Asymptotic Behavior of a Leray Solution around a Rotating Obstacle

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Abstract

We consider a body, $\mathfrak{B}$, that rotates, without translating, in a Navier-Stokes liquid that fills the whole space exterior to $\mathfrak{B}$. We analyze asymptotic properties of steady-state motions, that is, time-independent solutions to the equation of motion written in a frame attached to the body. We prove that "weak" steady-state solutions in the sense of J. Leray that satisfy the energy inequality are Physically Reasonable in the sense of R. Finn, provided the "size" of the data is suitably restricted

1 Introduction

Consider a rigid body, $\mathfrak{B}$, whose particles move with prescribed (Eulerian) velocity $\omega \times x$ in a Navier-Stokes liquid. Here, $\omega \in \mathbb{R}^3$, $\omega \neq 0$, and x is the spatial variable. It is well known that a prescribed velocity field of this form corresponds to a uniform rotation of $\mathfrak{B}$ with angular velocity ω.

We assume the liquid fills the whole exterior of $\mathfrak{B}$. More precisely, we assume that, at each time t, $\mathfrak{B}$ occupies a compact set of $\mathbb{R}^3$ with a connected boundary, so that, at each time t, the liquid fills an exterior domain, $\mathfrak{D} = \mathfrak{D}(t)$, of $\mathbb{R}^3$. As customary in this problem, it is convenient to refer the motion of the liquid to a

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frame, $\mathfrak{S}$, attached to $\mathfrak{B}$. In this way, the region occupied by the liquid becomes a time-independent domain, Ω, of $\mathbb{R}^3$. We shall suppose that, with respect to $\mathfrak{S}$, the motion of the liquid is steady and that it reduces to rest at large spatial distances. Thus, the equations governing the motion of the liquid in $\mathfrak{S}$ can be written in the following non-dimensional form (see, *e.g.* [8])

$$
\left\{
\begin{aligned}
\Delta v - \nabla p - \mathrm{Re}\, v\cdot\nabla v + \mathrm{Ta}\left(\mathrm{e}_1\times x\cdot\nabla v - \mathrm{e}_1\times v\right) &= f \quad &&\text{in } \Omega,\\
\operatorname{div} v &= 0 \quad &&\text{in } \Omega,\\
v &= v_* \quad &&\text{on } \partial\Omega,
\end{aligned}
\right.
\tag{1.1}
$$

with

$$
\lim_{|x|\to\infty} v = 0. \tag{1.2}
$$

Here, v and p are velocity and pressure fields of the liquid in $\mathfrak{S}$, while f and v_* are prescribed functions of x. The Reynolds number Re and Taylor number Ta are dimensionless constants with $\mathrm{Re}\,, \mathrm{Ta}\, > 0$.

Mostly over the past decade, the study of the properties of solutions to (1.1), (1.2) has attracted the attention of many mathematicians, who have investigated basic issues like existence, uniqueness and asymptotic (in space) behavior; see, *e.g.* [2, 4, 3, 9, 10, 11, 12, 13, 14] and the literature cited therein.

We wish to recall and to emphasize that the characteristic difficulty related to the investigation of (1.1), (1.2) is the presence of the term $\omega\times x\cdot\nabla v$, whose coefficient becomes unbounded as $|x|\to\infty$. For this reason, the above problem can *not* be treated as a "perturbation" to the analogous one with $\omega = 0$, even for "small" $|\omega|$.

Concerning the *existence* of solutions, there are, basically, two types of results.

On one hand, one can show that, for any f and v_* in a suitable (and quite large) class with $\int_{\partial\Omega} v_*\cdot n = 0$, there corresponds a pair (v, p), such that

$$
v\in L^6(\Omega), \quad \nabla v\in L^2(\Omega)\,, \tag{1.3}
$$

and $p\in L^2_{loc}(\Omega)$ satisfying (1.1) in the sense of distribution, and (1.2) in an appropriate generalized sense; see [1]. In addition, v and p obey the energy *inequality*:

$$
\begin{aligned}
2\int_\Omega |\mathbf{D}(v)|^2\,\mathrm{d}x \le -\int_\Omega f\cdot v\,\mathrm{d}x + \int_{\partial\Omega}\big(\mathrm{T}(v,p)\cdot n\big)\cdot v_*\,\mathrm{d}S\\
-\frac{\mathrm{Re}}{2}\int_{\partial\Omega}|v_*|^2\, v_*\cdot n\,\mathrm{d}S + \frac{\mathrm{Ta}}{2}\int_{\partial\Omega}|v_*|^2\;\mathrm{e}_1\times x\cdot n\,\mathrm{d}S\,,
\end{aligned}
\tag{1.4}
$$

where $\mathrm{T}(v,p)$ and $\mathbf{D}(v)$ are the Cauchy stress and stretching tensor, respectively; see (2.1). Finally, if Ω and the data are sufficiently smooth, then v and p are likewise smooth and satisfy both (1.1) and (1.2) in the ordinary sense; see [8]. This type of solution is usually called *Leray solution*, in that they were first found by J. Leray in the case $\omega = 0$; see [15]. It must be emphasized that a Leray solution carries very little information about the behavior of v as $|x| \to \infty$, namely, (1.3), while no information at all is available for the pressure field p. It is just for this reason that in (1.4) there appears an inequality sign (instead of an equality sign) that may cast shadows about the physical meaning of Leray solution.

On the other hand, if f is sufficiently smooth and decays sufficiently fast as $|x| \to \infty$, and provided the size of the data is suitably restricted, one can show the existence of a solution (v, p) with a suitable asymptotic behavior that, in fact, verifies the energy *equality*

$$
\begin{aligned}
2\int_\Omega |\mathbf{D}(v)|^2\,\mathrm{d}x = &-\int_\Omega f\cdot v\,\mathrm{d}x + \int_{\partial\Omega}\big(\mathrm{T}(v,p)\cdot n\big)\cdot v_*\,\mathrm{d}S \\
&- \frac{\mathrm{Re}}{2}\int_{\partial\Omega}|v_*|^2\,v_*\cdot n\,\mathrm{d}S + \frac{\mathrm{Ta}}{2}\int_{\partial\Omega}|v_*|^2\,\mathrm{e}_1\times x\cdot n\,\mathrm{d}S\,,
\end{aligned}
\tag{1.5}
$$

see [10, 3]. In particular, in [10] it is shown the existence of a solution that (besides satisfying (1.5)) decays like the Stokes fundamental solution as $|x| \to \infty$, namely,

$$
\begin{aligned}
v(x) = O\big(|x|^{-1}\big)\,, \quad \nabla v(x) = O\big(|x|^{-2}\big)\,,\\
p(x) = O\big(|x|^{-2}\big)\,, \quad \nabla p(x) = O\big(|x|^{-3}\big)\,.
\end{aligned}
\tag{1.6}
$$

Keeping the nomenclature introduced by R. Finn [5] for the case $\omega = 0$, solutions possessing this type of properties are called *Physically Reasonable*.

Now, while it is quite obvious that a Physically Reasonable solution is also a Leray solution, the converse is by no means obvious, even in the case of small data.

