Homogenization of Viscoplastic Models of Monotone Type with Positive Semi-Definite Free Energy

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Abstract

Using the periodic unfolding method we construct the homogenization theory for the quasistatic initial boundary value problems with internal variables, which model the deformation behavior of viscoplastic materials with a periodic microstructure. The free energy associated with models is assumed to be positive semi-definite only.

Key words: homogenization, plasticity, unfolding method, viscoplasticity, maximal monotone operator, periodic microstructure.

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1 Introduction and statement of results

In this work we are concerned with the homogenization of the initial boundary value problem describing the deformation behavior of inelastic materials with a periodic microstructure, in particular for plastic and viscoplastic materials. The associated free energy is assumed to be positive semi-definite only.

The formulation of the problem is based on the assumption that only small strains occur: Let Ω be an open bounded set, the set of material points of the body, with C^1-boundary $\partial\Omega$. T_e denotes a positive number (time of existence) and for $0 < t \leq T_e$

$$\Omega_t = \Omega \times (0, t).$$

Let $\mathcal{S}^3$ denote the set of symmetric 3×3-matrices, and let $u(x,t) \in \mathbb{R}^3$ be the unknown displacement of the material point x at time t , $T(x,t) \in \mathcal{S}^3$ is the unknown Cauchy stress tensor and $z(x,t) \in \mathbb{R}^N$ denotes the unknown vector of

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internal variables. The model equations of the problem (a microscopic problem) are

$$
\begin{aligned}
-\operatorname{div}_x T_\eta(x,t) &= b(x,t), && (1)\\
T_\eta(x,t) &= \mathcal{D}\,[x/\eta]\,(\varepsilon(\nabla_x u_\eta(x,t)) - Bz_\eta(x,t)), && (2)\\
\frac{\partial}{\partial t} z_\eta(x,t) &\in g\,(x/\eta, -\nabla_z \psi\,(x/\eta, \varepsilon(\nabla_x u_\eta(x,t)), z_\eta(x,t))) && (3)\\
&= g\left(x/\eta, B^T T_\eta(x,t) - L\,[x/\eta]\, z_\eta(x,t)\right),
\end{aligned}
$$

which must hold for $x \in \Omega$ and $t \in [0,\infty)$. The initial value for $z(x,t)$ is taken in the form

$$z_\eta(x,0) = 0, \qquad (4)$$

which must hold for $x \in \Omega$. We consider the homogeneous Dirichlet boundary condition

$$u_\eta(x,t) = 0, \qquad (5)$$

which must be satisfied for $(x,t) \in \partial\Omega \times [0,\infty)$. Here

$$\varepsilon(\nabla_x u_\eta(x,t)) = \frac{1}{2}(\nabla_x u_\eta(x,t) + (\nabla_x u_\eta(x,t))^T) \in \mathcal{S}^3,$$

is the strain tensor, $B : \mathbb{R}^N \to \mathcal{S}^3$ is a linear mapping, which assigns to the vector $z_\eta(x,t)$ the plastic strain tensor $\varepsilon_p(x,t) = Bz_\eta(x,t)$. For every $y \in \mathbb{R}^3$ we denote by $\mathcal{D}[y] : \mathcal{S}^3 \to \mathcal{S}^3$ a linear symmetric mapping, the elasticity tensor. The mapping $y \to \mathcal{D}[y]$ is assumed to be measurable and periodic with a periodicity cell $Y \subset \mathbb{R}^3$. We suppose that there exist two positive constants $0 < \alpha \le \beta$ satisfying

$$\alpha|\xi|^2 \le \mathcal{D}_{ijkl}[y]\xi_{kl}\xi_{ij} \le \beta|\xi|^2 \quad \text{for any } \xi \in \mathcal{S}^3.$$

The function $b : \Omega \times [0,\infty) \to \mathbb{R}^3$ is the volume force. The positive semi-definite quadratic form

$$\psi(y,\varepsilon,z) = \frac{1}{2}\mathcal{D}[y](\varepsilon - Bz)\cdot(\varepsilon - Bz) + \frac{1}{2}(L[y]z)\cdot z \qquad (6)$$

represents the free energy (see Appendix [1]), and for all $y \in \mathbb{R}^3$ the function $z \to g(y,z) : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is monotone[1] satisfying $0 \in g(y,0)$; $y \to g(y,z)$ is periodic with the periodicity cell $Y \subset \mathbb{R}^3$. The symmetric positive semi-definite $N \times N-$matrix $L[y]$ is measurable and periodic with the same periodicity cell Y. The number $\eta > 0$ is the scaling parameter of the microstructure.

Definition 1.1. *The system of equations (1) - (5) with the mappings B and L introduced above is called a problem/model of monotone type iff the symmetric $N \times N-$matrix $M := L + B^T\mathcal{D}B$ is positive definite and the nonlinear function $g : \mathbb{R}^3 \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ satisfying for a.e. $y \in \mathbb{R}^3$ the inequality $v^* \cdot v \ge 0$ for all $v^* \in g(y,v)$ is monotone.*

[1] See Section 2 for definitons of monotone and maximal monotone mappings.

The class of problems of monotone type was introduced by Alber in [1] and generalized naturally the class of generalized standard materials defined by Halphen and Nguyen Quoc Son in [17]. The function g in (3) for generalized standard materials is a subdifferential of a convex function and, since the subdifferential of a convex function is monotone, the class of generalized standard materials is a sub-class of problems of monotone type. Typical application of such models is elasto/visco-plasticity with or without hardening effects at small strains. Such classical models of Prandtl-Reuss and Norton-Hoff belong to this class and are examples from rate-independent and rate-dependent sub-classes of monotone problems, respectively. It is worth to mention here that the initial boundary value problem (1) - (5) is written in the most general form and, describing thermodynamically admissible processes, includes all elasto/visco-plastic models at small strains used in engineering (see [1]), yet the function g is not monotone quite often. In the rate independent case, i.e. when $g = \partial I_K$ for a closed convex set K, an alternative approach for such models was proposed by Mielke and Theil in [19], a so-called energetic formulation. In the setting of Mielke and Theil the effects like damage, fracture and hysteretic behavior in ferroelectic materials at finite strains can be also analyzed.

The construction of the homogenization theory for the quasistatic initial boundary value problem (1) - (5) has started with the work [2], where the homogenized system of equations has been derived using the formal asymtotic ansatz. In the followed-up work [3], for the case of positive definite free energy, the author was able to justify the formal asymtotic ansatz constructed in [2] employing the energy method of Murat-Tartar, but for locally smooth solutions of the homogenized problem only. It was shown there that the solutions of (1) - (5) can be successfully approximated in the $L^2(\Omega)-$norm by the functions defined with the smooth solutions of the homogenized problem. Under the assumption that free energy is positive definite, what corresponds to linear kinematic hardening behavior of materials, in [22] it is proved that the difference of the solutions of the microscopic problem and the homogenized one tends to zero in the $L^2(\Omega \times Y)-$norm, where Y is the periodicity cell. This type of convergence is called there the phase shift convergence. Shortly afterwards, based on the results in [22], the convergence of the mentioned difference with respect to the $L^2(\Omega)-$norm was derived in [6]. In the meantime, for the rate-independent problems similar results were obtained in [20] using the unfolding operator method and methods of energetic solutions. For special rate-dependent models of monotone type the two-scale convergence of the solutions of the microscopic problem to the solutions of the homogenized problem was proved in [29, 30]. In this work we consider only the rate-dependent problems of monotone type with the constitutive function g, which belongs to a special class of function introduced in Section 3. For this class of functions, using the unfolding operator method (see Section 4), we are able to construct easily the homogenization theory for the problem (1) - (5) based on the results obtained in [16] (see Section 5). But to be able to apply the results from [16], we need to show first that the solutions of the problem (1) - (5) are slightly smoother than the existence theory in [21] provides. Therefore, Section 3 is devoted to the derivation of the additional regularity for solutions of the microscopic problem, which is obtained by a time-discretization technique and imposing more regularity on the given data.

Notation. For $m \in \mathbb{N}$, $q \in [1, \infty]$, we denote by $W^{m,q}(\Omega, \mathbb{R}^k)$ the Banach space of Lebesgue integrable functions having q-integrable weak derivatives up to order m. This space is equipped with the norm $\| \cdot \|_{m,q,\Omega}$. If $m = 0$ we also write $\| \cdot \|_{q,\Omega}$. If m is not integer, then the corresponding Sobolev-Slobodeckij space is denoted by $W^{m,q}(\Omega, \mathbb{R}^k)$. We set $H^m(\Omega, \mathbb{R}^k) = W^{m,2}(\Omega, \mathbb{R}^k)$.

The space $W^{m,p}_{per}(Y, \mathbb{R}^k)$ denotes the Banach space of Y-periodic functions in $W^{m,p}_{loc}(\mathbb{R}^k, \mathbb{R}^k)$ equipped with the $W^{m,p}(Y, \mathbb{R}^k)$-norm.

We choose the numbers p, q satisfying $1 < p, q < \infty$ and $1/p + 1/q = 1$. For such p and q one can define the bilinear form on the product space $L^p(\Omega, \mathbb{R}^k) \times L^q(\Omega, \mathbb{R}^k)$ by

$$(\xi, \zeta)_\Omega = \int_\Omega \xi(x) \cdot \zeta(x) dx.$$

Finally, we frequently use the spaces $W^{k,p}(0, T_e; X)$, which consist of Bochner measurable functions with a p-integrable weak derivatives up to order k.

2 Maximal monotone operators

In this section we recall some basics about monotone and maximal monotone operators. For more details see [9, 18, 25], for example.

2.1 Preliminaries

Let V be a reflexive Banach space with the norm $\| \cdot \|$, V^* be its dual space with the norm $\| \cdot \|_*$. The brackets $\langle \cdot, \cdot \rangle$ denotes the dual pairing between V and V^*. Under V we shall always mean a reflexive Banach space throughout this section. For a multivalued mapping $A : V \to 2^{V^*}$ the sets

$$D(A) = \{v \in V \mid Av \neq \emptyset\}$$

and

$$GrA = \{[v, v^*] \in V \times V^* \mid v \in D(A), \; v^* \in Av\}$$

are called the *effective domain* and the *graph* of A, respectively.

Definition 2.1. *A mapping $A : V \to 2^{V^*}$ is called* monotone *iff the inequality holds*

$$\langle v^* - u^*, v - u \rangle \geq 0 \qquad (\forall) \; [v, v^*], [u, u^*] \in GrA.$$

A monotone mapping $A : V \to 2^{V^*}$ is called maximal monotone *iff the inequality*

$$\langle v^* - u^*, v - u \rangle \geq 0 \qquad (\forall) \; [u, u^*] \in GrA$$

implies $[v, v^*] \in GrA$.
A mapping $A : V \to 2^{V^*}$ is called generalized pseudomonotone *iff the set Av is closed, convex and bounded for all $v \in D(A)$ and for every pair of[16, Proposition 2.16] sequences $\{v_n\}$ and $\{v_n^*\}$ such that $v_n^* \in Av_n$, $v_n \rightharpoonup v_0$, $v_n^* \rightharpoonup v_0^* \in V^*$ and*

$$\limsup_{n \to \infty} \langle v_n^*, v_n - v_0 \rangle \leq 0,$$

we have that $[v_0, v_0^*] \in GrA$ *and* $\langle v_n^*, v_n\rangle \to \langle v_0^*, v_0\rangle$.
A mapping $A : V \to 2^{V^*}$ *is called* strongly coercive *iff either* $D(A)$ *is bounded or* $D(A)$ *is unbounded and the condition*

$$\frac{\langle v^*, v - w\rangle}{\|v\|} \to +\infty \quad \text{as } \|v\| \to \infty, \quad [v, v^*] \in GrA,$$

is satisfied for each $w \in D(A)$.

