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# Modified QR Decomposition to Avoid Non-Uniqueness in Water Supply Networks with Extension to Adjoint Calculus

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## Abstract

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The dynamic simulation and optimization of water supply networks contains various difficulties. One of them is that a classical modelling may yield singularities in the form of non-unique solutions. In [1], the application of singular value decomposition (SVD) is proposed in the context of water supply networks. Since the SVD of a matrix is computationally very expensive, we introduce an approach based on the QR decomposition of a suitably modified matrix. The properties of the resulting solution are analysed and we show the applicability of our method on a real life water supply network. Further, we incorporate the presented approach in an adjoint calculus to provide sensitivity information such that gradient-based optimization methods can be applied.

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## 1 Introduction

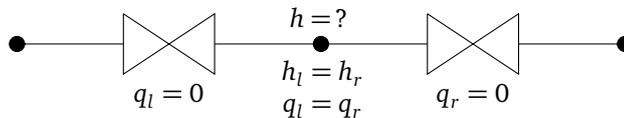
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Simulation and optimization of water supply networks is a topic of great common interest. The first algorithm to determine pressure heads and flows for a networked system in the steady state case was published in 1936 [2]. Meanwhile, a variety of software packages has been implemented, e.g. KANET [3], STANET [4] and EPANET [5]. The latter one is released as freeware by the United States Environmental Protection Agency, broadly accepted, and often a core part of proprietary packages. But EPANET and also other codes have difficulties with certain constellations of control devices. Several problem cases have been published by Simpson in 1999 [6]. Meanwhile, the EPANET software copes with all of them but many recent publications still report about new cases where it fails or computes wrong results, e.g. [7, 8].

One underlying problem can be explained by a very simple example: Consider two closed valves as shown in Fig. 1. The equations modelling the pressure heads  $h_l$  and  $h_r$  and the flow rates  $q_l$  and  $q_r$  at the connection of the two valves are given as follows,

$$F(h_l, q_l, h_r, q_r) = \underbrace{\begin{pmatrix} h_l - h_r \\ q_l - q_r \\ q_l \\ q_r \end{pmatrix}}_{=: b(h_l, q_l, h_r, q_r)} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (1)$$

Here, the pressure head between the two valves is not uniquely determined by the model equations. We only claim that  $h_l$  equals  $h_r$ . From the practical point of view, we might not be interested in the “real” pressure values between the two closed valves, but in a dynamic or quasi-stationary numerical simulation, we would expect the pressure variables to keep the same or at least similar values as in the previous time step.



**Figure 1:** Two closed valves – the model equations do not yield a unique solution

Of course, one could cope with the non-uniqueness in the mentioned example but the situation becomes more difficult for large networks and especially when devices are state-controlled so that “truncated” parts are not known a priori. Anyway, we expect getting into trouble when solving the in general nonlinear model equations of a water supply network with Newton’s method in situations like above.

The nature of the non-uniqueness in our small example is in close correlation with the Jacobian matrix of the model equations:

$$A(h_l, q_l, h_r, q_r) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2)$$

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The Jacobian matrix is singular, and obviously, the vector  $\mathbf{v}_0 = (1, 0, 1, 0)$  is an element of the kernel of  $A$ . Thus, when solving  $A\delta = b$  in Newton's method, we have to cope with the singularity of  $A$ , and moreover, an arbitrary multiple of  $\mathbf{v}_0$  could be added to any solution  $\delta$ , which corresponds to arbitrary but equal values for  $h_l$  and  $h_r$ .

As already mentioned, from the practical point of view, we might not be interested in the pressure values between the two closed valves, but at least, we have to ensure that the underlying singularity does not impede the solution process of the whole system of model equations. Moreover, we would prefer the pressure variables to change as little as possible.

In literature, there are different approaches to handle the shown problem in general. Alvarez et al. [7] propose to add virtual tanks in the network. In [9], Deuerlein proposes a reformulation of the model equations in the form of a minimization problem. For the analysis of the resulting model, he also uses game theory. Hähnlein [1] uses basically the same modelling as we do. For solving the sets of linear equations in Newton's method, he applies singular value decomposition.

