Steady-State Navier-Stokes Flows Past a Rotating Body: Leray Solutions are Physically Reasonable

Giovanni P. Galdi
Department of Mechanical Engineering and Materials Science
University of Pittsburgh, U.S.A
email:galdi@engr.pitt.edu

Mads Kyed
Institut für Mathematik
RWTH-Aachen, Germany
email:kyed@instmath.rwth-aachen.de

28.01.2010

Abstract

A rigid body, $\mathcal{B}$, moves in a Navier-Stokes liquid, $\mathcal{L}$, filling the whole space outside $\mathcal{B}$. We assume that, when referred to a frame attached to $\mathcal{B}$, the nonzero translational velocity, ξ, and the angular velocity, ω, of $\mathcal{B}$ are constant and that the flow of $\mathcal{L}$ is steady. Our main theorem implies that every "weak" steady-state solutions in the sense of Leray is, in fact, physically reasonable in the sense of Finn, for data of arbitrary "size". Such a theorem improves and generalizes an analogous famous result of K.I. Babenko [1], obtained in the case $\omega = 0$.

1 Introduction

Consider a rigid body, $\mathcal{B}$, moving in a Navier-Stokes liquid, $\mathcal{L}$, that fills the whole three-dimensional space, Ω, outside $\mathcal{B}$. We assume that the translational velocity, ξ, and the angular velocity, ω, characterizing the motion of $\mathcal{B}$, are constant when referred to a frame, $\mathcal{F}$, attached to $\mathcal{B}$. We also assume that the flow of $\mathcal{L}$ in $\mathcal{F}$ is time-independent, and that $\mathcal{L}$ is quiescent at large (infinite) distance from $\mathcal{B}$. Therefore, denoting by $v = v(x)$, $p = p(x)$ the velocity and pressure fields of $\mathcal{L}$, respectively, referred to $\mathcal{F}$, we obtain that the generic flow of $\mathcal{L}$ is governed by the boundary-value problem (see, *e.g.* [13, §1])

$$
\left\{
\begin{aligned}
\mu\Delta v - \nabla p - v\cdot\nabla v + \xi\cdot\nabla v + \omega\wedge x\cdot\nabla v - \omega\wedge v = f \quad & \text{in } \Omega,\\
\operatorname{div}(v) = 0 \quad & \text{in } \Omega,\\
v = v_* \quad & \text{on } \partial\Omega,\\
\lim_{|x|\to\infty} v(x) = 0. \quad &
\end{aligned}
\right.
\tag{1.1}
$$

In these equations, Ω is the "region of flow", that is, an open, connected set complement to a compact set of $\mathbb{R}^3$, the "body" $\mathcal{B}$. Moreover, μ denotes the (constant) kinematic viscosity coefficient of $\mathcal{L}$, f is the body force acting on it, and v_* is a velocity distribution at the boundary of Ω. Both f and v_* are prescribed functions of $x \in \Omega$.

The first contribution to the solvability of (1.1) traces back to the pioneering work of J. Leray [21]. Specifically, Leray investigated the case $\omega = 0$ (no spin), and showed that, under suitable assumptions on f and on v_* satisfying the zero-flux condition $\int_{\partial\Omega} v_*\cdot n = 0$, problem (1.1) has, for any $\xi \in \mathbb{R}^3$, at least one solution. This solution is characterized by the fact that the velocity field has a finite Dirichlet integral:

$$
\int_{\Omega} |\nabla v|^2\, \mathrm{d}x \le C_1, \tag{1.2}
$$

where C_1 is a positive constant depending only on the data. Moreover, if f is smooth, then v and the corresponding pressure field p are equally smooth in Ω. A solution to (1.1) satisfying (1.2) is called a *Leray solution*.

The most important feature of a Leray solution is that it exists for data of arbitrary "size". However, its main drawback resides in the fact that it is not clear if it satisfies all the basic physical properties that a solution should possess like, for example, (i) showing a "wake-like" behavior if $\xi \neq 0$, (ii) obeying the equation of energy balance, and (iii) being unique for "small" data. Notice that the proof of all these properties can be reduced to a detailed knowledge of the *structure of the solution at large* $|x|$. For that matter, the only asymptotic information that, *at the outset*, we have on a Leray solution is that it verifies (1.2) along with

$$
\int_{\Omega} |v|^6\, \mathrm{d}x \le C_2 \tag{1.3}
$$

(C_2 depending only on the data), which is proved as a corollary to (1.2), $(1.1)_4$, and the Sobolev inequality.

For this reason, in 1959, R. Finn introduced the definition of a *Physically Reasonable* solution, that is, a solution that satisfies the properties (i)–(iii) listed above and, later on, in 1965, he was able to prove their existence; see [9]. However, Finn's existence result – unlike Leray's – is *local*, namely, it holds if the size of the data is appropriately restricted. Thus, the natural and fundamental question that remained open was whether or not a Leray solution – that exists for data of arbitrary size – is also physically reasonable in the sense of FINN [1]. This outstanding question, in the case $\xi \neq 0$, was positively answered in 1973 by BABENKO [1] [2], even though a complete and more direct proof was available only much later [10, 8].

All results described above refer to the case $\omega = 0$. However, over the past decade, there has been a growing interest, in both mathematical and applied science communities, aimed at the study of the properties of solutions to (1.1) in the general case, when *also* $\omega \neq 0$; for a motivation, see, *e.g.*, [13] and the references there included. The most significant analytical results concern existence, uniqueness, asymptotic behavior (in space) and stability, and the list of their contributors is quite extended. Without claiming to be exhaustive, we refer the reader to [16, 7, 5, 6, 15, 14, 17, 18, 19, 20, 4] and to the bibliography cited therein.

It is important to emphasize that the case $\omega \neq 0$ presents a fundamental challenge, due to the presence of the term $\omega \wedge x \cdot \nabla v$ in the linear momentum equation, whose coefficient becomes arbitrarily large at large distances from the boundary $\partial\Omega$. One consequence of this fact is that problem (1.1) can *not*, by any means, be considered as a perturbation to the analogous problem when $\omega = 0$, and that, in particular, the linearized operator – obtained by disregarding $v \cdot \nabla v$ in (1.1) – is *not* a perturbation to the well-studied *Oseen operator*; see, *e.g.* [11, Chapter VII]. Another and not less important consequence is that, as shown in [7], the fundamental tensor solution, $\mathfrak{E} = \mathfrak{E}(x, y)$, of the linearized operator associated to (1.1) does *not* satisfy the fundamental estimate

$$|\mathfrak{E}(x,y)| \leq \frac{C}{|x-y|}, \quad \forall x, y \in \mathbb{R}^3, \tag{1.4}$$

with a constant C *independent* of x and y. The lack of property (1.4) strongly suggests that one might not be able to obtain asymptotic estimates (at large distances, that is) via the standard method [3] of representing the solution by convolution integrals with kernels involving $\mathfrak{E}$, even for the *linearized* problem.

Notwithstanding this difficulty, thanks to the remarkable fact that the total power of the "rotational term", $\int_\Omega (\omega \wedge x \cdot \nabla u - \omega \wedge u) \cdot u$, vanishes identically along differentiable vector functions u of compact support in Ω, one can prove that all solutions (in a suitable class) to problem (1.1) satisfy an *a priori* estimate analogous to (1.2). As a consequence, by appropriately modifying the procedure used for the case $\omega = 0$, one can show existence of a Leray solution for data of *arbitrary "size"*, also in the case when $\omega \neq 0$; see [3, 25].

The question of existence of Physically Reasonable solutions was initiated in [16], and continued and, to some extent, completed, in [14, 15]. Specifically, by an entirely different approach than the one adopted by Finn in [9], in those papers it is proved that *if the data f and v_* are "small"* in a suitable sense,

[1] Notice that the converse is always true [9].
[2] If $\xi = 0$, to this day, the question is still open.
[3] See, *e.g.*, [11, Chapters V, VII].

then problem (1.1) possess one (and, in fact, only one) Physically Reasonable solution. As in the case $\omega = 0$, while it is immediate to show that these solutions are also Leray solutions, the converse property is in no way obvious, even for "small" data.

Objective of this paper is to prove that, if

$$\xi \cdot \omega \neq 0, \tag{1.5}$$

then *every Leray solution to problem* (1.1) *is Physically Reasonable*. In particular, we show in Theorem 5.1 that if (v, p) is a solution to (1.1) with f mildly regular and of bounded support [4] and with v satisfying conditions (1.2) and (1.3), then, for all sufficiently large $|x|$, $|v(x)| \leq C_3/|x|$, with C_3 independent of x. More precisely, $v(x)$ is bounded by a function of $|x|$ that decays as $|x|^{-3/2+\delta}$, arbitrary $\delta > 0$, if x is outside a "downstream" cone, $\mathcal{C}$, with its axis having the direction of ω, and as $|x|^{-1}$, otherwise. Likewise, in Theorem 5.2, we show that $\nabla v(x)$ is bounded by a function that decays like $|x|^{-3/2}$, uniformly, and as $|x|^{-2+\eta}$, arbitrary $\eta > 0$, for all x outside $\mathcal{C}$. This "anisotropic behavior" is representative of the "wake-like" behavior of the flow. Finally, in Theorem 5.3 we prove that, for some $p_0 \in \mathbb{R}$, we have $|p(x) - p_0| \leq C_4 \ln(|x|)/|x|^2$, where C_4 is independent of x.

Notice that, for the above results to hold, no condition is imposed on the flux of v_* through $\partial\Omega$.

Simple consequences of these theorems are, on one hand, that *every Leray solution is unique in its own class, if the data are "sufficiently small"* (see Theorem 5.4), and, on the other hand, that *every Leray solution satisfies the balance of energy equation* (see Theorem 5.5).

Thus, an immediate corollary to the above results is the following *global existence theorem*: for any given f and v_* in suitable function classes, and for any $\xi, \omega \in \mathbb{R}^3$ satisfying (1.5), there exists at least one corresponding Physically Reasonable solution to (1.1).

The proofs of the above theorems rely upon two fundamental results that we would like to present next.

The first one (see Theorem 4.4) states that even though, at the outset, v only satisfies the summability conditions (1.2) and (1.3), in fact, v and p possess the following additional properties:

$$\begin{aligned}
&v \in L^{q_1}(\Omega^R)\ \forall q_1 \in (2, \infty], \quad \nabla v \in L^{q_2}(\Omega^R)\ \forall q_2 \in (\tfrac{4}{3}, 6), \\
&\nabla^2 v \in L^{q_3}(\Omega^R)\ \forall q_3 \in (1, 2), \\
&(p - p_0) \in L^{\frac{3s}{s-3}}(\Omega^R), \quad \nabla p \in L^s(\Omega^R)\ \forall s \in (1, 2),
\end{aligned} \tag{1.6}$$

where $\Omega^R := \Omega \cap \{|x| > R\}$, for a sufficiently large R, and L^q denotes the usual Lebesgue space. This result is, in turn, established by combining an existence

[4] This latter assumption can be fairly weakened, by imposing only that f decays "sufficiently fast" at large distances.

theorem for the *linearized* problem [5] in the whole space $\mathbb{R}^3$, due to FARWIG [6], along with a corresponding new uniqueness result. The novelty of (and the difficulty in obtaining) this latter result relies in the fact that it is established in a class of functions *merely* satisfying the summability property (1.3).

The second result states that the properties (1.6) lead to the desired *pointwise decay* for v, ∇v, and p, at large distances. However, due to the lack of basic estimates, such as (1.4), for the fundamental solution to the linearized problem, it appears very unlikely to prove this decay by means of the representation of the solution in terms of $\mathfrak{E}$, as it happens in the case $\omega = 0$. Thus, we need a new idea. This new idea is inspired by the approach proposed in [16]. Precisely, by a suitable *orthogonal* and *time-dependent* change of variables combined with a standard "cut-off" procedure, we transform problem (1.1), formally, into a time-dependent Oseen-like Cauchy problem, with suitable initial data. The solution to this latter can then be represented in terms of the well-known Oseen (time-dependent) fundamental solutions, for which appropriate estimates, *uniform* in time, were already established in [14]. Our main result, given in Theorem 5.1, is thus obtained by utilizing, in the representation, these estimates along with (1.6), and then by transforming back to the original variables.

In order to prove the results mentioned above, it is convenient to rewrite the original problem (1.1) in a suitable non-dimensional form, based on the Mozzi-Chasles transformation; see [15]. In doing so, it will also become clear why we need the assumption on ξ and ω made in (1.5). To this end, we take, without loss of generality, the direction of ω to be the unit vector e_1 in the x_1-axis direction [6], and set $\mathrm{e} := \xi/|\xi|$. Thus, denoting by d a length scale [7], and defining

$$
\begin{aligned}
&x^* := x - \lambda\, \mathrm{e}_1 \wedge \mathrm{e}, \quad \lambda := |\xi| d/|\omega|,\\
&\Omega^* = \{x^* \in \mathbb{R}^3 \mid x^* = x - \lambda\, \mathrm{e}_1 \wedge \mathrm{e}, \text{ for some } x \in \Omega\},\\
&v^*(x^*) := v(x^* + \lambda\, \mathrm{e}_1 \wedge \mathrm{e}), \quad p^*(x^*) := p(x^* + \lambda\, \mathrm{e}_1 \wedge \mathrm{e}),\\
&f^*(x^*) := f(x^* + \lambda\, \mathrm{e}_1 \wedge \mathrm{e}),\\
&\mathcal{R} := \left(\frac{|\xi| d}{\nu}\right) \mathrm{e}\cdot \mathrm{e}_1 \ \text{(Reynolds number)}, \quad \mathcal{T} := \frac{|\omega|\, d^2}{\nu} \ \text{(Taylor number)},
\end{aligned}
$$

the original system $(1.1)_{1,2,3}$ becomes (stars omitted)

$$
\text{(1.7)} \quad \left\{
\begin{aligned}
\Delta v - \nabla p + \mathcal{R}\big(\partial_1 v - v\cdot\nabla v\big) + \mathcal{T}\big(\mathrm{e}_1 \wedge x \cdot \nabla v - \mathrm{e}_1 \wedge v\big) = f \quad &\text{in } \Omega,\\
\operatorname{div}(v) = 0 \quad &\text{in } \Omega,\\
v = v_* \quad &\text{on } \partial\Omega.
\end{aligned}
\right.
$$

For our method to work, it is crucial that the linearized system, obtained from (1.7) by disregarding the term $v \cdot \nabla v$, contains the term $\partial_1 v \equiv \mathrm{e}_1 \cdot \nabla v$, and this

[5] That is, the problem that results from (1.1) by disregarding the term $v \cdot \nabla v$ in $(1.1)_1$.

[6] We are, of course, assuming $\omega \neq 0$, because, otherwise, as we already remarked, our main theorem is well-known [12, Chapter IX].

[7] For example, the diameter of the body.

happens if and only if $\mathcal{R} \neq 0$, that is, if and only if $\xi \neq 0$ *and* $\mathrm{e} \cdot \mathrm{e}_1 \neq 0$, which explains the assumption (1.5). In more physical terms, we need that the body produces a "wake", and this happens if the translational velocity of the body has a nonzero component along the direction of the spin. Only for the sake of definiteness, we shall take throughout $\mathcal{R} > 0$.

The plan of the paper is the following. After introducing certain basic definitions and properties in Section 2, in Section 3 we prove some relevant properties of solutions to the Cauchy problem associated to the time-dependent Oseen equations. Successively, in Section 4, we show that a generic Leray solution enjoys a number of summability properties in any neighborhood of infinity. Finally, in Section 5, we prove our main result that every Leray solution is Physically Reasonable.

2 Notation

As customary, given a scalar field $p : \mathbb{R}^3 \to \mathbb{R}$, and a base $\{\mathrm{e}_i\}_{i=1,2,3}$ in $\mathbb{R}^3$, we define $\nabla p = \partial_i p \, \mathrm{e}_i$ [8]. Likewise, for a vector field $v : \mathbb{R}^3 \to \mathbb{R}^3$, we let $\nabla v : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ be the tensor field with components in $\{e_i \otimes \mathrm{e}_j\}_{i,j=1,2,3}$ given by

$$\big(\nabla v\big)_{ij} := \partial_i v_j \,, \ (i, j = 1, 2, 3).$$

Moreover, we define $\nabla' v : \mathbb{R}^3 \to \mathbb{R}^{3\times 2}$ by

$$\big(\nabla' v\big)_{ij} := \partial_i v_j \,, \ (i = 2, 3; j = 1, 2, 3).$$

By the Greek letter Ω, we will denote an *exterior domain* of $\mathbb{R}^3$, namely, an open connected set of $\mathbb{R}^3$ whose complement is a non-empty compact set. For $\rho > 0$, we put $\mathrm{B}_\rho := \{x \in \mathbb{R}^3 \mid |x| < \rho\}$, $\mathrm{B}^\rho := \{x \in \mathbb{R}^3 \mid |x| \geq \rho\}$, and set $\Omega_\rho := \Omega \cap \mathrm{B}_\rho$ and $\Omega^\rho := \Omega \cap \mathrm{B}^\rho$. Also, we define $\mathrm{B}_{\rho_2,\rho_1} := \mathrm{B}_{\rho_2} \setminus \mathrm{B}_{\rho_1}$.