It is well known ([25, p. 105]) that if A is a maximal monotone operator, then for any $v \in D(A)$ the image Av is closed convex subset of V^* and the graph GrA is demiclosed[2]. A maximal monotone operator is also generalized pseudomonotone (see [9, 18, 25]).

Remark 2.1. We recall that the subdifferential of a lower semi-continuous and convex function is maximal monotone (see [26, Theorem 2.25]).

Definition 2.2. *The* duality mapping $J : V \to 2^{V^*}$ *is defined by*

$$J(v) = \{v^* \in V^* \mid \langle v^*, v\rangle = \|v\|^2 = \|v^*\|_*^2\ \}$$

for all $v \in V$.

For maximal monotone operators we have the following characterization in reflexive Banach spaces.

Theorem 2.1. *Let* $A : V \to 2^{V^*}$ *be a monotone mapping. Then* A *is maximal monotone iff for any* $\lambda > 0$ *the following surjectivity result holds*

$$R(A + \lambda J) = V^*.$$

Proof. See [9, Theorem II.1.2]. □

Without loss of generality (due to Asplund's theorem) we can assume that both V and V^* are strictly convex, i.e. if the unit ball in the corresponding space is strictly convex. In virtue of Theorem 2.1, the equation

$$J(v_\lambda - v) + \lambda A v_\lambda \ni 0$$

has a solution $v_\lambda \in D(A)$ for every $v \in V$ and $\lambda > 0$ if A is maximal monotone. The solution is unique (see [9, p. 41]).

Definition 2.3. *Setting*

$$v_\lambda = j_\lambda^A v \quad \text{and} \quad A_\lambda v = -\lambda^{-1} J(v_\lambda - v)$$

we define two single valued operators: the Yosida approximation $A_\lambda : V \to V^*$ *and the* resolvent $j_\lambda^A : V \to D(A)$ *with* $D(A_\lambda) = D(j_\lambda^A) = V$.

By the definition, one immediately sees that $A_\lambda v \in A\big(j_\lambda^A v\big)$. For the main properties of the Yosida approximation we refer to [9, 18, 25] and mention only that both are continuous operators and that A_λ is bounded and maximal monotone.

Proposition 2.1. *If* $v \in \overline{\text{conv} D(A)}$, *then* $j_\lambda^A v \to v$ *in* V *as* $\lambda \to 0$.

Proof. See [9, Proposition II.1.1] or [25, Proposition III.3.1]. □

[2] A set $A \in V \times V^*$ is demiclosed if v_n converges strongly to v_0 in V and v_n^* converges weakly to v_0^* in V^* (or v_n converges weakly to v_0 in V and v_n^* converges strongly to v_0^* in V^*) and $[v_n, v_n^*] \in GrA$, then $[v, v^*] \in GrA$

2.2 Convergence of maximal monotone graphs

In the presentation of the next subsections we follow the work [16], where the reader can also find the proofs of the results mentioned here.

The derivation of the homogenized equitions for the initial boundary value problem (1) - (5) is based on the notion of the convergence of the graphs of maximal monotone operators.

According to Brezis [10] and Attouch [8], the convergence of the graphs of maximal monotone operators is defined as follows.

Definition 2.4. *Let A^n, $A : V \to 2^{V^*}$ be maximal monotone operators. The sequence A^n converges to A as $n \to \infty$, ($A^n \rightarrowtail A$), if for every $[v, v^*] \in GrA$ there exists a sequence $[v_n, v_n^*] \in GrA^n$ such that $[v_n, v_n^*] \to [v, v^*]$ strongly in $V \times V^*$ as $n \to \infty$.*

Obviously, if A^n and A are everywhere defined, continuous and monotone, then the pointwise convergence, i.e. if for every $v \in V$, $A^n(v) \to A(v)$, implies the convergence of the graphs. The converse is true in finite-dimensional spaces.

The next theorem is the main mathematical tool in the derivation of the homogenized equations for the problem (1) - (5).

Theorem 2.2. *Let A^n, $A : V \to 2^{V^*}$ be maximal monotone operators, and let $[v_n, v_n^*] \in GrA^n$ and $[v, v^*] \in V \times V^*$. If, as $n \to \infty$, $A^n \rightarrowtail A$, $v_n \rightharpoonup v_0$, $v_n^* \rightharpoonup v_0^* \in V^*$ and*

$$\limsup_{n\to\infty} \langle v_n^*, v_n \rangle \leq \langle v_0^*, v_0 \rangle ,$$

then $[v_0, v_0^*] \in GrA$ and

$$\liminf_{n\to\infty} \langle v_n^*, v_n \rangle = \langle v_0^*, v_0 \rangle .$$

Proof. See [16, Theorem 2.8]. □

The convergence of the graphs of multi-valued maximal monotone operators can be equivalently stated in term of the pointwise convergence of the corresponding single-valued Yosida approximations and resolvents as the following result shows.

Theorem 2.3. *Let A^n, $A : V \to 2^{V^*}$ be maximal monotone operators and $\lambda > 0$. The following statements are equivalent:*

(a) $A^n \rightarrowtail A$ *as* $n \to \infty$;

(b) for every $v \in V$, $j_\lambda^{A^n} v \to j_\lambda^A v$ *as* $n \to \infty$;

(c) for every $v \in V$, ${A^n}_\lambda v \to A_\lambda v$ *as* $n \to \infty$;

(d) $A_\lambda^n \rightarrowtail A_\lambda$ *as* $n \to \infty$.

Moreover, the convergences $j_\lambda^{A^n} v \to j_\lambda^A v$ and ${A^n}_\lambda v \to A_\lambda v$ are uniform on strongly compact subsets of V.

Proof. See [16, Theorem 2.9]. □

2.3 Measurability of multi-valued mappings

In this subsection we present briefly some facts about measurable multi-valued mappings. We assume that V, and hence V^*, is separable and denote the set of maximal monotone operators from V to V^* by $\mathcal{M}(V \times V^*)$. Further, let $(\omega, \Sigma(\omega), \mu)$ be a $\sigma-$finite $\mu-$complete measurable space.

Definition 2.5. *A function $A : \omega \to \mathcal{M}(V \times V^*)$ is measurable iff for every open set $U \in V \times V^*$ (resp closed set, Borel set, open ball, closed ball),*

$$\{x \in \omega \mid A(x) \cap U \neq \emptyset\}$$

is measurable in ω.

The next result states that the notion of measurability for maximal monotone mappins can be equivalently defined in terms of the measurability for appropriate single-valued mappings.

Proposition 2.2. *Let $A : \omega \to \mathcal{M}(V \times V^*)$, let $\lambda > 0$ and let E be dense in V. The following are equivalent:*

(a) A is measurable;

(b) for every $v \in E$, $x \mapsto j_\lambda^{A(x)} v$ is measurable;

(c) $v \in E$, $x \mapsto A_\lambda(x)v$ is measurable.

Proof. See [16, Proposition 2.11]. □

For further reading on measurable multi-valued mappings we refer the reader to [11, 18, 24].

2.4 Canonical extensions of maximal monotone operators

Given a mapping $A : \omega \to \mathcal{M}(V \times V^*)$, one can define a monotone graph from $L^p(\omega, V)$ to $L^q(\omega, V^*)$, where $1/p + 1/q = 1$, as follows:

Definition 2.6. *Let $A : \omega \to \mathcal{M}(V \times V^*)$, the canonical extension of A from $L^p(\omega, V)$ to $L^q(\omega, V^*)$, where $1/p + 1/q = 1$, is defined by:*

$$Gr\mathcal{A} = \{[v, v^*] \in L^p(\omega, V) \times L^q(\omega, V^*) \mid [v(x), v^*(x)] \in GrA(x) \; for \; a.e. \; x \in \omega\}.$$

Monotonicity of $\mathcal{A}$ defined in Definition 2.6 is immediate, while its maximality follows from the next proposition.

Proposition 2.3. *Let $A : \omega \to \mathcal{M}(V \times V^*)$ be measurable. If $Gr\mathcal{A} \neq \emptyset$, then $\mathcal{A}$ is maximal monotone.*

Proof. See [16, Proposition 2.13]. □

We have to point out here that the maximality of $A(x)$ for almost every $x \in \omega$ does not imply the maximality of $\mathcal{A}$ as the latter can be empty [16]: $\omega = (0, 1)$, and $GrA(x) = \{[v, v^*] \in \mathbb{R}^m \times \mathbb{R}^m \mid v^* = t^{-1/q}\}$.

For given mappings $A, A^n : \omega \to \mathcal{M}(V \times V^*)$ and their canonical extensions $\mathcal{A}, \mathcal{A}^n$, one can ask whether the pointwise convergence $A^n(x) \rightarrowtail A(x)$ implies the convergence of the graphs of corresponding canonical extensions $\mathcal{A}^n \rightarrowtail \mathcal{A}$. The answer is given by the next theorem.

Theorem 2.4. *Let $A, A^n : \omega \to \mathcal{M}(V \times V^*)$ be measurable. Assume*

(a) for almost every $x \in \omega$, $A^n(x) \rightarrowtail A(x)$ as $n \to \infty$,

(b) $\mathcal{A}$ and $\mathcal{A}^n$ are maximal monotone,

(c) there exists $[\alpha_n, \beta_n] \in Gr\mathcal{A}^n$ and $[\alpha, \beta] \in L^p(\omega, V) \times L^q(\omega, V^*)$ such that $[\alpha, \beta] \to [\alpha, \beta]$ strongly in $L^p(\omega, V) \times L^q(\omega, V^*)$ as $n \to \infty$,

then $\mathcal{A}^n \rightarrowtail \mathcal{A}$.

Proof. See [16, Proposition 2.16]. □

We note that assumption (c) in Theorem 2.4 can not be dropped in virtue of [16, Remark 2.16].

3 Existence of solutions

The existence of solutions for initial boundary value problem (1) - (5) under quite general assumptions has been already shown in [21]. Unfortunately, the existence theory constructed in [21] does not provide us with enough uniform estimates for the solutions of the problem (1) - (5) in order to the homogenization procedure could be performed smoothly. However, we have noticed that the homogenized equations can be easily derived once we increase slightly the regularity of given data. The additional regularity of the data, except desired estimates, yields the additional regularity of the solutions. This section is therefore devoted to the derivation of the additional uniform estimates and to the proof of the existence of more regular solutions.

We define a class of maximal monotone functions, which we are dealing with in this work.

Definition 3.1. *For $m \in L^1(\omega, \mathbb{R})$, $\alpha \in \mathbb{R}_+$ and $p > 1$, $\mathcal{M}(\omega, \mathbb{R}^k, p, \alpha, m)$ is the set of multi-valued functions $g : \omega \times \mathbb{R}^k \to 2^{\mathbb{R}^k}$ with following properties*

- *$v \mapsto g(x, v)$ is maximal monotone for almost all $x \in \omega$,*
- *the mapping $x \mapsto j_\lambda(x, v) : \omega \to \mathbb{R}^k$ is measurable for all $\lambda > 0$, where $j_\lambda(x, v)$ is the inverse of $v \mapsto v + \lambda g(x, v)$,*
- *for a.e. $x \in \omega$ and every $v^* \in g(x, v)$*

$$\alpha \left(\frac{\|v\|^p}{p} + \frac{\|v^*\|^q}{q} \right) \leq (v, v^*) + m(x), \tag{7}$$

where $1/p + 1/q = 1$.

