On the energy equality of Navier-Stokes equations in general unbounded domains

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Abstract

We present a sufficient condition for the energy equality of Leray-Hopf's weak solutions to the Navier-Stokes equations in general unbounded 3-dimensional domains.

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1 Introduction

In this paper, we consider a viscous incompressible fluid in general (unbounded) 3-dimensional domains Ω. The motion of such a fluid is governed by the Navier-Stokes equations

$$\text{(N-S)} \qquad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p &= f, \qquad t > 0, \quad x \in \Omega, \\ \operatorname{div} u &= 0, \qquad t > 0, \quad x \in \Omega, \\ u|_{\partial\Omega} &= 0, \\ u(0) &= a, \end{cases}$$

where $u = (u^1(x,t), u^2(x,t), u^3(x,t))$, and $p = p(x,t)$ denote the velocity vector and the pressure, respectively, of the fluid at the point $(x,t) \in \Omega \times \mathbb{R}_+$. Here a is the given initial data and f the external force. For simplicity, we assume that the coefficient of viscosity equals 1. Leray [10] and Hopf [6] showed the global existence of weak solutions $u \in L^\infty(0,\infty; L^2_\sigma(\Omega)) \cap L^2_{\mathrm{loc}}([0,\infty); H^1_0(\Omega))$ to (N-S) satisfying the energy inequality. However,

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uniqueness and regularity of Leray-Hopf's weak solutions are still open problems. Even the question whether or not every Leray-Hopf's solution satisfies the energy equality

$$\|u(t)\|_2^2 + 2\int_{t_0}^{t} \|\nabla u(\tau)\|_2^2\, d\tau = \|u(t_0)\|_2^2 + 2\int_{t_0}^{t} (u(\tau), f(\tau))\, d\tau, \quad 0 \le t_0 < t < T, \tag{1.1}$$

is still an open problem.

Serrin [11] showed – in addition to a uniqueness result – that if a weak solution u belongs to $L^s(0,T;L^q(\Omega))$ for some $q > 3$, $s > 2$ with

$$3/q + 2/s \le 1,$$

then u satisfies the energy equality (1.1). Later, Shinbrot [12] derived the same conclusion if the weak solution u belongs to $L^s(0,T;L^q(\Omega))$ for some $q, s > 1$ with

$$3/q + 2/s \le 1 + 1/q, \quad q \ge 4, \tag{1.2}$$

see also [13]. By using the exponents $s = q = 4$ in (1.2) and Hölder's inequality in space-time, the same result holds when

$$3/q + 2/s \le 1 + 1/s, \quad s \ge 4. \tag{1.3}$$

Recently, Cheskidov–Friedlander–Shvydkoy [2] proved the energy equality in a function class with better scaling properties than that of Shinbrot. They showed that if Ω is a bounded domain with C^2-boundary and if a Leray–Hopf weak solution u belongs to $L^3(0,T;V^{5/6})$, then u satisfies the energy equality. Here $V^{5/6} := D(A_2^{5/12}) \subset H^{5/6,2}(\Omega) \subset L^{9/2}(\Omega)$, and A_2 denotes the Stokes operator on $L^2_\sigma(\Omega)$. Moreover, if $\Omega = \mathbb{R}^3$, Cheskidov–Constantin–Friedlander–Shvydkoy [3] proved the same conclusion if $u \in L^3(0,T;B^{1/3}_{3,\infty}(\mathbb{R}^3))$ $(\supset L^3(0,T;H^{5/6,2}(\mathbb{R}^3)))$. For the Euler equation they also prove the same result under the slightly stronger condition $u \in L^3(0,T;B^{1/3}_{3,c}(\mathbb{R}^3))$; here, $B^{1/3}_{3,c}(\mathbb{R}^3)$ denotes the closure of $C_0^\infty(\mathbb{R}^3)$ in $B^{1/3}_{3,\infty}(\mathbb{R}^3)$.

In the present paper we generalize the result of Cheskidov–Friedlander–Shvydkoy [2] to arbitrary unbounded domains and Sobolev or Besov spaces of W^k_q-type with order of differentiability $k \ge \frac{1}{2}$ only, see Theorem 1 below. By using properties of Stokes semigroups, roughly speaking, we prove that the energy equality holds if u satisfies some conditions in terms of intermediate spaces between $H^{5/6,2}$ and $B^{1/3}_{3,\infty}$ such that (1.3) is satisfied with $s = 3$, $q = \frac{9}{2}$. Moreover, we consider more general domains Ω of $C^{1,1}$-type.

