Solutions to a model with Neumann boundary conditions for phase transitions driven by configurational forces

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Abstract

We study an initial boundary value problem to a model describing the evolution in time of diffusive phase interfaces in solid materials, in which martensitic phase transformations driven by configurational forces take place. The model was proposed earlier by the authors and consists of the partial differential equations of linear elasticity coupled to a nonlinear, degenerate parabolic equation of second order for an order parameter. In a previous paper global existence of weak solutions in one space dimension was proved under Dirichlet boundary conditions for the order parameter. Here we show that global solutions also exist for Neumann boundary conditions. Again, the method of proof is only valid in one space dimension.

1 Introduction

In [3] we have investigated a system of partial differential equations modeling the evolution of a phase interface in solid bodies and proved that in the case of one space dimension an initial boundary value problem to this system has global solutions.

This system has been derived in [2, 4] from a sharp interface model for martensitic phase transformations in a solid body. The sharp interface model consists of the equations of linear elasticity theory coupled with an equation posed on the interface, which determines the normal speed of the interface. To find the phase field model, the interface condition was transformed in a first step by rigorous mathematical arguments into a Hamilton-Jacobi transport equation for a smooth order parameter. In a second step a regularizing term was inserted into the Hamilton-Jacobi equation to avoid that the order parameter develops singularities after a finite time. This regularizing term, which

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consists of the Laplace operator with a small positive parameter ν, was inserted such that the second law of thermodynamics holds. For details of this procedure, for mathematical investigations of phase field models and for the background in continuum mechanics we refer to [1], [2] – [7], [8, 9, 10, 12, 13, 14].

This derivation suggests that solutions of this system of partial differential equations converge to solutions of the original sharp interface model for $\nu \to 0$. The usage of the new system of partial differential equations as phase field model for martensitic transformations depends on this asymptotic behavior. Yet, it is not obvious whether this convergence really holds. To verify it, we construct in [7] an asymptotic solution for the system of partial differential equation, which indeed converges to a solution of the sharp interface model for $\nu \to 0$.

The asymptotic behavior of the new phase field model differs in an essential way from the asymptotic behavior of the standard model, which consists of the equations of linear elasticity theory coupled with the Allen-Cahn equation. The asymptotic behavior of this standard model is studied in [11] by formal methods. The result given there shows that in the limit sharp interface model the driving force of the interface motion contains a term with the mean curvature of the interface, which cannot be avoided. On the other hand, the limit model of the new phase field model does not contain such a curvature term. It is possible to make the constant multiplying the mean curvature term in the limit model of the Allen-Cahn model small by choosing a parameter in the Allen-Cahn model appropriately, but in [7] it is shown by analytical considerations and numerical examples, that in this case the numerical solution of the Allen-Cahn model becomes very ineffective, and that when the same physical problem is simulated with the new phase field model the computing time is smaller by a large factor.

This property makes the new phase field model interesting and justifies further investigation. It would be important to prove rigorously that solutions converge to solutions of the sharp interface model for $\nu \to 0$; the result in [7] is formal, since the asymptotic solutions constructed there satisfy the new phase field model only up to an error term in the right hand side of the equations. For a rigorous proof it is necessary to show that the error in the solution caused by this error term in the right hand side tends to zero for $\nu \to \infty$. Such a proof needs an existence result for the phase field model. In this paper we do not estimate this error term, but we continue the investigation of the existence theory, which we started in [3]. There we proved that an initial-boundary value problem to the new phase field model in one space dimension has solutions, if the displacement field and the order parameter both satisfy Dirichlet boundary conditions. Here we show that solutions exist for the one-dimensional problem when the order parameter satisfies homogeneous Neumann boundary conditions.

We next formulate the initial-boundary value problem in one space dimension and the main result of the paper. For the original form of the phase field model in three space dimensions we refer the reader to [3].