We let $L^q(\Omega)$ and $W^{m,q}(\Omega)$ denote Lebesgue and Sobolev spaces, respectively, and $\|\cdot\|_q$, $\|\cdot\|_{m,q}$ the associated norms. We write $D^{m,q}(\Omega)$ and $|\cdot|_{m,q}$ to denote homogeneous Sobolev spaces and their (semi-)norms, respectively. We will typically indicate when a function space consists of vector- or tensor-valued functions, for example $L^q(\Omega)^3$, but may omit the indication when no confusion arises.

For $G \subset \mathbb{R}^3$ an exterior domain or $G = \mathbb{R}^3$, and $q \in (1, 2)$, we put

$$\begin{aligned}\mathrm{X}_q(G) := \big\{(v, p) \mid v \in L^{\frac{2q}{2-q}}(G)^3,\, \nabla' v \in L^{\frac{4q}{4-q}}(G),\, \partial_1 v \in L^q(G),\\ \nabla^2 v \in L^q(G),\, p \in L^{\frac{3q}{3-q}}(G),\, \nabla p \in L^q(G)\big\}\end{aligned}$$

and

$$\|(v, p)\|_{\mathrm{X}_q} := \|v\|_{\frac{2q}{2-q}} + \|\nabla' v\|_{\frac{4q}{4-q}} + \|\partial_1 v\|_q + \|\nabla^2 v\|_q + \|p\|_{\frac{3q}{3-q}} + \|\nabla p\|_q.$$

Clearly, $\big(\mathrm{X}_q(G), \|\cdot\|_{\mathrm{X}_q}\big)$ is a Banach space.

[8] Throughout this paper we shall use the summation convention over repeated indices.

If $T \in (0,\infty]$ we set $G_T := G \times (0,T)$, and for $q>1$ we define

$$\mathcal{W}_q^{(1,2)}(\Omega_T) := \big\{ w \in L^q(\Omega_T)^3 \mid \partial_t w, \nabla w, \nabla^2 w \in L^q \big\}$$

and introduce the norm

$$\|w\|_{\mathcal{W}_q^{(1,2)}(\Omega_T)} := \Bigg(\int_0^T \|\partial_t w(\cdot,t)\|_q^q + \sum_{|\alpha|=0}^{2} \|\mathrm{D}^\alpha w(\cdot,t)\|_q^q \,\mathrm{d}t \Bigg)^{\frac{1}{q}}.$$

Moreover, we put

$$\begin{aligned}
&\mathcal{W}_{q,loc}^{(1,2)}(\Omega_T) := \big\{ w \in L^q_{loc}(\Omega_T)^3 \mid \partial_t w, \nabla w, \nabla^2 w \in L^q_{loc} \big\},\\
&\mathcal{D}_q^{(0,1)}(\Omega_T) := \big\{ \pi \in L^q_{loc}(\Omega_T) \mid \nabla \pi \in L^q \big\},\\
&\mathcal{D}_{q,loc}^{(0,1)}(\Omega_T) := \big\{ \pi \in L^q_{loc}(\Omega_T) \mid \nabla \pi \in L^q_{loc} \big\}, \text{ and}\\
&L^{\infty,q}(\Omega_T) := \big\{ u \in L^1_{loc}(\Omega_T)^3 \mid \operatorname*{ess\,sup}_{t\in(0,T)} \|u(\cdot,t)\|_q < \infty \big\}.
\end{aligned}$$

For all functions $f(x,t)$ depending on time, we always denote $\nabla f := \nabla_x f$ and $\operatorname{div} f := \operatorname{div}_x f$. In general, coordinates of domains that are subsets of $\mathbb{R}^3 \times \mathbb{R}$ are denoted by (x,t) and (y,τ).

For $x \in \mathbb{R}^3$ we put $s(x) := |x| + x_1$.

Finally, we use small letters $(c_1, c_2, \dots)$ for constants that appear only in a single proof, and capital letters $(C_1, C_2, \dots)$ for constants appearing in the statement of a result.

3 Time-Dependent Oseen Problem

As emphasized in the introduction, our method of proof relies on the validity of basic properties of the solutions to the linearization of (1.1). These properties, in turn, will be a direct consequence of analogous ones that we will prove for the solutions to a suitable *unsteady* Oseen problem.

The objective of this section is to reproduce such properties, some of which are well known.

We start by recalling the fundamental solution to the time-dependent Oseen operator, *i.e.*, the tensor $\Gamma(x,t)$ and vector $\gamma(x,t)$ satisfying (in the sense of distributions)

$$\begin{cases} \partial_t \Gamma_{ij} = \Delta \Gamma_{ij} - \partial_j \gamma_i + \mathcal{R}\partial_1 \Gamma_{ij} + \delta_{ij}\delta(t)\delta(x), \\ \partial_k \Gamma_{ik} = 0, \end{cases}$$

for $i,j = 1,2,3$, where δ_{ij} denotes the Kronecker delta and $\delta(\cdot)$ the Dirac delta distribution. The fundamental solution takes the form (see [23])

$$\Gamma_{ij} := -\delta_{ij}\Delta\Psi + \partial_i\partial_j\Psi, \quad \gamma_i := \partial_i(\Delta - \partial_t)\Psi,$$

with

$$\Psi(x,t) := \frac{1}{4\pi^{\frac{3}{2}} t^{\frac{1}{2}}} \int_0^1 \mathrm{e}^{-\frac{|x+\mathcal{R}t\,\mathrm{e}_1|^2 r^2}{4t}}\,\mathrm{d}r \text{ for } t>0 \quad \text{and} \qquad \Psi(x,t)=0 \text{ for } t\le 0.$$

The following estimates hold.

Lemma 3.1. *There exist constants* $C_5, C_6, C_7, C_8, C_9 > 0$ *so that*

$$|\Gamma(x,t)| \le C_5 \left(t + |x+\mathcal{R}t\,\mathrm{e}_1|^2\right)^{-\frac{3}{2}} \quad \forall (x,t)\in\mathbb{R}^3_\infty, \tag{3.1}$$

$$|\nabla\Gamma(x,t)| \le C_6 \left(t + |x+\mathcal{R}t\,\mathrm{e}_1|^2\right)^{-2} \quad \forall (x,t)\in\mathbb{R}^3_\infty, \tag{3.2}$$

$$|\nabla^2\Gamma(x,t)| \le C_7 \left(t + |x+\mathcal{R}t\,\mathrm{e}_1|^2\right)^{-\frac{5}{2}} \quad \forall (x,t)\in\mathbb{R}^3_\infty, \tag{3.3}$$

$$\int_0^\infty |\nabla\Gamma(x,\tau)|\,\mathrm{d}\tau \le C_8 \begin{cases} \mathcal{R}^{\frac{1}{2}}|x|^{-\frac{3}{2}}\big(1+\mathcal{R}s(x)\big)^{-\frac{3}{2}} & \text{for } |x|\ge \dfrac{1}{4\mathcal{R}}, \\ |x|^{-2} & \text{for } |x| < \dfrac{1}{4\mathcal{R}}. \end{cases} \tag{3.4}$$

$$\int_0^\infty |\nabla^2\Gamma(x,\tau)|\,\mathrm{d}\tau \le C_9 \begin{cases} \mathcal{R}|x|^{-2}\big(1+\mathcal{R}s(x)\big)^{-2} & \text{for } |x|\ge \dfrac{1}{4\mathcal{R}}, \\ |x|^{-3} & \text{for } |x| < \dfrac{1}{4\mathcal{R}}. \end{cases} \tag{3.5}$$

Proof. For the proof of (3.1), (3.2), and (3.3), we refer to [23, §55 and §73] (see also [24]). A proof of (3.4) can be found in [15, Lemma 1]. We shall now prove (3.5). Consider first the case $|x| < \frac{1}{4\mathcal{R}}$. In this case,

$$t + |x+\mathcal{R}t\,\mathrm{e}_1|^2 = t + |x|^2 + 2\mathcal{R}tx_1 + \mathcal{R}^2t^2 \ge \frac{1}{2}t + |x|^2,$$

and we thus find, applying (3.3),

$$\int_0^\infty |\nabla^2\Gamma(x,\tau)|\,\mathrm{d}\tau \le C_7 \int_0^\infty \left(\frac{1}{2}\tau + |x|^2\right)^{-\frac{5}{2}} \mathrm{d}\tau \le \frac{1}{3} C_7\, |x|^{-3}. \tag{3.6}$$

Next, let $|x| \ge \frac{1}{4\mathcal{R}}$ and $1 + 2\mathcal{R}x_1 \ge 0$. Then,

$$t + |x+\mathcal{R}t\,\mathrm{e}_1|^2 = |x|^2 + (1+2\mathcal{R}x_1)\,t + \mathcal{R}^2t^2 \ge \mathcal{R}^2t^2 + |x|^2,$$

and thus

$$\begin{aligned} \int_0^\infty |\nabla^2\Gamma(x,\tau)|\,\mathrm{d}\tau &\le C_7 \int_0^\infty \left(\mathcal{R}^2\tau^2 + |x|^2\right)^{-\frac{5}{2}} \mathrm{d}\tau \\ &= C_7\,|x|^{-5} \int_0^\infty \left(\left(\frac{\mathcal{R}\tau}{|x|}\right)^2 + 1\right)^{-\frac{5}{2}} \mathrm{d}\tau \le \frac{c_1}{\mathcal{R}}\,|x|^{-4}, \end{aligned}$$

which, since $|x| \geq \frac{1}{4\mathcal{R}}$ implies $1+\mathcal{R}s(x) \leq 6\mathcal{R}|x|$, yields

$$\int_0^\infty |\nabla^2\Gamma(x,\tau)|\,\mathrm{d}\tau \leq c_2\,\mathcal{R}\,|x|^{-2}\big(1+\mathcal{R}s(x)\big)^{-2}. \tag{3.7}$$

Finally, consider the case $|x| \geq \frac{1}{4\mathcal{R}}$ and $1+2\mathcal{R}x_1 < 0$. Utilizing that

$$4\mathcal{R}^2|x|^2 - (1+2\mathcal{R}x_1)^2 = \big(2\mathcal{R}s(x)+1\big)\big(2\mathcal{R}|x| - (2\mathcal{R}x_1+1)\big),$$

we find

$$\begin{aligned}
\int_0^\infty |\nabla^2\Gamma(x,\tau)|\,\mathrm{d}\tau &\leq C_7 \int_0^\infty \big(|x|^2 + (1+2\mathcal{R}x_1)\,\tau + \mathcal{R}^2\tau^2\big)^{-\frac{5}{2}}\,\mathrm{d}\tau \\
&= C_7 \int_0^\infty \Big(\big(\mathcal{R}\tau + \frac{1+2\mathcal{R}x_1}{2\mathcal{R}}\big)^2 + \frac{1}{4\mathcal{R}^2}\big(4\mathcal{R}^2|x|^2 - (1+2\mathcal{R}x_1)^2\big)\Big)^{-\frac{5}{2}}\,\mathrm{d}\tau \\
&\leq \frac{C_7}{\mathcal{R}} \int_{-\infty}^\infty \Big(r^2 + \frac{1}{4\mathcal{R}^2}\big(2\mathcal{R}s(x)+1\big)\big(2\mathcal{R}|x| - (2\mathcal{R}x_1+1)\big)\Big)^{-\frac{5}{2}}\,\mathrm{d}r \\
&\leq \frac{C_7}{\mathcal{R}} \int_{-\infty}^\infty \Big(r^2 + \frac{1}{2\mathcal{R}}\big(\mathcal{R}s(x)+1\big)|x|\Big)^{-\frac{5}{2}}\,\mathrm{d}r \\
&\leq c_3\,\mathcal{R}\,|x|^{-2}\big(1+\mathcal{R}s(x)\big)^{-2}.
\end{aligned} \tag{3.8}$$

Combining (3.6), (3.7), and (3.8), we obtain (3.5). □ □

We also need the following lemma.

Lemma 3.2. *Let $1 \leq q_1 < \frac{3}{2}$ and $\frac{3}{2} < q_2$. Then*

$$\int_0^\infty \Big(\int_{\mathbb{R}^3\cap\{x_2^2+x_3^2<1\}} (t+|x+t\,\mathrm{e}_1|^2)^{-2q_1}\,\mathrm{d}x\Big)^{\frac{1}{q_1}}\,\mathrm{d}t < \infty, \tag{3.9}$$

$$\int_0^\infty \Big(\int_{\mathbb{R}^3\cap\{x_2^2+x_3^2\geq 1\}} (t+|x+t\,\mathrm{e}_1|^2)^{-2q_2}\,\mathrm{d}x\Big)^{\frac{1}{q_2}}\,\mathrm{d}t < \infty. \tag{3.10}$$

Proof. Using polar coordinates with respect to (x_2, x_3), one can verify (3.9) and (3.10) by a direct calculation (see also [22, Lemma 3]). □ □

The next lemma furnishes the existence of solutions to the unsteady Oseen problem and corresponding estimates.

Lemma 3.3. *Let $q>1$ and $f \in L^q(\mathbb{R}^3_T)^3$. There exists a solution (w,π) with*

$$(w,\pi) \in \mathcal{W}^{(1,2)}_q(\mathbb{R}^3_T) \times \mathcal{D}^{(0,1)}_q(\mathbb{R}^3_T) \tag{3.11}$$

to

$$\begin{cases} \partial_t w = \Delta w - \nabla \pi + \mathcal{R}\, \partial_1 w + f & \text{in } \mathbb{R}^3_T,\\ \operatorname{div}(w) = 0 & \text{in } \mathbb{R}^3_T,\\ \lim\limits_{t\to 0} \|w(\cdot,t)\|_q = 0. \end{cases} \tag{3.12}$$

If, in addition, $f \in L^r(\mathbb{R}^3_T)$ for some $r \in (1,\infty)$, then

$$(w,\pi) \in \mathcal{W}^{(1,2)}_r(\mathbb{R}^3_T) \times \mathcal{D}^{(0,1)}_r(\mathbb{R}^3_T). \tag{3.13}$$

Finally, if $q>3$ and $f \in L^{\infty,q}(\mathbb{R}^3_\infty)$ with $\operatorname{supp}\big(f(\cdot,t)\big) \subset \mathrm{B}_\rho$ for all $t>0$, then

$$|w(x,t)| \leq C_{10}\, \frac{\operatorname{ess\,sup}_{t>0} \|f(\cdot,t)\|_q}{\big(1+|x|\big)\big(1+\mathcal{R}s(x)\big)}, \tag{3.14}$$

where $C_{10} := C_{10}(\rho,q,\mathcal{R})$.