Regarding the example problem, solving  $A\delta = b$  with an SVD has exactly the desired effect: We get the solution of the linear system of equations with minimal norm – the solution component in the nullspace of  $A$  (multiples of the vector  $\mathbf{v}_0$ ) equals zero. Since this property of using an SVD for solving sets of linear equations holds in a more general setting, singular value decomposition seems to be a good approach. While we get a solution of the linear system of equations (if one exists), the correction terms for the “critical variables” (here  $h_l$  and  $h_r$ ) are kept small. Moreover, we may eliminate small singular values in the SVD of the matrix to stabilize the solution process. Besides all those advantages of using singular value decomposition, the price to pay is the huge computational effort.

Typically, there are various points in water supply networks where the pressure variables may not be uniquely determined and thus may be “critical” in the solution process. Without proper treatment, one gets very large correction terms  $\delta$  in Newton's method or no solution at all for the linear system of equations, and either usually leads to a failure of the method.

In the next section, we will present our approach to the solution of the introduced class of problems. In the general setting, parts of the solution of the underlying sets of linear equations are uniquely determined while there may be degrees of freedom in other parts. We want to keep those solution components as small as possible while maintaining a useful solution.

Our approach is based on the QR decomposition of a modified matrix  $\tilde{A}$ . Since we know the critical variables in our applications, the original matrix  $A$  is extended with additional rows in such a way that the resulting matrix has full rank. The basic idea behind this approach is to penalize corrections in critical variables. Since various choices for the additional rows are possible that meet the assumptions made in the following theoretical analysis, we will give an example afterwards to demonstrate the influence of the choice.

Although we increase the size of the underlying matrix, the application of a QR decomposition to the modified matrix compared to the SVD of the original matrix results in an enormous speed-up. But this is not the only contribution we make to the simulation task of water supply networks. Additionally, we use the decomposition of the modified matrix to efficiently compute sensitivity information with respect to a given target functional in Sect. 3.

In contrast to the modelling used in EPANET, our software is not based on a quasi-steady-state model. The dynamics of the water in the pipes are modelled by the so-called water hammer equations, a hyperbolic system of partial differential equations (PDEs). In this context, the application of adjoint equations to derive sensitivity information is widely used. We make use of them in a first-discretize manner. For the discretization of the PDE, we apply an implicit box scheme [10], which allows for coarse grids. Numerical results are given in Sect. 4.

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## 2 Theoretical Analysis - Forward Direction

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The discretization of the model equations of the whole water supply network yields a coupled system of nonlinear algebraic equations  $E(\mathbf{y}, \mathbf{u})$ , which depends on state variables  $\mathbf{y} = (y(t_0), \dots, y(t_N))$ , like pressure heads and flow rates, and control variables  $\mathbf{u} = (u(t_0), \dots, u(t_N))$ , e.g. the speed of pumps. For a given initial state  $y(t_0) \in \mathbb{R}^n$  and

control variables for all time steps, the set of equations can be partitioned and solved for  $y(t_j)$  time step by time step ( $j = 1, \dots, N$ ) with Newton's method by solving subsets of  $E$  of the form

$$F(t_j, t_{j+1}, y(t_j), y(t_{j+1}), u(t_j), u(t_{j+1})) = 0. \quad (3)$$

In every step of Newton's method, we have to compute a correction  $\delta$  by solving a linear system of equations of the form

$$A\delta = b \Leftrightarrow A\delta - b = 0 \quad (4)$$

with  $A$  being an  $n \times n$  matrix. If  $A$  is singular (or ill-conditioned), we cannot make use of an LU decomposition of  $A$  in order to solve (4). Instead, we reformulate (4) as linear least squares problem

$$\min_{\delta} \|A\delta - b\|_2^2. \quad (5)$$

This problem can always be solved with a singular value decomposition of  $A$ . The SVD yields the (unique) solution  $\delta^*$  of (5) where additionally  $\|\delta^*\|_2$  is minimal among all solutions. In general, there is a residual  $r^* = A\delta^* - b$ .

Linear least squares problems can also be solved via a QR decomposition of the underlying matrix if it has full rank. Therefore, we consider the modified problem

$$\min_{\delta} \|\tilde{A}\delta - \tilde{b}\|_2^2 \quad (6)$$

with  $\tilde{A} = \begin{pmatrix} A \\ B_s \end{pmatrix}$  and  $\tilde{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}$ .