Let $\Omega = (a, d)$ be a bounded open interval, which represents the material points of a solid bar. T_e is a positive constant, which can be chosen arbitrarily large. We write $Q_{T_e} = (0, T_e) \times \Omega$ and define

$$(v, \varphi) = \int_Z v(y)\, \varphi(y)\, dy,$$

where $Z = \Omega$ or $Z = Q_{T_e}$. If v is a function defined on Q_{T_e}, we denote the mapping

$x \mapsto v(t,x)$ by $v(t)$. If no confusion is possible we sometimes drop the argument t and write $v = v(t)$. The crystallographic structure of the material can vary in space and time. We assume that two different structures, called phases, are possible. The different phases are characterized by the order parameter $S(t,x) \in \mathbb{R}$. A value of $S(t,x)$ near to zero indicates that the material is in the matrix phase at the point $x \in \Omega$ at time t, a value near to one indicates that the material is in the second phase. The other unknowns are the displacement $u(t,x) \in \mathbb{R}^3$ of the material point x at time t and the Cauchy stress tensor $T(t,x) \in \mathcal{S}^3$, where $\mathcal{S}^3$ denotes the set of symmetric 3×3-matrices. If we denote the first column of the matrix $T(t,x)$ by $T_1(t,x)$ and set

$$\varepsilon(u_x) = \frac{1}{2}\left((u_x,0,0) + (u_x,0,0)^T\right) \in \mathcal{S}^3,$$

then the unknowns must satisfy the quasistatic equations

$$-T_{1x} = b, \tag{1.1}$$
$$T = D(\varepsilon(u_x) - \bar{\varepsilon}S), \tag{1.2}$$
$$S_t = -c\left(\psi_S\big(\varepsilon(u_x),S\big) - \nu S_{xx}\right)|S_x| \tag{1.3}$$

for $(t,x) \in Q_{T_e}$. Since the equations (1.1), (1.2) are linear, the inhomogeneous Dirichlet boundary condition for u can be reduced in the standard way to the homogeneous condition. For simplicity we thus assume that u satisfies homogeneous Dirichlet boundary conditions. The initial and boundary conditions therefore are

$$u(t,x) = 0, \qquad (t,x) \in [0,T_e] \times \partial\Omega, \tag{1.4}$$
$$S_x(t,x) = 0, \qquad (t,x) \in [0,T_e] \times \partial\Omega, \tag{1.5}$$
$$S(0,x) = S_0(x), \qquad x \in \Omega. \tag{1.6}$$

Here $\bar{\varepsilon} \in S^3$ is a given matrix, the misfit strain, and $D : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, positive definite mapping, the elasticity tensor. In the free energy

$$\psi^*(\varepsilon, S, \nabla_x S) = \psi(\varepsilon, S) + \frac{\nu}{2}|\nabla_x S|^2, \tag{1.7}$$

where

$$\psi(\varepsilon,S) = \psi(\varepsilon(\nabla_x u),S) = \frac{1}{2}\big(D(\varepsilon - \bar{\varepsilon}S)\big)\cdot(\varepsilon - \bar{\varepsilon}S) + \hat{\psi}(S), \tag{1.8}$$

we choose for $\hat{\psi} \in C^2(\mathbb{R},[0,\infty))$ a double well potential with minima at $S = 0$ and $S = 1$. The scalar product of two matrices is $A \cdot B = \sum a_{ij}b_{ij}$. Also,

$$\psi_S(\varepsilon,S) = \partial_S\,\psi(\varepsilon,S) = -T\cdot\bar{\varepsilon} + \hat{\psi}'(S)$$

is the partial derivative, $c > 0$ is a constant and ν is a small positive constant. Given are the volume force $b : [0,\infty) \times \Omega \to \mathbb{R}^3$ and the initial data $S_0 : \Omega \to \mathbb{R}$. This completes the formulation of the initial-boundary value problem.

To define weak solutions of this initial-boundary value problem we note that because of $\frac{1}{2}(|y|y)' = |y|$ equation (1.3) is equivalent to

$$S_t - c\nu\frac{1}{2}(|S_x|S_x)_x - c\left(T\cdot\bar{\varepsilon} - \hat{\psi}'(S)\right)|S_x| = 0. \tag{1.9}$$

Definition 1.1. *Let $b \in L^\infty(0,T_e,L^2(\Omega))$, $S_0 \in L^1(\Omega)$. A function (u,T,S) with*

$$u \in L^\infty(0,T_e;W_0^{1,\infty}(\Omega)), \tag{1.10}$$
$$T \in L^\infty(Q_{T_e}), \tag{1.11}$$
$$S \in L^\infty(Q_{T_e}) \cap L^2(0,T_e,H^1(\Omega)), \tag{1.12}$$

is a weak solution to the problem (1.1) – (1.6), if the equations (1.1), (1.2), (1.4) are satisfied weakly and if for all $\varphi \in C_0^\infty((-\infty,T_e)\times\Omega)$

$$(S,\varphi_t)_{Q_{T_e}} - c\nu\frac{1}{2}(|S_x|S_x,\, \varphi_x)_{Q_{T_e}} + c\left(\big(T\cdot\bar{\varepsilon} - \hat{\psi}'(S)\big)|S_x|,\, \varphi\right)_{Q_{T_e}} + (S_0,\varphi(0))_\Omega = 0. \tag{1.13}$$