Objective of this paper is to prove that every Leray solution corresponding to data of restricted size, with f decaying sufficiently fast at large distances, is Physically Reasonable; see Theorem 4.1. The proof of this theorem exploits the method introduced in [6] for the case $\omega = 0$, and it is based on a uniqueness argument. Precisely, we shall show that a Physically Reasonable solution is *unique* (for small data) in the class of Leray solutions (see Lemma 3.3), so that the desired result follows from the existence result proved in [10]. However, for this argument to work, it is crucial to show that the pressure, p, associated to a Leray solution possesses the summability property $p \in L^3(\Omega)$. Now, while in the case $\omega = 0$ the proof of this property is quite straightforward [6], in the case at hand the proof is far from being obvious, due to the presence of the term $\omega \times x \cdot \nabla v$. Actually, it requires a detailed analysis that we develop through Lemma 3.1 and 3.2.

The plan of the paper is the following. After recalling some standard notation in Section 2, in Section 3 we begin to establish appropriate global summability property for the pressure of a Leray solution. Successively, using also this property, we show the uniqueness of a Physically Reasonable solution corresponding to "small" data in the class of Leray solutions. Finally, in Section 4, as a corollary to this latter result and with the help of the existence theorem established in [10], we prove that every Leray solution corresponding to "small" data is, in fact, Physically Reasonable.

2 Notation

We let $L^q(\Omega)$ and $W^{m,q}(\Omega)$ denote Lebesgue and Sobolev spaces, respectively, and $\|\cdot\|_q$, $\|\cdot\|_{m,q}$ the associated norms. We write $D^{m,q}(\Omega)$ and $|\cdot|_{m,q}$ to denote homogeneous Sobolev spaces and their (semi-)norms, respectively. We will initially explicitly indicate when a function space consists of vector- or tensor-valued functions, for example $L^q(\Omega)^3$, but may omit the indication when no confusion can arise.

We will make use of the weighted norms

$$[\![f]\!]_{\alpha,A} := \operatorname*{ess\,sup}_{x\in A} \left[(1+|x|^{\alpha})|f(x)|\right]$$

for A a domain of $\mathbb{R}^3$, and $f : A \to \mathbb{R}^3$ measurable and $\alpha \in \mathbb{N}$. If no confusion arises, we will omit the subscript "A".

We denote by

$$\mathrm{T}(v,p) := 2\mathbf{D}(v) - pI, \quad \mathbf{D}(v) := \frac{1}{2}\big(\nabla v + \nabla v^T\big) \tag{2.1}$$

the usual Cauchy stress and stretching tensors, respectively, of a Navier-Stokes liquid corresponding to the non-dimensional form of the equations (1.1).

In what follows, $\Omega \subset \mathbb{R}^3$ will denote an exterior domain of class C^2. Without loss of generality, we assume $0 \in \mathbb{R}^3 \setminus \overline{\Omega}$. For $\rho > 0$, we put $\mathrm{B}_\rho := \{x \in \mathbb{R}^3 \mid |x| < \rho\}$, $\mathrm{B}^\rho := \{x \in \mathbb{R}^3 \mid |x| \geq \rho\}$, and set $\Omega_\rho := \Omega \cap \mathrm{B}_\rho$ and $\Omega^\rho := \Omega \cap \mathrm{B}^\rho$. Moreover, we put $\mathrm{B}_{\rho_2,\rho_1} := \mathrm{B}_{\rho_2} \setminus \mathrm{B}_{\rho_1}$.

As noted in the introduction, Re and Ta are positive real constants.

We use small letters for constants $(c_1, c_2, \ldots)$ that appear only in a single proof, and capital letters $(C_1, C_2, \ldots)$ for global constants.

3 Preliminaries

In this section, we will establish, in a series of preliminary lemmas, some properties of weak solutions to (1.1).

We start by recalling the well-known inequality

$$\|v\|_6 \leq C_1 |v|_{1,2} \tag{3.1}$$

which holds for all $v \in D^{1,2}(\Omega) \cap L^6(\Omega)$ (see [7, Theorem II.5.1]). We shall frequently use (3.1) without reference.

In the first lemma, we establish (global) higher order regularity of a weak solution.

Lemma 3.1. *Let $f \in L^2(\Omega)^3$, $v_* \in W^{\frac{3}{2},2}(\partial\Omega)^3$, and $(v,p) \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3 \times L^2_{loc}(\Omega)$ be a solution to (1.1). Then $v \in D^{2,2}(\Omega)$.*

Proof. By standard regularity theory for elliptic systems, $v \in W^{2,2}_{loc}(\overline{\Omega})$ and $p \in W^{1,2}_{loc}(\overline{\Omega})$. We therefore only need to show $v \in D^{2,2}(\Omega^\rho)$ for some $\rho > 0$.

Choose $r > 0$ so that $\mathbb{R}^3 \setminus \Omega \subset \mathrm{B}_r$. Moreover, choose for any $R > 2r$ a function $\psi_R \in C^\infty(\mathbb{R}^3;\mathbb{R})$ with $0 \leq \psi_R \leq 1$, $\psi_R = 0$ in B_r, $\psi_R = 1$ in $\mathrm{B}_{R,2r}$, $\psi_R = 0$ in B^{2R}, and $|\mathrm{D}^\alpha \psi_R| \leq \frac{c_1}{|x|^{|\alpha|}}$ with c_1 independent of R.

We shall test $(1.1)_1$ with $-\nabla \times (\psi_R^2 \nabla \times v)$. Note that $-\nabla \times (\psi_R^2 \nabla \times v) \in L^2(\mathbb{R}^3)$, has bounded support,

$$\operatorname{div}\big[-\nabla \times (\psi_R^2 \nabla \times v)\big] = 0, \tag{3.2}$$

and

$$-\nabla \times (\psi_R^2 \nabla \times v) = \psi_R^2 \Delta v + (\nabla \times v) \times \nabla[\psi_R^2]. \tag{3.3}$$

Thus, we compute

$$\begin{aligned}
&\Big|\int_\Omega (\mathrm{e}_1 \times v) \cdot \big(-\nabla \times (\psi_R^2 \nabla \times v)\big)\,\mathrm{d}x\Big| \\
&\qquad = \Big|\int_\Omega \psi_R^2 (\mathrm{e}_1 \times v) \cdot (\nabla \times (\nabla \times v)) + (\mathrm{e}_1 \times v) \cdot \big((\nabla \times v) \times \nabla[\psi_R^2]\big)\,\mathrm{d}x\Big| \\
&\qquad = \Big|\int_\Omega -(\nabla \times \psi_R^2 (\mathrm{e}_1 \times v)) \cdot (\nabla \times v) + \nabla[\psi_R^2] \cdot \big((\mathrm{e}_1 \times v) \times (\nabla \times v)\big)\,\mathrm{d}x\Big| \\
&\qquad \leq c_2 \bigg(\int_\Omega |\nabla v|^2\,\mathrm{d}x + \int_{\mathrm{B}_{2R,R}} \frac{1}{R} |v||\nabla v|\,\mathrm{d}x + \int_{\mathrm{B}_{2r,r}} \frac{1}{r} |v||\nabla v|\,\mathrm{d}x \bigg) \\
&\qquad \leq c_3 \bigg(\int_\Omega |\nabla v|^2\,\mathrm{d}x + \|v\|_6 \|\nabla v\|_2 \bigg) \leq c_4\, |v|_{1,2}^2.
\end{aligned} \tag{3.4}$$

