Leray's inequality for fluid flow in symmetric multi-connected two-dimensional domains

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We consider the stationary Navier-Stokes equations with nonhomogeneous boundary condition in a domain with several boundary components. If the boundary value satisfies only the necessary flux condition (GOC), Leray's inequality does not holds true in general and we cannot prove the existence of a solution. But for a 2-D domain which is symmetric with respect to a line and where the data is also symmetric, Amick [1] showed the existence of solutions by reduction to absurdity; later Fujita [4] proved Leray-Fujita's inequality and hence the existence of symmetric solutions. In this paper we give a new short proof of Leray-Fujita's inequality.

Key words: 2-dimensional steady Navier-Stokes flow, general outflow condition, symmetry, regularized distance
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1 Introduction

Suppose Ω is a two-dimensional Lipschitz bounded domain symmetric with respect to the x_2 -axis and such that the boundary $\partial\Omega$ consists of several connected components, $\Gamma_0, \Gamma_1, \cdots, \Gamma_N$ $(N \geq 1)$. Consider the stationary

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Navier-Stokes equations

$$\begin{cases} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= \nu\Delta\boldsymbol{u}-\nabla p & \text{in} & \Omega, \\ \operatorname{div}\boldsymbol{u} &= 0 & \text{in} & \Omega, \\ \boldsymbol{u} &= \boldsymbol{\beta} & \text{on} & \partial\Omega, \end{cases} \qquad (NS)$$

where $\boldsymbol{u}=(u_1,u_2)$ is the fluid velocity, p the pressure, $\nu>0$ the kinematic viscosity constant, and $\boldsymbol{\beta}$ is a given vector function on $\partial\Omega$.

Suppose the boundary value $\boldsymbol{\beta}$ satisfies the *stringent outflow condition*

$$\int_{\Gamma_i}\boldsymbol{\beta}\cdot\boldsymbol{n}d\sigma=0 \quad (0\le i\le N) \qquad (SOC)$$

where $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$. Then, for every $\varepsilon>0$, we can find a solenoidal extension $\boldsymbol{b}_\varepsilon$ of $\boldsymbol{\beta}$ which satisfies the inequality (*Leray's inequality*)

$$|((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{b}_\varepsilon)|\le\varepsilon||\nabla\boldsymbol{v}||^2 \quad \text{for all } \boldsymbol{v}\in V(\Omega), \qquad (L)$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$, $\|\cdot\|$ the L^2-norm and $V(\Omega)=\{u\in H^1_0(\Omega);\ \operatorname{div}u=0\}$. Using this inequality, we obtain an *a priori* estimate of solutions to (NS), and the Leray-Schauder principle assures the existence of solutions. See Leray [9], Hopf [6], Fujita [3], Ladyzhenskaya [8].

If the boundary value $\boldsymbol{\beta}$ satisfies only the *general outflow condition*

$$\int_{\partial\Omega}\boldsymbol{\beta}\cdot\boldsymbol{n}\,d\sigma=0 \qquad (GOC)$$

the inequality (L) does not hold: in many cases, the validity of (L) for all $\varepsilon>0$ implies (SOC), cf. Takeshita [13], Farwig-Kozono-Yanagisawa [2].

Nevertherless, if the two-dimensional domain is symmetric with respect to a line, with all the boundary components intersecting the line of symmetry, and if $\boldsymbol{\beta}$ is also symmetric, then, firstly Amick [1] proved the existence of stationary solutions by reduction to absurdity. Later, Fujita [4] succeeded to construct an extension of $\boldsymbol{\beta}$ which satisfies an estimate similar to (L) for symmetric functions and to prove the existence of solutions by the Leray-Schauder principle. In [10], there is a simple approach to prove Leray's inequality yielding a solution with a decomposition into a weak part (in H^1) and very weak part (in L^2).

As for the nonsymmetric case, Morimoto-Ukai [11] and Fujita-Morimoto [5] considered boundary values of the form $\mu\nabla h+\beta_1$. Here h is a harmonic function, $\mu \in \mathbb{R}$, and β_1 satisfies (GOC). They obtained, using properties of compact operators, an existence result for all $\mu \in \mathbb{R}\backslash\mathcal{M}$ with small β_1, where $\mathcal{M}$ is an at most countable set. Recently, Kozono-Yanagisawa [7] proved a more precise result in terms of a smallness condition using harmonic vector fields.