For general unbounded domains $\Omega \subset \mathbb{R}^3$ of uniform C^1-class (cf. Definition 1 below) Farwig-Kozono-Sohr [4] showed the existence of the Helmholtz projection by replacing the space L^q by

$$\tilde{L}^q(\Omega) := \begin{cases} L^q(\Omega) \cap L^2(\Omega), & 2 \le q < \infty \\ L^q(\Omega) + L^2(\Omega), & 1 < q < 2 \end{cases}.$$

Definition 1 [4, 5] *A domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is called a uniform $C^{1,1}$-domain of type (α, β, K) (where $\alpha > 0, \beta > 0, K > 0$) if for each $x_0 \in \partial\Omega$ we can choose a Cartesian coordinate system with origin at x_0 and coordinates $y = (y', y_n)$, $y' = (y_1, y_2, \cdots, y_{n-1})$, and a $C^{1,1}$-function $h(y')$, $|y'| \le \alpha$, with $C^{1,1}$-norm $\|h\|_{C^{1,1}} \le K$ such that the neighborhood*

$$U_{\alpha,\beta,h}(x_0) := \{(y', y_n) \in \mathbb{R}^n;\ h(y') - \beta < y_n < h(y') + \beta,\ |y'| < \alpha\}$$

of x_0 satisfies

$$\Omega \cap U_{\alpha,\beta,h}(x_0) = \{(y', y_n);\ h(y') - \beta < y_n < h(y'),\ |y'| < \alpha\}$$

and $\partial\Omega \cap U_{\alpha,\beta,h}(x_0) = \{(y', h(y')); |y'| < \alpha\}$. Similarly, uniform C^k-domains, $k \in \mathbb{N}$, of type (α, β, K) are defined.

Let $L^q_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$, $G^q(\Omega) := \{\nabla p \in (L^q(\Omega))^3\ ;\ p \in L^q_{\mathrm{loc}}(\Omega)\}$, $1 < q < \infty$, and define the spaces

$$\tilde{L}^q_\sigma(\Omega) := \begin{cases} L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega), & 2 \le q < \infty \\ L^q_\sigma(\Omega) + L^2_\sigma(\Omega), & 1 < q < 2 \end{cases}, \quad \tilde{G}^q(\Omega) := \begin{cases} G^q(\Omega) \cap G^2(\Omega), & 2 \le q < \infty \\ G^q(\Omega) + G^2(\Omega), & 1 < q < 2 \end{cases}.$$

Then Farwig-Kozono-Sohr [4] showed that the Helmholtz decomposition

$$\tilde{L}^q(\Omega) = \tilde{L}^q_\sigma(\Omega) + \tilde{G}^q(\Omega), \quad 1 < q < \infty,$$

holds for unbounded domains $\Omega \subset \mathbb{R}^n$ of uniform C^1-class. When Ω is a uniform $C^{1,1}$-domain, using the corresponding Helmholtz projection $\tilde{P}_q$: $\tilde{L}^q(\Omega) \to \tilde{L}^q_\sigma(\Omega)$, they defined the Stokes operator $\tilde{A}_q$ in $\tilde{L}^q_\sigma(\Omega)$ as the linear operator with domain

$$\mathcal{D}(\tilde{A}_q) := \begin{cases} D_q(\Omega) \cap D_2(\Omega), & 2 \le q < \infty \\ D_q(\Omega) + D_2(\Omega), & 1 < q < 2 \end{cases},$$

where $D_q(\Omega) := W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$, by setting

$$\tilde{A}_q u := -\tilde{P}_q \Delta u, \quad u \in \mathcal{D}(\tilde{A}_q),$$