Here,  $B_s$  is a  $k \times n$  matrix (with  $k \leq n$ ) where in each row, there is exactly one nonzero entry  $s > 0$  and at most one entry in every column, for example,

$$B_s = \begin{pmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & s & 0 \end{pmatrix}. \quad (7)$$

Let  $I_B$  be the set of column indices of the nonzero entries in  $B_s$  (in the example  $I_B = \{1, 2, 4\}$ ). By adding additional rows to  $A$ , we can achieve that  $\tilde{A}$  has full rank, and accordingly, the modified minimization problem (6) can be solved via a QR decomposition of  $\tilde{A}$ .

There are several advantages of using a singular value decomposition for solving the original problem (5):

1. If  $A$  is regular,  $\delta^* = A^{-1}b$  and  $r^* = 0$ .
2. If  $A$  is not regular,  $\|\delta^*\|_2$  is minimal among all solutions.
3. By eliminating small singular values, the solution process can be stabilized.

The main disadvantage of using SVD is the computational effort. Typically, the singular value decomposition is computed in two steps. First, the matrix is reduced to a bidiagonal matrix, and afterwards, the SVD of the bidiagonal matrix is computed by an iterative method up to a certain precision. In one of our real life applications, we have a  $766 \times 766$  matrix with 1774 nonzero entries. For the computation of the SVD, 9.68 seconds are needed using MATLAB [11].

For the same example, the QR factorization of the corresponding modified matrix ( $1018 \times 766$  with 2026 nonzero entries) takes only 13 milliseconds. Hence, from the computational point of view, we prefer a QR decomposition to solve the modified problem (6) instead of solving (5) with an SVD. The results computed for the modified task (6) have to be measured in comparison to the three points given above. This is done in the following.

Let  $\tilde{\delta}^*$  be the unique solution of (6) and  $\tilde{r}^* = A\tilde{\delta}^* - b$ . With

$$f(\delta) = \|\tilde{A}\delta - \tilde{b}\|_2^2 = \|A\delta - b\|_2^2 + s^2 \sum_{j \in I_B} \delta_j^2 = \|A\delta - b\|_2^2 + s^2 \|\delta_{I_B}\|_2^2 \quad (8)$$

we have

$$f(\tilde{\delta}^*) \leq f(\delta^*). \quad (9)$$

Inequality (9) yields for the corresponding residuals

$$\|\tilde{r}^*\|_2^2 \leq \|r^*\|_2^2 + s^2 \left( \|\delta_{I_B}^*\|_2^2 - \|\tilde{\delta}_{I_B}^*\|_2^2 \right) \leq \|r^*\|_2^2 + s^2 \|\delta_{I_B}^*\|_2^2. \quad (10)$$

Thus, the maximum deviation of the Euclidean norm of the residual term  $\tilde{r}^*$  from the possible minimum  $\|r^*\|_2$  is limited and can be reduced by reducing  $s$ . In particular, if  $A$  is regular (or at least  $b$  is in the range of  $A$ ), we have  $\|r^*\|_2 = 0$  and

$$\|\tilde{r}^*\|_2 \leq s \|\delta_{I_B}^*\|_2. \quad (11)$$

Moreover, we get in the regular case:

$$A(\tilde{\delta}^* - \delta^*) = \tilde{r}^* \iff \tilde{\delta}^* - \delta^* = A^{-1} \tilde{r}^*. \quad (12)$$

Taking the Euclidean norm on both sides yields

$$\|\tilde{\delta}^* - \delta^*\|_2 \leq \|A^{-1}\|_2 \|\tilde{r}^*\|_2 \stackrel{(11)}{\leq} \|A^{-1}\|_2 s \|\delta_{I_B}^*\|_2 \leq \|A^{-1}\|_2 s \|\delta^*\|_2 \quad (13)$$

and finally<sup>1</sup>

$$\frac{\|\tilde{\delta}^* - \delta^*\|_2}{\|\delta^*\|_2} \leq s \|A^{-1}\|_2 \quad (14)$$

for the relative error of  $\tilde{\delta}^*$  compared to  $\delta^* = A^{-1}b$ .