2 Notation and Results

In order to state our results, we need for a bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary the function spaces $C^\infty_{0,\sigma}(\Omega) = \{\boldsymbol{v} \in C^\infty_0(\Omega);\ \mathrm{div}\boldsymbol{v} = 0\}$ and

$$V(\Omega) = \text{completion of } C^\infty_{0,\sigma}(\Omega) \text{ under the Dirichlet norm } \|\nabla\ \cdot\|.$$

Assume that Ω is symmetric with respect to the x_2-axis, i.e., $x = (x_1, x_2) \in \Omega$ if and only if $(-x_1, x_2) \in \Omega$. The vector function $\boldsymbol{v} = (v_1, v_2)$ is called symmetric with respect to the x_2-axis ("symmetric" in short) if and only if v_1 is an odd function of x_1 and v_2 an even function of x_1, i.e.,

$$v_1(-x_1, x_2) = -v_1(x_1, x_2),\ \ v_2(-x_1, x_2) = v_2(x_1, x_2)$$

hold true.

Remark 1. *If $\boldsymbol{v} = (v_1, v_2)$ is smooth and symmetric, then $v_1(0, x_2) = 0$ for $(0, x_2) \in \Omega$.*

Then we need the following symmetric function spaces:

$$C^{\infty,S}_{0,\sigma}(\Omega) = \{\boldsymbol{v} \in C^\infty_0(\Omega);\ \boldsymbol{v} \text{ is symmetric},\ \ \mathrm{div}\,\boldsymbol{v} = 0\},$$

$$V^S(\Omega) = \text{completion of } C^{\infty,S}_{0,\sigma}(\Omega) \text{ under the Dirichlet norm.}$$

Our main theorem is as follows.

Theorem 1. *Let Ω be a 2-dimensional bounded Lipschitz domain, symmetric with respect to the x_2-axis such that every boundary component intersects the x_2-axis. Further assume that the boundary value $\boldsymbol{\beta} \in H^{\frac{1}{2}}(\partial\Omega)$ is symmetric with respect to the x_2-axis satisfying (GOC). Then, for every positive ε, there exists a symmetric solenoidal extension $\boldsymbol{b}_\varepsilon$ of $\boldsymbol{\beta}$ such that the inequality*

$$|((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{b}_\varepsilon)| \le \varepsilon||\nabla\boldsymbol{v}||^2 \quad (\boldsymbol{v} \in V^S(\Omega)) \qquad (LF)$$

holds true.

Remark 2. *An a priori estimate for symmetric solutions to (NS) follows from the inequality (LF). Then the Leray-Schauder principle assures the existence of symmetric solutions to the stationary Navier-Stokes system, see, e.g., Ladyzhenskaya [8]. We can also obtain the solution using the Galerkin method, cf., e.g., Fujita [3].*

3 Proof of Theorem 1

Let

$$\Omega_+ = \{(x_1, x_2) \in \Omega;\ x_1 > 0\}, \quad \Omega_- = \{(x_1, x_2) \in \Omega;\ x_1 < 0\}.$$

Suppose that $\boldsymbol{\beta} \in H^{\frac{1}{2}}(\partial\Omega)$ is symmetric with respect to the x_2-axis and satisfies (GOC). Then there exists a solenoidal extension $\boldsymbol{b} = (b_1, b_2)$ in $H^1(\Omega)$, symmetric with respect to the x_2-axis, i.e.,

$$\operatorname{div} \boldsymbol{b} = 0 \ \text{ in } \ \Omega, \quad \boldsymbol{b}|_{\partial\Omega} = \boldsymbol{\beta}.$$

Remark 3. *Note that $b_1(0, x_2) = 0$ for $(0, x_2) \in \Omega$, and*

$$\int_{\partial\Omega_+} \boldsymbol{b} \cdot \boldsymbol{n} \, d\sigma = \int_{\partial\Omega_-} \boldsymbol{b} \cdot \boldsymbol{n} \, d\sigma = 0$$

where $\boldsymbol{n}$ is the unit outward normal vector to the boundary of Ω_+, or Ω_-.