see [5]. They also proved that $-\tilde{A}_q$ generates an analytic semigroup in $\tilde{L}^q_\sigma(\Omega)$ and $\tilde{A}_q$ has maximal L^s-regularity for $1 < s < \infty$. Recently, Kunstmann [8] showed that the operator $\epsilon + \tilde{A}_q$ has a bounded H^∞-calculus in $\tilde{L}^q_\sigma(\Omega)$, $1 < q < \infty$, for $\epsilon > 0$, hence admits bounded imaginary powers and the domain of $(\epsilon+\tilde{A}_q)^s$, $0<s<1$, coincides with the complex interpolation space $[\tilde{L}^q_\sigma(\Omega), \mathcal{D}(\tilde{A}_q)]_s$. In particular,

$$D((\epsilon+\tilde{A}_q)^{1/2}) = \tilde{W}^{1,p}_0(\Omega)\cap \tilde{L}^q_\sigma(\Omega), \tag{1.4}$$

where $\tilde{W}^{1,q}_0(\Omega) = W^{1,q}_0(\Omega)\cap W^{1,2}_0(\Omega)$ for $q\geq 2$ and $\tilde{W}^{1,q}_0(\Omega) = W^{1,q}_0(\Omega)+W^{1,2}_0(\Omega)$ for $1<q<2$, see [8, Corollary 1.2]. For simplicity, we use the notation

$$\tilde{D}^{2s}_q := D((1+\tilde{A}_q)^s) \ = [\tilde{L}^q_\sigma(\Omega), \mathcal{D}(\tilde{A}_q)]_s.$$

Using real interpolation, let

$$\tilde{L}^{q,r}(\Omega) := (\tilde{L}^{q_0}(\Omega), \tilde{L}^{q_1}(\Omega))_{\theta,r}, \tag{1.5}$$

where $1\leq r\leq\infty$ and

$$\begin{array}{l} 1<q_0<q<q_1<\infty,\ 0<\theta<1 \ \text{ with } \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}\,, \\ q_0,q_1>2 \ \text{ if } \ q>2, \quad q_0<2<q_1 \ \text{ if } \ q=2, \quad q_0,q_1<2 \ \text{ if } \ q<2\,, \end{array} \tag{1.6}$$

see [9, Section 4]. Then, by interpolation, the Helmholtz projection $\tilde{P}_{q,r}$ is defined as a bounded operator from $\tilde{L}^{q,r}(\Omega)$ to

$$\tilde{L}^{q,r}_\sigma(\Omega) := (\tilde{L}^{q_0}_\sigma(\Omega), \tilde{L}^{q_1}_\sigma(\Omega))_{\theta,r},$$

where q_0, q_1, θ, r are as in (1.6). We can also define the Stokes operator $\tilde{A}_{q,r}$ by

$$\tilde{A}_{q,r}u = -\tilde{P}_{q,r}\Delta u, \quad u\in\mathcal{D}(\tilde{A}_{q,r}) := [\mathcal{D}(\tilde{A}_{q_0}), \mathcal{D}(\tilde{A}_{q_1})]_{\theta,r},$$

with q_0, q_1, θ, r as in (1.6). Then, by real interpolation, we see that $-\tilde{A}_{q,r}$ generates an analytic semigroup in $\tilde{L}^{q,r}_\sigma(\Omega)$ and the operator $\epsilon+\tilde{A}_{q,r}$ has a bounded H^∞-calculus in $\tilde{L}^{q,r}_\sigma(\Omega)$ for $\epsilon>0$. Consequently the domain of $(\epsilon+\tilde{A}_{q,r})^s$, $0<s<1$, coincides with the complex interpolation space $[\tilde{L}^{q,r}_\sigma(\Omega), \mathcal{D}(\tilde{A}_{q,r})]_s$. In particular, for $s\in(0,1)$,

$$D((\epsilon+\tilde{A}_{q,r})^s) = (D((\epsilon+\tilde{A}_{q_0})^s), D((\epsilon+\tilde{A}_{q_1})^s))_{\theta,r}, \tag{1.7}$$

where q_0, q_1, θ, r are as in (1.6), see Kunstmann [9, Section 4]. We also denote this space by

$$\tilde{D}^{2s}_{q,r} := D((1+\tilde{A}_{q,r})^s).$$

Note that $\tilde{D}^k_{q,1}\subset\tilde{D}^k_q\subset\tilde{D}^k_{q,\infty}$. From now on we will write $\tilde{A}$ instead of $\tilde{A}_{q,r}$ for simplicity.