Proof. Consider the volume potential

$$w(x,t) = \int_0^t \int_{\mathbb{R}^3} \Gamma(x-y,t-\tau) \cdot f(y,\tau)\, \mathrm{d}y \mathrm{d}\tau\,. \tag{3.15}$$

It is well known that w solves (3.12), for an appropriate choice of the associated pressure π, and that (w,π) is in both classes (3.11) and (3.13) if $f \in L^q(\mathbb{R}^3_T)^3 \cap L^r(\mathbb{R}^3_T)^3$; see [24, §13]. This shows the first part of the lemma. Next put

$$H(x,t) := \nabla\big[\mathcal{E} * f(\cdot,t)\big](x),$$

where $\mathcal{E}(x) := \frac{1}{4\pi\,|x|}$ is the fundamental solution to the Laplace equation in $\mathbb{R}^3$ and the convolution is with respect to the spatial variable only. One may easily verify, by means of the Hölder inequality, that $H \in L^\infty(\mathbb{R}^3_\infty)$ with

$$\operatorname{div}(H) = f, \text{ and } |H(x,t)| \leq c_1 \|f(\cdot,t)\|_q, \tag{3.16}$$

where $c_1 = c_1(q)$. Inserting $\operatorname{div}(H)$ for f in (3.15) and integrating by parts, we obtain

$$\begin{aligned} w(x,t) &= \int_0^t \int_{\mathbb{R}^3} \Gamma(x-y,\tau) \cdot f(y,\tau)\, \mathrm{d}y \mathrm{d}\tau \\ &= \int_0^t \int_{\mathrm{B}_\rho} \partial_k \Gamma_{ij}(x-y,t)\, H_{kj}(y,\tau)\, \mathrm{d}y \mathrm{d}\tau \\ &\quad + \int_0^t \int_{\partial \mathrm{B}_\rho} \Gamma(x-y,t) \cdot \big(H(y,\tau)\cdot n\big)\, \mathrm{d}S \mathrm{d}\tau =: I_1 + I_2. \end{aligned} \tag{3.17}$$

Using Lemma 3.1, we get

$$I_1 \leq c_2\,|H(x,t)|\,\Bigg(\int\limits_{\mathrm{B}_\rho} |x-y|^{-2}\,\mathrm{d}y + \int\limits_{\mathrm{B}_\rho} |x-y|^{-\frac{3}{2}}\,\big(1+\mathcal{R}s(x-y)\big)^{-\frac{3}{2}}\,\mathrm{d}y\Bigg),$$

with $c_2 = c_2(\mathcal{R})$. Examining the cases $|x| > 2\rho$ and $|x| \leq 2\rho$ separately, one now verifies that

$$I_1 \leq c_3\,\frac{|H(x,t)|}{\big(1+|x|\big)\big(1+\mathcal{R}s(x)\big)}, \tag{3.18}$$

with $c_3 = c_3(\rho,\mathcal{R})$. Similarly, using again Lemma 3.1 and assuming, without loss of generality, $\rho > \frac{1}{4\mathcal{R}}$, we deduce

$$\begin{aligned} I_2 &\leq c_4\,|H(x,t)| \int\limits_{\partial\,\mathrm{B}_\rho} |x-y|^{-\frac{3}{2}}\,\big(1+\mathcal{R}s(x-y)\big)^{-\frac{3}{2}}\,\mathrm{d}y \\ &\leq c_5\,\frac{|H(x,t)|}{\big(1+|x|\big)\big(1+\mathcal{R}s(x)\big)}, \end{aligned} \tag{3.19}$$

with $c_5 = c_5(\rho,\mathcal{R})$. Combining now (3.16), (3.17), (3.18), and (3.19), we obtain the desired estimate (3.14). □ □

Lemma 3.4. *Let $q > 1$ and $u_0 \in L^q(\mathbb{R}^3)^3$ with $\operatorname{div}(u_0) = 0$ (in the sense of distributions). Then*

$$w(x,t) := (4\pi t)^{-\frac{3}{2}} \int\limits_{\mathbb{R}^3} \mathrm{e}^{-|x-y+\mathcal{R}\,t\,\mathrm{e}_1|^2/4t}\,u_0(y)\,\mathrm{d}y \tag{3.20}$$

is a solution to the initial-value problem

$$\begin{cases} \partial_t w = \Delta w + \mathcal{R}\,\partial_1 w & \text{in } \mathbb{R}^3_\infty, \\ \operatorname{div}(w) = 0 & \text{in } \mathbb{R}^3_\infty, \\ \lim\limits_{t\to 0} \|w(\cdot,t) - u_0(\cdot)\|_q = 0, \end{cases} \tag{3.21}$$

with

$$w \in L^q(\mathbb{R}^3_T) \ \text{and} \ \ w \in \mathcal{W}^{(1,2)}_q\big(\mathbb{R}^3\times(\varepsilon,T)\big),\ \ \forall T,\varepsilon > 0. \tag{3.22}$$

Furthermore, $\mathrm{D}^\alpha_x w(\cdot,t) \in L^r(\mathbb{R}^3)$, $|\alpha| = 0,1$, for all $r \in [q,\infty]$ and all $t > 0$, and the following estimate holds:

$$\|\mathrm{D}^\alpha_x w(\cdot,t)\|_r \leq C_{11}\,t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{|\alpha|}{2}}\,\|u_0\|_q, \quad \text{for all } t > 0,\ |\alpha| = 0,1, \tag{3.23}$$

where $C_{11} := C_{11}(r,q,\mathcal{R})$.

Proof. A standard calculation shows that (3.20) satisfies (3.21) and (3.22). A direct application of Young's inequality yields (3.23). □ □

Classical results for the heat equation ensure that the solution (3.20) is unique in the class (3.22). However, for our purposes, we need a more general uniqueness result. To this end, we begin to prove the following.

Lemma 3.5. *Let $q>1$ and $w\in\mathcal{W}_q^{(1,2)}(\mathbb{R}^3_T)$ with $\mathrm{div}(w)=0$ and $w(\cdot,T)=0$* [9]*. Then, there is a sequence $\{w_n\}_{n=1}^{\infty}$ with $w_n\in C_0^\infty\big(\mathbb{R}^3\times[0,T)\big)^3$, $\mathrm{div}(w_n)=0$, and $w_n\to w$ in $\mathcal{W}_q^{(1,2)}(\mathbb{R}^3_T)$ as $n\to\infty$. This sequence may be taken independent of q.*

Proof. We begin with the following simple remark. Suppose

$$w\in\mathcal{W}_q^{(1,2)}(\mathbb{R}^3_T),\quad \mathrm{supp}(w)\subset[0,T)\times\mathbb{R}^3,\quad \mathrm{div}(w)=0. \tag{3.24}$$

Then, for any $\varepsilon>0$ there is $w_\varepsilon\in C_0^\infty\big(\mathbb{R}^3\times[0,T)\big)^3$ so that

$$\|w-w_\varepsilon\|_{\mathcal{W}_q^{(1,2)}(\mathbb{R}^3_T)}<\varepsilon\,,\ \ \mathrm{div}(w_\varepsilon)=0. \tag{3.25}$$

In fact, it is enough to pick

$$w_\varepsilon;=j_\varepsilon*\tilde{w},$$

where j_ε is a standard mollifier, while $\tilde{w}(\cdot,t)=w(\cdot,t)$ if $t\ge0$, and $\tilde{w}(\cdot,t)=w(\cdot,-t)$ if $t<0$ [10]. The property (3.25) then follows from standard properties of mollifiers. Thus, in order to show the result, it suffices to show that a function w satisfying the assumption of the lemma can be approximated in the space $\mathcal{W}_q^{(1,2)}(\mathbb{R}^3_T)$ by functions satisfying (3.24). To this end, let $\varphi_\eta\in C^\infty(\mathbb{R};\mathbb{R})$, $\eta>0$, be a smooth "cut-off" function satisfying the properties:

(i) There is $\gamma=\gamma(\eta)$, with $0<\gamma<1$ and $\gamma\to0$ as $\eta\to0$, so that $\varphi_\eta(t)=0$ if $t\in[T-\gamma^2/2,T)$, while $\varphi_\eta(t)=1$ if $t\in[0,T-2\gamma]$,

(ii) $|\varphi_\eta(t)|\le1$ for all $t\in[0,T]$,

(iii) $|\varphi'_\eta(t)|\le\frac{\eta}{T-t}$.

The existence of such a function is well known; see, *e.g.*, [11, Lemma III.6.2]. Likewise, by $\psi_\eta\in C^\infty(\mathbb{R}^3;\mathbb{R})$ we denote another smooth (spatial) "cut-off" function with

$$\psi_\eta(x)=1\text{ for }|x|\le1/(2\eta),\quad \psi_\eta(x)=0\text{ for }|x|\ge1/\eta.$$

Next, we set

$$z_\eta(x,t):=\varphi_\eta(t)\,\psi_\eta(x)\,w(x,t),\ \ (x,t)\in\mathbb{R}^3_T.$$

For all sufficiently small $\eta>0$, we have, clearly, $\mathrm{supp}(z_\eta)\subset \mathrm{B}_{1/\eta}\times[0,T)$, and, in addition, $z_\eta\in\mathcal{W}_q^{(1,2)}(\mathbb{R}^3_T)$. Actually, in view of the properties of φ_η and ψ_η,

[9] Notice that, by the Sobolev embedding theorem, $w(\cdot,t)$ is well defined for all $t\in[0,T]$.
[10] See footnote 9.

it is immediate to show that $\mathrm{D}^\alpha z_\eta \in L^q(\mathbb{R}^3_T)$, $|\alpha| = 0, 1, 2$. Furthermore, by the property (iii) of the function φ_η, we find, in particular,

$$\|\partial_t z_\eta(\cdot,t)\|_q^q < c_1 \frac{\|w(\cdot,t)\|_q^q}{|T-t|^q} + c_2 \|\partial_t w(\cdot,t)\|_q^q.$$

However, since $w(\cdot,T)=0$ and thus $\|w(\cdot,T)\|_q = 0$, we obtain, by Hardy's and Hölder's inequalities,

$$\int_0^T \frac{\|w(\cdot,t)\|_q^q}{|T-t|^q}\,\mathrm{d}t \le c_3 \int_0^T |\partial_t \|w(\cdot,t)\|_q^q|\,\mathrm{d}t \le c_4 \|w\|^q_{\mathcal{W}_q^{(1,2)}(\mathbb{R}^3_T)}, \tag{3.26}$$

which proves $z_\eta \in \mathcal{W}_q^{(1,2)}(\mathbb{R}^3_T)$. It is now simple to show that

$$\|z_\eta - w\|_{\mathcal{W}_q^{(1,2)}(\mathbb{R}^3_T)} \to 0 \quad \text{as } \eta \to 0. \tag{3.27}$$

Actually, again by the properties of the functions φ_η and ψ_η, one can easily show that

$$\mathrm{D}^\alpha z_\eta \to \mathrm{D}^\alpha w \quad \text{in} \;\; L^q(\mathbb{R}^3_T) \text{ as } \eta \to 0,\; |\alpha| = 0,1,2. \tag{3.28}$$

Moreover, by the property (iii)

$$\int_0^T \|\partial_t (z_\eta - w)(\cdot,t)\|_q^q \le c_5\, \eta \int_0^T \frac{\|w(\cdot,t)\|_q^q}{|T-t|^q}\,\mathrm{d}t + o(1) \quad \text{as } \eta \to 0,$$

which, in turn, combined with (3.26), delivers

$$\partial_t z \to \partial_t w \quad \text{in} \;\; L^q(\mathbb{R}^3_T) \text{ as } \eta \to 0.$$

From this latter relation and (3.28), we conclude the proof of (3.27). Thus far, we have proved that the functions z_η satisfy the following properties: (i) they are in $\mathcal{W}_q^{(1,2)}(\mathbb{R}^3_T)$, (ii) they have support contained in $[0,T) \times \mathrm{B}_{1/\eta}$, and (iii) they tend to w in $\mathcal{W}_q^{(1,2)}(\mathbb{R}^3_T)$ as $\eta \to 0$. We shall next modify z_η appropriately to obtain solenoidal fields w_η satisfying the same above properties. To this end, we recall a representation formula due to Bogovskiĭ's (see [2] or [11, Proof of Lemma III.3.1]) based on the kernel

$$N(x,y) := \frac{x-y}{|x-y|^3} \int_{|x-y|}^{\infty} \omega\left(y + \xi \frac{x-y}{|x-y|^3}\right) \mathrm{d}\xi,$$

where $\omega \in C_0^\infty(\mathbb{R}^3)$ is some function with $\mathrm{supp}(\omega) \subset \mathrm{B}_1$ and $\int_{\mathrm{B}_1} \omega \mathrm{d}x = 1$. Utilizing that $\int_{\mathrm{B}_{R_\eta}} \mathrm{div}(z_\eta)\,\mathrm{d}x = 0$ (note that $\mathrm{supp}\, z_\eta \subset \mathrm{B}_\eta$), one can show (see for example [11, Proof of Lemma III.3.1]) that

$$k_\eta(x,t) := \int_{\mathbb{R}^3} \mathrm{div}(z_\eta)(y,t)\, N(x,y)\,\mathrm{d}y$$

satisfies

$$\begin{cases} \operatorname{div}(k_\eta) = \operatorname{div}(z_\eta), \ k_\eta \in C_0^\infty\big(\mathbb{R}^3 \times [0,T]\big), \ \operatorname{supp} k_\eta \subset \mathrm{B}_\eta \times [0,T), \\ \|\nabla k_\eta(\cdot,t)\|_{q,\mathbb{R}^3} \le \|\operatorname{div}(z_\eta)\|_{q,\mathbb{R}^3}, \ \|\nabla k_\eta(\cdot,t)\|_{1,q,\mathbb{R}^3} \le c_6 \|\operatorname{div}(z_\eta)\|_{q,\mathbb{R}^3}, \end{cases}$$

with c_6 independent on t and η. It can be checked that $k_\eta \to 0$ in $\mathcal{W}_q^{(1,2)}\big(\mathbb{R}^3_T\big)$ as $\eta \to \infty$. For example, we have, by Poincaré's inequality and the properties of z_η,

$$\begin{aligned} \|k_\eta\|^q_{L^q\left(\mathbb{R}^3_T\right)} &= \int_0^T \int_{\mathrm{B}_{1/\eta}} |k_\eta(x,t)|^q \,\mathrm{d}x\mathrm{d}t \\ &\le c_7 \int_0^T \big(\|\nabla k_n\|_{q,\mathbb{R}^3}/\eta\big)^q \,\mathrm{d}t \\ &\le c_7 \int_0^T \big(\|\operatorname{div}(z_\eta)\|_{q,\mathbb{R}^3}/\eta\big)^q \,\mathrm{d}t \le c_8 \int_0^T \|w\|_{q,B_{1/\eta,1/(2\eta)}} \,\mathrm{d}t \to 0 \end{aligned}$$

as $\eta \to \infty$. Similarly, one shows that $\partial_t k_\eta \to 0$, $\nabla k_\eta \to 0$, and $\nabla^2 k_\eta \to 0$ in $L^q\big(\mathbb{R}^3_T\big)^3$ as $\eta \to 0$. We conclude that $w_\eta := z_\eta - k_\eta$ satisfies the conditions of the lemma for a fixed q. However, if $w \in \mathcal{W}_r^{(1,2)}(\mathbb{R}^3_T)$, for some $r \neq q$, then by repeating exactly the above argument, we show $w_n \to w$ also in $\mathcal{W}_r^{(1,2)}(\mathbb{R}^3_T)$, which completes the proof of the lemma. □ □

We are now in a position to prove the following uniqueness result.

Lemma 3.6. *Let* (z, Π) *be a solution to*

$$\begin{cases} \partial_t z = \Delta z - \nabla \Pi + \mathcal{R}\,\partial_1 z & \text{in } \mathbb{R}^3_T, \\ \operatorname{div}(z) = 0 & \text{in } \mathbb{R}^3_T, \end{cases} \tag{3.29}$$

with the properties [11]

$$\begin{aligned} &z = z_1 + z_2, \ z_i \in L^{q_i}(\mathbb{R}^3_T) \ \text{for some } q_i \in (1,\infty) \ (i=1,2), \\ &z \in \mathcal{W}^{(1,2)}_{s,loc}(\mathbb{R}^3_T), \ \Pi \in \mathcal{D}^{(0,1)}_{s,loc}(\mathbb{R}^3_T) \ \text{for some } s \in (1,\infty). \end{aligned} \tag{3.30}$$

Then, if

$$\lim_{t\to 0} \|z(\cdot,t)\|_{r,\mathrm{B}_\rho} = 0, \quad \text{for some } r \in (1,\infty) \text{ and all } \rho > 0, \tag{3.31}$$

necessarily $z \equiv \nabla\Pi \equiv 0$ *a.e. in* $\mathbb{R}^3_T$.

[11] The assumption made on Π in (3.30) is redundant, in that it is a consequence of those made on z, and of the fact that z satisfies (3.29).

Proof. Let $H \in C_0^\infty(\mathbb{R}^3_T)$. From Lemma 3.3, we deduce that there exists a solution (v,p) to the problem

$$
\begin{cases} \partial_t v = -\Delta v - \nabla p + \mathcal{R}\,\partial_1 v + H & \text{in } \mathbb{R}^3_T, \\ \operatorname{div}(v) = 0 & \text{in } \mathbb{R}^3_T, \\ \lim\limits_{t\to T} \|v(\cdot,t)\|_q = 0, \end{cases} \tag{3.32}
$$

with

$$
(v,p) \in \mathcal{W}_q^{(1,2)}(\mathbb{R}^3_T) \times \mathcal{D}_q^{(0,1)}(\mathbb{R}^3_T) \text{ for all } q \in (1,\infty). \tag{3.33}
$$

Now, by Lemma 3.5, there is a sequence $\{v_n\}_{n=1}^\infty$ with $v_n \in C_0^\infty\big(\mathbb{R}^3 \times [0,T)\big)$ and $\operatorname{div}(v_n) = 0$ satisfying $v_n \to v$ in $\mathcal{W}_q^{(1,2)}(\mathbb{R}^3_T)$ as $n \to \infty$ for all $q \in (1,\infty)$. Multiplying both sides of (3.29) by v_n and integrating by parts (note that for any compact subdomain $K \subset \mathbb{R}^3$ we have $z \in W^{1,s}\big((\varepsilon, T-\varepsilon); L^s(K)\big)$ and thus, by the Sobolev embedding theorem, $z \in C\big([\varepsilon, T-\varepsilon]; L^s(K)\big)$ for all $\varepsilon > 0$), we obtain, for fixed n and sufficiently small ε,

$$
0 = \int_{\mathbb{R}^3} z(x,\varepsilon) \cdot v_n(x,\varepsilon)\,\mathrm{d}x + \int_\varepsilon^{T-\varepsilon} \int_{\mathbb{R}^3} z \cdot \big(-\partial_t v_n - \Delta v_n + \mathcal{R}\,\partial_1 v_n\big)\,\mathrm{d}x\mathrm{d}t.
$$

Letting $\varepsilon \to 0$ and exploiting (3.31), we infer that

$$
\int_0^T \int_{\mathbb{R}^3} z \cdot \big(-\partial_t v_n - \Delta v_n + \mathcal{R}\,\partial_1 v_n\big)\,\mathrm{d}x\mathrm{d}t = 0.
$$

We next let $n \to \infty$ into this relation, and employ the assumption $(3.30)_1$ along with the property that $v_n \to v$ in $\mathcal{W}_q^{(1,2)}(\mathbb{R}^3_T)$ for *all* $q \in (1,\infty)$. Thus, taking also into account $(3.32)_1$, we conclude

$$
0 = \int_0^T \int_{\mathbb{R}^3} z \cdot (H - \nabla p)\,\mathrm{d}x\mathrm{d}t.
$$

Utilizing again $(3.30)_1$ and the fact that $\operatorname{div}(z) = 0$, and that $\nabla p(\cdot,t) \in L^q(\mathbb{R}^3)$ for a.a. $t \in [0,T]$ and all $q \in (1,\infty)$, we obtain

$$
0 = \int_0^T \int_{\mathbb{R}^3} z \cdot (H - \nabla p)\,\mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbb{R}^3} z \cdot H\,\mathrm{d}x\mathrm{d}t.
$$

Since $H \in C_0^\infty(\mathbb{R}^3_T)$ was arbitrary, $z = 0$ follows. From $(3.29)_1$ we then also obtain $\nabla \Pi = 0$. □ □

Remark 3.7. The uniqueness result just shown possesses three important features, each of which is crucial to our further purposes. The first, is that no assumption is made on the behavior of the pressure at large distances. The second, is that the velocity is assumed to vanish at spatial infinity only in the L^q sense, and, the third, is that the initial (zero) value is attained only in a "local" L^r fashion (see (3.31)). The authors were not able to find such a result in the existing related literature where, typically, other extra assumptions are required.