The main result of this article is

Theorem 1.1 *Assume that there exists a constant $M > 0$ such that the double well potential $\hat{\psi} \in C^1(\mathbb{R},[0,\infty))$ satisfies*

$$S^2 \leq M(\hat{\psi}(S)+1). \tag{1.14}$$

Then to all $S_0 \in H^1(\Omega)$ and $b \in C(\overline{Q}_{T_e})$ with $b_t \in C(\overline{Q}_{T_e})$ there exists a weak solution (u,T,S) of the problem (1.1) – (1.6), which, in addition to (1.10) – (1.13), satisfies

$$S_t \in L^{\frac{4}{3}}(Q_{T_e}), \quad S_x \in L^{\frac{8}{3}}(0,T_e;L^\infty(\Omega)), \tag{1.15}$$

and

$$(|S_x|S_x)_x \in L^{\frac{4}{3}}(Q_{T_e}), \quad S_{xt} \in L^{\frac{4}{3}}(0,T_e;W^{-1,\frac{4}{3}}(\Omega)). \tag{1.16}$$

The remaining sections are devoted to the proof of this theorem. The main difficulty in the proof stems from the fact that the coefficient $\nu|S_x|$ of the highest order derivative S_{xx} in the equation (1.3) is not bounded away from zero and that it is not differentiable with respect to S_x. The equation (1.3) is therefore degenerate parabolic. As in [3], to overcome this difficulty we approximate (1.3) by a uniformly parabolic equation, where the regularizing term depends on a parameter $\kappa > 0$. We derive a-priori estimates for solutions of the regularized initial-boundary value problem and use these estimates to show that a sequence of solutions $(u^\kappa, T^\kappa, S^\kappa)$ of the regularized problem converges to a solution of the original problem. The convergence proof is based on the generalized version of the Aubin-Lions Lemma valid in $L^1(0,T_e;H^{-2}(\Omega))$. We need this generalized version since one of the a-priori estimtes is valid only in this space.

In the existence proof for the inital-boundary value problem with Dirichlet boundary condition given in [3] in use the maximum principle to estimate the L^∞-norm of the functions S^κ. In the case of the Neumann problem this is not possible. We use instead an energy estimate, which is essentially the Clausius-Duhem inequality, to estimate these functions. Also, we use a different regularization of the evolution equation (1.3) and apply a new method to prove Lemma 2.3. Together both modifications lead to a simplification of the proof of this lemma and of the subsequent convergence proof given in Section 3.

The method of proof is limited to one space dimension, since for the a-priori estimates it is crucial that the term $|S_x|S_{xx}$ in (1.3) can be written in the form $\frac{1}{2}(|S_x|S_x)_x$. In higher space dimensions the evolution equation for S, which is stated in [3], contains the term $|\nabla_x S|\Delta_x S$, which cannot be written in this form.

2 A-priori estimates for approximate solutions

In this section we study a modified problem obtained by regularization of the equation (1.3), which we use to construct approximation solutions of (1.1) – (1.6) depending on a small parameter κ.

2.1 Existence of solutions to a modified problem

In this subsection we formulate the modified initial-boundary value problem and show that it has a Hölder continuous classical solution. To this end we choose a function $\chi \in C_0^\infty(\mathbb{R}^+, [0, \infty))$, which satisfies $\int_{-\infty}^{\infty} \chi(t)dt = 1$. For $\kappa > 0$, we set

$$\chi_\kappa(t) := \frac{1}{\kappa}\chi\left(\frac{t}{\kappa}\right),$$

and for $S \in L^\infty(Q_{T_e}, \mathbb{R})$ we define

$$(\chi_\kappa * S)(t, x) = \int_0^{T_e} \chi_\kappa(t - s)S(s, x)ds. \tag{2.1}$$