Furthermore, we have

$$\begin{aligned}
&\int_\Omega (\mathrm{e}_1 \times x \cdot \nabla v) \cdot \big(-\nabla \times (\psi_R^2 \nabla \times v)\big)\,\mathrm{d}x \\
&\qquad = \int_\Omega \psi_R^2\, (\mathrm{e}_1 \times x \cdot \nabla v) \cdot \Delta v\,\mathrm{d}x + \int_\Omega (\mathrm{e}_1 \times x \cdot \nabla v) \cdot \big((\nabla \times v) \times \nabla[\psi_R^2]\big)\,\mathrm{d}x \\
&\qquad = -\int_\Omega \nabla[\psi_R^2] \otimes (\mathrm{e}_1 \times x \cdot \nabla v) : \nabla v\,\mathrm{d}x - \int_\Omega \psi_R^2\, \nabla(\mathrm{e}_1 \times x \cdot \nabla v) : \nabla v\,\mathrm{d}x \\
&\qquad\quad + \int_\Omega (\mathrm{e}_1 \times x \cdot \nabla v) \cdot \big((\nabla \times v) \times \nabla[\psi_R^2]\big)\,\mathrm{d}x.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_\Omega \psi_R^2\, \nabla(\mathrm{e}_1 \times x \cdot \nabla v) : \nabla v\,\mathrm{d}x \\
&\qquad = \int_\Omega \psi_R^2\, \partial_j \partial_k v_i\, (\mathrm{e}_1 \times x)_k\, \partial_j v_i\,\mathrm{d}x + \int_\Omega \psi_R^2\, \partial_k v_i\, \partial_j [\mathrm{e}_1 \times x]\, \partial_j v_i\,\mathrm{d}x \\
&\qquad = -\frac{1}{2} \int_\Omega \partial_k \big[\psi_R^2\, (\mathrm{e}_1 \times x)_k\big]\, (\partial_j v_i)^2\,\mathrm{d}x + \int_\Omega \psi_R^2\, \partial_k v_i\, \partial_j [\mathrm{e}_1 \times x]\, \partial_j v_i\,\mathrm{d}x,
\end{aligned}$$

and observing that $|\partial_i \psi_R\, (\mathrm{e}_1 \times x)_j| \leq c_5$ for any $i,j=1,2,3$, we may conclude

$$|\int_\Omega (\mathrm{e}_1 \times x \cdot \nabla v) \cdot \big(-\nabla \times (\psi_R^2 \nabla \times v)\big)\,\mathrm{d}x| \leq c_6\, |v|_{1,2}^2. \tag{3.5}$$

Next, we estimate

$$\begin{aligned} |\int_\Omega (v\cdot\nabla v)\cdot(\psi_R^2 \Delta v)\,\mathrm{d}x| &\leq \int_\Omega |\psi_R \nabla v|\,|v|\,|\psi_R \Delta v|\,\mathrm{d}x \\ &\leq \|\psi_R \nabla v\|_3\, \|v\|_6\, \|\psi_R \Delta v\|_2 \\ &= \|\nabla[\psi_R v] - v\otimes \nabla\psi_R\|_3\, \|v\|_6\, \|\psi_R\Delta v\|_2 \\ &\leq \big(\|\nabla[\psi_R v]\|_3 + \|v\otimes\nabla\psi_R\|_3\big)\, \|v\|_6\, \|\psi_R \Delta v\|_2. \end{aligned} \tag{3.6}$$

By the Nirenberg inequality, we have

$$\begin{aligned} \|\nabla[\psi_R v]\|_{3,\mathbb{R}^3} &\leq c_7\, \|\nabla[\psi_R v]\|_{2,\mathbb{R}^3}^{\frac12}\, \|\nabla^2[\psi_R v]\|_{2,\mathbb{R}^3}^{\frac12} \\ &\leq c_8\, \|\nabla[\psi_R v]\|_{2,\mathbb{R}^3}^{\frac12}\, \|\Delta[\psi_R v]\|_{2,\mathbb{R}^3}^{\frac12} \\ &\leq c_8\, \|\nabla[\psi_R v]\|_{2,\mathbb{R}^3}^{\frac12}\, \big(\|\psi_R\Delta v\|_2 + 2\|\nabla v\cdot\nabla\psi_R\|_2 + \|\Delta\psi_R v\|_2\big)^{\frac12}. \end{aligned}$$

Since

$$\begin{aligned} \|\nabla[\psi_R v]\|_2 &\leq \|v\otimes\nabla\psi_R\|_2 + \|\psi_R\nabla v\|_2 \\ &\leq c_9 \Big(\int_{\mathrm{B}_{2R,R}} \frac{|v|^2}{R^2}\,\mathrm{d}x + \int_{\mathrm{B}_{2r,r}} \frac{|v|^2}{r^2}\,\mathrm{d}x\Big)^{\frac12} + \|\nabla v\|_2 \\ &\leq c_{10}\|v\|_6 + \|\nabla v\|_2 \leq c_{11}\, |v|_{1,2}, \end{aligned}$$

and similarly

$$\|\Delta\psi_R v\|_2 \leq c_{12}\, |v|_{1,2},$$

we see that

$$\|\nabla[\psi_R v]\|_{3,\mathbb{R}^3} \leq c_{13}\big(|v|_{1,2}^{\frac12}\, \|\psi_R\Delta v\|_2^{\frac12} + |v|_{1,2}\big).$$

Also,

$$\|v\otimes\nabla\psi_R\|_3 \leq c_{14}\, \Big(\int_{\mathrm{B}_{2R,R}} \frac{|v|^3}{R^3}\,\mathrm{d}x + \int_{\mathrm{B}_{2r,r}} \frac{|v|^3}{r^3}\,\mathrm{d}x\Big)^{\frac13} \leq c_{15}\, \|v\|_6 \leq c_{16}\, |v|_{1,2}.$$