Since Ω_+ is simply connected, there exists a scalar function (stream function) $\varphi \in H^2(\Omega_+)$ such that

$$\boldsymbol{b} = \nabla^\perp \varphi = \Big(\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1}\Big) \ \text{ in } \Omega_+.$$

Let $h(t) = h(t; \kappa, \delta)$ be a C^∞ function in $t \geq 0$, depending on the parameters $\delta > 0$ and $1/4 > \kappa > 0$, and satisfying

$$h(t) = \begin{cases} 1 & (0 \leq t \leq \kappa\delta) \\ 0 & (t \geq (1-\kappa)\delta) \end{cases}, \quad 0 \leq h \leq 1,$$

(1) $\quad \sup_{0 \leq t \leq \delta} |t\, h'(t)| \to 0 \ (\kappa \to 0) \quad$ uniformly in $\delta > 0$.

Furthermore, let $d(x)$ be the regularized distance from $\partial\Omega$, i.e., $d(x)$ is a smooth function on Ω, equivalent to the Euclidean distance function to $\partial\Omega$, and its gradient $\nabla d(x)$ is bounded; see Stein [12, p.171, Theorem 2] . Therefore, there exists a constant M such that

$$0 \le d(x) < M, \quad |\nabla d(x)| < M \ (x \in \Omega)\,.$$

Finally, we define

$$\rho(x) = x_1 d(x) \quad (x \in \Omega_+).$$

Then, $\rho(x)$ is smooth, $\rho(x) > 0$ for $x \in \Omega_+$, $\rho(x) = 0$ for $x \in \partial\Omega_+$ and its first order derivatives are

$$\text{(2)} \quad \frac{\partial}{\partial x_1}\rho(x) = d(x) + x_1 \frac{\partial}{\partial x_1} d(x)$$

$$\text{(3)} \quad \frac{\partial}{\partial x_2}\rho(x) = x_1 \frac{\partial}{\partial x_2} d(x).$$

Let $0 < \delta$ be small and $r_0 = \sup\{x_1;\ (x_1, x_2) \in \Omega_+\}$. Put

$$\Omega_{+,2} = \left\{x \in \Omega_+;\ d(x) < \frac{\sqrt{\delta}}{r_0}\right\}$$

$$\Omega_{+,1} = \{x \in \Omega_+ \setminus \Omega_{+,2};\ x_1 < r_0\sqrt{\delta}\}.$$

Then, we have

$$\text{(4)} \quad \frac{\sqrt{\delta}}{r_0} x_1 \le \rho(x) = x_1 d(x) < x_1 M \quad (x \in \Omega_{+,1}),$$

$$\text{(5)} \quad \rho(x) = x_1 d(x) \ge r_0\sqrt{\delta} \cdot \frac{\sqrt{\delta}}{r_0} = \delta \quad (x \in \Omega_+ \setminus \overline{\Omega_{+,1} \cup \Omega_{+,2}}).$$

Therefore, $\rho(x) \sim x_1$ in $\Omega_{+,1}$ and $h(\rho(x)) = 0$ in $\Omega_+ \setminus \overline{\Omega_{+,1} \cup \Omega_{+,2}}$.

Using (2) and (3), we see,

$$\left|\frac{\partial}{\partial x_1}\rho(x)\right| \le d(x) + x_1 \left|\frac{\partial}{\partial x_1} d(x)\right| \le M(1 + r_0\sqrt{\delta}) \quad (x \in \Omega_{+,1})$$

$$\left|\frac{\partial}{\partial x_2}\rho(x)\right| \le x_1 M \le r_0\sqrt{\delta} M \quad (x \in \Omega_{+,1})$$

Put

(6) $\quad \tilde{\boldsymbol{b}}(x) = \nabla^{\perp}\{h(\rho(x))\varphi(x)\} \quad (x \in \Omega_+)$

where the derivative is taken in the sense of distribution. Then $\operatorname{div} \tilde{\boldsymbol{b}} = 0$,

(7) $\quad \tilde{\boldsymbol{b}}(x) = h(\rho(x))\nabla^{\perp}\varphi(x) + h'(\rho)\{\nabla^{\perp}\rho(x)\}\varphi(x),$

and we see $\tilde{\boldsymbol{b}} \in H^1(\Omega_+)$. Furthermore, we have

$$\tilde{\boldsymbol{b}}|_{\partial\Omega_+} = \boldsymbol{b}|_{\partial\Omega_+}$$

because $h'(t) \equiv 0$ in a neighbourhood of $t = 0$.