Since  $\|\tilde{\delta}_{I_B}^* - \delta_{I_B}^*\|_2 \leq \|\tilde{\delta}^* - \delta^*\|_2$ , we also get from (13):<sup>2</sup>

$$\frac{\|\tilde{\delta}_{I_B}^* - \delta_{I_B}^*\|_2}{\|\delta_{I_B}^*\|_2} \leq s \|A^{-1}\|_2. \quad (15)$$

In addition to the given results, (9) also yields

$$\|\tilde{\delta}_{I_B}^*\|_2^2 \leq \|\delta_{I_B}^*\|_2^2 - \frac{1}{s^2} \underbrace{(\|\tilde{r}^*\|_2^2 - \|r^*\|_2^2)}_{\geq 0} \leq \|\delta_{I_B}^*\|_2^2. \quad (16)$$

This means that regarding the indices  $I_B$  of the ‘‘correction terms’’  $\tilde{\delta}^*$  and  $\delta^*$ , the correction induced by the QR decomposition of the modified matrix  $\tilde{A}$  is not greater than the one induced by the SVD of the original matrix  $A$ . This is an important property since the set of indices  $I_B$  typically refers to ‘‘critical’’ variables of the problem, while the rest of the variables is supposed to be determined anyway.

<sup>1</sup> Note that  $\tilde{\delta}^* = \delta^*$  if  $\|\delta^*\|_2 = 0$ .

<sup>2</sup> Analogously,  $\tilde{\delta}_{I_B}^* = \delta_{I_B}^*$  if  $\|\delta_{I_B}^*\|_2 = 0$ .

So far, we have given quantitative results for our QR decomposition approach related to the first two advantages of using singular value decomposition. To give a quantitative result related to the third point, we consider the case  $I_B = \{1, \dots, n\}$  with  $B_s = sI_n$ , where  $I_n$  is the  $n$ -dimensional identity matrix.

Let  $A = U\Sigma V^T$  be a singular value decomposition of  $A$  with

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}. \quad (17)$$

For the modified matrix  $\tilde{A}$  we get

$$\tilde{A}^T \tilde{A} = \begin{pmatrix} A^T & B_s^T \end{pmatrix} \begin{pmatrix} A \\ B_s \end{pmatrix} = A^T A + B_s^2 = V(\Sigma^2 + B_s^2)V^T. \quad (18)$$

Hence, the singular values  $\tilde{\sigma}_j$  ( $j = 1, \dots, n$ ) of  $\tilde{A}$  are given by

$$\tilde{\sigma}_j^2 = \sigma_j^2 + s^2. \quad (19)$$

In particular, we have  $\tilde{\sigma}_j > \sigma_j$  and  $\tilde{\sigma}_j \geq s > 0$ .

While in the results above a smaller  $s$  is always preferred, here, the opposite is the case since an increase of the singular values leads to more stability. Thus, in practice, a trade-off has to be made.

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### 3 Theoretical Analysis - Backward Direction

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Let  $f(\mathbf{y}, \mathbf{u})$  be a (scalar) quantity of interest. We would like to gain sensitivity information, that is, the dependency of  $f$  with respect to the control variables. In this context, the state variables  $\mathbf{y}(\mathbf{u})$  are considered as a function of the control variables, induced by the model equations  $E$ . The desired information can be efficiently computed by solving the (linear) adjoint equation

$$\left( \frac{\partial}{\partial \mathbf{y}} E(\mathbf{y}(\mathbf{u}), \mathbf{u}) \right)^T \xi = - \left( \frac{\partial}{\partial \mathbf{y}} f(\mathbf{y}(\mathbf{u}), \mathbf{u}) \right)^T. \quad (20)$$

Due to the special structure of  $E$ , this can also be done time step wise, but backwards in time, and we finally have to solve linear systems of equations with the same matrices as in the forward direction, but transposed.

Thus, we have to solve systems of the form

$$A^T \xi = c. \quad (21)$$

In the whole section, we postulate that  $c$  is in the range of  $A^T$ . This has the following reason: The solution of the simulation process has degrees of freedom in the kernel  $\ker(A)$  of  $A$ . Thus, regarding the quantity of interest  $f$ , it is reasonable to claim that the partial derivatives of  $f$  with respect to the state variables (in each time step) are perpendicular to  $\ker(A)$ , which is equivalent to being in the range of  $A^T$ . Additionally to the partial derivatives of  $f$ ,  $c$  also may contain components from the preceding time step. This can only occur in parts of the network where the modeling contains temporal derivatives, but those parts do not suffer from the described problem of non-uniqueness since the state variables of consecutive time steps are linked here.