The modified initial-boundary value problem consists of the equations

$$-T_{1x} = b, \tag{2.2}$$

$$T = D(\varepsilon(u_x) - \bar{\varepsilon}\chi_\kappa * S), \tag{2.3}$$

$$S_t = c\left(\nu S_{xx} + T \cdot \bar{\varepsilon} - \hat{\psi}'(S)\right) |S_x|_\kappa\,, \tag{2.4}$$

which must hold in Q_{T_e}, and of the boundary and initial conditions

$$u(t, x) = 0, \quad (t, x) \in [0, T_e] \times \partial\Omega, \tag{2.5}$$

$$S_x(t, x) = 0, \quad (t, x) \in [0, T_e] \times \partial\Omega, \tag{2.6}$$

$$S(0, x) = S_0(x), \quad x \in \Omega. \tag{2.7}$$

To formulate an existence theorem for this problem we need some function spaces: For nonnegative integers m, n and a real number $\alpha \in (0, 1)$ we denote by $C^{m+\alpha}(\overline{\Omega})$ the space of m-times differentiable functions on $\overline{\Omega}$, whose m-th derivative is Hölder continuous with exponent α. The space $C^{\alpha,\alpha/2}(\overline{Q}_{T_e})$ consists of all functions on $\overline{Q}_{T_e}$, which are Hölder continuous in the parabolic distance

$$\mathrm{d}((t, x), (s, y)) := \sqrt{|t - s| + |x - y|^2}.$$

$C^{m,n}(\overline{Q}_{T_e})$ and $C^{m+\alpha,n+\alpha/2}(\overline{Q}_{T_e})$, respectively, are the spaces of functions, whose x–derivatives up to order m and t–derivatives up to order n belong to $C(\overline{Q}_{T_e})$ or to $C^{\alpha,\alpha/2}(\overline{Q}_{T_e})$, respectively.

Theorem 2.1 *Let $\nu, \kappa > 0$, $T_e > 0$. Suppose that the function $b \in C(\overline{Q}_{T_e})$ has the derivative $b_t \in C(\overline{Q}_{T_e})$ and that the initial data S_0 belong to the space $C^{2+\alpha}(\overline{\Omega})$ and satisfy the compatibility condition $S_{0,x}(x) = 0$ for $x \in \partial\Omega$. Then there exists a solution*

$$(u, T, S) \in C^{2,1}(\overline{Q}_{T_e}) \times C^{1,1}(\overline{Q}_{T_e}) \times C^{2+\alpha,1+\alpha/2}(\overline{Q}_{T_e})$$

to the modified initial-boundary value problem (2.2) – (2.7). This solution satisfies $S_{tx} \in L^2(Q_{T_e})$.

Sketch of the proof of Theorem 2.1. Note that if S is given then for every t the equations (2.2), (2.3), (2.5) form a linear elliptic boundary value problem for the unknown function $x \mapsto (u(t,x), T(t,x))$. In [4] the authors have shown that the unique solution of this problem is given by

$$u(t,x) \;=\; u^* \left(\int_a^x (\chi_\kappa * S)(t,y)dy - \frac{x-a}{d-a} \int_a^d (\chi_\kappa * S)(t,y)dy \right) + w(t,x), \tag{2.8}$$

$$T(t,x) \;=\; D(\varepsilon^* - \bar{\varepsilon})(\chi_\kappa * S)(t,x) - \frac{D\varepsilon^*}{d-a} \int_a^d (\chi_\kappa * S)(t,y)dy + \sigma(t,x), \tag{2.9}$$

where $u^* \in \mathbb{R}^3, \varepsilon^* \in \mathcal{S}^3$ are suitable constants only depending on $\bar{\varepsilon}$ and D, and where for every $t \in [0, T_e]$ the function $(w(t), \sigma(t)) : \Omega \to \mathbb{R}^3 \times \mathcal{S}^3$ is the solution to the boundary value problem

$$\begin{aligned} -\sigma_{1x}(t) &= b(t), \\ \sigma(t) &= D\varepsilon(w_x(t)), \\ w(t)_{|\partial\Omega} &= 0. \end{aligned}$$

Insertion of (2.9) into (2.4) yields the parabolic equation

$$S_t = a_1(S_x)S_{xx} + a_2\left(t, x, S, S_x, \chi_\kappa * S, \frac{1}{d-a}\int_a^d (\chi_\kappa * S)(t,y)dy\right), \tag{2.10}$$

which contains a non-local term. Here the coefficients are defined by

$$\begin{aligned} a_1(p) &= c\nu |p|_\kappa\,, \\ a_2(t,x,S,p,r,s) &= c\left(\bar{\varepsilon}\cdot D(\varepsilon^* - \bar{\varepsilon})r - \bar{\varepsilon}\cdot D\varepsilon^* s + \bar{\varepsilon}\cdot\sigma(t,x) - \hat{\psi}'(S)\right)|p|_\kappa\,. \end{aligned}$$

The statements of Theorem 2.1 follow if we can show that the inital-boundary value problem to the equation (2.10) has a solution with appropriate regularity properties. To verify this we combine an existence theorem for nonlinear parabolic equations given in [15] with some obvious considerations concerning the non-local term in (2.10). Details of the proof can be found in [3].

