity of (4) for all $n \in \mathbb{N}$. Firstly, if (4) holds for all $n \in \mathbb{N}$ and $0 \in \text{int}(\Pi)$ then

$$\bigcup_{n \in \mathbb{N}} n * (\Pi \cap \mathbb{Z}^d) \;=\; \bigcup_{n \in \mathbb{N}} (n\Pi) \cap \mathbb{Z}^d \;=\; \left(\bigcup_{n \in \mathbb{N}} n\Pi \right) \cap \mathbb{Z}^d \;=\; \mathbb{Z}^d$$

so that (7) holds. On the other hand, from (7) it follows, by the trivial inclusion "$\supseteq$" in (4), that

$$\mathbb{Z}^d \;=\; \bigcup_{n \in \mathbb{N}} n * (\Pi \cap \mathbb{Z}^d) \;\subseteq\; \bigcup_{n \in \mathbb{N}} (n\Pi) \cap \mathbb{Z}^d \;=\; \left(\bigcup_{n \in \mathbb{N}} n\Pi \right) \cap \mathbb{Z}^d \;\subseteq\; \mathbb{Z}^d,$$

whence, by the convexity of Π, $\cup n\Pi = \mathbb{R}^d$ and hence $0 \in \text{int}(\Pi)$. $\square$

3 Boundaries of subsets of a group Γ

Let Γ be a finitely generated discrete group with identity element e. We are going to introduce some notions of topological type. Note that the standard topology on Γ is the discrete one; so every subset of Γ is open with respect to this topology.

Let Ω be a finite subset of Γ which contains the identity element e and which generates Γ as a semi-group, i.e., if we set $\Omega_0 := \{e\}$ and if we let Ω_n denote the set of all words of length at most n with letters in Ω for $n \geq 1$, then $\cup_{n \geq 0} \Omega_n = \Gamma$. Note also that the sequence (Ω_n) is increasing; so the operators P_{Ω_n} can play the role of the finite section projections P_{Y_n} from the introduction, and in fact we will obtain some of the subsequent results exactly for this sequence.

With respect to Ω, we define the following "algebro-topological" notions. Let $A \subseteq \Gamma$. A point $a \in A$ is called an Ω-*inner* point of A if $\Omega a := \{\omega a : \omega \in \Omega\} \subseteq A$. The set $\text{int}_\Omega A$ of all Ω-inner points of A is called the Ω-*interior* of A, and the set $\partial_\Omega A := A \setminus \text{int}_\Omega A$ is the Ω-*boundary* of A. Note that we consider the Ω-boundary of a set always as a part of that set. (In this point, the present definition of a boundary differs from other definitions in the literature; see [1] for instance.) One easily checks that

$$\Omega_{n-1} \subseteq \text{int}_\Omega \Omega_n \subseteq \Omega_n \quad \text{and} \quad \partial_\Omega \Omega_n \subseteq \Omega_n \setminus \Omega_{n-1} \tag{8}$$

for each $n \geq 1$.

Recall from [19] that there are at least two reasons for the interest in the boundaries $\partial_\Omega \Omega_n$:

- The sequence $(P_{\partial_\Omega \Omega_n})_{n \geq 1}$ belongs to the C^*-algebra $\mathcal{S}(\mathsf{Sh}(\Gamma))$ which is generated by all finite sections sequences $(P_{\Omega_n} A P_{\Omega_n})_{n \geq 1}$ where A runs through the operators on $l^2(\Gamma)$ of left shift by elements in Γ (i.e., they are given by the left-regular representation of Γ on $l^2(\Gamma)$), and it generates the quasicommutator ideal of that algebra.

- There is a criterion for the stability of sequences in $\mathcal{S}(\mathsf{Sh}(\Gamma))$ which can be formulated by means of limit operators, and it turns out that it is sufficient to consider limit operators with respect to sequences taking their values in the boundaries $\partial_\Omega \Omega_n$.

For details, see [19]. In many instances one observes that the "algebraic" boundary $\partial_\Omega \Omega_n$ coincides with the "numerical" boundary $\Omega_n \setminus \Omega_{n-1}$; in fact, one inclusion holds in general as mentioned in (8). We will see now that the reverse inclusion can be guaranteed if $\Gamma = \mathbb{Z}^d$ and if Ω arises from a lattice polytope Π such that (4) holds for all $n \in \mathbb{N}$.

Proposition 3.1 *Let Π be a lattice polytope in $\mathbb{Z}^d$ which satisfies (4) and set $\Omega := \Pi \cap \mathbb{Z}^d$. Then*

$$\partial_\Omega(n * \Omega) = (n * \Omega) \setminus ((n-1) * \Omega) \tag{9}$$

holds for all positive integers n.