4 Global Summability Properties of Leray Solutions

As already pointed out, a Leray solution, (v, p), possesses, at the outset, only the summability properties:

$$\nabla v \in L^2(\Omega), \quad v \in L^6(\Omega).$$

Objective of this section is to show that, under suitable assumptions on f, in fact, $(v, p) \in \mathrm{X}_q(\Omega^\rho)$ for some sufficiently large $\rho > 0$, and *for all* $q \in (1, 2)$; see Theorem 4.4.

The proof of this theorem has two main ingredients: a result due to Farwig [6], on existence and associate L^q-estimates of solutions to the corresponding *linear* problem and recalled in the first part of Lemma 4.2, and a very general uniqueness result proved in the following Lemma 4.1.

Lemma 4.1. *Let* $1 < s, q_1, q_2 < \infty$. *For every* $f \in L^s_{loc}(\mathbb{R}^3)^3$ *and any two solutions* $(v_i, p_i) \in \big(L^{q_i}(\mathbb{R}^3)^3 \cap W^{2,s}_{loc}(\mathbb{R}^3)^3\big) \times W^{1,s}_{loc}(\mathbb{R}^3)$ $(i = 1, 2)$ *to*

$$\begin{cases} \Delta v - \nabla p + \mathcal{R}\partial_1 v + \mathcal{T}\big(\mathrm{e}_1 \wedge x \cdot \nabla v - \mathrm{e}_1 \wedge v\big) = f & \text{in } \mathbb{R}^3, \\ \mathrm{div}(v) = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

there exists a constant c *so that* $(v_1, p_1) = (v_2, p_2 + c)$.

Proof. Let

$$Q(t) := \exp(\mathcal{T} E_1 t), \text{ with } E_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{4.1}$$

and set

$$\begin{aligned} w(x,t) &:= Q(t)\Big(v_1\big(Q^T(t)x\big) - v_2\big(Q^T(t)x\big)\Big), \text{ and} \\ \pi(x,t) &:= p_1\big(Q^T(t)x\big) - p_2\big(Q^T(t)x\big). \end{aligned} \tag{4.2}$$

Then

$$
(4.3) \qquad \begin{cases} \partial_t w = \Delta w - \nabla\pi + \mathcal{R}\,\partial_1 w & \text{in } \mathbb{R}^3_\infty, \\ \operatorname{div}(w) = 0 & \text{in } \mathbb{R}^3_\infty, \\ \lim\limits_{t\to 0} \|w(\cdot,t) - \big(v_1(\cdot,t) - v_2(\cdot,t)\big)\|_{\underline{q},\mathrm{B}_\rho} = 0 & \text{for all } \rho > 0, \end{cases}
$$

where $\underline{q} = \min\{q_1, q_2\}$. From (4.2) and the assumptions of the lemma, it follows at once that (w, π) satisfies the properties

$$
(4.4) \quad w = w_1 + w_2,\ w_i \in L^{q_i}(\mathbb{R}^3_T)\ (i = 1, 2),\ w \in \mathcal{W}^{(1,2)}_{s,loc}(\mathbb{R}^3_T),\ \pi \in \mathcal{D}^{(0,1)}_{s,loc}(\mathbb{R}^3_T),
$$

for any $T > 0$. From Lemma 3.4 we find solutions W_i $(i = 1, 2)$ to

$$
(4.5) \qquad \begin{cases} \partial_t W_i = \Delta W_i + \mathcal{R}\,\partial_1 W_i & \text{in } \mathbb{R}^3_\infty, \\ \operatorname{div}(W_i) = 0 & \text{in } \mathbb{R}^3_\infty, \\ \lim\limits_{t\to 0} \|W_i(\cdot,t) - v_i\|_{q_i} = 0, \end{cases}
$$

with

$$
(4.6) \qquad W_i \in \mathcal{W}^{(1,2)}_{q_i,loc}(\mathbb{R}^3_T) \cap L^{q_i}(\mathbb{R}^3_T)
$$

and, further, satisfying the inequality

$$
(4.7) \qquad \|W_1(\cdot,t)\|_r + \|W_2(\cdot,t)\|_r \le c_9\, t^{3(1/r - 1/\bar{q})/2}\big(\|v_1\|_{q_1} + \|v_2\|_{q_2}\big)
$$

for all $t > 1$ and $r \ge \bar{q}$, where $\bar{q} = \max\{q_1, q_2\}$. With a view to equations (4.2) through (4.6) we find that the pair (z, π) with $z := w - (W_1 - W_2)$ satisfies all assumptions of Lemma 3.6. Consequently, $w = W_1 - W_2$. Thus, from (4.2) and (4.7), we conclude that

$$
\|v_1 - v_2\|_r = \|w(\cdot,t)\|_r \le\ c_9 t^{3(1/r - 1/\bar{q})/2}\big(\|v_1\|_{q_1} + \|v_2\|_{q_2}\big)
$$

for all $t > 1$ and $r \ge \bar{q}$. Fixing $r > \bar{q}$ and letting $t \to \infty$ in the above, we recover $v_1 = v_2$, which implies, $\nabla p_1 = \nabla p_2$, that is, $p_1 = p_2 + c$ for some constant c. $\square$

In the next lemma, we combine a result of Farwig [6] with that of the previous Lemma 4.1.

Lemma 4.2. *Let $1 < q < 2$. For every $f \in L^q(\mathbb{R}^3)^3$ there exists at least one solution $(v, p) \in \mathrm{X}_q(\mathbb{R}^3)$ to*

$$
(4.8) \qquad \begin{cases} \Delta v - \nabla p + \mathcal{R}\partial_1 v + \mathcal{T}\big(\mathrm{e}_1 \wedge x \cdot \nabla v - \mathrm{e}_1 \wedge v\big) = f & \text{in } \mathbb{R}^3, \\ \operatorname{div}(v) = 0 & \text{in } \mathbb{R}^3, \end{cases}
$$

which satisfies

$$
(4.9) \qquad \|(v, p)\|_{\mathrm{X}_q} \le C_{12}\, \|f\|_q,
$$

with $C_{12} = C_{12}(\mathcal{R}, \mathcal{T})$. Moreover, if (v_1, p_1) is any other solution corresponding to f with $(v_1, p_1) \in \Big(L^{q_1}(\mathbb{R}^3)^3 \cap W^{2,s}_{loc}(\mathbb{R}^3)^3\Big) \times W^{1,s}_{loc}(\mathbb{R}^3)$, for some $s, q_1 > 1$, then necessarily $v = v_1$, $p = p_1 + c$, for some constant c.

Proof. Existence of a solution $(v,p) \in D^{2,q}(\mathbb{R}^3) \times D^{1,q}(\mathbb{R}^3)$ to (4.8) with

$$\|\nabla^2 v\|_q + \|\partial_1 v\|_q + \|\nabla p\|_q \le c_1 \|f\|_q \tag{4.10}$$

follows directly from [6, Theorem 1.1], where $c_1 = c_1(\mathcal{R},\mathcal{T})$. Moreover, by [6, Corollary 1.2] there exist $\alpha, \beta \in \mathbb{R}$ so that [12]

$$\|\nabla'\big(v - \beta\, \mathrm{e}_1 \wedge x - \alpha\, \mathrm{e}_1\big)\|_{\frac{4q}{4-q}} \le c_2 \|f\|_q \tag{4.11}$$

and

$$\|v - \beta\, \mathrm{e}_1 \wedge x - \alpha\, \mathrm{e}_1\|_{\frac{2q}{2-q}} \le c_3 \|f\|_q. \tag{4.12}$$

Put $v^* := v - \beta\, \mathrm{e}_1 \wedge x - \alpha\, \mathrm{e}_1$ and $p^* = p + c$, with the constant c chosen so that $p^* \in L^{\frac{3q}{3-q}}$ (see for example [11, Theorem II.5.1]). One may easily verify that (v^*, p^*) solves (4.8). We have thus established the existence of a solution $(v^*, p^*) \in \mathrm{X}_q(\mathbb{R}^3)$ to (4.8). Furthermore, (4.9) follows from (4.10), (4.11), and (4.12). Finally, uniqueness is an immediate consequence of Lemma 4.1. □

We now extend the previous lemma to a more general case.

Lemma 4.3. *Let $f \in L^{q_i}(\mathbb{R}^3)$, $1 < q_i < 2$ ($i = 1,2$), and $A \in L^2(\mathbb{R}^3)^{3\times 3}$ be given. There exists $\varepsilon_0 := \varepsilon_0(\mathcal{R},\mathcal{T},q_1,q_2) > 0$ so that if $\|A\|_2 \le \varepsilon_0$ there is a unique solution $(v,q) \in \mathrm{X}_{q_1}(\mathbb{R}^3) \cap \mathrm{X}_{q_2}(\mathbb{R}^3)$ to the problem*

$$\begin{cases} \Delta v - \nabla p + \mathcal{R}\partial_1 v + \mathcal{T}\big(\mathrm{e}_1 \wedge x \cdot \nabla v - \mathrm{e}_1 \wedge v\big) + Av = f & \text{in } \mathbb{R}^3, \\ \mathrm{div}(v) = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{4.13}$$

Moreover,

$$\|(v,p)\|_{\mathrm{X}_{q_1}} + \|(v,p)\|_{\mathrm{X}_{q_2}} \le C_{13}\big(\|f\|_{q_1} + \|f\|_{q_2}\big), \tag{4.14}$$

where $C_{13} = C_{13}(\mathcal{R},\mathcal{T},q_1,q_2)$.

Proof. Let $(v,p) \in \mathrm{X}_{q_i}(\mathbb{R}^3)$ ($i = 1,2$). Since $v \in L^{\frac{2q_i}{2-q_i}}(\mathbb{R}^3)$, by the Hölder inequality, we see that $Av \in L^{q_1}(\mathbb{R}^3) \cap L^{q_2}(\mathbb{R}^3)$ with

$$\|Av\|_{q_1} + \|Av\|_{q_2} \le \|A\|_2 \left(\|v\|_{\frac{2q_1}{2-q_1}} + \|v\|_{\frac{2q_2}{2-q_2}}\right). \tag{4.15}$$

By Lemma 4.2, under the stated assumptions for f, there exists a unique solution $(z,\pi) \in \mathrm{X}_{q_1}(\mathbb{R}^3) \cap \mathrm{X}_{q_2}(\mathbb{R}^3)$ to

$$\begin{cases} \Delta z - \nabla \pi + \mathcal{R}\partial_1 z + \mathcal{T}\big(\mathrm{e}_1 \wedge x \cdot \nabla z - \mathrm{e}_1 \wedge z\big) = f - Av & \text{in } \mathbb{R}^3, \\ \mathrm{div}(z) = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

[12] In principle, the constant β in [6, Corollary 1.2 (1)] may differ from the β in [6, Corollary 1.2 (2)]. It is, however, easy to verify that these two β's must coincide.

Putting $\mathcal{L}(v,p) := (z,\pi)$, we define a mapping $\mathcal{L} : \mathrm{X} \to \mathrm{X}$, where $\mathrm{X} := \mathrm{X}_{q_1}(\mathbb{R}^3) \cap \mathrm{X}_{q_2}(\mathbb{R}^3)$. Clearly, X with the norm $\|\cdot\|_{\mathrm{X}} := \|\cdot\|_{\mathrm{X}_{q_1}} + \|\cdot\|_{\mathrm{X}_{q_2}}$ is a Banach space. Moreover, utilizing (4.9) and (4.15), we find that

$$\begin{aligned}\|\mathcal{L}(v_1,p_1) - \mathcal{L}(v_2,p_2)\|_{\mathrm{X}} &= \|(z_1,\pi_1) - (z_2,\pi_2)\|_{\mathrm{X}} \\ &\leq C_{12}\left(\|A(v_1-v_2)\|_{q_1} + \|A(v_1-v_2)\|_{q_2}\right) \\ &\leq C_{12}\|A\|_2\,\|(v_1,p_1)-(v_2,p_2)\|_{\mathrm{X}}.\end{aligned}$$

Consequently, if we choose $\varepsilon_0 < \frac{1}{C_{12}(\mathcal{R},\mathcal{T})}$, then $\mathcal{L}$ is a contraction. In this case we obtain, by Banach's fixed point theorem, a unique fixed point of $\mathcal{L}$. Clearly, this fixed point is a solution to (4.13). Moreover, by (4.9) and (4.15), we obtain (4.14). □

We are now able to prove the main result of this section.

Theorem 4.4. *Let* $1 < q < 2$ *and* $f \in L^q(\Omega)^3 \cap L^{\frac{3}{2}}(\Omega)^3 \cap L^2_{loc}(\Omega)^3$. *Any solution* $v \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3 \cap W^{2,2}_{loc}(\Omega)^3$ *and* $p \in W^{1,2}_{loc}(\Omega)$ *to*

$$\text{(4.16)}\qquad \begin{cases}\Delta v - \nabla p + \mathcal{R}\big(\partial_1 v - v\cdot\nabla v\big) + \mathcal{T}\big(\mathrm{e}_1\wedge x\cdot\nabla v - \mathrm{e}_1\wedge v\big) = f & \text{in } \Omega,\\ \operatorname{div}(v) = 0 & \text{in } \Omega\end{cases}$$

satisfies for sufficiently large $\rho > 0$ *and some constant c:*

$$(v, p+c) \in \mathrm{X}_q(\Omega^\rho).$$

Proof. Let $\psi_\rho \in C^\infty(\mathbb{R}^3;\mathbb{R})$ be a "cut-off" function with $\psi_\rho = 0$ on B_ρ and $\psi_\rho = 1$ on $\mathbb{R}^3\setminus\mathrm{B}_{2\rho}$. We put

$$v_\rho := \psi_\rho v - \psi_\rho\sigma - H, \quad p_\rho := \psi_\rho p,$$

where

$$\sigma(x) := \Big(-\int_{\partial\mathrm{B}_{2\rho}} v\cdot n\,\mathrm{d}S\Big)\cdot\nabla\mathcal{E}(x), \quad \mathcal{E}(x) := \frac{1}{4\pi\,|x|},$$

and

$$H \in W^{3,2}(\mathbb{R}^3),\ \operatorname{supp}(H) \subset \mathrm{B}_{2\rho},\ \operatorname{div}(H) = \nabla\psi_\rho\cdot(v-\sigma).$$

The existence of such a H follows from [11, Theorem III.3.2] since

$$\int_{\mathrm{B}_{2\rho}} \nabla\psi_\rho\cdot(v-\sigma)\,\mathrm{d}x = \int_{\partial\mathrm{B}_{2\rho}} v\cdot n\,\mathrm{d}S - \int_{\partial\mathrm{B}_{2\rho}} \sigma\cdot n\,\mathrm{d}S = 0.$$