2.2 A-priori estimates which are uniform in κ

We next construct a family of solutions $(u^\kappa, T^\kappa, S^\kappa)$ of (2.2) – (2.7) and derive a-priori estimates for these solutions, which hold uniformly with respect to the positive parameter κ. To define the solution family, let T_e be a fixed positive number and choose for every κ a function $S_0^\kappa \in C^\infty(\Omega)$ such that

$$\|S_0^\kappa - S_0\|_{H^1(\Omega)} \to 0, \quad \kappa \to 0, \tag{2.11}$$

where $S_0 \in H^1(\Omega)$ are the initial data given in Theorem 1.1. We insert for S_0 in (2.7) the function S_0^κ and choose for b in (2.2) the function given in Theorem 1.1. These functions satisfy the assumptions of Theorem 2.1, hence there is a solution $(u^\kappa, T^\kappa, S^\kappa)$ of the modified problem (2.2) – (2.7), which exists in Q_{T_e}.

In what follows we assume, without loss of generality, that

$$0 < \kappa \le 1, \tag{2.12}$$

since we consider the limit $\kappa \to 0$. The $L^2(\Omega)$-norm is denoted by $\|\cdot\|$, and the letter C stands for universal positive constants independent of κ. The first a-priori estimates are

Lemma 2.1 *There holds for any $t \in [0, T_e]$*

$$\|S^\kappa_x(t)\|^2 + \|S^\kappa(t)\|^2 + \|\varepsilon(u^\kappa_x(t))\|^2 \le C, \tag{2.13}$$

$$\|S^\kappa\|_{L^\infty(Q_{T_e})} \le C, \tag{2.14}$$

$$\|T^\kappa \cdot \bar\varepsilon - \hat\psi'(S^\kappa)\|_{L^\infty(Q_{T_e})} \le C. \tag{2.15}$$

Proof. It is easy to see that (2.14) is a direct consequence of (2.13) by the Sobolev embedding theorem. To prove (2.15) we use (2.13) to conclude that $S^\kappa_x \in L^\infty(0, T_e; L^2(\Omega))$. From this information and from the elliptic regularity theory for the linear elasticity system (2.2) – (2.3) we obtain that

$$\|u^\kappa\|_{L^\infty(0,T_e;H^2(\Omega))} \le C,$$

which implies $u^\kappa_x \in L^\infty(Q_{T_e})$. By (2.3) we then arrive at (2.15).

It remains to prove (2.13). We denote $\varepsilon^\kappa = \varepsilon(u^\kappa_x)$. Let the free energy ψ^* and the flux q be defined by

$$\begin{aligned}
\psi^*(\varepsilon^\kappa, S^\kappa, S^\kappa_x) &= \psi(\varepsilon^\kappa, S^\kappa) + \frac{\nu}{2}|S^\kappa_x|^2 = \frac{1}{2}D(\varepsilon^\kappa - \bar\varepsilon S^\kappa)\cdot(\varepsilon^\kappa - \bar\varepsilon S^\kappa) + \hat\psi(S^\kappa) + \frac{\nu}{2}|S^\kappa_x|^2, \\
q &= q(u^\kappa_t, T^\kappa, S^\kappa_x, S^\kappa_t) = T^\kappa \cdot u^\kappa_t + \nu S^\kappa_t S^\kappa_x .
\end{aligned}$$

If $(u^\kappa, T^\kappa, S^\kappa)$ satisfies (2.2) – (2.4), then a straightforward computation yields

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \psi^*(\varepsilon^\kappa, S^\kappa, S^\kappa_x)dx - \int_\Omega \mathrm{div}_x\, q^\kappa dx - \int_\Omega b\cdot u^\kappa_t dx \\
= & -\int_\Omega \Big(\nu S^\kappa_{xx} + T^\kappa \cdot \bar\varepsilon - \hat\psi_S(S^\kappa)\Big)^2 |S^\kappa_x|_\kappa dx \\
\le & \ 0.
\end{aligned} \tag{2.16}$$