Before stating our result, we introduce the definition of weak solutions to the Navier-Stokes equations. Let $H^1_{0,\sigma}(\Omega) = H^1_0(\Omega)\cap L^2_\sigma(\Omega)$; for simplicity, we assume $f=0$.

Definition 2. Let $a\in L^2_\sigma$. A measurable function u on $\Omega\times(0,T)$, $0<T\leq\infty$, is called a weak solution of (N-S) on $(0,T)$ if

(i) $u \in L^\infty(0,T; L^2_\sigma) \cap L^2_{\mathrm{loc}}([0,T); H^1_{0,\sigma})$;

(ii) $u(t)$ is continuous on $[0,T]$ in the weak topology of L^2_σ and $u(0) = a$;

(iii) for every $0 \le s \le t < T$ and every $\Phi \in H^1((s,t); H^1_{0,\sigma})$

$$(1.8)\quad \int_s^t \{-(u, \partial_\tau \Phi) + (\nabla u, \nabla \Phi) + (u \cdot \nabla u, \Phi)\}\, d\tau = -(u(t), \Phi(t)) + (u(s), \Phi(s)).$$

Now our main result reads as follows:

Theorem 1. *Let u be a weak solution to the Navier-Stokes equations in a uniform $C^{1,1}$-domain on $(0,T)$. Assume that*

$$(1.9)\quad u \in L^3(0,T; \tilde{D}^{1/2}_{18/7}),\ \text{or}$$

$$(1.10)\quad u \in L^3(0,T; \tilde{D}^k_{p,\infty})\ \text{for some } k,p \text{ with}\quad \frac{2}{9} = \frac{1}{p} - \frac{k}{3},\ \ \frac{1}{2} < k < \frac{5}{6},\ \ 2 < p < \frac{18}{7}.$$

Then the energy equality

$$(1.11)\qquad \|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2\, d\tau = \|u(0)\|_2^2$$

holds for all $t \in (0,T)$.

Remarks. 1. On the one hand, the space $L^3(0,T; \tilde{D}^{1/2}_{18/7})$ (1.9) is not contained in $L^3(0,T; \tilde{D}^k_{p,\infty})$ when $k > \frac{1}{2}$. On the other hand, when $k > \frac{1}{2}$ is sufficiently close to $\frac{1}{2}$, the space $\tilde{D}^k_{p,\infty} (\subset L^{9/2,\infty})$ in (1.10) contains some functions having the singularity of $|x|^{-2/3}$ at the origin, while $\tilde{D}^{1/2}_{18/7} (\subset L^{9/2})$ does not contain such functions. Indeed, let $\phi \in C_0^\infty(B_2(0))$ satisfy $\phi \equiv 1$ in $B_1(0)$ and define

$$a(x) := \mathrm{rot}\ (0, 0, \phi(x)|x|^{1/3}).$$

Obviously, $a(x) \sim |x|^{-2/3}$ and $(1-\Delta)a \sim |x|^{-8/3}$ for $x \sim 0$ so that $(1-\Delta)a \in L^{9/8,\infty}(\mathbb{R}^3)$. Moreover, $\mathrm{div}\, a = 0$ in the sense of distributions on $\mathbb{R}^3$ Let θ, k, p_0 and p_1 satisfy $\frac{1}{2} < k < \frac{5}{6}$, $2 < p_0 < p < p_1$, $\frac{2}{9} = \frac{1}{p} - \frac{k}{3}$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then

$$\begin{aligned}\tilde{D}^k_{p,\infty}(\mathbb{R}^3) &= (1+A)^{-k/2} \tilde{L}^{p,\infty}_\sigma(\mathbb{R}^3) = (1-\Delta)^{-k/2} P(\tilde{L}^{p_0}(\mathbb{R}^3), \tilde{L}^{p_1}(\mathbb{R}^3))_{\theta,\infty} \\ &= P(1-\Delta)^{-k/2}(L^{p_0} \cap L^2, L^{p_1} \cap L^2)_{\theta,\infty} \supset P(1-\Delta)^{-1}(L^{d_0} \cap L^h, L^{d_1} \cap L^h)_{\theta,\infty},\end{aligned}$$

where

$$\frac{1}{p_0} = \frac{1}{d_0} - \frac{2-k}{3},\quad \frac{1}{p_1} = \frac{1}{d_1} - \frac{2-k}{3},\quad \frac{1}{2} = \frac{1}{h} - \frac{2-k}{3}.$$