Let  $\xi^*$  be the solution of (21) of minimal Euclidean norm. Similar to Sect. 2, this can be computed by a singular value decomposition of  $A$  respectively  $A^T$ . It is natural to apply the QR decomposition of  $\tilde{A}$  to solve the modified problem

$$\tilde{A}^T \begin{pmatrix} \xi \\ \mu \end{pmatrix} = c. \quad (22)$$

With the QR decomposition

$$\tilde{A} = \begin{pmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{pmatrix} \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix}, \quad (23)$$

where  $\tilde{Q}_{11}$  and  $\tilde{R}$  are  $n \times n$  matrices, the solution of (22) of minimal norm can be written as

$$\begin{pmatrix} \tilde{\xi}^* \\ \tilde{\mu}^* \end{pmatrix} = \begin{pmatrix} \tilde{Q}_{11} \\ \tilde{Q}_{21} \end{pmatrix} \tilde{R}^{-T} c. \quad (24)$$

This results from the well-known fact that the columns of  $\begin{pmatrix} \tilde{Q}_{11} \\ \tilde{Q}_{21} \end{pmatrix}$  form a basis of the kernel of  $\tilde{A}^T$ . With  $\tilde{q}^* = A^T \tilde{\xi}^* - c$  we have

$$\|\tilde{\xi}^*\|_2^2 + \frac{1}{s^2} \|\tilde{q}^*\|_2^2 = \|\tilde{\xi}^*\|_2^2 + \|\tilde{\mu}^*\|_2^2 = \left\| \begin{pmatrix} \tilde{\xi}^* \\ \tilde{\mu}^* \end{pmatrix} \right\|_2^2 \leq \left\| \begin{pmatrix} \xi^* \\ 0 \end{pmatrix} \right\|_2^2 = \|\xi^*\|_2^2. \quad (25)$$

This yields

$$\|\tilde{q}^*\|_2^2 \leq s^2(\|\xi^*\|_2^2 - \|\tilde{\xi}^*\|_2^2) \leq s^2\|\xi^*\|_2^2 \quad (26)$$

as upper bound for the residual with respect to the original equation (21). Analogously to Sect. 2, we get in the regular case

$$\frac{\|\tilde{\xi}^* - \xi^*\|_2}{\|\xi^*\|_2} \leq s \|A^{-T}\|_2 \quad (27)$$

for the relative error of  $\tilde{\xi}^*$  compared to  $\xi^* = A^{-T}c$ .

## 4 Numerical Results

In our current implementation, we put the indices of all pressure variables in front of and behind valves and pumps into the set  $I_B$  to ensure full rank of the underlying matrices without taking a closer look at the network structure. As already mentioned in the introduction, various choices for  $I_B$  are possible. To give an impression of the influence of the choice, let us first consider the introductory example with the following (initial) values for the state variables:

$$h_l^0 = 10.01, \quad q_l^0 = 0.01, \quad h_r^0 = 10, \quad q_r^0 = 0. \quad (28)$$

First, we compute  $\delta^*$  by applying singular value decomposition,<sup>3</sup> which yields

$$h_l^1 = 10.005, \quad q_l^1 = 0, \quad h_r^1 = 10.005, \quad q_r^1 = 0. \quad (29)$$

Thus, the two pressure variables “meet” in the middle while the flow rates are set to zero.

Next, we apply our QR decomposition approach. Since the two pressure variables (indices 1 and 3) are the critical variables in this example, we try  $I_B = \{1\}$ ,  $I_B = \{3\}$  and  $I_B = \{1, 3\}$  with  $s = 10^{-8}$ . We expect all three choices to deliver useful solutions because the two critical variables are connected by an equality constraint. Therefore, the penalization of the correction of one of these variables should be sufficient.

In the first case, we get

$$h_l^1 = 10.01, \quad q_l^1 = 0, \quad h_r^1 = 10.01, \quad q_r^1 = 0. \quad (30)$$

Analogous to the solution induced by the SVD, the flow rates are set to zero. Since we only penalized corrections in the first variable, both pressure variables end up at the “old” value of  $h_l$ , 10.01.

In the second case, we get

$$h_l^1 = 10, \quad q_l^1 = 0, \quad h_r^1 = 10, \quad q_r^1 = 0. \quad (31)$$

<sup>3</sup> The computations were made using MATLAB [11].

While the flow rates are set to zero, we now end up at the “old” value of  $h_r$  for both pressure variables since we have only penalized corrections in the third variable.