Remark 3.1. We note that the condition (7) is equivalent to the following two inequalities

$$\|v^*\|^q \leq m_1(x) + \alpha_1 \|v\|^p, \tag{8}$$
$$(v, v^*) \geq m_2(x) + \alpha_2 \|v\|^p, \tag{9}$$

for a.e. $x \in \omega$ and every $v^* \in g(x, v)$ and with suitable functions $m_1, m_2 \in L^1(\omega, \mathbb{R})$ and numbers $\alpha_1, \alpha_2 \in \mathbb{R}_+$.

The main properties of the class $\mathcal{M}(\omega, \mathbb{R}^k, p, \alpha, m)$ are stated in the following proposition.

Proposition 3.1.

(a) Let $\mathcal{G}$ be a canonical extension of a function $g \in \mathcal{M}(\omega, \mathbb{R}^k, p, \alpha, m)$. Then $\mathcal{G}$ is maximal monotone, surjective and $D(\mathcal{G}) = L^p(\omega, \mathbb{R}^k)$.

(b) Let $\mathcal{G}_\eta$ be canonical extensions of functions $g_\eta \in \mathcal{M}(\omega, \mathbb{R}^k, p, \alpha, m_\eta)$ and $g : \omega \to \mathcal{M}(\mathbb{R}^k \times \mathbb{R}^k)$ is measurable. If m_η converges to m strongly in $L^1(\omega, \mathbb{R})$ and if for a.e. $x \in \omega$ the convergence $g_\eta(x) \rightarrowtail g(x)$ holds, then $g \in \mathcal{M}(\omega, \mathbb{R}^k, p, \alpha, m)$ and $\mathcal{G}_\eta \rightarrowtail \mathcal{G}$.

Proof. See Corollary 2.15 and Corollary 2.17 in [16]. □

Now, we can state the main result of this section.

Theorem 3.1. *Assume that the matrix $L_\eta \in L^\infty(\Omega, \mathbb{R}^{N\times N})$ in (6) is positive semi-definite and that the mappings[3] $g_\eta \in \mathcal{M}(\Omega, \mathbb{R}^N, p, \alpha, m_\eta)$ are given. Suppose that $b \in L^p(0, T_e; W^{-1,p}(\Omega, \mathbb{R}^3))$. Then*

$$(u_\eta, T_\eta) \in L^q(0, T_e; W^{1,q}(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathcal{S}^3)),$$
$$z_\eta \in W^{1,q}(0, T_e; L^q(\Omega, \mathbb{R}^N)), \quad B^T T_\eta - L_\eta z_\eta \in L^p(\Omega_{T_e}, \mathbb{R}^N)$$

is a solution of the microscopic intial boundary value problem (1) - (5). If, additionally, $b \in W^{1,p}(0, T_e; W^{-1,p}(\Omega, \mathbb{R}^3))$ and

$$g_\eta(x, B^T T_\eta^{(0)}) \cap L^2(\Omega, \mathbb{R}^N) \neq \emptyset, \tag{10}$$

where $(u_\eta^{(0)}, T_\eta^{(0)})$ is a solution of the problem (85) - (87) to the data $\hat{b} = b(t)$, $\hat{\gamma} = \hat{\varepsilon}_p = 0$, then the solutions possess also the following regularity properties

$$(T_\eta, L_\eta^{1/2} z_\eta) \in H^1(0, T_e; L^2(\Omega, \mathcal{S}^3 \times \mathbb{R}^N)).$$

Moreover, if the next condition holds

$$\sup_{\eta>0} |\mathcal{G}_\eta T_\eta^{(0)}|_\Omega < \infty, \tag{11}$$

where $|\mathcal{G}_\eta T_\eta^{(0)}|_\Omega = \inf\{\|\zeta\|_\Omega \mid \zeta \in L^2(\Omega, \mathbb{R}^N), \ \ \zeta(x) \in g_\eta(x, B^T T_\eta^{(0)}(x))\}$, then all inclusions mentioned above are uniform with respect to η.

Proof. To simplify the notations we drop η. The first part of the proof is shown in [21]. Therefore, it is left to verify the additional regularity of solutions of (1) - (5) only. We show this by the Rothe method (a time-discretization method, see [27] for details). In order to introduce a time-discretized problem, let us fix any $m \in \mathbb{N}$ and set

$$h := \frac{T_e}{2^m}, \ z_m^0 := 0, \ b_m^n := \frac{1}{h} \int_{(n-1)h}^{nh} b(s) ds \in W^{-1,p}(\Omega, \mathbb{R}^3), \quad n = 1, ..., 2^m.$$

[3]Here, $g_\eta(x, z) := g(x/\eta, z)$ and $L_\eta[x] := L[x/\eta]$ for $x \in \Omega, z \in \mathbb{R}^N$.

Then, we are looking for functions $u^n_m \in H^1(\Omega, \mathbb{R}^3)$, $T^n_m \in L^2(\Omega, \mathcal{S}^3)$ and $z^n_m \in L^2(\Omega, \mathbb{R}^N)$ with

$$\Sigma_{n,m} := B^T T^n_m - \frac{1}{m} z^n_m - L z^n_m \in L^p(\Omega, \mathbb{R}^N)$$

solving the following problem

$$-\operatorname{div}_x T^n_m(x) = b^n_m(x), \tag{12}$$

$$T^n_m(x) = \mathcal{D}(\varepsilon(\nabla_x u^n_m(x)) - B z^n_m(x)), \tag{13}$$

$$\frac{z^n_m(x) - z^{n-1}_m(x)}{h} \in g\big(x, \Sigma_{n,m}(x)\big), \tag{14}$$

together with the boundary conditions

$$u^n_m(x) = 0, \quad x \in \partial\Omega\,. \tag{15}$$

Next, we adopt the reduction technique in [4] to the above equations. Let (u^n_m, T^n_m, z^n_m) be a solution of the boundary value problem (12) - (15). The equations (12), (13) and (15) form a boundary value problem for the components (u^n_m, T^n_m) of the solution, the problem of linear elasticity theory. Due to linearity of this problem we can write these components of the solution in the form

$$(u^n_m, T^n_m) = (\tilde{u}^n_m, \tilde{\sigma}^n_m) + (v^n_m, \sigma^n_m),$$

with the solution (v^n_m, σ^n_m) of the Dirichlet boundary value problem (85) - (87) to the data $\hat{b} = b^n_m$, $\hat{\gamma} = \hat{\varepsilon}_p = 0$, and with the solution $(\tilde{u}^n_m, \tilde{\sigma}^n_m)$ of the problem (85) - (87) to the data $\hat{b} = \hat{\gamma} = 0$, $\hat{\varepsilon}_p = B z^n_m$. We thus obtain

$$\varepsilon(\nabla_x u^n_m) - B z^n_m = (P_2 - I) B z^n_m + \varepsilon(\nabla_x v^n_m).$$

We insert this equation into (13) and get that (14) can be rewritten in the following form

$$\frac{z^n_m - z^{n-1}_m}{h} \in \mathcal{G}\big(-M_m z^n_m + B^T \sigma^n_m\big), \tag{16}$$

where

$$M_m := B^T(\mathcal{D} Q_2 + L) B + \frac{1}{m} I : L^2(\Omega, \mathbb{R}^N) \to L^2(\Omega, \mathbb{R}^N)$$

with the Helmholtz projection Q_2 and $\mathcal{G}$ is a canonical extension of g. Next, the problem (16) reads

$$\Psi(z^n_m) \ni B^T \sigma^n_m, \tag{17}$$

where

$$\Psi(v) = \mathcal{G}^{-1}\Big(\frac{v - z^{n-1}_m}{h}\Big) + M_m v.$$

We note first that M_m is bounded and positive definite operator (see Corollary 5.1.1 and the definition of M_m). Thus, it is maximal monotone by Theorem II.1.3 in [9]. Since M_m is everywhere defined and $\mathcal{G}^{-1}$ is maximal monotone, by Theorem III.3.6 in [9] the sum $\mathcal{G}^{-1} + M_m$ is maximal monotone too. Since M_m

is coercive in $L^2(\Omega, \mathbb{R}^N)$, what obviously yields the coercivity of Ψ, the operator Ψ is surjective by Theorem III.2.10 in [25]. Thus, we may conclude that the equation (17) as well as the problem (16) have solutions. By the constructions this implies that the boundary value problem (12) - (15) is solvable as well (for more details we refer the reader to [4]).

Rothe approximation functions: For any family $\{\xi_m^n\}_{n=0,\dots,m}$ of functions in a reflexive Banach space X, we define *the piecewise affine interpolant* $\xi_m \in C([0,T_e],X)$ by

$$\xi_m(t) := \left(\frac{t}{h} - (n-1)\right)\xi_m^n + \left(n - \frac{t}{h}\right)\xi_m^{n-1} \quad \text{for } (n-1)h \le t \le nh \tag{18}$$

and *the piecewise constant interpolant* $\bar{\xi}_m \in L^\infty(0,T_e;X)$ by

$$\bar{\xi}_m(t) := \xi_m^n \text{ for } (n-1)h < t \le nh,\ n = 1, ..., 2^m, \text{ and } \bar{\xi}_m(0) := \xi_m^0. \tag{19}$$

For the further analysis we recall the following property of $\bar{\xi}_m$ and ξ_m:

$$\|\xi_m\|_{L^P(0,T_e;X)} \le \|\bar{\xi}_m\|_{L^P(-h,T_e;X)} \le \left(h\|\xi_m^0\|_X^p + \|\bar{\xi}_m\|^p_{L^P(0,T_e;X)}\right)^{1/p}, \tag{20}$$

where $\bar{\xi}_m$ is formally extended to $t \le 0$ by ξ_m^0 and $1 \le p \le \infty$ (see [27]).