Let E_0 be the 0-extension operator from functions defined on $B_2(0)$ to functions on $\mathbb{R}^3$:

$$E_0 f(x) = \begin{cases} f(x) & \text{if } x \in B_2(0) \\ 0 & \text{otherwise} \end{cases} .$$

Since $d_0, d_1 > h$, we have $L^{d_i}(\mathbb{R}^3) \cap L^h(\mathbb{R}^3) \supset E_0 L^{d_i}(B_2(0))$ for $i = 0, 1$. Hence,

$$\tilde{D}^k_{p,\infty}(\mathbb{R}^3) \supset P(1-\Delta)^{-1} E_0 (L^{d_0}(B_2(0)), L^{d_1}(B_2(0)))_{\theta,\infty} = P(1-\Delta)^{-1} E_0 L^{9/8,\infty}(B_2(0)).$$

Let $b := (1-\Delta)a$ ($\in L^{9/8,\infty}$). Since $\operatorname{div} b = 0$ and $\operatorname{supp} b \subset B_2(0)$, we have $a = P(1-\Delta)^{-1} E_0 b$, which implies $a \in \tilde{D}^k_{p,\infty}(\mathbb{R}^3)$. Therefore, there is no inclusion between conditions (1.9) and (1.10).

2. When Ω is a bounded domain,

$$\tilde{D}^{1/2}_{18/7} = [\tilde{L}^{18/7}_\sigma, \mathcal{D}(\tilde{A}_{18/7})]_{1/4} = [L^{18/7}_\sigma, \mathcal{D}(A_{18/7})]_{1/4} = D(A^{1/4}_{18/7}) \supset V^{5/6} = D(A_2^{5/12}) \,.$$

Hence (1.9) is a better condition than that in [2] requiring less regularity. Moreover, note that $L^3(0,T; \tilde{D}^{1/2}_{18/7}) \subset L^3(0,T; L^{9/2})$; here, with $s = 3$, $q = \frac{9}{2}$ we have $2/s + 3/q = 1 + 1/s$, cf. (1.3).

2 Preliminaries

In this section we introduce some lemmata.

Lemma 2.1. *Let u be a weak solution to (N-S) on $(0,T)$ and $\mathbb{S}$ be a bounded, linear and self-adjoint operator in L^2_σ with $\|\mathbb{S}v\|_{D(A_2)} \le C\|v\|_2$. Assume $A^{1/2}\mathbb{S}v = \mathbb{S}A^{1/2}v$ for $v \in D(A_2)$. Then, it holds that*

$$(u(t), \mathbb{S}u(t)) + 2\int_s^t (\mathbb{S}A^{1/2}u, A^{1/2}u)\, d\tau = (u(s), \mathbb{S}u(s)) + 2\int_s^t (u \cdot \nabla u, \mathbb{S}u)\, d\tau \tag{2.1}$$

for all $s, t \in [0,T)$.

Proof. We follow Serrin [11]. Let $\rho \ge 0$ be a function in $C_0^\infty(0,1)$ with $\rho(t) = \rho(|t|)$ and $\int_{-1}^1 \rho\, dt = 1$, $\rho_\epsilon(t) := \frac{1}{\epsilon}\rho(\frac{t}{\epsilon})$ for $\epsilon > 0$, fix arbitrary $0 \le s < t < T$ and let

$$u_\epsilon(\tau) := \int_s^t \rho_\epsilon(|\tau - \sigma|) u(\sigma)\, d\sigma, \quad u^l_\epsilon(\tau) := \int_s^t \rho_\epsilon(|\tau-\sigma|)\mathbb{S}u(\sigma)\, d\sigma (= \mathbb{S}u_\epsilon(\tau)).$$