Clearly,

$$\begin{cases}\Delta v_\rho - \nabla p_\rho + \mathcal{R}\,\partial_1 v_\rho + \mathcal{T}\big(\mathrm{e}_1\wedge x\cdot\nabla v_\rho - \mathrm{e}_1\wedge v_\rho\big) = F_\rho + \mathcal{R}\psi_\rho v\cdot\nabla v & \text{in } \mathbb{R}^3,\\ \operatorname{div}(v_\rho) = 0 & \text{in } \mathbb{R}^3,\end{cases}$$

with

$$F_\rho := \psi_\rho f + v\Delta\psi_\rho + 2\nabla v\nabla\psi_\rho - \nabla\psi_\rho p + \mathcal{R}\partial_1\psi_\rho v + \mathcal{T} v\otimes(\nabla\psi_\rho)\cdot(\mathrm{e}_1\wedge x)$$
$$+ \sigma\Delta\psi_\rho + 2\nabla\sigma\nabla\psi_\rho + \mathcal{R}\psi_\rho\partial_1\sigma + \mathcal{R}\partial_1\psi_\rho\sigma + \mathcal{T}\sigma\otimes(\nabla\psi_\rho)\cdot(\mathrm{e}_1\wedge x)$$
$$+ \Delta H + \mathcal{R}\partial_1 H + \mathcal{T}\big(\mathrm{e}_1\wedge x\cdot\nabla H - \mathrm{e}_1\wedge H\big).$$

Here, we have used that $(\mathrm{e}_1\wedge x\cdot\nabla\sigma - \mathrm{e}_1\wedge\sigma) = 0$. The decay properties of σ ensure that $\partial_1\sigma\in L^r(\Omega)$, $\forall r>1$. Consequently, we have $F_\rho\in L^q(\mathbb{R}^3)\cap L^{\frac{3}{2}}(\mathbb{R}^3)$. Next, we observe that

$$\psi_\rho v\cdot\nabla v = \big(\psi_{\rho/2}\nabla v\big)\,(v_\rho + \psi_\rho\sigma + H).$$

Putting

$$A_\rho := -\mathcal{R}\psi_{\rho/2}\nabla v,$$

we thus have

(4.17)
$$\begin{cases} \Delta v_\rho - \nabla p_\rho + \mathcal{R}\partial_1 v_\rho + \mathcal{T}\big(\mathrm{e}_1\wedge x\cdot\nabla v_\rho - \mathrm{e}_1\wedge v_\rho\big) + A_\rho v_\rho = \tilde{F}_\rho & \text{in } \mathbb{R}^3,\\ \operatorname{div}(v_\rho) = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

with $\tilde{F}_\rho\in L^q(\mathbb{R}^3)\cap L^{\frac{3}{2}}(\mathbb{R}^3)$. Since $\nabla v\in L^2(\Omega)$, we see that $\lim_{\rho\to\infty}\|A_\rho\|_2 = 0$. Hence, for sufficiently large ρ, there exists, by Lemma 4.3, a solution $(V_\rho, P_\rho)\in \mathrm{X}_q(\mathbb{R}^3)\cap\mathrm{X}_{\frac{3}{2}}(\mathbb{R}^3)$ to (4.17). We will show that $(v_\rho, p_\rho) = (V_\rho, P_\rho)$. To this end, consider

$$(z,\pi) := (v_\rho - V_\rho, p_\rho - P_\rho)$$

and note that $z\in L^6(\mathbb{R}^3)$ and solves

$$\text{(4.18)}\quad \begin{cases} \Delta z - \nabla\pi + \mathcal{R}\partial_1 z + \mathcal{T}\big(\mathrm{e}_1\wedge x\cdot\nabla z - \mathrm{e}_1\wedge z\big) = A_\rho(v_\rho - V_\rho) & \text{in } \mathbb{R}^3,\\ \operatorname{div}(z) = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

Observe that $A_\rho v_\rho\in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $A_\rho V_\rho\in L^{\frac{3}{2}}(\mathbb{R}^3)$. Thus, from Lemma 4.2 we obtain a solution $(\tilde{z},\tilde{\pi})\in\mathrm{X}_{\frac{3}{2}}(\mathbb{R}^3)$ to (4.18), which satisfies (4.9). Since also $\tilde{z}\in L^6(\mathbb{R}^3)$, Lemma 4.1 (invoked with $q_1 = q_2 = 6$, $s = \frac{3}{2}$) yields $(\tilde{z},\tilde{\pi}) = (z,\pi + c)$ for some constant c. Consequently, $(z,\pi + c)\in\mathrm{X}_{\frac{3}{2}}(\mathbb{R}^3)$ and

$$\|(z,\pi + c)\|_{\mathrm{X}_{\frac{3}{2}}} \le C_{12}\|A_\rho z\|_{\frac{3}{2}} \le C_{12}\|A_\rho\|_2\|(z,\pi + c)\|_{\mathrm{X}_{\frac{3}{2}}}.$$

If we choose ρ sufficiently large so that $\|A_\rho\|_2 < \frac{1}{C_{12}}$, we deduce that $(z,\pi+c) = 0$. It then follows that $(v_\rho, p_\rho + c) = (V_\rho, P_\rho)$ and thereby $(v_\rho, p_\rho + c)\in\mathrm{X}_q(\mathbb{R}^3)$. Combining this latter with the fact that $\operatorname{supp}(H)$ is bounded and, clearly, $\psi_\rho\sigma\in\mathrm{X}_q(\mathbb{R}^3)$, we conclude the proof of the lemma. □

Remark 4.5. Note that Lemma 4.4 holds regardless of the regularity of $\partial\Omega$ and the boundary values of v on $\partial\Omega$.

Remark 4.6. An immediate corollary to Theorem 4.4 is that any Leray solution satisfies $\nabla v \in L^s(\Omega^\rho)$ for all $s \in (\frac{4q}{4-q}, \frac{3q}{3-q})$. Actually, since $\nabla^2 v \in L^q(\Omega^\rho)$ for all $1 < q < 2$, by [11, Theorem II.5.1] we infer $\nabla v \in L^{\frac{3q}{3-q}}(\Omega^\rho)$, so that the stated property follows by this latter and by the fact that $\nabla' v \in L^{\frac{4q}{4-q}}(\Omega^\rho)$, $\partial_1 v \in L^q(\Omega^\rho)$ combined with elementary interpolation inequalities.

We end this section by proving an important decay estimate.

Lemma 4.7. *Let* $v \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3 \cap W^{2,2}_{loc}(\Omega)^3$ *and* $p \in W^{1,2}_{loc}(\Omega)$ *be a solution to*

$$\text{(4.19)} \qquad \begin{cases} \Delta v - \nabla p + \mathcal{R}\big(\partial_1 v - v\cdot\nabla v\big) + \mathcal{T}\big(\mathrm{e}_1 \wedge x \cdot \nabla v - \mathrm{e}_1 \wedge v\big) = 0 & \text{in } \Omega, \\ \mathrm{div}(v) = 0 & \text{in } \Omega. \end{cases}$$

Then for all $\varepsilon > 0$*:*

$$\|\nabla v\|^2_{2,\mathrm{B}^R} \le C_{14}\, R^{-1+\varepsilon},$$

with $C_{14} := C_{14}(\varepsilon, \mathcal{R}, \mathcal{T}, v, p)$.

Proof. We multiply $(4.19)_1$ by v, integrate over $\mathrm{B}_{R,R^*} := \mathrm{B}_{R^*} \cap \mathrm{B}^c_R$, $(R^* > R)$, and obtain

$$\begin{aligned} 0 &= \int_{\mathrm{B}_{R,R^*}} \big(\Delta v - \nabla p + \mathcal{R}(\partial_1 v - v\cdot\nabla v) + \mathcal{T}(\mathrm{e}_1\wedge x\cdot\nabla v - \mathrm{e}_1 \wedge v)\big)\cdot v \,\mathrm{d}x \\ &= \int_{\mathrm{B}_{R,R^*}} -\nabla v : \nabla v + \mathcal{T}(\mathrm{e}_1 \wedge x \cdot \nabla v)\cdot v\,\mathrm{d}x \\ &\quad + \int_{\partial \mathrm{B}_R \cup \partial \mathrm{B}_{R^*}} (\nabla v\cdot n)\cdot v + \frac{1}{2}\mathcal{R}|v|^2 n_1 - \frac{1}{2}\mathcal{R}|v|^2 v\cdot n - p(v\cdot n)\,\mathrm{d}S. \end{aligned}$$

We observe that on $\partial\,\mathrm{B}_R \cup \partial\,\mathrm{B}_{R^*}$ we have $n := \frac{x}{|x|}$ and thus

$$\int_{\mathrm{B}_{R,R^*}} \mathrm{e}_1 \wedge x\cdot\nabla v\cdot v\,\mathrm{d}x = \frac{1}{2}\int_{\partial \mathrm{B}_R\cup\partial \mathrm{B}_{R^*}} |v|^2(\mathrm{e}_1\wedge x)\cdot n\,\mathrm{d}S = 0.$$

We thus conclude

$$\int_{\mathrm{B}_{R,R^*}} \nabla v : \nabla v\,\mathrm{d}x = \int_{\partial \mathrm{B}_R\cup\partial \mathrm{B}_{R^*}} (\nabla v\cdot n)\cdot v + \frac{1}{2}\mathcal{R}|v|^2 n_1 - \frac{1}{2}\mathcal{R}|v|^2 v\cdot n - p(v\cdot n)\,\mathrm{d}S.$$

The rest of the proof follows precisely the proof of [12, Lemma IX.8.2], and will be, therefore, omitted. □

5 Leray Solutions are Physically Reasonable

Objective of this section is to prove point-wise asymptotic estimates for v, ∇v and p. These estimates will, in particular, furnish that every Leray solution is, in fact, Physically Reasonable.

We begin to show the point-wise behavior of the velocity.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain. Moreover, let $\mathcal{R}, \mathcal{T} > 0$ and $f \in L^2_{loc}(\Omega)^3$ with $\operatorname{supp}(f)$ bounded. Then a solution $v \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3$ with $p \in L^2_{loc}(\Omega)$ to the system of equations:*

$$\begin{cases} \Delta v - \nabla p + \mathcal{R}\big(\partial_1 v - v \cdot \nabla v\big) + \mathcal{T}\big(\mathrm{e}_1 \wedge x \cdot \nabla v - \mathrm{e}_1 \wedge v\big) = f & \text{in } \Omega, \\ \operatorname{div}(v) = 0 & \text{in } \Omega, \end{cases} \tag{5.1}$$

satisfies for any $\delta > 0$ and sufficiently large $|x|$:

$$v(x) = \mathcal{V}_1(x) + \mathcal{V}_2(x)$$

with $\mathcal{V}_1 = O\Big(|x|^{-1}\big(1 + \mathcal{R}\, s(x)\big)^{-1}\Big)$ and $\mathcal{V}_2 = O\Big(|x|^{-3/2+\delta}\Big)$.

Proof. Choose ρ sufficiently large so that $\operatorname{supp}(f) \subset \mathrm{B}_\rho$. By elliptic regularity theory, we then have $v \in C^\infty(\Omega^\rho)$ and $p \in C^\infty(\Omega^\rho)$. Furthermore, by Lemma 4.4 (after possibly adding a constant to p), we obtain

$$(v, p) \in \mathrm{X}_q(\Omega^\rho), \ \forall q \in (1, 2). \tag{5.2}$$

From (5.2), we deduce $v \in L^\infty(\Omega^\rho)$, by Sobolev embedding.

Let $\psi_\rho \in C^\infty(\mathbb{R}^3; \mathbb{R})$ be a "cut-off" function with $\psi_\rho = 0$ in B_ρ and $\psi_\rho = 1$ in $\mathbb{R}^3 \setminus \mathrm{B}_{2\rho}$. We put

$$u =: \psi_\rho v - \psi_\rho \sigma - Z, \quad d := \psi_\rho p, \tag{5.3}$$

where

$$\sigma(x) := \Bigg(-\int_{\partial \mathrm{B}_{2\rho}} v \cdot n \,\mathrm{d}S\Bigg) \cdot \nabla\mathcal{E}(x), \quad \mathcal{E}(x) := \frac{1}{4\pi\,|x|},$$

and $Z \in C_0^\infty(B_{2\rho})$ with $\operatorname{div}(Z) := \nabla\psi_\rho \cdot (v - \sigma)$. The existence of such a Z follows from [11, Theorem III.3.2] since

$$\int_{\mathrm{B}_{2\rho}} \nabla\psi_\rho \cdot (v - \sigma)\,\mathrm{d}x = \int_{\partial \mathrm{B}_{2\rho}} v \cdot n \,\mathrm{d}S - \int_{\partial \mathrm{B}_{2\rho}} \sigma \cdot n \,\mathrm{d}S = 0.$$

Note at this point that

$$|\sigma(x)| \le c_1\,|x|^{-2}, \ \text{and} \ \ |\nabla\sigma(x)| \le c_2\,|x|^{-3}. \tag{5.4}$$

Using the fact that $(\mathrm{e}_1 \wedge x \cdot \nabla\sigma - \mathrm{e}_1 \wedge \sigma) = 0$, we see that

$$
(5.5)\qquad \begin{cases} \Delta u - \nabla d + \mathcal{R}\partial_1 u + \mathcal{T}\big(\mathrm{e}_1 \wedge x \cdot \nabla u - \mathrm{e}_1 \wedge u\big) = \\ \qquad\qquad \mathcal{R}\,\mathrm{div}\,\big((\psi_\rho v)\otimes(\psi_\rho v)\big) - \mathcal{R}\,\partial_1[\psi_\rho\sigma] + F_c & \text{in } \mathbb{R}^3, \\ \mathrm{div}(u) = 0 & \text{in } \mathbb{R}^3, \end{cases}
$$

with $F_c \in C_0^\infty(\mathbb{R}^3)$. Now choose $Q(t)$ as in (4.1) and put

$$
(5.6)\qquad \begin{aligned} & w(y,t) := Q(t)u\big(Q^T(t)y\big), && \pi(y,t) := d\big(Q^T(t)y\big), \\ & V(y,t) := Q(t)[\psi_\rho v]\big(Q^T(t)y\big), && \Sigma(y,t) := Q(t)[\psi_\rho\sigma]\big(Q^T(t)y\big), \\ & H_c(y,t) := Q(t)F_c\big(Q^T(t)y\big). \end{aligned}
$$

From (5.2) and (5.4) we obtain

$$
(5.7)\qquad u \in L^r(\mathbb{R}^3),\ \forall r > 2,
$$

and hence we have

$$
(5.8)\qquad \begin{cases} \partial_t w = \Delta w - \nabla\pi + \mathcal{R}\,\partial_1 w \\ \qquad\qquad - \mathcal{R}\,\mathrm{div}(V\otimes V) + \mathcal{R}\,\partial_1\Sigma - H_c & \text{in } \mathbb{R}^3_\infty, \\ \mathrm{div}(w) = 0 & \text{in } \mathbb{R}^3_\infty, \\ \lim_{t\to 0}\|w(\cdot,t) - u\|_r = 0, \end{cases}
$$

for all $r > 2$. Utilizing again (5.2), we deduce

$$
\mathrm{div}(V\otimes V) \in L^{\infty,r}\big(\mathbb{R}^3_\infty\big),\ \forall r \in (1,4).
$$

Moreover, due to (5.4) we have

$$
\partial_1\Sigma \in L^{\infty,r}\big(\mathbb{R}^3_\infty\big),\ \forall r > 1.
$$

Also,

$$
\mathrm{supp}\,\big(H_c(\cdot,t)\big) \subset \mathrm{B}_{2\rho} \ \text{ and } \ H_c \in L^{\infty,r}\big(\mathbb{R}^3_\infty\big),\ \forall r > 1.
$$

We may now combine Lemma 3.3 and Lemma 3.4 to obtain a solution $(\breve{w},\breve{\pi})$ to (5.8) with

$$
\begin{cases} \breve{w} \in \mathcal{W}^{(1,2)}_r\big(\mathbb{R}^3\times(\varepsilon,T)\big),\ \breve{w} \in L^r\big(\mathbb{R}^3_T\big), \text{ and} \\ \breve{\pi} \in \mathcal{D}^{(0,1)}_r\big(\mathbb{R}^3_T\big),\ \forall r \in (2,4),\ \forall\varepsilon > 0,\ \forall T > 0, \end{cases}
$$

given by

(5.9)

$$
\begin{aligned} \breve{w}(x,t) &= (4\pi t)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathrm{e}^{-|x-y+\mathcal{R}\,t\,\mathrm{e}_1|^2/4t}\ u(y)\,\mathrm{d}y \\ &\quad + \int_0^t \int_{\mathbb{R}^3} \Gamma(x-y,\tau)\cdot\big(-\mathcal{R}\,\mathrm{div}(V\otimes V)(y,\tau) + \mathcal{R}\,\partial_1\Sigma - H_c(y,\tau)\big)\,\mathrm{d}y\mathrm{d}\tau. \end{aligned}
$$