Thus, from (3.6) we conclude that

$$\begin{aligned} |\int_\Omega (v\cdot\nabla v)\cdot(\psi_R^2\Delta v)\,\mathrm{d}x| &\leq c_{17}\big(|v|_{1,2}^{\frac12}\, \|\psi_R\Delta v\|_2^{\frac12} + |v|_{1,2}\big)\, |v|_{1,2}\, \|\psi_R\Delta v\|_2 \\ &\leq c_{18}\big(|v|_{1,2}^{\frac32}\, \|\psi_R\Delta v\|_2^{\frac32} + |v|_{1,2}^2\, \|\psi_R\Delta v\|_2\big) \\ &\leq c_{19}(\varepsilon)\, \big(|v|_{1,2}^6 + |v|_{1,2}^4\big) + \varepsilon\, \|\psi_R\Delta v\|_2^2 \end{aligned} \tag{3.7}$$

for any $\varepsilon > 0$. In a similar manner, we estimate

$$
\begin{aligned}
&|\int_\Omega (v\cdot\nabla v)\cdot\big((\nabla\times v)\times\nabla[\psi_R^2]\big)\,\mathrm{d}x| \\
&\qquad \le c_{20}\int_\Omega |v|\,|\nabla v|\,|\nabla\psi_R||\psi_R\nabla v|\,\mathrm{d}x \\
&\qquad \le c_{21}\,\|v\|_6\,\|\nabla v\|_2\,\|\psi_R\nabla v\|_3 \\
&\qquad \le c_{22}\,|v|_{1,2}^2\,\|\psi_R\nabla v\|_2^{\frac12}\,\|\nabla[\psi_R\nabla v]\|_2^{\frac12} \\
&\qquad \le c_{23}\,|v|_{1,2}^{\frac52}\,\big(\|\psi_R\Delta v]\|_2+|v|_{1,2}\big)^{\frac12} \\
&\qquad \le c_{23}\,\big(|v|_{1,2}^{\frac52}\,\|\psi_R\Delta v\|_2^{\frac12}+|v|_{1,2}^3\big) \\
&\qquad \le c_{24}(\varepsilon)|v|_{1,2}^{\frac{10}{3}}+\varepsilon\,\|\psi_R\Delta v\|_2^2+c_{23}|v|_{1,2}^3
\end{aligned}
\tag{3.8}
$$

for any $\varepsilon > 0$. We also have

$$
\begin{aligned}
|\int_\Omega \Delta v\cdot\big((\nabla\times v)\times\nabla[\psi_R^2]\big)\,\mathrm{d}x| &\le \int_\Omega |\psi_R\Delta v|\,|\nabla v|\,|\nabla\psi_R|)\,\mathrm{d}x \\
&\le \varepsilon\,\|\psi_R\Delta v\|_2^2+c_{25}(\varepsilon)|v|_{1,2}^2
\end{aligned}
\tag{3.9}
$$

for any $\varepsilon > 0$. Finally, we can estimate

$$
\begin{aligned}
&|\int_\Omega f\cdot\big(-\nabla\times(\psi_R^2\nabla\times v)\big)\,\mathrm{d}x| \\
&\qquad \le \int_\Omega |f\cdot\psi_R^2\Delta v|\,\mathrm{d}x+\int_\Omega |f\cdot\big((\nabla\times v)\times\nabla[\psi_R^2]\big)|\,\mathrm{d}x \\
&\qquad \le c_{26}(\varepsilon)\,\|f\|_2^2+\varepsilon\,\|\psi_R\Delta v\|_2^2+c_{27}|v|_{1,2}\,\|f\|_2
\end{aligned}
\tag{3.10}
$$

for any $\varepsilon > 0$. Combining now (3.4), (3.5), (3.7), (3.8), (3.9), (3.10) and recalling (3.2) and (3.3), we conclude that multiplication of $(1.1)_1$ by $-\nabla\times(\psi_R^2\nabla\times v)$ and subsequent integration over Ω yields

$$
\int_\Omega \psi_R^2\,|\Delta v|^2\,\mathrm{d}x \le c_{28}(\varepsilon)\,\big(|v|_{1,2}^2+|v|_{1,2}^6+\|f\|_2^2\big)+\varepsilon\,\|\psi_R\Delta v\|_2^2
\tag{3.11}
$$

for any $\varepsilon > 0$. Hence, by choosing $0 < \varepsilon < 1$ and letting $R \to \infty$ in (3.11), we infer that $\Delta v \in L^2(\Omega^r)$. It follows that $v \in D^{2,2}(\Omega^\rho)$ for $\rho > r$. In fact, by an easy calculation that takes into account the properties of the "cut-off" ψ_R, we obtain

$$
\sum_{|\alpha|=2}\|\psi_R\,\mathrm{D}^\alpha v\|_{2,\Omega^r}^2 \le c_{29}\left(\|\frac{v}{|x|^2}\|_{2,\Omega^r}^2+\|\frac{\nabla v}{|x|}\|_{2,\Omega^r}^2+\sum_{|\alpha|=2}\|\mathrm{D}^\alpha(\psi_R\,v)\|_{2,\Omega^r}^2\right).
$$

However, since $\psi_R v$ is of compact support, we have

$$
\sum_{|\alpha|=2}\|\mathrm{D}^\alpha(\psi_R\,v)\|_{2,\mathbb{R}^3}^2 \le c_{30}\|\Delta(\psi_R\,v)\|_{2,\mathbb{R}^3}^2,
$$

with c_{30} independent of R, and so, the previous inequality implies

$$\sum_{|\alpha|=2} \|\psi_R \, \mathrm{D}^\alpha v\|_{2,\Omega^r}^2 \leq c_{31} \left(\|\frac{v}{|x|^2}\|_{2,\Omega^r}^2 + \|\frac{\nabla v}{|x|}\|_{2,\Omega^r}^2 + \|\psi_R(\Delta v)\|_{2,\Omega^r}^2 \right).$$

where c_{31} is independent of R. If we use the assumption $v \in D^{1,2}(\Omega) \cap L^6(\Omega)$ in this relation, along with a Hardy-type inequality (see for example [7, Theorem II.5.1]) and the fact that $\Delta v \in L^2(\Omega^r)$, we deduce

$$\sum_{|\alpha|=2} \|\psi_R \, \mathrm{D}^\alpha v\|_{2,\Omega^r}^2 \leq c_{32}, \tag{3.12}$$

where c_{32} is independent of R. The desired property for $\mathrm{D}^2 v$ then follows by letting $R \to \infty$ in (3.12). □

In the next lemma, we establish $L^3(\Omega)$-summability of the pressure. More precisely, we have:

Lemma 3.2. *Let $f \in L^2(\Omega)^3 \cap L^{\frac{3}{2}}(\Omega)^3$, $v_* \in W^{\frac{3}{2},2}(\partial\Omega)^3$, and let $(v,p) \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3 \times L^2_{loc}(\Omega)$ be a corresponding solution to (1.1). Then $p + c \in L^3(\Omega)$ for some constant $c \in \mathbb{R}$.*

Proof. Standard regularity theory for elliptic systems again yields $p \in W^{1,2}_{loc}(\overline{\Omega})$. Consequently, by Sobolev embedding, we have $p \in L^3_{loc}(\overline{\Omega})$. We therefore only need to show $p + c \in L^3(\Omega^\rho)$ for some $\rho > 0$ and $c \in \mathbb{R}$.