Proof. As mentioned above, it is sufficient to show that

$$(n * \Omega) \setminus ((n-1) * \Omega) \subseteq \partial_\Omega(n * \Omega).$$

We start with working on the continuous level and check first the implication

$$\text{If} \quad x \in n\Pi \setminus (n-1)\Pi, \quad \text{then} \quad \Pi + x \not\subset n\Pi. \tag{10}$$

Indeed, write x as $t\omega$ with $\omega \in \partial\Pi$ (= the usual topological boundary of Π) and $n - 1 < t \leq n$. Then $x + \omega = (t+1)\omega$ with $t + 1 > n$, whence $x + \omega \notin n\Pi$.

In the next step we show that ω can be chosen such that $x + \omega$ becomes a grid point for x a grid point. Indeed, let $x \in (n\Pi \setminus (n-1)\Pi) \cap \mathbb{Z}^d$. Consider the points $x + \omega_i$ where the ω_i, $i = 1, \ldots, k$, run through the (integer) vertices of Π. If we would have $x + \omega_i \in n\Pi$ for each i, then we would have

$$x + \Pi = \text{conv}\{x + \omega_i : i = 1, \ldots, k\} \subseteq n\Pi$$

by convexity of Π, which contradicts (10). Hence, for each $x \in (n\Pi \setminus (n-1)\Pi) \cap \mathbb{Z}^d$, there is a vertex ω_i of Π such that

$$x + \omega_i \in ((n+1)\Pi \setminus n\Pi) \cap \mathbb{Z}^d.$$

Employing the assumption (4) we conclude that, for each $x \in n * \Omega \setminus (n-1) * \Omega$ there is a $\omega_i \in \Omega$ such that $x + \omega_i \in (n+1) * \Omega \setminus n * \Omega$. Hence, x is in the Ω-boundary of $n * \Omega$. ∎

We proceed with an example which shows that the generalized version of (9),

$$\partial_\Omega \Omega_n = \Omega_n \setminus \Omega_{n-1}, \tag{11}$$

does not hold for general subsets Ω of a finitely generated discrete group Γ and $n \in \mathbb{N}$. Consider the matrices

$$\omega_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \omega_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \omega_2 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$\omega_3 := \omega_2\omega_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \omega_4 := \omega_2^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \omega_5 := \omega_4\omega_1 = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $\Omega := \{\omega_i : i = 0, \ldots, 5\}$ generates the group $GL(2, \mathbb{Z})$ as a semi-group (clearly, Ω is not minimal as a generating system: $\omega_0, \omega_1, \omega_2, \omega_4$ already generate this group). One easily checks that $\omega_0\omega_1 = \omega_1$, $\omega_1\omega_1 = \omega_0$, $\omega_2\omega_1 = \omega_3$, $\omega_3\omega_1 = \omega_2$, $\omega_4\omega_1 = \omega_5$ and $\omega_5\omega_1 = \omega_4$, whence $\Omega\omega_1 \subseteq \Omega$. Thus, $\omega_1 \in \Omega_1 \setminus \Omega_0$, but $\omega_1 \notin \partial_\Omega\Omega_1$. So, (11) is violated already for $n = 1$.

Let us conclude with a curious consequence of the coincidence (9) of the boundaries. We mentioned above that the sequence $(P_{\partial_\Omega \Omega_n})_{n\geq 1}$ always belongs to the C^*-algebra $\mathcal{S}(\mathsf{Sh}(\Gamma))$ generated by the finite sections sequences $(P_{\Omega_n} A P_{\Omega_n})_{n\geq 1}$ where A is constituted by operators of left shift by elements of Γ. Under the conditions of Proposition 3.1, we conclude that the sequence $(P_{\Omega_n} - P_{\Omega_{n-1}})$ belongs to $\mathcal{S}(\mathsf{Sh}(\mathbb{Z}^d))$. In particular, the sequence $(P_{\Omega_{n-1}}) = (P_{\Omega_n}) - (P_{\Omega_n} - P_{\Omega_{n-1}})$ belongs to $\mathcal{S}(\mathsf{Sh}(\mathbb{Z}^d))$. Consequently, with each sequence $(P_{\Omega_n} A P_{\Omega_n})_{n\geq 1}$, the sequence

$$(P_{\Omega_{n-1}})\,(P_{\Omega_n} A P_{\Omega_n})\,(P_{\Omega_{n-1}}) = (P_{\Omega_{n-1}} A P_{\Omega_{n-1}})$$

(with the operators $P_{\Omega_{n-1}} A P_{\Omega_{n-1}}$ considered as acting on the range of P_{Ω_n}) also belongs to $\mathcal{S}(\mathsf{Sh}(\mathbb{Z}^d))$. In particular, the algebra $\mathcal{S}(\mathsf{Sh}(\mathbb{Z}^d))$ contains a shifted copy (hence, infinitely many shifted copies) of itself. The same fact clearly holds for every algebra which is generated by finite sections sequences $(Q_n A Q_n)$ and contains the sequence $(Q_n - Q_{n-1})$. A less trivial example where this happens is the algebra of the finite sections method for operators in (a concrete representation of) the Cuntz algebra O_N with $N \geq 2$ [18].

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