A-priori estimates. Multiplying (12) by $(u_m^n - u_m^{n-1})/h$ and then integrating over Ω we get

$$\left(\sigma_m^n, \varepsilon(\nabla(u_m^n - u_m^{n-1})/h)\right)_\Omega = \left(b_m^n, (u_m^n - u_m^{n-1})/h\right)_\Omega.$$

Multiplying (14) by $w_m^n := (z_m^n - z_m^{n-1})/h$ and then integrating the obtained equation over Ω yield

$$\int_\Omega \left(\mathcal{G}^{-1}(w_m^n), w_m^n\right)dx = (T_m^n, Bw_m^n)_\Omega - \frac{1}{mh}\left(z_m^n - z_m^{n-1}, z_m^n\right)_\Omega - \frac{1}{h}\left(z_m^n - z_m^{n-1}, Lz_m^n\right)_\Omega.$$

With (13) we get that

$$\frac{1}{h}\left(\mathcal{D}^{-1}T_m^n, T_m^n - T_m^{n-1}\right)_\Omega + \frac{1}{h}\left(L^{1/2}(z_m^n - z_m^{n-1}), L^{1/2}z_m^n\right)_\Omega$$
$$+\frac{1}{m}\frac{1}{h}\left(z_m^n - z_m^{n-1}, z_m^n\right)_\Omega + \int_\Omega \left(\mathcal{G}^{-1}(w_m^n), w_m^n\right)dx = \frac{1}{h}\left(b_m^n, u_m^n - u_m^{n-1}\right)_\Omega.$$

Multiplying by h and summing the obtained relation for $n = 1, ..., l$ for any fixed $l \in [1, 2^m]$ we derive the following inequality (here $\mathbb{B} := \mathcal{D}^{-1}$)

$$\frac{1}{2}\left(\|\mathbb{B}^{1/2}T_m^l\|^2_\Omega + \|L^{1/2}z_m^l\|^2_\Omega + \frac{1}{m}\|z_m^l\|^2_\Omega\right) + h\sum_{n=1}^{l}\int_\Omega \left(\mathcal{G}^{-1}(w_m^n), w_m^n\right)dx$$
$$\le C^{(0)} + h\sum_{n=1}^{l}\left(b_m^n, \frac{u_m^n - u_m^{n-1}}{h}\right)_\Omega,$$

where[4]

$$2C^{(0)} := \|\mathbb{B}^{1/2}T_m^0\|^2_\Omega + \|L^{1/2}z_m^0\|^2_\Omega + \frac{1}{m}\|z_m^0\|^2_\Omega.$$

[4]Here we use the following inequality (see [27])

$$\sum_{n=1}^{l}(\phi_m^n - \phi_m^{n-1}, \phi_m^n)_\Omega = \frac{1}{2}\sum_{n=1}^{l}\left(\|\phi_m^n\|^2_\Omega - \|\phi_m^{n-1}\|^2_\Omega\right)$$

We estimate now the right hand side of the last inequality. Since u_m^n is a solution of the linear elliptic problem formed by the equations (12) - (13), (15), it satisfies (see [28], if needed) the inequality

$$\|u_m^n\|_{1,q,\Omega} \leq C\big(\|b_m^n\|_{q,\Omega} + \|z_m^n\|_{q,\Omega}\big), \tag{21}$$

where C is a positive constant independent of n and m. Therefore, using the linearity of the problem formed by (12) - (13), (15), the inequality (21) and Young's inequality with $\epsilon > 0$ we get that

$$\begin{aligned}\left(b_m^n, \frac{u_m^n - u_m^{n-1}}{h}\right)_\Omega &\leq \|b_m^n\|_{p,\Omega}\|(u_m^n - u_m^{n-1})/h\|_{1,q,\Omega} \leq CC_\epsilon\|b_m^n\|_{p,\Omega}^p \\ &+C\epsilon\|(b_m^n - b_m^{n-1})/h\|_{q,\Omega}^q + C\epsilon\|(z_m^n - z_m^{n-1})/h\|_{q,\Omega}^q,\end{aligned} \tag{22}$$

where C_ϵ is a positive constant appearing in the Young inequality. Combining the inequalities (21) and (22), applying (7) and choosing an appropriate value for $\epsilon > 0$ we obtain the following estimate

$$\begin{aligned}\frac{1}{2}\Big(\|\mathbb{B}^{1/2}T_m^l\|_\Omega^2 + \|L^{1/2}z_m^l\|_\Omega^2 + \frac{1}{m}\|z_m^l\|_\Omega^2\Big) + h\hat{C}_\epsilon\sum_{n=1}^{l}\int_\Omega \Big\|\frac{z_m^n - z_m^{n-1}}{h}\Big\|^q dx \\ \leq C^{(0)} + h\tilde{C}_\epsilon\sum_{n=1}^{l}\Big(\|b_m^n\|_{p,\Omega}^p + \|(b_m^n - b_m^{n-1})/h\|_{q,\Omega}^q\Big),\end{aligned} \tag{23}$$

where $\tilde{C}, \tilde{C}_\epsilon$ and $\hat{C}_\epsilon$ are some positive constants. Now, taking Remark 8.15 in [27] and the definition of Rothe's approximation functions into account we may rewrite (23) as follows

$$\begin{aligned}&\|\mathbb{B}^{1/2}\bar{T}_m(t)\|_\Omega^2 + C_1\|L^{1/2}\bar{z}_m(t)\|_\Omega^2 + \frac{1}{m}\|\bar{z}_m(t)\|_\Omega^2 \\ &+2\hat{C}_\epsilon\int_0^{T_e}\int_\Omega \big\|\partial_t z_m(x,t)\big\|^q dxdt \leq 2C^{(0)} + 2\tilde{C}_\epsilon\|b\|_{W^{1,p}(0,T_e;L^p(\Omega,\mathbb{R}^3))}^p.\end{aligned} \tag{24}$$

From the estimate (25) we get then that

$$\{z_m\}_m \text{ is uniformly bounded in } W^{1,q}(0,T_e;L^q(\Omega,\mathbb{R}^N)), \tag{25}$$

$$\{L^{1/2}\bar{z}_m\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathbb{R}^N)), \tag{26}$$

$$\{\bar{T}_m\}_m, \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{S}^3)), \tag{27}$$

$$\left\{\frac{1}{\sqrt{m}}\bar{z}_m\right\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathbb{R}^N)). \tag{28}$$

In particular, the uniform boundness of the sequences in (25) - (28) yields

$$\big\{\bar{\Sigma}_m\big\}_m \text{ is uniformly bounded in } L^p(0,T_e;L^p(\Omega,\mathbb{R}^N)), \tag{29}$$

$$\{u_m\}_m \text{ is uniformly bounded in } W^{1,q}(0,T_e;W^{1,q}(\Omega,\mathbb{R}^3)). \tag{30}$$

$$+\frac{1}{2}\sum_{n=1}^{l}\|\phi_m^n - \phi_m^{n-1}\|_\Omega^2 \geq \frac{1}{2}\|\phi_m^l\|_\Omega^2 - \frac{1}{2}\|\phi_m^0\|_\Omega^2$$

for any family of functions $\phi_m^0, \phi_m^1, ..., \phi_m^m$.

Employing (20) the estimates (26) - (29) further imply that sequnces $\{T_m\}_m$, $\{L^{1/2}z_m\}_m$, $\{z_m/\sqrt{m}\}_m$ and $\{\Sigma_m\}_m$ are also uniformly bounded in the corresponding spaces.GraphConvCanExten Moreover, due to (25) and the following obvious relation

$$z_m^l = z_m^0 + h\sum_{n=1}^{l}\left(\frac{z_m^n - z_m^{n-1}}{h}\right)$$

we may conclude that $\{\bar{z}_m\}_m$ is uniformly bounded in $L^q(0,T_e;L^q(\Omega,\mathbb{R}^N))$.

Weak limits of approximating sequences. By estimates (25) - (30) and at the expense of extracting a subsequence, we have that sequences in (25) - (30) converge with respect to weak and weakly star topologies in corresponding spaces, respectively. Next, we claim that weak limits of $\{\bar{z}_m\}_m$ and $\{z_m\}_m$ coincide. Indeed, using (25) it can be shown as follows

$$\|z_m - \bar{z}_m\|_{q,\Omega_{T_e}}^q = \sum_{n=1}^{m}\int_{(n-1)h}^{nh}\left\|(z_m^n - z_m^{n-1})\frac{t-nh}{h}\right\|_{q,\Omega}^q dt$$

$$= \frac{h^{q+1}}{q+1}\sum_{n=1}^{m}\left\|\frac{z_m^n - z_m^{n-1}}{h}\right\|_{q,\Omega}^q = \frac{h^q}{q+1}\left\|\frac{dz_m}{dt}\right\|_{q,\Omega_{T_e}}^q,$$

which implies that $\bar{z}_m - z_m$ converges strongly to 0 in $L^q(\Omega_{T_e},\mathbb{R}^N)$. The proof of the fact that the difference $\bar{T}_m - T_m$ converges weakly to 0 in $L^2(\Omega_{T_e},\mathcal{S}^3)$ can be performed as in [27, p. 210]. For reader's convenience we reproduce here the reasoning from there. Let us choose some appropriate number $d\in\mathbb{N}$ and then fix any integer $n_0\in[1,2^d]$. Let $h_0 = T_e/2^{n_0}$. Consider functions $I_{[h_0(n_0-1),h_0n_0]}v$ with $v\in L^2(\Omega,\mathcal{S}^3)$, where I_K denotes the indicator function of a set K. We note that, according to [27, Proposition 1.36], the linear conbinations of all such functions are dense in $L^2(\Omega_{T_e},\mathcal{S}^3)$. Then for any $h\le h_0$ [5]

$$\left(T_m - \bar{T}_m, I_{[h_0(n_0-1),h_0n_0]}v\right)_{\Omega_{T_e}} = \int_{h_0(n_0-1)}^{h_0n_0}\left(T_m(t) - \bar{T}_m(t), v\right)_\Omega dt$$

$$= \sum_{n=h_0(n_0-1)/h+1}^{h_0n_0/h}\int_{(n-1)h}^{nh}\left((T_m^n - T_m^{n-1})\frac{t-nh}{h}, v\right)_\Omega dt$$

$$= -\frac{h}{2}\left(T_m^{h_0n_0/h} - T_m^{h_0(n_0-1)/h}, v\right)_\Omega = -\frac{h}{2}\left(\bar{T}_m(h_0n_0) - \bar{T}_m(h_0(n_0-1)), v\right)_\Omega.$$

Employing (27) we get that $\bar{T}_m - T_m$ converges weakly to 0 in $L^2(\Omega_{T_e},\mathcal{S}^3)$. Next, by (28) the sequence $\{z_m/m\}_m$ converges strongly to 0 in $L^2(\Omega_{T_e},\mathbb{R}^N)$. Summarizing all observations made above we may conclude that the limit functions denoted by u, T, z and Σ have the following properties

$$u\in W^{1,q}(\Omega_{T_e},\mathbb{R}^3),\quad (T, L^{1/2}z)\in L^\infty(0,T_e;L^2(\Omega,\mathcal{S}^3\times\mathbb{R}^N)),$$

and

$$z\in W^{1,q}(0,T_e;L^q(\Omega,\mathbb{R}^N)),\quad \Sigma = B^TT - Lz\in L^p(\Omega_{T_e},\mathbb{R}^N).$$

To be able to pass to the weak limit, we need to derive further a'priori estimates.

[5] We recall that h is chosen to be equal to $T_e/2^m$ for some $m\in\mathbb{N}$.

Existence of solutions. In order to get the additional a'priori estimates, we extend the function b for $t<0$ by setting $b(t)=b(0)$. The extended function b is still in the space $W^{1,p}(-2h,T_e;W^{-1,p}(\Omega,\mathbb{R}^3))$. Then, we set $b_m^0=b_m^{-1}:=b(0)$. Let us further set

$$z_m^{-1}:=z_m^0-h\mathcal{G}(\Sigma_{0,m}).$$

For given z_m^{-1} and z_m^0, we can define functions (u_m^{-1},T_m^{-1}) and (u_m^0,T_m^0) as solutions of linear elasticity problem (85) - (87) to the data $\hat{b}=b_m^{-1}$, $\hat{\gamma}=0$, $\hat{\varepsilon}_p=z_m^{-1}$ and $\hat{b}=b_m^0$, $\hat{\gamma}=0$, $\hat{\varepsilon}_p=z_m^0$, respectively. Due to the assumption (10), the following estimate holds

$$\left\{\left\|\frac{z_m^0-z_m^{-1}}{h}\right\|_\Omega,\left\|\frac{u_m^0-u_m^{-1}}{h}\right\|_\Omega,\left\|\frac{T_m^0-T_m^{-1}}{h}\right\|_\Omega\right\}\le C,\tag{31}$$

where C is some positive constant independent of m.