Then $u_\epsilon \to u$ in $L^2(s,t; L^2(\Omega))$ as $\epsilon \to 0+$. Since $\mathbb{S}$ is a self adjoint operator,

$$\begin{aligned} \int_s^t (u(\tau), \partial_\tau u^l_\epsilon(\tau))\, d\tau &= \int_s^t \int_s^t (u(\tau), \partial_\tau \rho_\epsilon(|\tau-\sigma|)\mathbb{S}u(\sigma))\, d\tau\, d\sigma \\ &= \int_s^t \int_s^t (\mathbb{S}u(\tau), -\partial_\sigma \rho_\epsilon(|\tau-\sigma|)u(\sigma))\, d\tau\, d\sigma =: K \end{aligned} \tag{2.2}$$

By replacing (σ,τ) by (τ,σ), we have

$$
\begin{aligned}
K &= \int_s^t \int_s^t (\mathbb{S}u(\sigma), -\partial_\tau \rho_\epsilon(|\sigma-\tau|)u(\tau))\, d\sigma\, d\tau \\
&= -\int_s^t \int_s^t (\partial_\tau \rho_\epsilon(|\tau-\sigma|)\mathbb{S}u(\sigma), u(\tau))\, d\sigma\, d\tau \qquad (2.3)\\
&= -\int_s^t (\partial_\tau u_\epsilon^l(\tau), u(\tau))\, d\tau = -K = 0\,.
\end{aligned}
$$

Clearly, since u is weakly continuous in L^2 and hence $u_\epsilon(t) \rightharpoonup \frac{1}{2}u(t)$ in L^2 as $\epsilon \to 0+$,

$$
(u(t), u_\epsilon^l(t)) = \int_s^t \rho_\epsilon(|t-\sigma|)(u(t), \mathbb{S}u(\sigma))\, d\sigma = \int_s^t \rho_\epsilon(|t-\sigma|)(\mathbb{S}u(t), u(\sigma))\, d\sigma \to \frac{1}{2}(\mathbb{S}u(t), u(t))
$$

as $\epsilon \to 0+$. Similarly we have

$$
(u(s), u_\epsilon^l(s)) \to \frac{1}{2}(\mathbb{S}u(s), u(s)).
$$

Since $A^{1/2}u_\epsilon \to A^{1/2}u$ in $L^2(s,t;L^2(\Omega))$,

$$
\int_s^t (\nabla u, \nabla u_\epsilon^l)\, d\tau = \int_s^t (A^{1/2}u, A^{1/2}u_\epsilon^l)\, d\tau = \int_s^t (\mathbb{S}A^{1/2}u, A^{1/2}u_\epsilon)\, d\tau \to \int_s^t (\mathbb{S}A^{1/2}u, A^{1/2}u)\, d\tau
$$

as $\epsilon \to 0+$. Finally,

$$
\begin{aligned}
\left| \int_s^t (u\cdot\nabla u, u_\epsilon^l - \mathbb{S}u)\, d\tau \right| &\le C \sup_{[s,t]} \|u\|_2 \|\nabla u\|_{L^2(0,T;L^2)} \left(\int_s^t \|\mathbb{S}u_\epsilon - \mathbb{S}u\|_{D(A_2)}^2\, d\tau \right)^{1/2} \\
&\le C \sup_{[s,t]} \|u\|_2 \|\nabla u\|_{L^2(0,T;L^2)} \|u_\epsilon - u\|_{L^2(s,t;L^2)}
\end{aligned}
$$

converges to 0 as $\epsilon \to 0+$. Substituting u_ϵ^l as test function in (1.8), and then letting $\epsilon \to 0$, we obtain (2.1). □

Lemma 2.2. *Let $\tilde{\mathcal{A}} := 1 + \tilde{A}$, $0 \le \alpha \le \theta$ and $1 \le r \le \infty$. Then, for all $0 < t < 1$, it holds that*

$$
\begin{aligned}
&\|e^{-t\tilde{\mathcal{A}}}f\|_{\tilde{D}^\theta_{q,1}} \le Ct^{\frac{\theta-\alpha}{2}-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{\tilde{D}^\alpha_{p,\infty}} && \text{for } 2<p<q<\infty,\\
&\|f - e^{-t\tilde{\mathcal{A}}}f\|_{\tilde{D}^\alpha_p} \le Ct^{\frac{\theta-\alpha}{2}}\|f\|_{\tilde{D}^\theta_p} && \text{for } 1<p<\infty. \qquad (2.4)\\
&\|f - e^{-t\tilde{\mathcal{A}}}f\|_{\tilde{D}^\alpha_{p,r}} \le Ct^{\frac{\theta-\alpha}{2}}\|f\|_{\tilde{D}^\theta_{p,r}} && \text{for } 1<p<\infty.
\end{aligned}
$$

Here $\tilde{D}^0_{p,r} := \tilde{L}^{p,r}$.