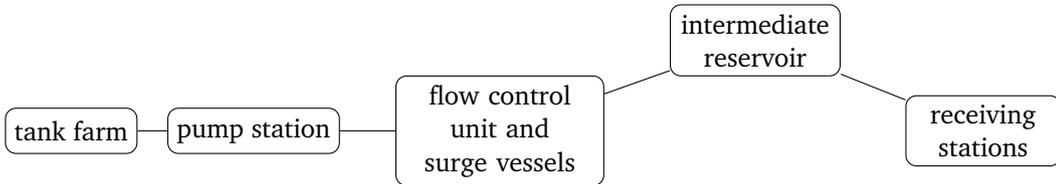
In the third case, we get

$$h_i^1 = 10.005, \quad q_i^1 = 0, \quad h_r^1 = 10.005, \quad q_r^1 = 0. \quad (32)$$

Up to roundoff errors, this solution is equal to the solution obtained by the SVD. The reason for this is that we have penalized corrections of both pressure variables, and therefore, they meet in the middle while the flow rates are determined to be zero anyway. Even if we choose  $I_B = \{1, 2, 3, 4\}$ , a short computation in MATLAB yields the same solution since the slight penalization ( $s = 10^{-8}$ ) does not impede the “necessary” corrections for the flow rate variables.

Altogether, this small example illustrates that the choice of  $I_B$  does have influence on the computed solution but mainly on those variables which are not uniquely determined by the model equations. The computed corrections for those variables were useful in all cases since enough indices of critical variables were included in  $I_B$ .

Now, let us consider a real life water supply network from one of our industry partners. The structure of the network is shown in Fig. 2. The fresh water stored in the tank farm is pressurized in the pump station and flows through a system of flow control valves and surge vessels. Up the hill, water can be stored in an intermediate reservoir or is delivered directly to several receiving stations in different valleys. Overall, the network consists of 8 pumps, 10 surge vessels, 20 pipes, 54 short pipes, 12 tanks and 118 valves. We consider a scenario with varying consumer demands over a period of 4 hours. The control used was the result of a gradient-based optimization [12], where the entire energy consumption of the pumps was minimized. As temporal step size we use one minute, resulting in 186,720 unknown state variables. A single run of our simulation software needs 2.4 seconds on an AMD Athlon™ 64 X2 Dual Core Processor 5000+. Since the computation of the corresponding adjoint state only demands the solution of a linear system of equations of the same size as the nonlinear model equations, the computational effort for this is dominated by the time needed for the simulation.



**Figure 2:** Structure of a real life water supply network

For the computations we used  $s = 10^{-8}$ . Note that the choice of  $s$  does not influence the accuracy of the computed solution with respect to the model equations since the termination criterion in Newton’s method is defined via the residual of the nonlinear equations. But the choice of  $s$  can affect the number of iterations needed to find such a solution. Table 1 shows the total number of iterations and the average per time step for the same scenario as above and various choices for the parameter  $s$ . As one can see, the number of iterations does not react very sensitively on  $s$  within a wide range.

**Table 1:** Number of iterations in Newton’s method depending on  $s$

parameter $s$	$10^{-5}$	$10^{-6}$	$10^{-7}$	$10^{-8}$	$10^{-9}$	$10^{-10}$	$10^{-11}$
number of iterations	456	392	373	373	373	373	373
average per time step	1.90	1.63	1.55	1.55	1.55	1.55	1.55

Finally, we would like to compare the computational effort of our approach to the application of singular value decomposition. The example given in Sect. 2 is based on the network presented in this section. Thus, even if we only need one iteration in Newton’s method per time step, the solution of the same simulation task would take us  $240 \times 9.68$  seconds  $\approx 39$  minutes with the SVD, compared to the 2.4 seconds needed with our new QR approach.

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## 5 Summary

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We have introduced a novel approach to tackle the problem of non-unique solutions in water supply networks. The discretized model equations are solved with Newton's method, where the solution of the underlying linear systems of equations is replaced by a modified minimization task, which can efficiently be solved via a QR decomposition. The resulting solutions feature similar properties as if the original systems were solved with a singular value decomposition. We illustrated the effect of our approach for a small example and demonstrated its applicability to a real life problem.

Additionally to solving the simulation task for water supply networks, we have introduced and analysed a technique to compute sensitivity information based on the presented approach. Adjoint equations are used to compute gradients of a given quantity of interest with respect to the control variables of the problem and are applied in a gradient-based optimization framework. Since the adjoint equations are linear and have a special structure, the additional computational effort to compute the sensitivity information is small compared to solving the simulation task.

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