We clearly have $(w,\pi) \in \mathcal{W}^{(1,2)}_{2,loc}(\mathbb{R}^3_T) \times \mathcal{D}^{(0,1)}_{2,loc}(\mathbb{R}^3_T)$ and, recalling (5.7), also $w \in L^r(\mathbb{R}^3_T)$, $\forall r > 2$. We thus conclude by Lemma 3.6 that $w = \breve{w}$. From (5.9), we can therefore derive the representation

$$w = w_1 + w_2 + w_3 + w_4 \tag{5.10}$$

with $(i = 1, 2, 3)$

$$w_{1i}(x,t) = \mathcal{R} \int_0^t \int_{\mathbb{R}^3} \partial_l \Gamma_{ij}(x-y, t-\tau)\, V_j(y,\tau)\, V_l(y,\tau)\, \mathrm{d}y\mathrm{d}\tau, \tag{5.11}$$

$$w_2(x,t) = -\int_0^t \int_{\mathbb{R}^3} \Gamma(x-y, t-\tau) \cdot H_c(y,\tau)\, \mathrm{d}y\mathrm{d}\tau, \tag{5.12}$$

$$w_3(x,t) = (4\pi t)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathrm{e}^{-|x-y+\mathcal{R}\, t\, \mathrm{e}_1|^2/4t}\, u(y)\, \mathrm{d}y, \tag{5.13}$$

$$w_4(x,t) = \mathcal{R} \int_0^t \int_{\mathbb{R}^3} \partial_1 \Gamma(x-y, t-\tau) \cdot \Sigma(y,\tau)\, \mathrm{d}y\mathrm{d}\tau. \tag{5.14}$$

We shall now give point-wise estimate of w_i, $i = 1, \dots, 4$, beginning with w_1. Since the numerical value of $\mathcal{R}$ and $\mathcal{T}$ is irrelevant in the proof (provided they are both positive, of course), in what follows we shall put, for simplicity, $\mathcal{R} = \mathcal{T} = 1$. By Lemma 3.1, we have

$$\begin{aligned} |w_1(x,t)| &\le \int_0^t \int_{\mathbb{R}^3} C_6\, (\tau + |x-y+\tau\, \mathrm{e}_1|^2)^{-2}\, |V(y,t-\tau)|^{-2}\, \mathrm{d}y\mathrm{d}\tau \\ &\le \int_0^t \int_{\mathrm{B}_R} C_6\, (\tau + |x-y+\tau\, \mathrm{e}_1|^2)^{-2} |V(y,t-\tau)|^{-2}\, \mathrm{d}y\mathrm{d}\tau \\ &\quad + \int_0^t \int_{\mathrm{B}^R} C_6\, (\tau + |x-y+\tau\, \mathrm{e}_1|^2)^{-2} |V(y,t-\tau)|^{-2}\, \mathrm{d}y\mathrm{d}\tau \\ &=: I_1 + I_2 \end{aligned} \tag{5.15}$$

for any $R > 0$. We fix $R = \frac{|x|}{3}$ and estimate, by Hölder's inequality, for any $r > 2$ and $r_0 := \frac{r}{r-2}$:

$$I_1 \le c_3 \int_0^\infty \Big(\int_{\mathrm{B}_R} (\tau + |x-y+\tau\, \mathrm{e}_1|^2)^{-2r_0} \mathrm{d}y \Big)^{\frac{1}{r_0}} \|V(\cdot, t-\tau)\|_r^2\, \mathrm{d}\tau.$$

From definition of V, we deduce

$$I_1 \leq c_3 \, \|v\|^2_{r,\Omega^\rho} \int\limits_0^\infty \left(\int\limits_{\mathrm{B}_R} (\tau + |x - y + \tau \, \mathrm{e}_1|^2)^{-2r_0} \mathrm{d}y \right)^{\frac{1}{r_0}} \mathrm{d}\tau.$$

Putting $z = y - \tau \, \mathrm{e}_1$, we have $|z| \leq 2R$ for $0 \leq \tau \leq R$ and $y \in \mathrm{B}_R$. Thus

$$\int\limits_0^R \left(\int\limits_{\mathrm{B}_R} (\tau + |x - y + \tau \, \mathrm{e}_1|^2)^{-2r_0} \mathrm{d}y \right)^{\frac{1}{r_0}} \mathrm{d}\tau \leq \int\limits_0^R \left(\int\limits_{\mathrm{B}_{2R}} (\tau + |x - z|^2)^{-2r_0} \mathrm{d}z \right)^{\frac{1}{r_0}} \mathrm{d}\tau.$$

Moreover, for $|z| \leq 2R$ we have $|x - z| \geq 3R - 2R = R$ and hence

$$\int\limits_0^R \left(\int\limits_{\mathrm{B}_R} (\tau + |x - y + \tau \, \mathrm{e}_1|^2)^{-2r_0} \mathrm{d}z \right)^{\frac{1}{r_0}} \mathrm{d}\tau \leq \int\limits_0^R \left(\int\limits_{\mathrm{B}_{2R}} (\tau + R^2)^{-2r_0} \mathrm{d}z \right)^{\frac{1}{r_0}} \mathrm{d}\tau \leq c_4 \int\limits_0^R \frac{R^{\frac{3}{r_0}}}{(\tau + R^2)^2} \, \mathrm{d}\tau \leq c_5 R^{-3 + \frac{3}{r_0}}.$$

Since

$$\int\limits_R^\infty \left(\int\limits_{\mathrm{B}_R} (\tau + |x - y + \tau \, \mathrm{e}_1|^2)^{-2r_0} \mathrm{d}y \right)^{\frac{1}{r_0}} \mathrm{d}\tau \leq \int\limits_R^\infty \left(\int\limits_{\mathrm{B}_R} \tau^{-2r_0} \mathrm{d}y \right)^{\frac{1}{r_0}} \mathrm{d}\tau \leq c_6 R^{-1 + \frac{3}{r_0}},$$

we conclude that for large $|x|$:

$$I_1 \leq c_7 R^{-1 + \frac{3}{r_0}} = c_8 |x|^{-1 + \frac{3(r-2)}{r}}, \ \forall r > 2, \tag{5.16}$$

where $c_8 = c_8(r)$. We will now estimate I_2. First, we put $\delta(x) := |(x_2, x_3)|$ and write

$$\begin{aligned} I_2 = & \int\limits_0^t \int\limits_{\{y \in \mathrm{B}^R \, |\delta(x-y) < 1\}} \frac{C_6}{(\tau + |x - y + \tau \, \mathrm{e}_1|^2)^2} \, |V(y, t - \tau)|^2 \, \mathrm{d}y \mathrm{d}\tau \\ & + \int\limits_0^t \int\limits_{\{y \in \mathrm{B}^R \, |\delta(x-y) \geq 1\}} \frac{C_6}{(\tau + |x - y + \tau \, \mathrm{e}_1|^2)^2} \, |V(y, t - \tau)|^2 \, \mathrm{d}y \mathrm{d}\tau \\ & =: I_{21} + I_{22}. \end{aligned}$$

By Hölder's inequality, we find for an arbitrary $q_0 > 6$, $q_1 := \frac{q_0}{q_0-2} \in (1, \frac{3}{2})$:

$$I_{21} \leq \int_0^t \bigg(\int_{\{\delta(x-y)<1\}} \frac{{C_6}^{q_1}}{(\tau + |x - y + \tau\,\mathrm{e}_1|^2)^{2q_1}}\,\mathrm{d}y \bigg)^{\frac{1}{q_1}} \|V(\cdot, t-\tau)\|^2_{q_0,\mathrm{B}^R}\,\mathrm{d}\tau,$$

from which we derive, using Lemma 3.2, for $R > 2\rho$:

$$I_{21} \leq c_9\,\|v\|^2_{q_0,\mathrm{B}^R} \leq c_9\,\|v\|_{\infty,\Omega^\rho}^{\frac{2(q_0-6)}{q_0}}\,\|v\|_{6,\mathrm{B}^R}^{2+\frac{2(6-q_0)}{q_0}}, \tag{5.17}$$

with $c_9 = c_9(q_0)$. We shall now use the inequality (see [11, Theorem 5.1])

$$\|v\|_{6,\mathrm{B}^R} \leq c_{10}\,\|\nabla v\|_{2,\mathrm{B}^R}, \tag{5.18}$$

where one may verify, by a simple scaling argument, that c_{10} does not depend on R. Furthermore, we shall use Lemma 4.7, which can applied to v since

$$\begin{cases} \Delta v - \nabla p + \partial_1 v - v\cdot\nabla v + \mathrm{e}_1\wedge x\cdot\nabla v - \mathrm{e}_1\wedge v = 0 & \text{in } \mathrm{B}^R, \\ \mathrm{div}(v) = 0 & \text{in } \mathrm{B}^R. \end{cases}$$

Combining (5.17), (5.18), and Lemma 4.7, we conclude that

$$I_{21} \leq c_{11}\,R^{-1+\varepsilon} = c_{12}\,|x|^{-1+\varepsilon},\ \forall\varepsilon > 0, \tag{5.19}$$

where $c_{12} = c_{12}(\varepsilon)$. We now move on to I_{22}. This time, we consider an arbitrary $2 < q_0 < 6$ and $q_2 := \frac{q_0}{q_0-2} > \frac{3}{2}$, and obtain

$$I_{22} \leq \int_0^t \bigg(\int_{\{\delta(x-y)\geq 1\}} \frac{{C_6}^{q_2}}{(\tau + |x - y + \tau\,\mathrm{e}_1|^2)^{2q_2}}\,\mathrm{d}y \bigg)^{\frac{1}{q_2}} \|V(\cdot, t-\tau)\|^2_{q_0,\mathrm{B}^R}\,\mathrm{d}\tau,$$

from which we derive, again applying Lemma 3.2, for $R > 2\rho$:

$$I_{22} \leq c_{13}\,\|v\|^2_{q_0,\mathrm{B}^R} \leq c_{13}\,\big(\|v\|^{1-\theta}_{6,\mathrm{B}^R}\,\|v\|^{\theta}_{3}\big)^2, \tag{5.20}$$

with $\frac{1}{q_0} = \frac{\theta}{3} + \frac{1-\theta}{6}$ and $c_{13} = c_{13}(q_0)$. Clearly, $\theta \to 0$ as $q_0 \to 6$. Combining (5.20), (5.18), and Lemma 4.7, we thus conclude that

$$I_{22} \leq c_{14}\,R^{-1+\varepsilon} = c_{15}\,|x|^{-1+\varepsilon},\ \forall\varepsilon > 0, \tag{5.21}$$

where $c_{15} = c_{15}(\varepsilon, v)$. Collecting now (5.15), (5.16), (5.19), and (5.21), we deduce

$$|w_1(x,t)| \leq c_{16}\,|x|^{-1+\varepsilon},\ \forall\varepsilon > 0, \tag{5.22}$$

for sufficiently large $|x|$, where $c_{16} = c_{16}(\varepsilon)$. This concludes, for the moment, the estimate of w_1.

Concerning an estimate of w_2, we obtain from Lemma 3.3 for all $r > 3$:

$$|w_2(x,t)| \le C_{10}\,\frac{\operatorname{ess\,sup}_{t>0}\|H_c(\cdot,t)\|_r}{\big(1+|x|\big)\big(1+s(x)\big)} \le C_{10}\,\frac{\|F_c\|_r}{\big(1+|x|\big)\big(1+s(x)\big)}. \tag{5.23}$$

As a consequence of (5.23), we note that

$$|w_2(x,t)| \le c_{17}\,|x|^{-1}$$

for sufficiently large $|x|$.

Concerning w_3, we deduce from Lemma 3.4 (more specifically (3.23)) that

$$|w_3(x,t)| \le c_{18}\,t^{-\frac{3}{2}\cdot\frac{1}{6}}\,\|u\|_6. \tag{5.24}$$

In order to estimate w_4, we consider, for $\varepsilon > 0$, the integral

$$J := \int\limits_{\mathbb{R}^3}\int\limits_0^\infty |\nabla\Gamma(x-y,\tau)|\,\mathrm{d}\tau\,(1+|y|)^{-2+\varepsilon}\,\mathrm{d}y. \tag{5.25}$$

Using Lemma 3.1, we conclude that

$$\begin{aligned} J \le c_{19}\Big(&\int\limits_{\{|x-y|<\frac{1}{4}\}} |x-y|^{-2}(1+|y|)^{-2+\varepsilon}\,\mathrm{d}y\,+ \\ &\int\limits_{\{|x-y|\ge\frac{1}{4}\}} |x-y|^{-\frac{3}{2}}\,(1+s(x-y))^{-\frac{3}{2}}(1+|y|)^{-2+\varepsilon}\,\mathrm{d}y\Big). \end{aligned} \tag{5.26}$$

The first integral on the right hand side in (5.26) can be estimated, for $|x| > \frac{1}{2}$, by

$$\begin{aligned} &\int\limits_{\{|x-y|<\frac{1}{4}\}} |x-y|^{-2}(1+|y|)^{-2+\varepsilon}\,\mathrm{d}y \\ &\qquad\le \int\limits_{\{|z|<\frac{1}{4}\}} |z|^{-2}(1+|x-z|)^{-2+\varepsilon}\,\mathrm{d}z \\ &\qquad\le \int\limits_{\{|z|<\frac{1}{4}\}} |z|^{-2}\big(1+\frac{1}{2}|x|\big)^{-2+\varepsilon}\,\mathrm{d}z \le c_{20}\,|x|^{-2+\varepsilon}, \end{aligned} \tag{5.27}$$

where $c_{20} = c_{20}(\varepsilon)$. We shall use [5, Lemma 3.1] to estimate the second integral on the right hand side in (5.26). More specifically, the proof of [5, Lemma 3.1] contains, as a particular case, the estimate

$$\int\limits_{\mathbb{R}^3}(1+|x-y|)^{-\frac{3}{2}}\,(1+s(x-y))^{-\frac{3}{2}}\,(1+|y|)^{-2+\varepsilon}\,\mathrm{d}y \le c_{21}\,|x|^{-\frac{3}{2}+\kappa(\varepsilon)}, \tag{5.28}$$

where $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $c_{21} = c_{21}(\varepsilon)$. To see this, choose (in the proof of [5, Lemma 3.1]) $a = \frac{3}{2}$, $b = \frac{3}{2}$, $c = 2 - \varepsilon$, $d = 0$ and utilize the estimates for all I_i with appropriately chosen constants e_i and f_i. Having established (5.28), we derive

$$\int\limits_{\{|x-y|\geq \frac{1}{4}\}} |x-y|^{-\frac{3}{2}} \left(1 + s(x-y)\right)^{-\frac{3}{2}} (1+|y|)^{-2+\varepsilon} \,\mathrm{d}y \leq c_{22}\, |x|^{-\frac{3}{2}+\kappa(\varepsilon)}, \tag{5.29}$$

where $c_{22} = c_{22}(\varepsilon)$. Collecting (5.26), (5.27), and (5.29), we obtain, for sufficiently large $|x|$, that

$$J \leq c_{23}\, |x|^{-\frac{3}{2}+\varepsilon},\ \forall \varepsilon > 0, \tag{5.30}$$

where $c_{23} = c_{23}(\varepsilon)$. Clearly, since $|\sigma(x)| \leq c_{24}\, |x|^{-2}$, we have

$$\begin{aligned} |w_4(x,t)| &\leq \int\limits_0^\infty \int\limits_{\mathbb{R}^3} |\nabla\Gamma(x-y,\tau)|\, (1+|y|)^{-2} \,\mathrm{d}y\mathrm{d}\tau \\ &\leq c_{25}\, J \leq c_{26}\, |x|^{-\frac{3}{2}+\varepsilon},\ \forall \varepsilon > 0, \end{aligned} \tag{5.31}$$

where $c_{26} = c_{26}(\varepsilon)$.

Finally, from (5.10), (5.22), (5.23), (5.24), (5.31), and (5.4) we conclude, for $|x|$ sufficiently large, that

$$\begin{aligned} |v(x)| \leq |u(x)| + |\sigma(x)| &\leq \left|w\big(Q(t)x,t\big)\right| + c_{27}|x|^{-2} \\ &\leq c_{28}\big(|x|^{-1+\varepsilon} + t^{-\frac{1}{4}}\, \|u\|_6\big),\ \forall \varepsilon > 0, \end{aligned}$$

where $c_{28} = c_{28}(\varepsilon)$. Letting $t \to \infty$, we obtain

$$|v(x)| \leq c_{28}\, |x|^{-1+\varepsilon},\ \forall \varepsilon > 0. \tag{5.32}$$

With the estimate (5.32) at hand, we return to the representation (5.11) of w_1. Clearly, as a consequence of (5.32), we have

$$|V(y,t)| \leq c_{29}\, (1+|y|)^{-1+\varepsilon}.$$

Thus, from (5.11) we find that

$$|w_1(x,t)| \leq c_{30} \int\limits_{\mathbb{R}^3} \int\limits_0^\infty |\nabla\Gamma(x-y,\tau)| \,\mathrm{d}\tau\, (1+|y|)^{-2+\varepsilon} \,\mathrm{d}y \leq c_{30}\, J.$$

From (5.30) we conclude, for sufficiently large $|x|$, that

$$|w_1(x,t)| \leq c_{31}\, |x|^{-\frac{3}{2}+\varepsilon},\ \forall \varepsilon > 0, \tag{5.33}$$

where $c_{31} = c_{31}(\varepsilon)$.