Let ε be an arbitrary positive number. Our aim is to show that if we choose $\delta > 0$ and $\kappa > 0$ sufficiently small, then the estimate

(8) $\quad |(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \tilde{\boldsymbol{b}})_{\Omega_+}| \leq \varepsilon ||\nabla \boldsymbol{v}||^2_{\Omega_+} \quad (\forall \boldsymbol{v} \in V^S(\Omega))$

holds. Since $C^{\infty,S}_{0,\sigma}(\Omega)$ is dense in $V^S(\Omega)$, we need prove (8) only for $C^{\infty,S}_{0,\sigma}(\Omega)$. Suppose $\boldsymbol{v} \in C^{\infty,S}_{0,\sigma}(\Omega)$. Using the formula $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \frac{1}{2}\nabla|\boldsymbol{v}|^2 - \omega \boldsymbol{v}^{\perp}$ where

$$\boldsymbol{v} = (v_1, v_2), \quad \omega = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \boldsymbol{v}^{\perp} = (v_2, -v_1),\ |\boldsymbol{v}|^2 = v_1^2 + v_2^2,$$

we have

(9) $\quad ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \tilde{\boldsymbol{b}})_{\Omega_+} = \int_{\Omega_+} \frac{1}{2}\nabla|\boldsymbol{v}|^2 \cdot \tilde{\boldsymbol{b}}\, dx - \int_{\Omega_+} \omega \boldsymbol{v}^{\perp} \cdot \tilde{\boldsymbol{b}}\, dx\,.$

Since $\tilde{\boldsymbol{b}}$ belongs to $L^2(\Omega_+)$ and $\operatorname{div} \tilde{\boldsymbol{b}} = 0$, it holds that

$$|\boldsymbol{v}|^2\tilde{\boldsymbol{b}} \in L^2(\Omega_+), \quad \operatorname{div}(|\boldsymbol{v}|^2\tilde{\boldsymbol{b}}) = \nabla|\boldsymbol{v}|^2 \cdot \tilde{\boldsymbol{b}} \in L^2(\Omega_+).$$

Furthermore, $|\boldsymbol{v}|^2\tilde{\boldsymbol{b}}\cdot\boldsymbol{n} = 0$ on the boundary $\partial\Omega_+$. Therefore Gauss' divergence theorem proves that the first term of the right-hand side of (9) vanishes. As for the second term of the right-hand side of (9), using the expression (7) for $\tilde{\boldsymbol{b}}$, we have

(10) $\quad \int_{\Omega_+} \omega \boldsymbol{v}^{\perp} \cdot \tilde{\boldsymbol{b}}\, dx = \int_{\Omega_+} \omega \boldsymbol{v}^{\perp} h(\rho)\nabla^{\perp}\varphi\, dx + \int_{\Omega_+} \omega \boldsymbol{v}^{\perp} h'(\rho)\varphi\nabla^{\perp}\rho\, dx.$

By virtue of (5) and the properties of h, it is sufficient to calculate the integration only on the domain $\Omega_{+,1} \cup \Omega_{+,2}$. Therefore,

$$\int_{\Omega_+} \omega \boldsymbol{v}^\perp h(\rho) \nabla^\perp \varphi \, dx = \int_{\Omega_{+,1} \cup \Omega_{+,2}} \omega \boldsymbol{v}^\perp h(\rho) \nabla^\perp \varphi \, dx =: I.$$

Using Poincaré's inequality for $v \in V^S(\Omega)$, we see that we may choose $\delta > 0$ sufficiently small so that $|I|$ is less than $\varepsilon \|\nabla v\|^2$. We fix this δ.

Using (2) and (3), we have

$$\begin{aligned} (11) \quad \omega \boldsymbol{v}^\perp h'(\rho) \varphi \nabla^\perp \rho &= \omega \varphi h'(\rho) \left(v_1 \frac{\partial \rho}{\partial x_1} + v_2 \frac{\partial \rho}{\partial x_2} \right) \\ &= \omega \varphi \left\{ v_1(x) \frac{d(x)}{\rho(x)} + \frac{x_1}{\rho} \left(v_1 \frac{\partial d}{\partial x_1} + v_2 \frac{\partial d}{\partial x_2} \right) \right\} \rho h'(\rho) \\ &= \omega \varphi \left\{ v_1(x) \frac{1}{x_1} + \frac{1}{d(x)} \left(v_1 \frac{\partial d}{\partial x_1} + v_2 \frac{\partial d}{\partial x_2} \right) \right\} \rho h'(\rho). \end{aligned}$$