Let $\rho > \operatorname{diam}(\Omega)$ and $\psi \in C^\infty(\mathbb{R}^3;\mathbb{R})$ be a "cut-off" function with $\psi = 0$ on B_ρ and $\psi = 1$ on $\mathbb{R}^3 \setminus \mathrm{B}_{2\rho}$. Moreover, let

$$\sigma(x) := \left(\int_{\partial \mathrm{B}_{2\rho}} v \cdot n \, dx \right) \nabla \mathfrak{E}, \quad \mathfrak{E}(x) := \frac{1}{4\pi|x|}. \tag{3.13}$$

Since

$$\begin{aligned} \int_{\mathrm{B}_{2\rho}} \nabla\psi \cdot (v + \sigma) \, dx &= \int_{\mathrm{B}_{2\rho}} \operatorname{div} \big[\psi(v + \sigma)\big] \, dx \\ &= \int_{\partial \mathrm{B}_{2\rho}} v \cdot n \, \mathrm{d}x + \int_{\partial \mathrm{B}_{2\rho}} \sigma \cdot n \, \mathrm{d}x = 0, \end{aligned}$$

there exists (see [7, Theorem III.3.2]) a field

$$H \in W^{3,2}(\mathbb{R}^3), \quad \operatorname{supp} H \subset \mathrm{B}_{2\rho}, \quad \operatorname{div} H = \nabla\psi \cdot (v + \sigma). \tag{3.14}$$

Put

$$w = \psi v + \psi\sigma - H, \quad \pi = \psi p.$$

Using the fact that $\mathrm{e}_1 \times x \cdot \nabla\sigma - \mathrm{e}_1 \times \sigma = 0$, we find that

$$\begin{cases} \Delta w - \nabla\pi + \mathrm{Ta}\left(\mathrm{e}_1 \times x \cdot \nabla w - \mathrm{e}_1 \times w\right) = \psi f + G + \mathrm{Re}\,\psi\, v \cdot \nabla v & \text{in } \mathbb{R}^3, \\ \operatorname{div} w = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{3.15}$$

where $G \in L^2(\mathbb{R}^3)$ with $\mathrm{supp}(G) \subset \mathrm{B}_{2\rho}$. Taking divergence on both sides in (3.15) yields

$$-\Delta\pi = \mathrm{div}\,\big[\psi f\big] + \mathrm{div}\, G + \mathrm{Re}\ \mathrm{div}\,\big[\psi\, v \cdot \nabla v\big] \quad \text{in } \mathbb{R}^3 \tag{3.16}$$

in the sense of distributions. We now observe that we can write f as follows (again in the sense of distributions)

$$f = \mathrm{div}\, F, \quad F \in L^3(\Omega). \tag{3.17}$$

In fact, it is enough to choose $F_k = \nabla\mathfrak{E} * f_k$, where $\{f_k\}_{k=1}^{\infty} \subset C_0^\infty(\Omega)$ converges to f in $L^3(\Omega)$, and then pass to the limit $k \to \infty$, in the sense of distributions. We can express, again in the sense of distributions,

$$\psi\, f = \psi\ \mathrm{div}\, F = \mathrm{div}[\psi F] - F \cdot \nabla\psi.$$

Thus, introducing

$$\tilde{G} := G - F \cdot \nabla\psi, \quad \tilde{F} := \psi F, \quad \text{and} \quad \tilde{f} := \mathrm{div}\, \tilde{F},$$

from (3.16) we have

$$-\Delta\pi = \mathrm{div}\, \tilde{f} + \mathrm{div}\, \tilde{G} + \mathrm{Re}\ \mathrm{div}\,\big[\psi\, v \cdot \nabla v\big] \quad \text{in } \mathbb{R}^3, \tag{3.18}$$

where $\tilde{f} = \mathrm{div}\, \tilde{F} \in L^2(\mathbb{R}^3)$, $\tilde{F} \in L^3(\mathbb{R}^3)$, and $\tilde{G} \in L^2(\mathbb{R}^3)$ with $\mathrm{supp}(\tilde{G}) \subset \mathrm{B}_{2\rho}$. Consider now the three separate equations

$$-\Delta\pi_1 = \mathrm{div}\, \tilde{f} \quad \text{in } \mathbb{R}^3, \tag{3.19}$$

$$-\Delta\pi_2 = \mathrm{div}\, \tilde{G} \quad \text{in } \mathbb{R}^3, \tag{3.20}$$

$$-\Delta\pi_3 = \mathrm{Re}\ \mathrm{div}\,\big[\psi\, v \cdot \nabla v\big] \quad \text{in } \mathbb{R}^3, \tag{3.21}$$

with respect to unknowns π_1,π_2,π_3. Using the Riesz transformations,

$$\mathfrak{R}_j : L^q(\mathbb{R}^3) \to L^q(\mathbb{R}^3),\ \forall q > 1, \quad \mathfrak{R}_j(u) := \mathfrak{F}^{-1}\bigg(\frac{\xi_j}{|\xi|}\mathfrak{F}(u)\bigg),$$

where $\mathfrak{F}$ denotes the Fourier transformation, we find that

$$\pi_1 := \mathfrak{F}^{-1}\bigg(\frac{i\xi_j}{|\xi|^2}\mathfrak{F}(\tilde{f}_j)\bigg) = \mathfrak{F}^{-1}\bigg(\frac{-\xi_j\xi_k}{|\xi|^2}\mathfrak{F}(\tilde{F}_{jk})\bigg) = -\mathfrak{R}_j \circ \mathfrak{R}_k(\tilde{F}_{jk}) \tag{3.22}$$

is a solution to (3.19) with $\pi_1 \in L^3(\mathbb{R}^3)$. Moreover, since clearly $\tilde{G} \in L^{\frac{3}{2}}(\mathbb{R}^3)$, we can use the Riesz potential

$$\mathfrak{I} : L^{\frac{3}{2}}(\mathbb{R}^3) \to L^3(\mathbb{R}^3), \quad \mathfrak{I}(u) := \mathfrak{F}^{-1}\bigg(\frac{1}{|\xi|}\mathfrak{F}(u)\bigg)$$

to obtain a solution

$$\pi_2 := \mathfrak{F}^{-1}\bigg(\frac{i\xi_j}{|\xi|^2}\mathfrak{F}\big(\tilde{G}_j\big)\bigg) = i\,\mathfrak{R}_j \circ \mathfrak{I}\big(\tilde{G}_j\big) \tag{3.23}$$

to (3.20) with $\pi_2 \in L^3(\mathbb{R}^3)$. Similarly, putting $h := \mathrm{Re}\,\psi\, v \cdot \nabla v$, we have $h \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and obtain by

$$\pi_3 := \mathfrak{F}^{-1}\left(\frac{i\xi_j}{|\xi|^2}\mathfrak{F}(h_j)\right) = i\,\mathfrak{R}_j \circ \mathfrak{I}(h_j) \tag{3.24}$$

a solution to (3.21) with $\pi_3 \in L^3(\mathbb{R}^3)$. We furthermore conclude that

$$\partial_k \pi_1 = \mathfrak{F}^{-1}\left(\frac{-\xi_k\xi_j}{|\xi|^2}\mathfrak{F}(\tilde{f}_j)\right) = -\mathfrak{R}_j \circ \mathfrak{R}_k(\tilde{f}_j) \in L^{\frac{3}{2}}(\mathbb{R}^3), \tag{3.25}$$

$$\partial_k \pi_2 = \mathfrak{F}^{-1}\left(\frac{-\xi_k\xi_j}{|\xi|^2}\mathfrak{F}\big(\tilde{G}_j\big)\right) = -\mathfrak{R}_j \circ \mathfrak{R}_k\big(\tilde{G}_j\big) \in L^{\frac{3}{2}}(\mathbb{R}^3), \tag{3.26}$$