Taking now the incremental ratio of (14) for $n=1,...,2^m$, we obtain[6]

$$\hat{z}_m^n-\hat{z}_m^{n-1}=\mathcal{G}(\Sigma_{n,m})-\mathcal{G}(\Sigma_{(n-1),m}).$$

Let us now multiply the last identity by $-(\Sigma_{n,m}-\Sigma_{(n-1),m})/h$. Then using the monotonicity of $\mathcal{G}$ we get

$$\frac{1}{m}\Big(\hat{z}_m^n-\hat{z}_m^{n-1},\hat{z}_m^n\Big)_\Omega+\big(\hat{z}_m^n-\hat{z}_m^{n-1},L\hat{z}_m^n\big)_\Omega\le\big(\hat{z}_m^n-\hat{z}_m^{n-1},B^T\hat{T}_m^n\big)_\Omega.$$

Further, (12) and (13) imply

$$\frac{1}{m}\Big(\hat{z}_m^n-\hat{z}_m^{n-1},\hat{z}_m^n\Big)_\Omega+\big(\hat{z}_m^n-\hat{z}_m^{n-1},L\hat{z}_m^n\big)_\Omega+\big(\hat{T}_m^n-\hat{T}_m^{n-1},\mathbb{C}^{-1}\hat{T}_m^n\big)_\Omega$$
$$\le\big(\hat{u}_m^n-\hat{u}_m^{n-1},\hat{b}_m^n\big)_\Omega.$$

As above, multiplying the last inequality by h and summing then for $n=1,...,l$ for any fixed $l\in[1,2^m]$ we get the estimate

$$\frac{h}{m}\|\hat{z}_m^l\|_\Omega^2+h\|L^{1/2}\hat{z}_m^l\|_\Omega^2+h\|\mathbb{B}^{1/2}\hat{T}_m^l\|_\Omega^2\le hC^{(0)}$$
$$+2h\sum_{n=1}^{l}\Big(\hat{b}_m^n,\hat{u}_m^n-\hat{u}_m^{n-1}\Big)_\Omega,\tag{32}$$

where now $C^{(0)}$ denotes

$$C^{(0)}:=\|\mathbb{B}^{1/2}\hat{T}_m^0\|_\Omega^2+\|L^{1/2}\hat{z}_m^0\|_\Omega^2+\frac{1}{m}\|\hat{z}_m^0\|_\Omega^2.$$

We note immediately that (31) yields the uniform boundness of $C^{(0)}$ with respect to m. The right hand side in (32) can be estimated as follows. First, we note that

$$\sum_{n=1}^{l}\Big(\hat{b}_m^n,\hat{u}_m^n-\hat{u}_m^{n-1}\Big)_\Omega=\sum_{n=1}^{l}\Big(\hat{b}_m^{n-1}-\hat{b}_m^n,\hat{u}_m^{n-1}\Big)_\Omega+\Big(\hat{b}_m^l,\hat{u}_m^l\Big)_\Omega-\Big(\hat{b}_m^0,\hat{u}_m^0\Big)_\Omega.$$

[6]For sake of simplicity we use the following notation $\hat{\phi}_m^n:=(\phi_m^n-\phi_m^{n-1})/h$, where $\phi_m^0,\phi_m^1,...,\phi_m^m$ is any family of functions.

Thus, by using (31) and Young's inequality one easily gets the following

$$h\sum_{n=1}^{l}\left(\hat{b}_m^n, \hat{u}_m^n - \hat{u}_m^{n-1}\right)_\Omega \le C\left(1 + \|b\|_{W^{1,p}(0,T_e,L^p(\Omega,\mathbb{R}^3))} + \|u_m\|_{W^{1,q}(\Omega_{T_e},\mathbb{R}^3)}\right),$$

where C is some positive constant. Summing now (32) for $l = 1, ..., 2^m$ we derive the estimate

$$\sum_{n=1}^{2^m}\left(\frac{h}{m}\|\hat{z}_m^l\|_\Omega^2 + h\|L^{1/2}\hat{z}_m^l\|_\Omega^2 + h\|\mathbb{B}^{1/2}\hat{T}_m^l\|_\Omega^2\right)$$
$$\le T_e\big(C^{(0)} + 2C\left(1 + \|b\|_{W^{1,p}(0,T_e,L^p(\Omega,\mathbb{R}^3))} + \|u_m\|_{W^{1,q}(\Omega_{T_e},\mathbb{R}^3)}\right)\big). \quad (33)$$

Since u_m is uniformly bounded in $W^{1,q}(\Omega_{T_e}, \mathbb{R}^3)$, the last estimate (33) gives

$$\{\partial_t L^{1/2} z_m\}_m \text{ is uniformly bounded in } L^2(0, T_e; L^2(\Omega, \mathbb{R}^N)), \quad (34)$$
$$\{\partial_t T_m\}_m \text{ is uniformly bounded in } L^2(0, T_e; L^2(\Omega, \mathcal{S}^3)), \quad (35)$$
$$\left\{\frac{1}{\sqrt{m}}\partial_t z_m\right\}_m \text{ is uniformly bounded in } L^2(0, T_e; L^2(\Omega, \mathbb{R}^N)). \quad (36)$$

To prove that the weak limits of sequences u_m, T_m and z_m solve the problem (1) - (5), we are goning to use the pseudo-monotonicity property of the maximal monotone mapping $\mathcal{G}$. To this end, we note first that one can easily pass to the weak limit in the equations (1), (2) and (5) and gets that

$$-\operatorname{div}_x T(x,t) = b(x,t), \quad (37)$$
$$T(x,t) = \mathcal{D}_\eta[x](\varepsilon(\nabla_x u(x,t)) - Bz(x,t)), \quad (38)$$
$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0, T_e). \quad (39)$$

To pass to the limit in (3), we proceed as follows:

$$\left(\frac{dz_m}{dt}, \bar{\Sigma}_m\right)_{\Omega_{T_e}} = \left(\frac{dz_m}{dt}, B^T\bar{T}_m\right)_{\Omega_{T_e}} - \frac{1}{m}\left(\frac{dz_m}{dt}, \bar{z}_m\right)_{\Omega_{T_e}} - \left(\frac{dz_m}{dt}, L\bar{z}_m\right)_{\Omega_{T_e}}.$$

With (12) - (13) we then have that

$$\left(\frac{dz_m}{dt}, \bar{\Sigma}_m\right)_{\Omega_{T_e}} = -\left(\frac{dT_m}{dt}, \bar{T}_m\right)_{\Omega_{T_e}} - \frac{1}{m}\left(\frac{dz_m}{dt}, \bar{z}_m\right)_{\Omega_{T_e}}$$
$$-\left(\frac{dz_m}{dt}, L\bar{z}_m\right)_{\Omega_{T_e}} + \left(\frac{du_m}{dt}, \bar{b}_m\right)_{\Omega_{T_e}}.$$

Thus, by Lemma 3.1,

$$\limsup_{m\to\infty}\left(\frac{dz_m}{dt}, \bar{\Sigma}_m\right)_{\Omega_{T_e}} \le -\left(\frac{d(L^{1/2}z)}{dt}, L^{1/2}z\right)_{\Omega_{T_e}} -$$
$$\left(\frac{dT}{dt}, T\right)_{\Omega_{T_e}} + (b, \partial_t u)_{\Omega_{T_e}}. \quad (40)$$

Finally, equations (37) - (39) together with (40) yield[7]

$$\limsup_{m\to\infty}\left(\frac{dz_m}{dt},\bar{\Sigma}_m\right)_{\Omega_{T_e}} \le \left(\frac{dz}{dt},B^T T - Lz\right)_{\Omega_{T_e}}, \tag{41}$$

and, by the pseudo-monotonicity property of $\mathcal{G}$, we conclude that

$$[\partial z(x,t), B^T T(x,t) - Lz(x,t)] \in Grg(x).$$

This, together with (37) - (39), completes the proof of Theorem 3.1. □

At the end of this section we prove the following lemma used in the proof of Theorem 3.1.

Lemma 3.1. *Let X be a reflexive Banach space embedded continuously into a Hilbert space H, functions $\phi_m, \bar{\phi}_m$ be defined by (18) and (19) for any family of functions $\phi_m^0, \phi_m^1, ..., \phi_m^m$, respectively, and ϕ be a weak limit of ϕ_m. Then, the following inequality holds*

$$\limsup_{m\to\infty}\left(\frac{d\phi_m}{dt},\bar{\phi}_m\right)_{L^q(X^*),L^p(X)} \ge \left(\frac{d\phi}{dt},\phi\right)_{L^q(X^*),L^p(X)}.$$

Proof. The inequality results from the next line by taking lim sup from both side and using the lower semi-continuity of the norm

$$\left(\frac{d\phi_m}{dt},\bar{\phi}_m\right)_{L^q(X^*),L^p(X)} = \sum_{n=1}^{m}\int_{h(n-1)}^{hn}\left(\frac{\phi_m^n-\phi_m^{n-1}}{h},\phi_m^n\right)_{X^*,X} dt$$

$$= \sum_{n=1}^{m}\left(\phi_m^n-\phi_m^{n-1},\phi_m^n\right)_{X^*,X} \ge \frac{1}{2}\|\phi_m^m\|_H^2 - \frac{1}{2}\|\phi_m^0\|_H^2.$$

The proof is completed by the application of the generalized integration-by-parts formula. □

4 The periodic unfolding

The derivation of the homogenized problem for (1) - (5) is based on the periodic unfolding operator introduced by Cioranescu, Damlamian and Griso [12]. For the reader unfamiliar with this method we recall the definitions and properties of this operator. The proofs can be found in [12, 13, 15].

4.1 Periodic unfolding method

In $\mathbb{Z}^k$, let Ω be an open set and Y a reference cell. For $z \in \mathbb{Z}^k$, $[z]_Y$ denotes the unique integer combination $\sum_{j=1}^k d_j b_j$ of the periods such that $z - [z]_Y$ belongs to Y, and set

$$\{z\}_Y := z - [z]_Y \in Y \quad z \in \mathbb{Z}^k.$$

[7]We observe that for a.e. (x,t) one has

$$\left(\frac{d(L^{1/2}z)}{dt}(x,t), L^{1/2}z(x,t)\right) = \left(\frac{dz}{dt}(x,t), Lz(x,t)\right),$$

and, since the first function is integrable, the second one is integrable as well.

Then, for each $x \in \mathbb{R}^k$, one has

$$x = \eta \left(\left[\frac{x}{\eta} \right]_Y + y \right).$$

We use the following notations:

$$\Xi_\eta = \{\xi \in \mathbb{Z}^k \mid \eta(\xi + Y) \subset \Omega\}, \quad \hat{\Omega}_\eta = \text{int} \left\{ \bigcup_{\xi \in \Xi_\eta} \left(\eta\xi + \eta\overline{Y} \right) \right\}, \quad \Lambda_\eta = \Omega \setminus \hat{\Omega}_\eta.$$

The set $\hat{\Omega}_\eta$ is the largest union of $\eta(\xi + \overline{Y})$ cells ($\xi \in \mathbb{Z}^k$) included in Ω, while Λ_η is the subset of Ω containing the parts from $\eta(\xi + \overline{Y})$ cells intersecting the boundary $\partial\Omega$.