Proof. Since $\|u\|_{\tilde{L}^q} \le C\|\tilde{\mathcal{A}}^\beta u\|_{\tilde{L}^p}$ for $1/q = 1/p - 2\beta/3,\ 0 < \beta < \min(1, 3/2p),\ p > 1$, (see [9, Corollary 2.7]), it holds that

$$\|u\|_{\tilde{L}^{q,r}} \le C\|\tilde{\mathcal{A}}^\beta u\|_{\tilde{L}^{p,r}} \tag{2.5}$$

for $1/q = 1/p - 2\beta/3$, $p > 2$, $1 \le r \le \infty$ and $0 < \beta < 3/2p$. Then we have

$$\|e^{-t\tilde{\mathcal{A}}}u\|_{\tilde{L}^{q,\infty}} \le C\|\tilde{\mathcal{A}}^\beta e^{-t\tilde{\mathcal{A}}}u\|_{\tilde{L}^{p,\infty}} \le Ct^{\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|u\|_{\tilde{L}^{p,\infty}}\,.$$

By the reiteration theorem (cf. [14, Sect. 1.10.2]) it holds that $\tilde{L}^{q,1} = (\tilde{L}^{q_0,\infty}, \tilde{L}^{q_1,\infty})_{\theta,1}$ for $2 < q_0 < q < q_1$ with $1/q = (1-\theta)/q_0 + \theta/q_1$. Then the interpolation inequality $\|f\|_{\tilde{L}^{q,1}} \le C\|f\|^{1-\theta}_{\tilde{L}^{q_0,\infty}}\|f\|^{\theta}_{\tilde{L}^{q_1,\infty}}$ (cf. [14, Sect. 1.3.3(5)]) and the above estimate yield

$$\|e^{-t\tilde{\mathcal{A}}}u\|_{\tilde{L}^{q,1}} \le Ct^{\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|u\|_{\tilde{L}^{p,\infty}}$$

which proves the first part of (2.4). The proofs of the two other inequalities are easy and are omitted. □

Lemma 2.3 (cf. [7]). *Let $1 < p_1, p_2 < \infty$ with $1/p := 1/p_1 + 1/p_2 < 1$ and let $1 \le q_1, q_2 \le \infty$ with $q := \min\{q_1, q_2\}$. Then, for all $f \in L^{p_1,q_1}(\Omega)$ and $g \in L^{p_2,q_2}(\Omega)$, it holds that*

$$\|f \cdot g\|_{L^{p,q}} \le C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}, \tag{2.6}$$

where $C = C(p_1, p_2, q_1, q_2) > 0$.

3 Proof of Theorem 1

Proof. We prove Theorem 1 by decomposing $u(t)$ into a low frequency part u^l and a high frequency part u^h, cf. [2]. Let $\mathbb{S} := e^{-\delta\tilde{\mathcal{A}}}$. Then by Lemma 2.1 we have for $0 \le s < t < T$

$$(u(t), e^{-\delta\tilde{\mathcal{A}}}u(t)) + 2\int_s^t (e^{-\delta\tilde{\mathcal{A}}}A^{1/2}u, A^{1/2}u)\,d\tau = (u(s), e^{-\delta\tilde{\mathcal{A}}}u(s)) + 2\int_s^t (u\cdot\nabla u, e^{-\delta\tilde{\mathcal{A}}}u)\,d\tau. \tag{3.1}$$

Since $\{e^{-\delta\tilde{\mathcal{A}}}\}_{\delta\ge 0}$ is a C_0-semigroup on L^2_σ and since $A^{1/2}u \in L^2(0,T;L^2_\sigma)$, letting $\delta \to 0+$, we obtain

$$\|u(t)\|_2^2 + 2\int_s^t (A^{1/2}u, A^{1/2}u)\,d\tau = \|u(s)\|_2^2 + \lim_{\delta\to 0+} 2\int_s^t (u\cdot\nabla u, e^{-\delta\tilde{\mathcal{A}}}u)\,d\tau. \tag{3.2}$$