$$\partial_k \pi_3 = \mathfrak{F}^{-1}\left(\frac{-\xi_k\xi_j}{|\xi|^2}\mathfrak{F}(h_j)\right) = -\mathfrak{R}_j \circ \mathfrak{R}_k(h_j) \in L^{\frac{3}{2}}(\mathbb{R}^3), \tag{3.27}$$

for $k = 1, 2, 3$. We therefore deduce that

$$\bar{\pi}(x) := \pi_1(x) + \pi_2(x) + \pi_3(x) \tag{3.28}$$

is a solution to (3.18) with $\bar{\pi} \in L^3(\mathbb{R}^3)$ and $\nabla\bar{\pi} \in L^{\frac{3}{2}}(\mathbb{R}^3)$. Since also π satisfies the same equation, it follows that $Z := \nabla(\bar{\pi} - \pi)$ is harmonic in $\mathbb{R}^3$, so that, by the mean-value theorem, we have for each fixed $x \in \mathbb{R}^3$,

$$Z(x) = \frac{c_1}{R^3}\int_{\mathrm{B}_R(x)} \nabla(\bar{\pi} - \pi)\,\mathrm{d}y =: \frac{c_1}{R^3}\big(I_1(R) + I_2(R)\big). \tag{3.29}$$

By the Hölder inequality we find

$$|I_1(R)| \leq \|\nabla\bar{\pi}\|_{\frac{3}{2}}\,|\mathrm{B}_R|^{\frac{1}{3}} \leq c_2\,R. \tag{3.30}$$

Moreover, from Lemma 3.1, we have $v \in D^{2,2}(\Omega)$. Thus, $\Delta w \in L^2(\mathbb{R}^3)$, and from $(3.15)_1$ we infer

$$\frac{\nabla\pi}{(1+|x|)} \in L^2(\mathbb{R}^3).$$

Therefore, by Schwarz inequality,

$$|I_2(R)| \leq c_3\,R\,\|\nabla\pi/(1+|y|)\|_2\,|\mathrm{B}_R|^{\frac{1}{2}} \leq c_4\,R^{\frac{5}{2}}. \tag{3.31}$$

Combining (3.29)–(3.31) and letting $R \to \infty$, we find $Z(x) = 0$ for all $x \in \mathbb{R}^3$. Hence, $\bar{\pi} = \pi + c$, for some constant c, which concludes the proof of the lemma. □

In next lemma, we show that a weak solution satisfying the energy inequality and a solution decaying like $\frac{1}{|x|}$ must coincide under a suitable smallness condition. The proof follows essentially that of the main theorem in [6].

Lemma 3.3. *Let $f \in L^2(\Omega)^3 \cap L^{\frac{6}{5}}(\Omega)^3$, and $v_* \in W^{\frac{3}{2},2}(\partial\Omega)^3$. Moreover, let $(v,p) \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3 \times L^2_{loc}(\Omega)$ be a solution to (1.1) that satisfies the energy inequality (1.4). If $(w,\pi) \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3 \times L^2(\Omega)$ is another solution to (1.1) and $[\![w]\!]_1 < \frac{1}{8\mathrm{Re}}$, then $(w,\pi) = (v,p)$. In this case, (v,p) satisfies the energy equality (1.5).*

Proof. By standard regularity theory for elliptic systems, we have $(v,p), (w,\pi) \in W^{2,2}_{loc}(\overline{\Omega}) \times W^{1,2}_{loc}(\overline{\Omega})$. We can thus multiply $(1.1)_1$ with w and integrate over Ω_R ($R > \operatorname{diam}\Omega$). By partial integration, we then obtain

$$\begin{aligned}
&-\int_{\Omega_R} \nabla v : \nabla w \,\mathrm{d}x + \int_{\partial \mathrm{B}_R} (\nabla v \cdot n)\cdot w \,\mathrm{d}S - \int_{\partial \mathrm{B}_R} p\,(w\cdot n)\,\mathrm{d}S \\
&\qquad - \mathrm{Re}\int_{\Omega_R} (v\cdot\nabla v)\cdot w\,\mathrm{d}x + \mathrm{Ta}\int_{\Omega_R} \big(\mathrm{e}_1 \times x\cdot\nabla v - \mathrm{e}_1\times v\big)\cdot w\,\mathrm{d}x \\
&\qquad = -\int_{\partial\Omega} \big((\nabla v - pI)\cdot n\big)\cdot w\,\mathrm{d}S + \int_{\Omega_R} f\cdot w\,\mathrm{d}x.
\end{aligned} \tag{3.32}$$

Analogously, by switching the roles of v and w, we get

$$\begin{aligned}
&-\int_{\Omega_R} \nabla v : \nabla w \,\mathrm{d}x + \int_{\partial \mathrm{B}_R} (\nabla w \cdot n)\cdot v \,\mathrm{d}S - \int_{\partial \mathrm{B}_R} \pi\,(v\cdot n)\,\mathrm{d}S \\
&\qquad - \mathrm{Re}\int_{\Omega_R} (w\cdot\nabla w)\cdot v\,\mathrm{d}x + \mathrm{Ta}\int_{\Omega_R} \big(\mathrm{e}_1 \times x\cdot\nabla w - \mathrm{e}_1\times w\big)\cdot v\,\mathrm{d}x \\
&\qquad = -\int_{\partial\Omega} \big((\nabla w - pI)\cdot n\big)\cdot v\,\mathrm{d}S + \int_{\Omega_R} f\cdot v\,\mathrm{d}x.
\end{aligned} \tag{3.33}$$

We shall now examine the integrals over $\partial\,\mathrm{B}_R$ in (3.32) and (3.33) in the limit as $R \to \infty$. For this purpose, we utilize Lemma 3.2 and obtain $p \in L^3(\Omega^\rho)$ for some $\rho > 0$. Consequently, we can find a sequence $\{R_n\}_{n=1}^\infty \subset [\rho,\infty]$ so that $\lim_{n\to\infty} R_n = \infty$ and

$$\lim_{n\to\infty}\left[R_n \int_{\partial \mathrm{B}_{R_n}} |p|^3 + |\nabla v|^2 + |v|^6 + |\pi|^2 + |\nabla w|^2 + |w|^6 \,\mathrm{d}x\right] = 0. \tag{3.34}$$

We conclude that

$$\begin{aligned}
\Big|\int_{\partial \mathrm{B}_{R_n}} (\nabla v\cdot n)\cdot w\,\mathrm{d}S\Big| &\leq c_1\,[\![w]\!]_1 \int_{\partial \mathrm{B}_{\mathbb{R}_n}} \frac{|\nabla v|}{R_n}\,\mathrm{d}S \\
&\leq c_2\,[\![w]\!]_1 \Big(\int_{\partial \mathrm{B}_{\mathbb{R}_n}} |\nabla v|^2\,\mathrm{d}S\Big)^{\frac{1}{2}} \to 0 \quad \text{as } n\to\infty
\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}
\Big|\int_{\partial \mathrm{B}_{R_n}} p\,(w\cdot n)\,\mathrm{d}S\Big| &\leq c_3\,[\![w]\!]_1 \int_{\partial \mathrm{B}_{R_n}} \frac{|p|}{R_n}\,\mathrm{d}S \\
&\leq c_4\,[\![w]\!]_1 \Big(R_n \int_{\partial \mathrm{B}_{R_n}} |p|^3\,\mathrm{d}S\Big)^{\frac{1}{3}} \to 0 \quad \text{as } n\to\infty.
\end{aligned} \tag{3.36}$$