Definition 4.1. *Let Y be a reference cell, η be a positive number and a map $v : \Omega \to \mathbb{R}^k$. The unfolding operator $\mathcal{T}_\eta(v) : \Omega \times Y \to \mathbb{R}^k$ is defined by*

$$\mathcal{T}_\eta(v) := \begin{cases} v\left(\eta \left[\frac{x}{\eta}\right]_Y + \eta y\right), & \text{a.e. } (x, y) \in \hat{\Omega}_\eta \times Y, \\ 0, & \text{a.e. } (x, y) \in \Lambda_\eta \times Y. \end{cases}$$

The next results are straightforward from Definition 4.1.

Proposition 4.1. *For $p \in [1, \infty[$, the operator $\mathcal{T}_\eta$ is linear and continuous from $L^p(\Omega, \mathbb{R}^k)$ to $L^p(\Omega \times Y, \mathbb{R}^k)$. For every ϕ in $L^1(\Omega, \mathbb{R}^k)$ one has*

(a) $\frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\eta(\phi)(x, y) dx dy = \int_{\hat{\Omega}_\eta} \phi(x) dx,$

(b) $\frac{1}{|Y|} \int_{\Omega \times Y} |\mathcal{T}_\eta(\phi)(x, y)| dx dy \leq \int_\Omega |\phi(x)| dx,$

(c) $\left| \int_{\hat{\Omega}_\eta} \phi(x) dx - \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\eta(\phi)(x, y) dx dy \right| \leq \int_{\Lambda_\eta} |\phi(x)| dx,$

(d) $\|\mathcal{T}_\eta(\phi)\|_{p, \Omega \times Y} = |Y|^{1/p} \|\phi I_{\hat{\Omega}_\eta}\|_{p, \Omega} \leq |Y|^{1/p} \|\phi\|_{p, \Omega}.$

Proof. See [13, Proposition 2.5]. □

Obviously, if $\phi_\eta \in L^1(\Omega, \mathbb{R}^k)$ satisfies

$$\int_{\Lambda_\eta} |\phi_\eta(x)| dx \to 0, \tag{42}$$

then

$$\int_\Omega \phi_\eta(x) dx - \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\eta(\phi_\eta)(x, y) dx dy \to 0.$$

If a sequence ϕ_η satisfies (42), we shall write

$$\int_\Omega \phi_\eta(x) dx \overset{\mathcal{T}_\eta}{\simeq} \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\eta(\phi_\eta)(x, y) dx dy.$$

Proposition 4.2. *Suppose $\partial\Omega$ is bounded. Let u_η be a bounded sequence in $L^p(\Omega, \mathbb{R}^k)$ and w_η be a bounded sequence in $L^q(\Omega, \mathbb{R}^k)$, $1/p + 1/q = 1$, then*

$$\int_\Omega u_\eta w_\eta dx \overset{\mathcal{T}_\eta}{\simeq} \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\eta(u_\eta) \mathcal{T}_\eta(w_\eta) dx dy.$$

Proof. See [13, Proposition 2.7]. □

Proposition 4.3. *Let p belong to $[1, \infty[$.*

(a) For any $v \in L^p(\Omega, \mathbb{R}^k)$, $\mathcal{T}_\eta(v) \to v$ strongly in $L^p(\Omega \times Y, \mathbb{R}^k)$,

(b) Let v_η be a bounded sequence in $L^p(\Omega, \mathbb{R}^k)$ such that $v_\eta \to v$ strongly in $L^p(\Omega, \mathbb{R}^k)$, then

$$\mathcal{T}_\eta(v_\eta) \to v, \quad \textit{strongly in } L^p(\Omega \times Y, \mathbb{R}^k).$$

(c) For every relatively weakly compact sequence v_η in $L^p(\Omega, \mathbb{R}^k)$, the corresponding $\mathcal{T}_\eta(v_\eta)$ is relatively weakly compact in $L^p(\Omega \times Y, \mathbb{R}^k)$. Furthermore, if

$$\mathcal{T}_\eta(v_\eta) \rightharpoonup \hat{v} \quad \textit{in } L^p(\Omega \times Y, \mathbb{R}^k),$$

then

$$v_\eta \rightharpoonup \frac{1}{|Y|} \int_Y \hat{v} dy \quad \textit{in } L^p(\Omega, \mathbb{R}^k).$$

Proof. See [13, Proposition 2.9]. □

Next results present some properties of the restriction of the unfolding operator to the space $W^{1,p}(\Omega, \mathbb{R}^k)$.

Proposition 4.4. *Let p belong to $]1, \infty[$.*

(a) Suppose that $v_\eta \in W^{1,p}(\Omega, \mathbb{R}^k)$ is bounded in $L^p(\Omega, \mathbb{R}^k)$ and satisfies

$$\eta \|\nabla v_\eta\|_{p,\Omega} \leq C.$$

Then, there exists a subsequence and $\hat{v} \in L^p(\Omega, W^{1,p}_{per}(Y, \mathbb{R}^k))$ such that

$$\mathcal{T}_\eta(v_\eta) \rightharpoonup \hat{v} \quad \textit{in } L^p(\Omega, W^{1,p}_{per}(Y, \mathbb{R}^k)),$$
$$\mathcal{T}_\eta(\nabla v_\eta) \rightharpoonup \nabla_y \hat{v} \quad \textit{in } L^p(\Omega \times Y, \mathbb{R}^k).$$

(b) Let v_η converge weakly in $W^{1,p}(\Omega, \mathbb{R}^k)$ to v. Then

$$\mathcal{T}_\eta(v_\eta) \rightharpoonup v \quad \textit{in } L^p(\Omega, W^{1,p}_{per}(Y, \mathbb{R}^k)).$$

Proof. See [13, Corollary 3.2, Corollary 3.3]. □

Proposition 4.5. *Let p belong to $]1, \infty[$. Let v_η converge weakly in $W^{1,p}(\Omega, \mathbb{R}^k)$ to some v. Then, up to a subsequence, there exists some $\hat{v} \in L^p(\Omega, W^{1,p}_{per}(Y, \mathbb{R}^k))$ such that*

$$\mathcal{T}_\eta(\nabla v_\eta) \rightharpoonup \nabla v + \nabla_y \hat{v} \quad \textit{in } L^p(\Omega \times Y, \mathbb{R}^k).$$

Proof. See [13, Theorem 3.5, (i)]. □

For a multi-valued function $g \in \mathcal{M}(\omega, \mathbb{R}^k, \alpha, m)$ the unfolding operator is defined as follows.

Definition 4.2. *Let Y be a reference cell, η be a positive number and a map $g \in \mathcal{M}(\omega, \mathbb{R}^k, p, \alpha, m)$. The unfolding operator $\mathcal{T}_\eta(g) : \Omega \times Y \times \mathbb{R}^k \to 2^{\mathbb{R}^k}$ is defined by*

$$\mathcal{T}_\eta(g)(x,y,z) := \begin{cases} g\left(\eta\left[\frac{x}{\eta}\right]_Y + \eta y, z\right), & \text{a.e. } (x,y) \in \hat{\Omega}_\eta \times Y,\ z \in \mathbb{R}^k, \\ \|z\|^{p-2} z, & \text{a.e. } (x,y) \in \Lambda_\eta \times Y,\ z \in \mathbb{R}^k. \end{cases}$$

We note that the periodic unfolding method described above is an alternative to the two-scale convergence method introduced in [23] and further developed in [7]. More precisely, the two-scale convergence of a bounded sequence v_η in $L^p(\Omega, \mathbb{R}^k)$ is equivalent to the weak convergence of the corresponding unfolded sequence $\mathcal{T}_\eta(v_\eta)$ in $L^p(\Omega \times Y, \mathbb{R}^k)$ (see [13, Proposition 2.14]).

4.2 Homogenization of linear elasticity system

Now we show how to apply the periodic unfolding method to the homogenization of linear elasticity systems with periodically highly oscillating coefficients (see [14] for properties of periodically oscillating functions). Our interest in this particular example is not only because of that the equations (1), (2) and (5) form an elasticity problem but also because of the strong convergence of the unfolded sequence of the gradients of the solutions of linear elasticity problem (see Theorem 4.1 below), what is of great importance in the derivation of the homogenized equations for the initial boundary value problem (1) - (5). The proof of the mentioned result applied to an elliptic partial differential equation is performed in [13] and can be carried over linear elasticity systems without significant modifications. Therefore, we present here only that part of the proof in [13], which might cause some difficulties at first glance when one tries to adopt the proof.

Let us consider the linear elasticity problem

$$-\operatorname{div} T_\eta(x) = \hat{b}(x), \quad x \in \Omega, \tag{43}$$

$$T_\eta(x) = \mathcal{D}\left[\frac{x}{\eta}\right] \varepsilon\left(\nabla_x u_\eta(x) - \varepsilon_{p,\eta}(x)\right), \quad x \in \Omega, \tag{44}$$

$$u_\eta(x) = 0, \qquad x \in \partial\Omega, \tag{45}$$

with given functions $\hat{b} \in H^{-1}(\Omega, \mathrm{IR}^3)$ and $\varepsilon_{p,\eta} \in L^2(\Omega, \mathcal{S}^3)$ such that $\varepsilon_{p,\eta}$ converges to $\varepsilon_{p,0}$ strongly in $L^2(\Omega, \mathcal{S}^3)$ as $\eta \to 0$. The following result holds.

Theorem 4.1. *There exist $u_0 \in H_0^1(\Omega, \mathrm{IR}^3)$, $T_0 \in L^2(\Omega \times Y, \mathcal{S}^3)$ and $u_1 \in L^2(\Omega, H^1_{per}(Y, \mathrm{IR}^3))$ such that*

$$u_\eta \rightharpoonup u_0 \quad \text{in } H_0^1(\Omega, \mathrm{IR}^3), \tag{46}$$

$$\mathcal{T}_\eta(u_\eta) \rightharpoonup u_0 \quad \text{in } L^2(\Omega, H^1(Y, \mathrm{IR}^3)), \tag{47}$$

$$\mathcal{T}_\eta(\nabla u_\eta) \rightharpoonup \nabla u_0 + \nabla_y u_1 \quad \text{in } L^2(\Omega \times Y, \mathrm{IR}^3), \tag{48}$$

$$\mathcal{T}_\eta(T_\eta) \rightharpoonup T_0 \quad \text{in } L^2(\Omega \times Y, \mathcal{S}^3), \tag{49}$$

and (u_0, T_0, u_1) *is the unique solution of the homogenized system:*

$$
\begin{aligned}
-\operatorname{div}_y T_0(x,y) &= 0, && (50)\\
T_0(x,y) &= \mathcal{D}[y]\big(\varepsilon(\nabla u_0(x) + \nabla_y u_1(x,y)) - \varepsilon_{p,0}(x)\big), && (51)\\
y &\mapsto u_1(x,y), \quad Y-\textit{periodic}, && (52)\\
-\operatorname{div}_x T_\infty(x) &= \hat{b}(x), && (53)\\
T_\infty(x) &= \frac{1}{|Y|}\int_Y T_0(x,y)dy && (54)\\
u_0(x) &= 0, \qquad x \in \partial\Omega. && (55)
\end{aligned}
$$

Moreover, the following convergences hold

$$
\lim_{\eta\to 0}\int_\Omega \mathcal{D}\left[\cdot/\eta\right]\varepsilon(\nabla_x u_\eta)\varepsilon(\nabla_x u_\eta)dx
$$

$$
= \frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{D}[y]\varepsilon\big(\nabla u_0 + \nabla_y u_1\big)\varepsilon\big(\nabla u_0 + \nabla_y u_1\big)dxdy, \quad (56)
$$

$$
\mathcal{T}_\eta(\nabla u_\eta) \to \nabla u_0 + \nabla_y u_1 \quad \textit{in } L^2(\Omega\times Y, \mathbb{R}^3), \quad (57)
$$

$$
\mathcal{T}_\eta(T_\eta) \to T_0 \quad \textit{in } L^2(\Omega\times Y, \mathcal{S}^3). \quad (58)
$$