Therefore,

$$\begin{aligned} (12) \quad & \left| \int_{\Omega_+} \omega \boldsymbol{v}^\perp h'(\rho) \varphi \nabla^\perp \rho \, dx \right| \\ & \leq \sup_\rho |\rho h'(\rho)| \, \|\varphi\|_\infty \|\omega\| \left(\left\| \frac{v_1}{x_1} \right\|_{L^2(\Omega_{+,1} \cup \Omega_{+,2})} + M \left\| \frac{v}{d} \right\|_{L^2(\Omega_{+,1} \cup \Omega_{+,2})} \right). \end{aligned}$$

As for the last term in (12) note that $1/d(x) \leq r_0/\sqrt{\delta}$ for $x \in \Omega_{+,1}$ so that

$$\left\| \frac{v}{d} \right\|_{L^2(\Omega_{+,1})} \leq C \|v\| .$$

Moreover, since $v_1 = v_2 = 0$ on $\partial\Omega$, we can apply Hardy's inequality to v in $\Omega_{+,2}$ and obtain

$$\left\| \frac{v}{d} \right\|_{L^2(\Omega_{+,2})} \leq C \|\nabla v\|_{L^2(\Omega_+)}.$$

Hence

$$\left\| \frac{v}{d} \right\|_{L^2(\Omega_{+,1} \cup \Omega_{+,2})} \leq C \|\nabla v\|_{L^2(\Omega_+)}.$$

Concerning the norm $\left\|v_1/x_1\right\|_{L^2(\Omega_{+,1}\cup\Omega_{+,2})}$ in (12) we use a slightly different decomposition of the set $\Omega_{+,1}\cup\Omega_{+,2}$ and define

$$\Omega_{+,12} = \{x \in \Omega_{+,2};\; x_1 < r_0\sqrt{\delta}\}.$$

Note that $\Omega_{+,1}\cup\Omega_{+,12}$ is a set of rectangular type with boundary components of class $C^{0,1}$ and that v_1 vanishes on the component $\{x_1 = 0\}$ of $\partial(\Omega_{+,1}\cup\Omega_{+,12})$. It is easy to see that using a change of variables in the x_2-variable for every $0 < x_1 < r_0\sqrt{\delta}$, we may apply Hardy's inequality to v_1 on several subsets of $\Omega_{+,1}\cup\Omega_{+,12}$. Hence we obtain the estimate

$$\left\|\frac{v_1}{x_1}\right\|_{L^2(\Omega_{+,1}\cup\Omega_{+,12})} \le C\|\nabla v_1\|.$$

On $\Omega_{+,2}\setminus\Omega_{+,12}$ we have $x_1 > r_0\sqrt{\delta}$ and it holds the estimate

$$\left\|\frac{v_1}{x_1}\right\|_{L^2(\Omega_{+,2}\setminus\Omega_{+,12})} \le \frac{1}{r_0\sqrt{\delta}}\|v_1\|\,.$$

Summing up the previous inequalities we see that (12) leads to the estimate

$$(13)\quad \left|\int_{\Omega_+} \omega \boldsymbol{v}^\perp h'(\rho)\varphi\nabla^\perp\rho\,dx\right| \le C\sup_\rho|\rho h'(\rho)|\|\varphi\|_\infty\|\omega\|\|\nabla v\|\,.$$

If we choose κ sufficiently small, we have

$$(14)\quad \left|\int_{\Omega_+} \omega \boldsymbol{v}^\perp h'(\rho)\varphi\nabla^\perp\rho\,dx\right| \le \varepsilon\|\nabla v\|^2\,,$$

and the estimate (8) holds true.

Put

$$\boldsymbol{b}_\varepsilon(x_1,x_2) = \begin{cases} (\tilde b_1(x_1,x_2),\tilde b_2(x_1,x_2)) & (x_1,x_2)\in\Omega_+ \\ (-\tilde b_1(-x_1,x_2),\tilde b_2(-x_1,x_2)) & (x_1,x_2)\in\Omega_- \end{cases}.$$

Then $\boldsymbol{b}_\varepsilon \in H^1(\Omega)$ is solenoidal in Ω, symmetric with respect to the x_2-axis, extends the boundary values $\boldsymbol{\beta}$ and satisfies (LF). $\square$

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