Finally, we can now combine (5.10), (5.23), (5.24), (5.31), (5.33), and (5.4), let $t \to \infty$, and thereby obtain, for $|x|$ sufficiently large,

$$|v(x)| \le c_{32} \left(1+|x|\right)^{-1}\left(1+s(x)\right)^{-1} + c_{33}\,|x|^{-\frac{3}{2}+\varepsilon},$$

where $c_{32} = c_{32}(\varepsilon)$ and $c_{33} = c_{33}(\varepsilon)$. The proof of the theorem is then accomplished. □

The following result concerns the asymptotic behavior of ∇v.

Theorem 5.2. *Let the assumptions of Theorem 5.1 be satisfied. Then, for any $\eta > 0$ and sufficiently large $|x|$:*

$$\nabla v(x) = \mathcal{G}_1(x) + \mathcal{G}_2(x),$$

where $\mathcal{G}_1 = O\Big(|x|^{-3/2}\big(1+\mathcal{R}\,s(x)\big)^{-3/2}\Big)$ *and* $\mathcal{G}_2 = O\Big(|x|^{-2+\eta}\Big)$.

Proof. We begin to prove that $\nabla v(x)$ is bounded for all large $|x|$. As in the proof of Theorem 5.1, we infer (5.2) and thus $\nabla^2 v \in L^q(\Omega^\rho)$ for all $q \in (1,2)$, which implies the same property for the function u defined in (5.3). Moreover, by Remark 4.6 and by the fact that $v \in L^\infty(\Omega^\rho)$, it follows that $v \cdot \nabla v \in L^r(\Omega^\rho)$ for all $r \in (1,6)$. Thus, recalling that $\partial_1 \sigma \in L^s(\Omega^\rho)$ for all $s > 1$, from (5.5) and [6, Theorem 1.1] we derive $\nabla^2 u \in L^r(\mathbb{R}^3)$ for all $r \in (1,6)$, that is, $\nabla^2 v \in L^r(\Omega^{2\rho})$, for all $r \in (1,6)$. The claimed boundedness of ∇v then follows from this property, Remark 4.6, and the Sobolev embedding theorem. Recalling definition (5.6), the boundedness of ∇v also implies

$$|\nabla V(y,t)| \le c_1, \tag{5.34}$$

with c_1 independent of y and t. Finally, from Theorem 5.1 we also obtain

$$|V(x,t)| \le c_2 |x|^{-1}, \tag{5.35}$$

with c_2 independent of t.

Our next step is to prove that $\nabla v(x)$ decays, at least, like $|x|^{-1}$. The starting point of our analysis will be, again, the representation (5.10)–(5.14), which yields

$$\partial_k w_{1i}(x,t) = \mathcal{R} \int_0^t \int_{\mathbb{R}^3} \partial_k \Gamma_{ij}(x-y,t-\tau)\,\partial_l\big[V_j(y,\tau)\,V_l(y,\tau)\big]\,\mathrm{d}y\mathrm{d}\tau, \tag{5.36}$$

$$\partial_k w_2(x,t) = -\int_0^t \int_{\mathbb{R}^3} \partial_k \Gamma(x-y,t-\tau)\cdot H_c(y,\tau)\,\mathrm{d}y\mathrm{d}\tau, \tag{5.37}$$

$$\partial_k w_3(x,t) = \partial_k\bigg[(4\pi t)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathrm{e}^{-|x-y+\mathcal{R}\,t\,\mathrm{e}_1|^2/4t}\; u(y)\,\mathrm{d}y\bigg], \tag{5.38}$$

$$\partial_k w_4(x,t) = \mathcal{R} \int_0^t \int_{\mathbb{R}^3} \partial_k \Gamma(x-y,t-\tau)\cdot \partial_1 \Sigma(y,\tau)\,\mathrm{d}y\mathrm{d}\tau. \tag{5.39}$$

As in the proof of Theorem 5.1, we fix, for simplicity $\mathcal{R} = \mathcal{T} = 1$. We shall give estimates for $\partial_k w_1$, $\partial_k w_2$, $\partial_k w_3$, and $\partial_k w_4$. Starting with $\partial_k w_2$, we choose $a > 0$ so that $\operatorname{supp}(F_c) \subset \mathrm{B}_a$, and deduce from (5.37), (3.4), and the properties of F_c that $(k = 1, 2, 3)$

$$|\partial_k w_2(x,t)| \le c_3 \int\limits_{\mathrm{B}_a} |x-y|^{-3/2} \big(1 + s(x-y)\big)^{-3/2} \mathrm{d}y. \tag{5.40}$$

For $x \in \mathbb{R}^3 \setminus \mathrm{B}_{2a}$ and $y \in \mathrm{B}_a$ we have, on one hand, $|x-y| \ge |x|/2$, and, on the other hand, $1 + s(x) \le 1 + s(x-y) + s(y) \le c_4(1 + s(x-y))$, from which we infer

$$|\partial_k w_2(x,t)| \le c_5 \, |x|^{-3/2} \big(1 + s(x)\big)^{-3/2}, \quad \forall |x| > 2a. \tag{5.41}$$

To estimate $\partial_k w_3$, we use inequality (3.23) with $r = \infty$, $q = 6$, and $|\alpha| = 1$ and deduce

$$\|\partial_k w_3(\cdot,t)\|_\infty \le c_6 \, t^{-\frac{3}{4}} \|u\|_6. \tag{5.42}$$

Concerning $\partial_k w_4$, we recall that $|\partial_1 \Sigma(y,t)| \le c_7 (1 + |y|)^{-3}$, $y \in \mathbb{R}^3$, with c_7 independent of $t \ge 0$, and deduce, utilizing (3.4), [11, Lemma II.7.2], and [5, Lemma 3.1] as in the estimate for w_4 in the proof of Theorem 5.1, that, for sufficiently large $|x|$,

$$\begin{aligned} |\partial_k w_4(x,t)| \le c_8 \Bigg(& \int\limits_{\{|x-y| < \frac{1}{4}\}} |x-y|^{-2} |y|^{-3+\varepsilon} \mathrm{d}y \\ & + \int\limits_{\{|x-y| \ge \frac{1}{4}\}} |x-y|^{-3/2} \big(1 + s(x-y)\big)^{-3/2} (1 + |y|)^{-3} \mathrm{d}y \Bigg) \\ \le c_9 \, & |x|^{-2+\varepsilon}, \end{aligned} \tag{5.43}$$

for all $\varepsilon > 0$ and $c_9 = c_9(\varepsilon)$. It remains to estimate $\partial_k w_1$. To this end, we split the integral in (5.36) into two parts:

$$\begin{aligned} \partial_k w_{1i}(x,t) = & \int\limits_0^t \int\limits_{\mathbb{R}^3 \setminus \mathrm{B}_{\frac{1}{4}}(x)} \partial_k \Gamma_{ij}(x-y, t-\tau) \, \partial_l \big[V_j(y,\tau) \, V_l(y,\tau)\big] \, \mathrm{d}y \mathrm{d}\tau \\ & + \int\limits_0^t \int\limits_{\mathrm{B}_{\frac{1}{4}}(x)} \partial_k \Gamma_{ij}(x-y, t-\tau) \, \partial_l \big[V_j(y,\tau) \, V_l(y,\tau)\big] \, \mathrm{d}y \mathrm{d}\tau \\ =: & \, I_1 + I_2. \end{aligned} \tag{5.44}$$

After a partial integration, we utilize (3.5) and (5.35) to obtain

$$
\begin{aligned}
|I_1| &\leq c_{10} \int\limits_{\mathbb{R}^3 \setminus \mathrm{B}_{\frac{1}{4}}(x)} |x-y|^{-2} \big(1+s(x-y)\big)^{-2} (1+|y|)^{-2} \,\mathrm{d}y\mathrm{d}\tau \\
&\quad + c_{11} \int\limits_{\partial\, \mathrm{B}_{\frac{1}{4}}(x)} |x-y|^{-3/2} \big(1+s(x-y)\big)^{-3/2} (1+|y|)^{-2} \,\mathrm{d}y \\
&=: I_{11} + I_{12}.
\end{aligned} \tag{5.45}
$$

Applying again [5, Lemma 3.1] as in the proof of Theorem 5.1, we find that

$$
|I_{11}| \leq c_{12} \, |x|^{-2+\varepsilon} \tag{5.46}
$$

for all $\varepsilon > 0$ and $c_{12} = c_{12}(\varepsilon)$. Clearly,

$$
|I_{12}| \leq c_{13} |x|^{-2}. \tag{5.47}
$$

Since $\operatorname{div}(V(y,t)) = 0$ for sufficiently large $|y|$, we observe that

$$
I_2 = \int\limits_0^t \int\limits_{\mathrm{B}_{\frac{1}{4}}(x)} \partial_k \Gamma_{ij}(x-y, t-\tau)\, \partial_l V_j(y,\tau)\, V_l(y,\tau)\,\mathrm{d}y\mathrm{d}\tau \tag{5.48}
$$

for sufficiently large $|x|$. Using (3.4) along with (5.34) and (5.35), we conclude

$$
|I_2| \leq c_{14}\, |x|^{-1} \int\limits_{\mathrm{B}_{\frac{1}{4}}(x)} |x-y|^{-2}\,\mathrm{d}y \leq c_{15}\, |x|^{-1} \tag{5.49}
$$

for sufficiently large $|x|$. Collecting (5.41), (5.42), (5.43), (5.44), (5.45), (5.46), (5.47), (5.49), we infer $(k = 1,2,3)$

$$
|\partial_k w(x,t)| \leq c_{16} \Big(|x|^{-1} + |x|^{-3/2} \big(1+s(x)\big)^{-3/2} + |x|^{-2+\varepsilon} + t^{-\frac{3}{4}} \|u\|_6 \Big) \tag{5.50}
$$

for all $\varepsilon > 0$ and sufficiently large $|x|$. Therefore, we get

$$
\begin{aligned}
|\nabla v(x)| &\leq |\nabla u(x)| + |\nabla \sigma(x)| \\
&\leq |\nabla w\big(Q(t)x,t\big)| + c_{17}|x|^{-3} \\
&\leq c_{18} \Big(|x|^{-1} + |x|^{-3/2} \big(1+s(x)\big)^{-3/2} + |x|^{-2+\varepsilon} + t^{-\frac{3}{4}} \|u\|_6 \Big),
\end{aligned}
$$

and thus, by letting $t \to \infty$,

$$
|\nabla v(x)| \leq c_{18} \Big(|x|^{-1} + |x|^{-3/2} \big(1+s(x)\big)^{-3/2} + |x|^{-2+\varepsilon} \Big) \tag{5.51}
$$

for large $|x|$ and arbitrary $\varepsilon > 0$. This latter estimate will furnish an improved estimate for ∇V, which then leads to the improved estimate $|I_2| \le c_{19}|x|^{-2}$. Hence we can improve (5.51) by replacing the term $|x|^{-1}$ with $|x|^{-2}$, which completes the proof. □

Our next result concerns the point-wise asymptotic behavior of the pressure.

Theorem 5.3. *Under the same assumption of Theorem 5.1, there is a constant p_0 so that the pressure p satisfies, for all sufficiently large $|x|$,*

$$p(x) - p_0 = O(|x|^{-2} \ln |x|).$$

Proof. If we apply the div operator on both sides of (5.1), on one hand, and, on the other hand, we evaluate the normal derivative of p at $\partial \mathrm{B}_\rho \equiv \partial\Omega^\rho$, we find

$$\begin{cases} \Delta p = \operatorname{div}(G) & \text{in } \Omega^\rho, \\ \dfrac{\partial p}{\partial n} = g & \text{on } \partial\Omega^\rho, \end{cases} \tag{5.52}$$

with

$$\begin{aligned} &G := \mathcal{R} v \cdot \nabla v, \text{ and} \\ &g := \Big(\Delta v + \mathcal{R}\big(\partial_1 v - v\cdot\nabla v\big) + \mathcal{T}\big(\mathrm{e}_1 \wedge x \cdot \nabla v - \mathrm{e}_1 \wedge v\big)\Big) \cdot n\big|_{\partial\Omega^\rho}, \end{aligned} \tag{5.53}$$

where we used that $\operatorname{div}(v) = 0$ along with the fact that $\operatorname{div}\big(\mathrm{e}_1 \wedge x \cdot \nabla v - \mathrm{e}_1 \wedge v\big) = 0$. Denoting by $\mathcal{E} = \mathcal{E}(\xi)$ the fundamental solution to Laplace's equation, from (5.52) we find, after a simple integration by parts, for all $x \in \mathrm{B}_{r,\rho}$, $r > \rho$,

$$\begin{aligned} p(x) = &- \int_{\mathrm{B}_{r,\rho}} G(y) \cdot \nabla\mathcal{E}(x-y)\,\mathrm{d}y + \int_{\partial \mathrm{B}_{r,\rho}} \mathcal{E}(x-y)\, G(y)\cdot n \,\mathrm{d}S(y) \\ &- \int_{\partial \mathrm{B}_\rho} \mathcal{E}(x-y)\, g(y)\,\mathrm{d}S(y) - \int_{\partial \mathrm{B}_r} \mathcal{E}(x-y)\,\frac{\partial p}{\partial n}(y)\,\mathrm{d}S(y) \\ &+ \int_{\partial \mathrm{B}_\rho} \frac{\partial \mathcal{E}}{\partial n}(x-y)\,p(y)\,\mathrm{d}S(y) + \int_{\partial \mathrm{B}_r} \frac{\partial \mathcal{E}}{\partial n}(x-y)\,p(y)\,\mathrm{d}S(y). \end{aligned} \tag{5.54}$$

Using that

$$|\mathrm{D}^\alpha \mathcal{E}(\xi)| \le c_1\, |\xi|^{-1-|\alpha|} \quad \text{for } |\alpha| \ge 0\,,\ \xi \in \mathbb{R}^3, \tag{5.55}$$

and recalling (5.2), we readily show, for fixed $x \in \Omega^\rho$, the existence of an unbounded sequence $\{r_n\}_{n=1}^\infty \subset \mathbb{R}_+$ so that (after possibly adding a constant to p)

$$\lim_{r_n\to\infty} \int_{\partial \mathrm{B}_{r_n}} \mathcal{E}(x-y)\,\big(G(y)\cdot n - \frac{\partial p}{\partial n}(y)\big) + \frac{\partial \mathcal{E}}{\partial n}(x-y)\,p(y)\,\mathrm{d}S(y) = 0.$$

Consequently, (5.54) furnishes, for $x \in \Omega^\rho$, the representation

$$p(x) = -\int_{\Omega^\rho} G(y) \cdot \nabla \mathcal{E}(x-y)\,\mathrm{d}y + \int_{\partial\,\mathrm{B}_\rho} \mathcal{E}(x-y)\,G(y)\cdot n\,\mathrm{d}S(y)$$
$$- \int_{\partial\,\mathrm{B}_\rho} \mathcal{E}(x-y)\,g(y)\,\mathrm{d}S(y) + \int_{\partial\,\mathrm{B}_\rho} \frac{\partial \mathcal{E}}{\partial n}(x-y)\,p(y)\,\mathrm{d}S(y).$$

We next use, in this latter relation, the property (5.55) along with the mean value theorem to obtain

$$\begin{aligned} p(x) &= -\int_{\Omega^\rho} G(y) \cdot \nabla \mathcal{E}(x-y)\,\mathrm{d}y \\ &\quad + \int_{\partial\,\mathrm{B}_\rho} \big(G(y)\cdot n - g(y)\big)\,\mathrm{d}S(y) \cdot \mathcal{E}(x) + O\big(|x|^{-2}\big) \\ &=: \; -P(x) + m\,\mathcal{E}(x) + O\big(|x|^{-2}\big). \end{aligned} \tag{5.56}$$

We shall now show that

$$P(x) = O\big(|x|^{-2} \ln |x|\big). \tag{5.57}$$

To this end, we set $|x| = R\,(> 2\rho)$ and write

$$\begin{aligned} P(x) &= \int_{\mathrm{B}_{R/2,\rho}} G(y)\cdot \nabla \mathcal{E}(x-y)\,\mathrm{d}y + \int_{\mathrm{B}_{2R,R/2}} G(y)\cdot\nabla\mathcal{E}(x-y)\,dy \\ &\quad + \int_{\Omega^{2R}} G(y)\cdot\nabla\mathcal{E}(x-y)\,\mathrm{d}y \\ &=: P_1(x) + P_2(x) + P_3(x). \end{aligned}$$

In view of the summability properties (5.2), and by means of the Hölder inequality, it is easy to show that $G \in L^1(\Omega^\rho)$. This, combined with (5.55), allows one to readily prove that

$$P_1(x) + P_3(x) = O\big(|x|^{-2}\big). \tag{5.58}$$

We next observe that

$$\begin{aligned} P_2(x) &= -\int_{\mathrm{B}_{2R,\frac{R}{2}}\setminus \mathrm{B}_1(x)} \partial_l\partial_k \mathcal{E}(x-y)\,v_l(y)\,v_k(y)\,\mathrm{d}y \\ &\quad + \int_{\mathrm{B}_1(x)} \partial_k\mathcal{E}(x-y)\,v_l(y)\,\partial_l v_k(y)\,\mathrm{d}y \\ &\quad + \int_{\partial\left(\mathrm{B}_{2R,\frac{R}{2}}\setminus \mathrm{B}_1(x)\right)} \partial_k \mathcal{E}(x-y)\,v_l(y)\,v_k(y)\,n_l\,\mathrm{d}S(y) \\ &=: P_{21}(x) + P_{22}(x) + P_{23}(x). \end{aligned} \tag{5.59}$$

From Theorem 5.1, Theorem 5.2, and (5.55), we readily obtain

$$\begin{aligned} &P_{22}(x)+P_{23}(x)=O\big(|x|^{-2}\big),\\ &|P_{21}|\le c_2|x|^{-2}\int\limits_{\mathrm{B}_{3R,1}(x)}|x-y|^{-3}\,\mathrm{d}y=O\big(|x|^{-2}\ln|x|\big). \end{aligned}\tag{5.60}$$

Thus, collecting (5.58), (5.59), and (5.60), we obtain (5.57), which, in turn, once replaced in (5.56), delivers

$$p(x)=m\,\mathcal{E}(x)+O\big(|x|^{-2}\ln|x|\big).\tag{5.61}$$

However, by (5.2), $p\in L^{3/2-\varepsilon}(\mathrm{B}^a)$, for sufficiently large a and all $\varepsilon>0$. Thus, in equation (5.61) we must have $m=0$, and the proof of the theorem is completed. □

We end this section by establishing further properties of Leray solutions that are a simple and direct consequence of Lemma 4.4, Theorem 5.1 and of the results of [15]. We begin with the following.