Proof. Convergences (46) - (49) follow from estimates of solutions of a Dirichlet boundary value problem from the linear elasticity theory (see [28], if needed) and the results from the revious subsection. The derivation of equations (50) - (55) are performed exactly as in Theorem 5.3 in [13]. Convergence (57) is an easy consequence of (56) (see [13, Corollary 5.11]) and (58) follows from (57). Therefore, it is left to check (56) only. By standard weak lower-semicontinuity, one obtains

$$
\begin{aligned}
&\frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{D}[y]\varepsilon\big(\nabla u_0 + \nabla_y u_1\big)\varepsilon\big(\nabla u_0 + \nabla_y u_1\big)dxdy\\
&\le \limsup_{\eta\to 0}\frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{T}_\eta(\mathcal{D}\left[\cdot/\eta\right])\mathcal{T}_\eta(\varepsilon(\nabla_x u_\eta))\mathcal{T}_\eta(\varepsilon(\nabla_x u_\eta))dxdy\\
&\le \limsup_{\eta\to 0}\int_\Omega \mathcal{D}\left[\cdot/\eta\right]\varepsilon(\nabla_x u_\eta)\varepsilon(\nabla_x u_\eta)dx = \limsup_{\eta\to 0}\left\langle \hat{b}, u_\eta\right\rangle\\
&+ \limsup_{\eta\to 0}\int_\Omega \mathcal{D}\left[\cdot/\eta\right]\varepsilon_{p,\eta}\varepsilon(\nabla_x u_\eta)dx \le \limsup_{\eta\to 0}\left\langle \hat{b}, u_\eta\right\rangle\\
&+ \limsup_{\eta\to 0}\frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{T}_\eta(\mathcal{D}\left[\cdot/\eta\right])\mathcal{T}_\eta(\varepsilon_{p,\eta})\mathcal{T}_\eta(\varepsilon(\nabla_x u_\eta))dxdy\\
&= \left\langle \hat{b}, u_0\right\rangle + \frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{D}[y]\varepsilon_{p,0}\varepsilon\big(\nabla u_0 + \nabla_y u_1\big)dxdy\\
&= \frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{D}[y]\varepsilon\big(\nabla u_0 + \nabla_y u_1\big)\varepsilon\big(\nabla u_0 + \nabla_y u_1\big)dxdy,
\end{aligned}
$$

which gives the convergence (56). □

5 Homogenization

In this section we prove the following theorem.

Theorem 5.1. *Suppose that all assumptions of Theorem 3.1 are satisfied. Let* (u_η, T_η, z_η) *be a solution of the initial-boundary value problem (1) - (5).*

Then, there exist

$$u_0 \in W^{1,q}(0,T_e; W_0^{1,q}(\Omega,\mathbb{R}^3)), \quad u_1 \in W^{1,q}(0,T_e; L^q(\Omega, W_{per}^{1,q}(Y,\mathbb{R}^3))),$$
$$T_0 \in H^1(0,T_e; L^2(\Omega\times Y,\mathcal{S}^3)), \quad z_0 \in W^{1,q}(0,T_e; L^q(\Omega\times Y,\mathbb{R}^{\mathrm{N}}))$$

such that

$$u_\eta \rightharpoonup u_0 \quad \text{in } W^{1,q}(0,T_e; W_0^{1,q}(\Omega,\mathbb{R}^3)), \tag{59}$$
$$\mathcal{T}_\eta(\nabla u_\eta) \rightharpoonup \nabla u_0 + \nabla_y u_1 \quad \text{in } W^{1,q}(0,T_e; L^q(\Omega\times Y,\mathbb{R}^3)), \tag{60}$$
$$\mathcal{T}_\eta(T_\eta) \rightharpoonup T_0 \quad \text{in } H^1(0,T_e; L^2(\Omega\times Y,\mathcal{S}^3)), \tag{61}$$
$$\mathcal{T}_\eta(z_\eta) \rightharpoonup z_0 \quad \text{in } W^{1,q}(0,T_e; L^q(\Omega\times Y,\mathbb{R}^{\mathrm{N}})), \tag{62}$$

and (u_0, u_1, T_0, z_0) *is a solution of the homogenized system:*

$$-\operatorname{div}_x T_\infty(x,t) = b(x,t), \tag{63}$$
$$T_\infty(x,t) = \frac{1}{|Y|}\int_Y T_0(x,y,t)dy, \tag{64}$$
$$-\operatorname{div}_y T_0(x,y,t) = 0, \tag{65}$$
$$T_0(x,y,t) = \mathcal{D}[y]\Big(\varepsilon(\nabla_y u_1(x,y,t)) - Bz_0(x,y,t) + \varepsilon(\nabla_x u_0(x,t))\Big), \tag{66}$$
$$\frac{\partial}{\partial t} z_0(x,y,t) \in g\big(y, B^T T_0(x,y,t) - L[y]z_0(x,y,t)\big), \tag{67}$$
$$z_0(x,y,0) = z^{(0)}(x), \tag{68}$$

which hold for $(x,y,t) \in \Omega\times\mathbb{R}^3\times[0,\mathrm{T_e}]$*, and with the boundary condition*

$$u_0(x,t) = 0, \quad (x,t)\in\partial\Omega\times[0,T_e]. \tag{69}$$

Moreover, the following convergences hold

$$\mathcal{T}_\eta(L^{1/2}z_\eta) \rightharpoonup L^{1/2}z_0 \quad \text{in } H^1(0,T_e; L^2(\Omega\times Y,\mathbb{R}^{\mathrm{N}})), \tag{70}$$
$$\mathcal{T}_\eta(B^T T_\eta - Lz_\eta) \rightharpoonup B^T T_0 - Lz_0 \quad \text{in } L^p(\Omega_{T_e}\times Y,\mathcal{S}^3). \tag{71}$$

Remark 5.1. *For fixed* x *the equations* (65) – (68) *together with the periodicity condition for* $y \mapsto (u_1, T_0)(x,y,t)$*, which can be considered as a boundary condition, define an initial-boundary problem in* $Y\times[0,T_e)$*, the so-called cell problem. The functions* u_0 *and* u_1 *in* (66) *and* (69) *can be interpreted as macro- and microdisplacements, respectively,* T_0 *as the microstress; the macrostress* T_∞ *is obtained by averaging of* T_0 *over the representative volume element* Y*.*

Remark 5.2. *We note the homogenized equations* (63) – (69) *coincide with those ones, which were formally obtained in [2].*

Proof. Our uniform estimates provide that

$$\{u_\eta\} \text{ is uniformly bounded in } W^{1,q}(0,T_e;W_0^{1,q}(\Omega,\mathbb{R}^3)), \tag{72}$$

$$\{T_\eta\} \text{ is uniformly bounded in } W^{1,1}(0,T_e;L^2(\Omega,\mathcal{S}^3)), \tag{73}$$

$$\{z_\eta\} \text{ is uniformly bounded in } W^{1,q}(0,T_e;L^q(\Omega,\mathbb{R}^N)), \tag{74}$$

$$\{L^{1/2}z_\eta\} \text{ is uniformly bounded in } W^{1,1}(0,T_e;L^2(\Omega,\mathbb{R}^N)), \tag{75}$$

$$\{B^T T_\eta - L z_\eta\} \text{ is uniformly bounded in } L^p(\Omega_{T_e},\mathcal{S}^3). \tag{76}$$

By Proposition 4.3 and Proposition 4.5, there exist functions u_0, u_1, T_0 and z_0 with the prescribed regularities in Theorem 5.1 such that the convergences in (59) - (62) hold. As a consequence of that and of the uniqueness of the weak limit, one can conclude that the convergences in (70) and (71) hold true as well. Note that (60)-(62) give the equation (66), i.e

$$T_0(x,y,t) = \mathcal{D}[y]\big(\varepsilon(\nabla_x u_0(x,t) + \nabla_y u_1(x,y,t)) - Bz_0(x,y,t)\big), \quad \text{a.e.} \tag{77}$$

Let us define

$$T_\infty(x,t) = \frac{1}{|Y|}\int_Y T_0(x,y,t)dy.$$

Note that T_∞ is the weak limit of T_η in $W^{1,1}(0,T_e;L^2(\Omega,\mathcal{S}^3))$ (see Proposition 4.3). As in [16], we consider any $\phi \in C_0^\infty(\Omega,\mathbb{R}^3)$. Then, by the weak convergence of T_η, the passage to the weak limit in (1) yields

$$\int_\Omega (T_\infty(x,t),\nabla\phi(x))dx = \int_\Omega (b(x,t),\phi(x))dx, \tag{78}$$

i.e $\operatorname{div}_x T_\infty = b$ in $(C_0^\infty(\Omega,\mathbb{R}^3))^*$. Next, define $\phi_\eta(x) = \eta\phi(x)\psi(x/\eta)$, where $\phi \in C_0^\infty(\Omega,\mathbb{R}^3)$ and $\psi \in C_{per}^\infty(Y,\mathbb{R}^3)$. Then, one obtains that

$$\phi_\eta \rightharpoonup 0, \ \text{ in } W_0^{1,p}(\Omega,\mathbb{R}^3), \ \text{ and } \ \mathcal{T}_\eta(\nabla\phi_\eta) \to \phi\nabla_y\psi, \ \text{ in } L^p(\Omega,W_{per}^{1,p}(Y,\mathbb{R}^3)).$$

Therefore, by Proposition 4.2

$$\frac{1}{|Y|}\int_{\Omega\times Y}(\mathcal{T}_\eta(T_\eta(t)),\mathcal{T}_\eta(\nabla\phi_\eta))dxdy \overset{\mathcal{T}_\eta}{=} \int_\Omega (b(t),\phi)dx. \tag{79}$$

The passage to the limit in (79) leads to

$$\frac{1}{|Y|}\int_{\Omega\times Y}(T_0(x,y,t),\phi(x)\nabla_y\psi(y))dxdy = 0.$$

Thus, in virtue of the arbitrariness of ϕ, one can conclude that

$$\frac{1}{|Y|}\int_{\Omega\times Y}(T_0(x,y,t),\nabla_y\psi(y))dxdy = 0. \tag{80}$$

i.e $\operatorname{div}_y T_0(x,\cdot,t) = 0$ in $(C_{per}^\infty(Y,\mathbb{R}^3))^*$. By Theorem 2.4, one has that $\mathcal{T}_\eta(\mathcal{G}_\eta) \rightarrowtail \mathcal{G}$ [8], where $\mathcal{T}_\eta(\mathcal{G}_\eta)$ and $\mathcal{G}$ denote canonical extensions of $\mathcal{T}_\eta(g_\eta)(x,y)$ and $g(y)$,

[8] The pointwise convergence $\mathcal{T}_\eta(g_\eta)(x,y) \rightarrowtail g(y)$ is immediate.