Furthermore,

(3.37)
$$
\begin{aligned}
|\int_{\partial \mathrm{B}_{R_n}} (\nabla w \cdot n) \cdot v \, \mathrm{d}S| &\le c_5 \left(\int_{\partial \mathrm{B}_{R_n}} |\nabla w|^2 \, dS \right)^{\frac{1}{2}} \left(\int_{\partial \mathrm{B}_{R_n}} |v|^6 \, dS \right)^{\frac{1}{6}} R_n^{\frac{2}{3}} \\
&= c_5 \left(R_n \int_{\partial \mathrm{B}_{R_n}} |\nabla w|^2 \, dS \right)^{\frac{1}{2}} \left(R_n \int_{\partial \mathrm{B}_{R_n}} |v|^6 \, dS \right)^{\frac{1}{6}} \\
&\to 0 \quad \text{as } n \to \infty
\end{aligned}
$$

and

$$
\begin{aligned}
|\int_{\partial \mathrm{B}_{R_n}} \pi \, (v \cdot n) \, \mathrm{d}S| &\le c_6 \left(\int_{\partial \mathrm{B}_{R_n}} |\pi|^2 \, dS \right)^{\frac{1}{2}} \left(\int_{\partial \mathrm{B}_{R_n}} |v|^6 \, dS \right)^{\frac{1}{6}} R_n^{\frac{2}{3}} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}
\tag{3.38}
$$

We now turn our attention to the limits as $R_n \to \infty$ of the integrals over Ω_{R_n} in (3.32) and (3.33). We begin to observe that, by using a Hardy-type inequality (see for example [7, Theorem II.5.1]), we find

$$
\int_\Omega |(v \cdot \nabla v) \cdot w| \, \mathrm{d}x \le [\![w]\!]_1 \left(\int_\Omega |\nabla v|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega \frac{|v|^2}{(1+|x|)^2} \, \mathrm{d}x \right)^{\frac{1}{2}} < \infty \, . \tag{3.39}
$$

Consequently,

$$
\lim_{n \to \infty} \int_{\Omega_{R_n}} (v \cdot \nabla v) \cdot w \, \mathrm{d}x = \int_\Omega (v \cdot \nabla v) \cdot w \, \mathrm{d}x. \tag{3.40}
$$

Similarly, we have

$$
\int_\Omega |(w \cdot \nabla w) \cdot v| \, \mathrm{d}x \le [\![w]\!]_1 \left(\int_\Omega |\nabla w|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega \frac{|v|^2}{(1+|x|)^2} \, \mathrm{d}x \right)^{\frac{1}{2}} < \infty
$$

and thus

$$
\lim_{n \to \infty} \int_{\Omega_{R_n}} (w \cdot \nabla w) \cdot v \, \mathrm{d}x = \int_\Omega (w \cdot \nabla w) \cdot v \, \mathrm{d}x. \tag{3.41}
$$

Concerning the integrals involving the data f, we observe that they are both well defined, in the sense of Lebesgue, because $f \in L^{\frac{6}{5}}(\Omega)$ and $w, v \in L^6(\Omega)$. We thus find

$$
\lim_{n \to \infty} \int_{\Omega_{R_n}} f \cdot v \, \mathrm{d}x = \int_\Omega f \cdot v \, \mathrm{d}x. \tag{3.42}
$$

and

$$
\lim_{n \to \infty} \int_{\Omega_{R_n}} f \cdot w \, \mathrm{d}x = \int_\Omega f \cdot w \, \mathrm{d}x. \tag{3.43}
$$

Now put $u := v - w$. Then

$$\begin{aligned}\int_{\Omega_{R_n}} \big(\mathrm{e}_1 \times x \cdot \nabla u - \mathrm{e}_1 \times u\big) \cdot u \,\mathrm{d}x &= \int_{\Omega_{R_n}} \big(\mathrm{e}_1 \times x \cdot \nabla u\big) \cdot u \,\mathrm{d}x \\ &= \frac{1}{2} \int_{\partial \mathrm{B}_{R_n}} |u|^2 \,(\mathrm{e}_1 \times x) \cdot n \,\mathrm{d}S = 0,\end{aligned} \tag{3.44}$$

where the last equality holds since $n = \frac{x}{|x|}$ on $\partial \mathrm{B}_{R_n}$. By the same argument, we also have

$$\int_{\Omega_{R_n}} \big(\mathrm{e}_1 \times x \cdot \nabla v - \mathrm{e}_1 \times v\big) \cdot v \,\mathrm{d}x = \frac{1}{2} \int_{\partial\Omega} |v_*|^2 \,(\mathrm{e}_1 \times x) \cdot n \,\mathrm{d}S \tag{3.45}$$

and

$$\int_{\Omega_{R_n}} \big(\mathrm{e}_1 \times x \cdot \nabla w - \mathrm{e}_1 \times w\big) \cdot w \,\mathrm{d}x = \frac{1}{2} \int_{\partial\Omega} |v_*|^2 \,(\mathrm{e}_1 \times x) \cdot n \,\mathrm{d}S \tag{3.46}$$

It follows from (3.44), (3.45), and (3.46) that

$$\begin{aligned}\int_{\Omega_{R_n}} \big(\mathrm{e}_1 \times x \cdot \nabla v - \mathrm{e}_1 \times v\big) \cdot w \,\mathrm{d}x + \int_{\Omega_{R_n}} \big(\mathrm{e}_1 \times x \cdot \nabla w - \mathrm{e}_1 \times w\big) \cdot v \,\mathrm{d}x \\ = \int_{\partial\Omega} |v_*|^2 \,(\mathrm{e}_1 \times x) \cdot n \,\mathrm{d}S.\end{aligned} \tag{3.47}$$

Adding together (3.32) and (3.33), utilizing (3.47), and finally letting $n \to \infty$, we find that

$$\begin{aligned}-2 \int_\Omega \nabla v : \nabla w \,\mathrm{d}x = \\ &\mathrm{Re} \left(\int_\Omega (v \cdot \nabla v) \cdot w \,\mathrm{d}x + \int_\Omega (w \cdot \nabla w) \cdot v \,\mathrm{d}x \right) \\ &+ \int_\Omega f \cdot v \,\mathrm{d}x - \int_{\partial\Omega} \big((\nabla v - pI) \cdot n\big) \cdot v_* \,\mathrm{d}S \\ &+ \int_\Omega f \cdot w \,\mathrm{d}x - \int_{\partial\Omega} \big((\nabla w - \pi I) \cdot n\big) \cdot v_* \,\mathrm{d}S \\ &- \mathrm{Ta} \int_{\partial\Omega} |v_*|^2 \,(\mathrm{e}_1 \times x) \cdot n \,\mathrm{d}S.\end{aligned} \tag{3.48}$$