Let $u^l := e^{-\delta\tilde{\mathcal{A}}}u$ for $0 < \delta < 1$ and let $u^h := u - u^l$. Since $\nabla \cdot u = 0$, by (1.4), Lemma 2.2 and (2.5) we have

$$
\begin{aligned}
|(u\cdot\nabla u, u^l)| &= |(u\cdot\nabla u^l, u^h)| \le \|u\|_{9/2}\|\nabla u^l\|_{18/7}\|u^h\|_{18/7} \\
&\le C\|u\|_{9/2}\|\tilde{\mathcal{A}}^{1/2}u^l\|_{\tilde{L}^{18/7}}\|u^h\|_{18/7} \\
&\le C\|\tilde{\mathcal{A}}^{1/4}u\|_{\tilde{L}^{18/7}}\,\delta^{-1/4}\|\tilde{\mathcal{A}}^{1/4}u\|_{\tilde{L}^{18/7}}\,\delta^{1/4}\|\tilde{\mathcal{A}}^{1/4}u\|_{\tilde{L}^{18/7}} \\
&\le C\|\tilde{\mathcal{A}}^{1/4}u\|^3_{\tilde{L}^{18/7}}.
\end{aligned}
\tag{3.3}
$$

Hence, under the assumption (1.9), Lebesgue's dominated convergence theorem yields

$$
\lim_{\delta\to 0+}\int_s^t (u\cdot\nabla u, u^l)\,d\tau = \int_s^t \lim_{\delta\to 0+}(u\cdot\nabla u, u^l)\,d\tau = 0,
$$

since by (2.4) we have $\|u-u^l\|_3 \le C\delta^{1/4}\|u\|_{\tilde{D}_3^{1/2}} \le C\delta^{1/4}\|u\|_{\tilde{D}_2^1}$ and consequently

$$
\lim_{\delta\to+0}(u\cdot\nabla u, u^l)(\tau) = (u\cdot\nabla u, u)(\tau) = 0 \quad \text{for a.e.} \quad \tau\in(0,T).
$$

This equation and (3.2) prove the energy equality under condition (1.9).

Next, we assume condition (1.10). Since by (1.7)

$$
D((\epsilon+\tilde{A}_{q,r})^{1/2}) = (D((\epsilon+\tilde{A}_{q_0})^{1/2}), D((\epsilon+\tilde{A}_{q_1})^{1/2}))_{\theta,r} \subset (\tilde{W}_0^{1,q_0}, \tilde{W}_0^{1/q_1})_{\theta,r},
$$

where q_0, q_1, θ are as in (1.6), we have $\|\nabla u\|_{\tilde{L}^{p,r}} \le \|\tilde{\mathcal{A}}^{1/2}u\|_{\tilde{L}^{p,r}}$ for all $1 < p < \infty$ and $1 \le r \le \infty$. Then, using Lemma 2.2, (2.5) and Lemma 2.3, we observe that

$$
\begin{aligned}
|(u\cdot\nabla u, u^l)| &\le C\|u\|_{L^{9/2,\infty}}\|\tilde{\mathcal{A}}^{1/2}u^l\|_{\tilde{L}^{(\frac{7}{9}-\frac{1}{p})^{-1},1}}\|u^h\|_{L^{p,\infty}} \\
&\le C\|u\|_{L^{9/2,\infty}}\,\delta^{-\frac{1-k}{2}-\frac{3}{2}[\frac{1}{p}-(\frac{7}{9}-\frac{1}{p})]}\|\tilde{\mathcal{A}}^{k/2}u\|_{\tilde{L}^{p,\infty}}\,\delta^{k/2}\|u\|_{\tilde{D}^k_{p,\infty}} \le C\|u\|^3_{\tilde{D}^k_{p,\infty}}.
\end{aligned}
\tag{3.4}
$$

Hence,

$$
|(u\cdot\nabla u, u^l)| \le C\|u\|^3_{\tilde{D}^k_{p,\infty}}.
$$

As in the above argument, we obtain the energy equality, which proves Theorem 1. □

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