respectively. Using equations (1) and (2), we successfully compute that

$$\begin{aligned}
&\frac{1}{|Y|}\int_{\Omega\times Y}(\mathcal{T}_\eta(\partial_t z_\eta(t)),\mathcal{T}_\eta(B^T T_\eta(t)-Lz_\eta(t)))dxdy\\
&=\frac{1}{|Y|}\int_{\Omega\times Y}\big(\mathcal{T}_\eta\big(\partial_t(\varepsilon(\nabla u_\eta(t))-\mathcal{D}^{-1}T_\eta(t))\big),\mathcal{T}_\eta(T_\eta(t))\big)dxdy\\
&\qquad-\frac{1}{|Y|}\int_{\Omega\times Y}\big(\mathcal{T}_\eta(\partial_t z_\eta(t)),\mathcal{T}_\eta(Lz_\eta(t))\big)dxdy\\
&=\int_\Omega(b(t),\partial_t u_\eta(t))dx-\frac{1}{|Y|}\int_{\Omega\times Y}\big(\mathcal{T}_\eta(\partial_t\mathcal{D}^{-1}T_\eta(t))),\mathcal{T}_\eta(T_\eta(t))\big)dxdy\\
&\qquad-\frac{1}{|Y|}\int_{\Omega\times Y}\big(\mathcal{T}_\eta(\partial_t L^{1/2}z_\eta(t)),\mathcal{T}_\eta(L^{1/2}z_\eta(t))\big)dxdy.
\end{aligned}$$

Integrating the last identity over $(0,t)$ and using the generalized integration-by-parts formula we get that

$$\begin{aligned}
&\frac{1}{|Y|}\int_0^t(\mathcal{T}_\eta(\partial_t z_\eta(t)),\mathcal{T}_\eta(B^T T_\eta(t)-Lz_\eta(t)))_{\Omega\times Y}dt \qquad (81)\\
&=\int_0^t(b(t),\partial_t u_\eta(t))_\Omega dt-\frac{1}{2}\|\mathcal{T}_\eta(\mathcal{B}^{1/2}T_\eta(t))\|^2_{\Omega\times Y}+\frac{1}{2}\|\mathcal{T}_\eta(\mathcal{B}^{1/2}T_\eta(0))\|^2_{\Omega\times Y}\\
&\qquad-\frac{1}{2}\|\mathcal{T}_\eta(L^{1/2}z_\eta(t))\|^2_{\Omega\times Y}+\frac{1}{2}\|\mathcal{T}_\eta(L^{1/2}z_\eta(0))\|^2_{\Omega\times Y},
\end{aligned}$$

where $\mathcal{B}=\mathcal{D}^{-1}$. Since $z_\eta(0)$ converges to $z^{(0)}$ strongly in $L^2(\Omega,\mathbb{R}^N)$, by Proposition 4.3, $\mathcal{T}_\eta(L^{1/2}z_\eta(0))$ converges to $z^{(0)}$ strongly in $L^2(\Omega\times Y,\mathbb{R}^N)$. Since $T_\eta(0)$ solves the linear elasticity problem (43) - (45) with $\varepsilon_{p,\eta}=z_\eta(0)$ and $\hat{b}=b(t)$, by Theorem 4.1, we can conclude that $\mathcal{T}_\eta(\mathcal{B}^{1/2}T_\eta(0))$ converges to $\mathcal{B}^{1/2}T_0(0)$ strongly in $L^2(\Omega\times Y,\mathcal{S}^3)$. Thus, passing to the limit in (81) yields

$$\begin{aligned}
&\limsup_{n\times\infty}\frac{1}{|Y|}\int_0^t(\mathcal{T}_\eta(\partial_t z_\eta(t)),\mathcal{T}_\eta(B^T T_\eta(t)-Lz_\eta(t)))_{\Omega\times Y}dt\\
&\le\int_0^t(b(t),\partial_t u_0(t))_\Omega dt-\frac{1}{2}\|\mathcal{T}_\eta(\mathcal{B}^{1/2}T_0(t))\|^2_{\Omega\times Y}+\frac{1}{2}\|\mathcal{T}_\eta(\mathcal{B}^{1/2}T_0(0))\|^2_{\Omega\times Y}\\
&\qquad-\frac{1}{2}\|\mathcal{T}_\eta(L^{1/2}z_0(t))\|^2_{\Omega\times Y}+\frac{1}{2}\|\mathcal{T}_\eta(L^{1/2}z_0(0))\|^2_{\Omega\times Y},
\end{aligned}$$

or

$$\begin{aligned}
&\limsup_{n\times\infty}\frac{1}{|Y|}\int_0^t(\mathcal{T}_\eta(\partial_t z_\eta(t)),\mathcal{T}_\eta(B^T T_\eta(t)-Lz_\eta(t)))_{\Omega\times Y}dt\\
&\le\int_0^t(b(t),\partial_t u_0(t))_\Omega dt-\frac{1}{|Y|}\int_0^t\big(\partial_t\mathcal{D}^{-1}T_0(t),T_0(t)\big)_{\Omega\times Y}dt\\
&\qquad-\frac{1}{|Y|}\int_0^t\big(\partial_t L^{1/2}z_0(t),L^{1/2}z_0(t)\big)_{\Omega\times Y}dt. \qquad (82)
\end{aligned}$$

We note that (78) and (80) imply

$$\begin{aligned}
\int_\Omega(b(t),\partial_t u_0(t))dx&=\frac{1}{|Y|}\int_{\Omega\times Y}\big(T_0,\partial_t\varepsilon(\nabla u_0+\nabla_y u_1)\big)dxdy\\
&\frac{1}{|Y|}\int_{\Omega\times Y}\big(T_0(t),\partial_t\varepsilon(\nabla u_0(t)+\nabla_y u_1(t))\big)dxdy. \qquad (83)
\end{aligned}$$

And, since for almost all $(x, y, t) \in \Omega \times Y \times (0, T_e)$ one has

$$\big(\partial_t L^{1/2}[y] z_0(x, y, t), L^{1/2}[y] z_0(x, y, t)\big) = \big(\partial_t z_0(x, y, t), L[y] z_0(x, y, t)\big),$$

the relations (82) and (83) together with (77) yield

$$\begin{aligned} \limsup_{n \times \infty} \frac{1}{|Y|} \int_0^t (\mathcal{T}_\eta(\partial_t z_\eta(t)), \mathcal{T}_\eta(B^T T_\eta(t) - L z_\eta(t)))_{\Omega \times Y} dt \\ \leq \frac{1}{|Y|} \int_0^t \big(\partial_t z_0(t), B^T T_0(t) - L z_0(t)\big)_{\Omega \times Y} dt. \end{aligned} \tag{84}$$

By Theorem 2.2, one immediately has that

$$[B^T T_0(x, y, t) - L z_0(x, y, t), \partial_t z_0(x, y, t)] \in Grg(y)$$

or, equivalently, that

$$\partial_t z_0(x, y, t) \in g(y, B^T T_0(x, y, t) - L z_0(x, y, t)).$$

Therefore, summarizing everything done above, we conclude that the functions $(u_0, u_1, T_\infty, T_0, z_0)$ satisfy the homogenized initial-boundary value problem formed by the equations (63) - (69). □

Appendix: Helmholtz projection on tensor fields

The construction of the solutions for the initial boundary value problem (1)–(5) is based on the existence theory for the evolution inclusions in a reflexive Banach space derived in Section 2. The construction procedure requires the introduction of projection operators to spaces of tensor fields, which are symmetric gradients and to spaces of tensor fields with vanishing divergence. All material for this section is taken from [4, 5], where more details and proofs of stated hier results can be found.

We recall ([28, Theorem 4.2]) that a Dirichlet boundary value problem from the linear elasticity theory formed by equations

$$-\mathrm{div}_x T(x) = \hat{b}(x), \qquad x \in \Omega, \tag{85}$$

$$T(x) = \mathcal{D}(\varepsilon(u(x)) - \hat{\varepsilon}_p(x)), \qquad x \in \Omega, \tag{86}$$

$$u(x) = \hat{\gamma}(x), \qquad x \in \partial\Omega, \tag{87}$$

to given $\hat{b} \in W^{-1,p}(\Omega, \mathbb{R}^3)$, $\hat{\varepsilon}_p \in L^p(\Omega, \mathcal{S}^3)$ and $\hat{\gamma} \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ has a unique weak solution $(u, T) \in W^{1,p}(\Omega, \mathbb{R}^3) \times L^p(\Omega, \mathcal{S}^3)$ with $1 < p < \infty$ and $1/p + 1/q = 1$. For $\hat{b} = \hat{\gamma} = 0$ the solution of (85) - (87) satisfies the inequality

$$\|\varepsilon(u)\|_{p,\Omega} \leq C \|\hat{\varepsilon}_p\|_{p,\Omega}$$

with some positive constant C depending on p and Ω.

Definition 5.1. *For every $\hat{\varepsilon}_p \in L^p(\Omega, \mathcal{S}^3)$ we define a linear operator $P_p : L^p(\Omega, \mathcal{S}^3) \to L^p(\Omega, \mathcal{S}^3)$ by*

$$P_p \hat{\varepsilon}_p = \varepsilon(u),$$

where $u \in W_0^{1,p}(\Omega, \mathbb{R}^3)$ is a unique weak solution of (85) - (87) to the given function $\hat{\varepsilon}_p$ and $\hat{b} = \hat{\gamma} = 0$.

A subset $\mathcal{G}^p$ of $L^p(\Omega, \mathcal{S}^3)$ is defined by

$$\mathcal{G}^p = \{\varepsilon(u) \mid u \in W_0^{1,p}(\Omega, \mathrm{I\!R}^3)\}.$$

The main properties of P_p are stated in the following lemma.

Lemma 5.1. *For every $1 < p < \infty$ the operator P_p is a bounded projector onto the subset $\mathcal{G}^p$ of $L^p(\Omega, \mathcal{S}^3)$. The projector $(P_p)^*$ adjoint with respect to the bilinear form $[\xi, \zeta]_\Omega$ on $L^p(\Omega, \mathcal{S}^3) \times L^q(\Omega, \mathcal{S}^3)$ satisfy*

$$(P_p)^* = P_q, \quad \textit{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

This implies $\ker(P_p) = H^p_{sol}$ *with*

$$H^p_{sol} = \{\xi \in L^p(\Omega, \mathcal{S}^3) \mid [\xi, \zeta]_\Omega = 0 \quad for\ all\ \zeta \in \mathcal{G}^q\}.$$

From the last lemma it follows that the projection operator

$$Q_p = (I - P_p) : L^p(\Omega, \mathcal{S}^3) \to L^p(\Omega, \mathcal{S}^3)$$

with $Q_p(L^p(\Omega, \mathcal{S}^3)) = H^p_{sol}$ is a generalization of the classical Helmholtz projection.

Corollary 5.1.1. *Let $(B^T \mathcal{D} Q_p B)^*$ denote the adjoint operator to $B^T \mathcal{D} Q_p B : L^p(\Omega, \mathrm{I\!R}^N) \to L^p(\Omega, \mathrm{I\!R}^N)$ with respect to the bilinear form $(\xi, \zeta)_\Omega$ on the product space $L^p(\Omega, \mathrm{I\!R}^N) \times L^q(\Omega, \mathrm{I\!R}^N)$. Then*

$$(B^T \mathcal{D} Q_p B)^* = B^T \mathcal{D} Q_q B : L^q(\Omega, \mathrm{I\!R}^N) \to L^q(\Omega, \mathrm{I\!R}^N).$$

Moreover, the operator $B^T \mathcal{D} Q_2 B$ is non-negative and self-adjoint.

Proof. This result is shown in [4, 5]. See also [21] for an alternative proof of this result. □

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