We can now write

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x = \int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega |\nabla w|^2 \,\mathrm{d}x - 2 \int_\Omega \nabla v : \nabla w \,\mathrm{d}x. \tag{3.49}$$

By assumption, (v, p) satisfies the energy inequality

$$\begin{aligned}\int_\Omega |\nabla v|^2 \,\mathrm{d}x \leq &- \int_\Omega f \cdot v \,\mathrm{d}x + \int_{\partial\Omega} \big((\nabla v - pI) \cdot n\big) \cdot v_* \,\mathrm{d}S \\ &- \frac{\mathrm{Re}}{2} \int_{\partial\Omega} |v_*|^2 \, v_* \cdot n \,\mathrm{d}S + \frac{\mathrm{Ta}}{2} \int_{\partial\Omega} |v_*|^2 \;\mathrm{e}_1 \times x \cdot n \,\mathrm{d}S.\end{aligned} \tag{3.50}$$

From the decay properties of (w, π), it is easy to verify that (w, π) satisfies the energy equality

$$\begin{aligned}\int_\Omega |\nabla w|^2 \,\mathrm{d}x = &-\int_\Omega f \cdot w \,\mathrm{d}x + \int_{\partial\Omega} \big((\nabla w - \pi I)\cdot n\big)\cdot v_* \,\mathrm{d}S \\ &- \frac{\mathrm{Re}}{2}\int_{\partial\Omega} |v_*|^2\, v_*\cdot n \,\mathrm{d}S + \frac{\mathrm{Ta}}{2}\int_{\partial\Omega} |v_*|^2\, \mathrm{e}_1 \times x \cdot n \,\mathrm{d}S.\end{aligned} \tag{3.51}$$

Combining now (3.48), (3.49), (3.50), and (3.51), we obtain

$$\begin{aligned}\int_\Omega |\nabla u|^2 \,\mathrm{d}x \leq \mathrm{Re}\bigg(\int_\Omega (v\cdot\nabla v)\cdot w \,\mathrm{d}x + \int_\Omega (w\cdot\nabla w)\cdot v \,\mathrm{d}x\bigg) \\ - \mathrm{Re}\int_{\partial\Omega} |v_*|^2\, v_*\cdot n \,\mathrm{d}S.\end{aligned}$$

Next, we observe that

$$\begin{aligned}&\int_\Omega (u\cdot\nabla u)\cdot w \,\mathrm{d}x - \int_\Omega (w\cdot\nabla u)\cdot u \,\mathrm{d}x = \\ &\qquad \int_\Omega (v\cdot\nabla v)\cdot w \,\mathrm{d}x + \int_\Omega (w\cdot\nabla w)\cdot v \,\mathrm{d}x - \int_{\partial\Omega} |v_*|^2\, v_*\cdot n \,\mathrm{d}S.\end{aligned}$$

By an argument similar to (3.39) and (3.40), all integrals above are well-defined and finite. We can now conclude that

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x \leq \mathrm{Re}\bigg(\int_\Omega (u\cdot\nabla u)\cdot w \,\mathrm{d}x - \int_\Omega (w\cdot\nabla u)\cdot u \,\mathrm{d}x\bigg)$$

and thus estimate, using again the Hardy-type inequality, this time in form

$$\int_\Omega \frac{|u|^2}{|x|^2} \,\mathrm{d}x \leq 4\int_\Omega |\nabla u|^2 \,\mathrm{d}x,$$

valid for all fields vanishing at the boundary $\partial\Omega$,

$$\begin{aligned}\int_\Omega |\nabla u|^2 \,\mathrm{d}x &\leq 2\,\mathrm{Re}\, [\![w]\!]_1 \bigg(\int_\Omega \frac{|u|}{1+|x|}\,|\nabla u| \,\mathrm{d}x\bigg) \\ &\leq 2\,\mathrm{Re}\, [\![w]\!]_1 \bigg(\int_\Omega \frac{|u|^2}{(1+|x|)^2} \,\mathrm{d}x\bigg)^{\frac{1}{2}} \bigg(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\bigg)^{\frac{1}{2}} \\ &\leq 8\,\mathrm{Re}\, [\![w]\!]_1 \int_\Omega |\nabla u|^2 \,\mathrm{d}x.\end{aligned} \tag{3.52}$$

We finally conclude that $u = 0$ when $8\,\mathrm{Re}\, [\![w]\!]_1 < 1$. □

4 Main Theorem

Our main theorem follows as a consequence of Lemma 3.3 and the fact that a solution (w, π) with the in Lemma 3.3 required properties exists, provided the data are suitably restricted [10].

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain of class C^2 and* $\mathrm{Re}\,,\mathrm{Ta} \in (0, B]$*, for some $B > 0$. Suppose $v_* \in W^{\frac{3}{2},2}(\partial\Omega)^3$ and $f = \operatorname{div} F$, with*

$$\mathbf{F} := \big([\![F]\!]_2 + [\![f]\!]_3 + [\![\operatorname{div}\operatorname{div} F]\!]_4\big) < \infty. \tag{4.1}$$

Then, there is a constant $M_1 = M_1(\Omega, B) > 0$ such that if

$$\mathrm{Re}\,\big(\mathbf{F} + \|v_*\|_{W^{\frac{3}{2},2}(\partial\Omega)}\big) < M_1, \tag{4.2}$$

then a weak solution $(v, p) \in D^{1,2}(\Omega)^3 \cap L^6(\Omega)^3 \times L^2_{loc}(\Omega)$ to (1.1) that satisfies the energy inequality (1.4), that is, a Leray solution*, also satisfies, for some constant $c \in \mathbb{R}$,*

$$|v|_{2,2} + [\![v]\!]_1 + [\![\nabla v]\!]_2 + [\![p + c]\!]_2 + [\![\nabla p]\!]_{3,\Omega^R} \le C_2\,\big(\mathbf{F} + \|v_*\|_{W^{\frac{3}{2},2}(\partial\Omega)}\big), \tag{4.3}$$

where $C_2 = C_2(\Omega, B, R)$. Moreover, (v, p) satisfies the energy equality (1.5). Finally, (v, p) is unique (up to addition of a constant to p) in the class of weak solutions satisfying (1.4).

Proof. The existence of a solution (w, π) satisfying the properties stated for (v, p) has been established in [10, Theorem 2.1 and Remark 2.1] in the case $v_* \equiv 0$. Moreover, in [9] the methods from [10] have been further developed to also consider this more general case. Now, from (4.3) – written with w and π in place of v and p – and from (4.2), it follows that, if M_1 is taken "sufficiently small", we find, in particular, $[\![w]\!]_1 < \frac{1}{8\mathrm{Re}}$. Therefore, the stated properties for (v, p) at once follow from the uniqueness Lemma 3.3. □

Remark 4.2. The properties satisfied by the Leray solution (v, p) in Theorem 4.1 imply that (v, p) is, in fact, physically reasonable in the sense of Finn [5].

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