Theorem 5.4 (Uniqueness)**.** *Let $\Omega\subset\mathbb{R}^3$ be an exterior domain of class C^2. Moreover, let $B_1,B_2>0$ be arbitrary constants and $\mathcal{R}\in(0,B_1)$, $\mathcal{T}\in(0,B_2)$. There is a constant $\varepsilon_1=\varepsilon_1(\Omega,B_1,B_2)>0$ so that if $f=\mathrm{div}(F)\in L^2(\Omega)^3$ with* $\mathrm{supp}(f)$ *bounded* [13] *and $v_*\in W^{\frac{3}{2},2}(\partial\Omega)$ satisfies*

$$\mathcal{R}\left(\operatorname{ess\,sup}\big[(1+|x|)^{-2}|F(x)|\big]\big)+\|f\|_2+\|v_*\|_{W^{\frac{3}{2},2}(\partial\Omega)}\right)\le\varepsilon_1,\tag{5.62}$$

then a solution (v,p) in the class

$$v\in D^{1,2}(\Omega)^3\cap L^6(\Omega)^3,\;\; p\in L^2_{loc}(\Omega)\tag{5.63}$$

to

$$\begin{cases}\Delta v-\nabla p+\mathcal{R}\big(\partial_1 v-v\cdot\nabla v\big)+\mathcal{T}\big(\mathrm{e}_1\wedge x\cdot\nabla v-\mathrm{e}_1\wedge v\big)=f & \text{in } \Omega,\\ \mathrm{div}(v)=0 & \text{in } \Omega,\\ v=v_* & \text{on } \partial\Omega,\end{cases}\tag{5.64}$$

is unique in the class (5.63).

Proof. Under the given assumptions, the existence of a "strong" solution (v,p) to (5.64) with, in particular, $\operatorname{ess\,sup}\big[(1+|x|)^{-1}|v(x)|\big]\big)$ bounded by the data was proved in [15, Theorem 1]. Using the decay properties established in Lemma 4.4 and Theorem 5.1 for a "weak" solution satisfying (5.63), it can be shown, by the same method as in [15, Theorem 1], that the "strong" and "weak" solution coincide. Thus, uniqueness of solutions in the class (5.63) follows. □

[13] This assumption can be fairly weakened, by requiring only that $f(x)$ decays to zero sufficiently fast for large $|x|$, as in [14, Theorem 1].

We also have the following result.

Theorem 5.5 (Energy Equation)**.** *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class C^2. Moreover, let $\mathcal{R}, \mathcal{T} > 0$, $1 < q < \frac{6}{5}$, $f \in L^q(\Omega)^3 \cap L^{\frac{2q}{3-q}}(\Omega)^3 \cap L^2_{loc}(\overline{\Omega})^3$, and $v_* \in W^{\frac{3}{2},2}(\partial\Omega)$. Then any solution (v,p) to (5.64) in the class (5.63) satisfies*

(5.65)
$$2\int_\Omega \mathbf{D}(v) : \mathbf{D}(v)\,\mathrm{d}x + \int_\Omega f \cdot v\,\mathrm{d}x = \int_{\partial\Omega} \big(\mathrm{T}(v,p)\cdot n\big)\cdot v_*\,\mathrm{d}S + \frac{1}{2}\int_{\partial\Omega} |v_*|^2 \big(\mathcal{R}\,\mathrm{e}_1 - v_* + \mathcal{T}\mathrm{e}_1 \wedge x\big)\cdot n\,\mathrm{d}S,$$

where $\mathrm{T}(v,p) := \mathbf{D}(v) - pI$ and $\mathbf{D}(v) := \frac{1}{2}\big(\nabla v + \nabla v^T\big)$.

Proof. Let $\Phi \in C^\infty(\mathbb{R};\mathbb{R})$ be a "cut-off" function with $\Phi(r) = 1$ for $|r| < 1$ and $\Phi(r) = 0$ for $|r| > 2$. Define $\varphi_\rho \in C^\infty(\mathbb{R}^3;\mathbb{R})$ by $\varphi_\rho(x) := \Phi\left(\frac{|x|}{\rho}\right)$. Note that, by elliptic regularity theory, $v \in W^{2,2}_{loc}(\overline{\Omega})$ and $p \in W^{1,2}_{loc}(\overline{\Omega})$. Thus, multiplying $(5.64)_1$ with $\varphi_\rho v$ and integrating over Ω, we obtain, for ρ sufficiently large,

$$\begin{aligned}&\int_{\partial\Omega} (\nabla v \cdot n)\cdot v - p\,v\cdot n\,\mathrm{d}S - \int_\Omega (\nabla v : \nabla v)\,\varphi_\rho + (\nabla v \cdot \nabla\varphi_\rho)\cdot v\,\mathrm{d}x \\ &\quad - \int_\Omega p\,(\nabla\varphi_\rho \cdot v)\,\mathrm{d}x + \mathcal{R}\Big(\int_\Omega \partial_1 v\cdot(\varphi_\rho v) - (v\cdot\nabla v)\cdot(\varphi_\rho v)\,\mathrm{d}x\Big) \\ &\quad + \mathcal{T}\Big(\int_\Omega (\mathrm{e}_1 \wedge x \cdot \nabla v)\cdot(\varphi_\rho v)\,\mathrm{d}x\Big) = \int_\Omega \varphi_\rho f\cdot v\,\mathrm{d}x.\end{aligned} \tag{5.66}$$

We next observe that

$$2\int_\Omega \partial_1 v\cdot(\varphi_\rho v)\,\mathrm{d}x = \int_{\partial\Omega} |v|^2\,\mathrm{e}_1\cdot n\,\mathrm{d}S - \int_\Omega |v|^2\,\partial_1\varphi_\rho\,dx. \tag{5.67}$$

Using Lemma 4.4 (note that $f \in L^q(\Omega) \cap L^{\frac{2q}{3-q}}(\Omega)$ implies $f \in L^{\frac{3}{2}}(\Omega)$ when $1 < q < \frac{6}{5}$), we have $(v,p) \in \mathrm{X}_q(\Omega^\rho)$. Consequently, by Hölder's inequality,

$$\int_\Omega |v|^2\,|\partial_1\varphi_\rho|\,dx \leq c_1\,\|v\|_{\frac{2q}{2-q},\Omega^\rho}\Big(\int_{\mathrm{B}_{2\rho,\rho}} |x|^{-r}\,\mathrm{d}x\Big)^{\frac{1}{r}},$$

where $r > 3$ (since $q < \frac{6}{5}$) and c_1 independent of ρ. Thus, letting $\rho \to \infty$ in (5.67) we see that

$$\lim_{\rho\to\infty} 2\int_\Omega \partial_1 v\cdot(\varphi_\rho v)\,\mathrm{d}x = \int_{\partial\Omega} |v|^2\,\mathrm{e}_1\cdot n\,\mathrm{d}S.$$

Similarly, we prove that

$$\lim_{\rho\to\infty} 2\int\limits_{\Omega}(v\cdot\nabla v)\cdot(\varphi_\rho v)\,\mathrm{d}x = \int\limits_{\partial\Omega}|v|^2\,v\cdot n\,\mathrm{d}x,$$

$$\lim_{\rho\to\infty}\int\limits_{\Omega}(\nabla v\cdot\nabla\varphi_\rho)\cdot v\,\mathrm{d}x = 0,\ \text{and}$$

$$\lim_{\rho\to\infty}\int\limits_{\Omega}p\,(\nabla\varphi_\rho\cdot v)\,\mathrm{d}x = 0.$$

Finally, we note that

$$2\int\limits_{\Omega}(\mathrm{e}_1\wedge x\cdot\nabla v)\cdot(\varphi_\rho v)\,\mathrm{d}x = \int\limits_{\partial\Omega}|v|^2(\mathrm{e}_1\wedge x)\cdot n\,\mathrm{d}S - \int\limits_{\Omega}|v|^2\,(\mathrm{e}_1\wedge x)\cdot(\nabla\varphi_\rho)\mathrm{d}x$$
$$= \int\limits_{\partial\Omega}|v|^2\,(\mathrm{e}_1\wedge x)\cdot n\,\mathrm{d}S,$$

since $(\mathrm{e}_1\wedge x)\cdot(\nabla\varphi_\rho) = (\mathrm{e}_1\wedge x)\cdot\big(\frac{x}{\rho\,|x|}\,\Phi'(\frac{|x|}{\rho})\big) = 0$. We may now conclude (5.65) by letting $\rho\to\infty$ in (5.66) and observing that

$$\int\limits_{\Omega}\nabla v:\nabla v^T\,\mathrm{d}x = \int\limits_{\partial\Omega}(\nabla v^T\cdot n)\cdot v\,\mathrm{d}S.$$

□

Acknowledgements: Most of this work was done while G.P. Galdi was tenured with a DFG Mercator Professorship at the Institute of Mathematics of the RWTH Aachen. He would like to thank Professor Josef Bemelmans for his kind invitation and warmest hospitality. His work was also partially supported by NSF Grant DMS-0707281. Finally, both authors would like to thank Professor Bemelmans for very helpful conversations.

References

[1] K.I. Babenko. On stationary solutions of the problem of flow past a body of a viscous incompressible fluid. *Math. USSR, Sb.*, 20:1–25, 1973. 1, 3

[2] M. E. Bogovskiĭ. Solutions of some problems of vector analysis, associated with the operators div and grad. In *Theory of cubature formulas and the application of functional analysis to problems of mathematical physics*, volume 1980 of *Trudy Sem. S. L. Soboleva, No. 1*, pages 5–40, 149. Akad. Nauk SSSR Sibirsk. Otdel. Inst. Mat., Novosibirsk, 1980. 13

[3] Wolfgang Borchers. Zur Stabilität und Faktorisierungsmethode für die Navier-Stokes Gleichungen inkompressibler viskoser Flüssigkeiten. Habilitationsschrift, Universität Paderborn., 1992. 3

[4] Deuring, Paul and Kračmar, Stanislav and Nečasová, Šárka. On pointwise decay of linearized stationary incompressible viscous flow around rotating and translating bodies. Preprint, 2009. 3

[5] Reinhard Farwig. The stationary exterior 3D-problem of Oseen and Navier-Stokes equations in anisotropically weighted spaces. *Math. Z.*, 211(3):409–448, 1992. 3, 27, 28, 30, 31

[6] Reinhard Farwig. An L^q-analysis of viscous fluid flow past a rotating obstacle. *Tohoku Math. J. (2)*, 58(1):129–147, 2006. 3, 5, 16, 17, 18, 29

[7] Reinhard Farwig, Toshiaki Hishida, and Detlef Muller. L^q-theory of a singular winding integral operator arising from fluid dynamics. *Pac. J. Math.*, 215(2):297–312, 2004. 3

[8] Reinhard Farwig and Hermann Sohr. Weighted estimates for the Oseen equations and the Navier-Stokes equations in exterior domains. Heywood, J. G. (ed.) et al., Theory of the Navier-Stokes equations. Proceedings of the third international conference on the Navier-Stokes equations: theory and numerical methods, Oberwolfach, Germany, June 5-11, 1994. Singapore: World Scientific. Ser. Adv. Math. Appl. Sci. 47, 11-30 (1998)., 1998. 3

[9] R. Finn. On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems. *Arch. Ration. Mech. Anal.*, 19:363–406, 1965. 2, 3

[10] Giovanni P. Galdi. On the asymptotic structure of D-solutions to steady Navier-Stokes equations in exterior domains. Galdi, Giovanni Paolo (ed.), Mathematical problems relating to the Navier-Stokes equation. River Edge, NJ: World Scientific Publishing Co. Ser. Adv. Math. Appl. Sci. 11, 81-104 (1992)., 1992. 3

[11] Giovanni P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I: Linearized steady problems.* Springer Tracts in Natural Philosophy. 38. New York: Springer-Verlag., 1994. 3, 12, 13, 18, 19, 21, 22, 26, 30

[12] Giovanni P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. II: Nonlinear steady problems.* Springer Tracts in Natural Philosophy. 39. New York: Springer-Verlag., 1994. 5, 21

[13] Giovanni P. Galdi. On the motion of a rigid body in a viscous liquid: A mathematical analysis with applications. Friedlander, S. (ed.) et al., Handbook of mathematical fluid dynamics. Vol. 1. Amsterdam: Elsevier. 653-791 (2002)., 2002. 2, 3

[14] Giovanni P. Galdi and Ana L. Silvestre. Further results on steady-state flow of a Navier-Stokes liquid around a rigid body. Existence of the wake. In *Kyoto Conference on the Navier-Stokes Equations and their Applications*, RIMS Kôkyûroku Bessatsu, B1, pages 127–143. Res. Inst. Math. Sci. (RIMS), Kyoto, 2007. 3, 5, 34

[15] Giovanni P. Galdi and Ana L. Silvestre. The steady motion of a Navier-Stokes liquid around a rigid body. *Arch. Ration. Mech. Anal.*, 184(3):371–400, 2007. 3, 5, 8, 34

[16] G.P. Galdi. Steady Flow of a Navier-Stokes Fluid around a Rotating Obstacle. *Journal of Elasticity*, 71:1–31, 2003. 3, 5

[17] Toshiaki Hishida. Steady motions of the Navier-Stokes fluid around a rotating body. In *Asymptotic analysis and singularities—hyperbolic and dispersive PDEs and fluid mechanics*, volume 47 of *Adv. Stud. Pure Math.*, pages 117–136. Math. Soc. Japan, Tokyo, 2007. 3

[18] Toshiaki Hishida and Yoshihiro Shibata. Decay estimates of the Stokes flow around a rotating obstacle. In *Kyoto Conference on the Navier-Stokes Equations and their Applications*, RIMS Kôkyûroku Bessatsu, B1, pages 167–186. Res. Inst. Math. Sci. (RIMS), Kyoto, 2007. 3

[19] Toshiaki Hishida and Yoshihiro Shibata. $L_p - L_q$ estimate of the Stokes operator and Navier-Stokes flows in the exterior of a rotating obstacle. *Arch. Ration. Mech. Anal.*, 193(2):339–421, 2009. 3

[20] Stanislav Kračmar, Šárka Nečasová, and Patrick Penel. L^q-approach to weak solutions of the Oseen flow around a rotating body. Rencławowicz, Joanna (ed.) et al., Parabolic and Navier-Stokes equations. Part 1. Proceedings of the confererence, Będlewo, Poland, September 10–17, 2006. Warsaw: Polish Academy of Sciences, Institute of Mathematics. Banach Center Publications 81, Pt. 1, 259-276 (2008)., 2008. 3

[21] Jean Leray. Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique. *J. Math. Pures Appl.*, 12:1–82, 1933. 2

[22] Ryûichi Mizumachi. On the asymptotic behavior of incompressible viscous fluid motions past bodies. *J. Math. Soc. Japan*, 36:497–522, 1984. 9

[23] C.W. Oseen. *Hydrodynamik*. Akademische Verlagsgesellschaft M.B.H., Leipzig, 1927. 7, 8

[24] V.A. Solonnikov. Estimates of the solutions of a nonstationary linearized system of Navier-Stokes equations. *Am. Math. Soc., Transl., II. Ser.*, 75:1–116, 1968. 8, 10

[25] H.F. Weinberger. On the steady fall of a body in a Navier-Stokes fluid. Partial diff. Equ., Berkeley 1971, Proc. Sympos. Pure Math. 23, 421-439 (1973)., 1973. 3