In fact, this inequality means that the Clausius-Duhem inequality holds. From the boundary conditions $u^\kappa_t = 0$ and $S^\kappa_x = 0$ we see that $q = 0$ at the boundary. Thus integrating (2.16) implies

$$\begin{aligned}
\int_\Omega \psi^*(\varepsilon^\kappa, S^\kappa, S^\kappa_x)dx\Big|_0^t &\le \int_0^t \int_\Omega b\cdot u^\kappa_t dx d\tau \\
&= \int_0^t \frac{d}{dt}\int_\Omega b\cdot u^\kappa dx d\tau - \int_{Q_t} b_t \cdot u^\kappa d(\tau, x) \\
&= \int_\Omega b\cdot u^\kappa dx\Big|_0^t - \int_{Q_t} b_t\cdot u^\kappa d(\tau, x).
\end{aligned} \tag{2.17}$$

Hence,

$$\begin{aligned}
&\int_\Omega \psi^*(\varepsilon^\kappa, S^\kappa, S^\kappa_x)(t,x)dx - \int_\Omega \psi^*(\varepsilon^\kappa, S^\kappa, S^\kappa_x)(0,x)dx \\
\le & \int_\Omega (b\cdot u^\kappa)(t,x)dx - \int_\Omega (b\cdot u^\kappa)(0,x)dx + \int_{Q_t}(b_t\cdot u^\kappa)d(\tau,x).
\end{aligned} \tag{2.18}$$

Now we deal with (2.18) term by term. From well known a-priori estimates for the linear elliptic system (2.2) – (2.3) and from the assumptions for S_0 we obtain

$$\|u^\kappa(0)\|_{H^1(\Omega)} \le C, \tag{2.19}$$

and the assumptions for b and Hölder's inequality yield

$$\left|\int_\Omega \psi^*(\varepsilon^\kappa, S^\kappa, S^\kappa_x)(0,x)dx\right| \leq C, \tag{2.20}$$

$$\left|\int_\Omega b(0,x)u^\kappa(0,x)dx\right| \leq C. \tag{2.21}$$

We next use the fact that $u^\kappa(t)$ vanishes at the boundary and that the definition of $\varepsilon(u^\kappa_x)$ implies $|\varepsilon(u^\kappa_x)|^2 \geq \frac{1}{2}|u^\kappa_x|^2$ to conclude from Poincaré's inequality and from the regularity assumptions of b for every $\mu > 0$

$$\left|\int_\Omega bu^\kappa\, dx\right| \leq \|b\|\ \|u^\kappa\| \leq C\|b\|\ \|u^\kappa_x\| \leq \frac{C^2}{2\mu}\|b\|^2 + \frac{\mu}{2}\|u^\kappa_x\|^2 \leq C_\mu + \mu\|\varepsilon^\kappa\|^2, \tag{2.22}$$

$$\left|\int_{Q_t} b_t u^\kappa dx d\tau\right| \leq \int_0^t \|b_t\|\ \|u^\kappa\| d\tau \leq C\int_0^t \left(\|b_t\|^2 + \|u^\kappa_x\|^2\right) d\tau \leq C + C\int_0^t \|\varepsilon^\kappa\|^2 d\tau. \tag{2.23}$$

Combining (2.18) – (2.23) we arrive at

$$\int_\Omega \psi^*(\varepsilon^\kappa, S^\kappa, S^\kappa_x)(t,x)dx \leq C_\mu + \mu\|\varepsilon^\kappa(t)\|^2 + C\int_0^t \|\varepsilon^\kappa(\tau)\|^2 d\tau. \tag{2.24}$$

In order to absorb the term $\mu\|\varepsilon\|^2$ in the right hand side we use assumption (1.14) to find

$$\begin{aligned}
\|\varepsilon^\kappa\|^2 &\leq 2\|\bar{\varepsilon}S^\kappa\|^2 + 2\|\varepsilon^\kappa - \bar{\varepsilon}S^\kappa\|^2 \\
&\leq C\int_\Omega \Big(M(\hat{\psi}(S^\kappa)+1) + \frac{1}{2}(D(\varepsilon^\kappa - \bar{\varepsilon}S^\kappa))\cdot(\varepsilon^\kappa - \bar{\varepsilon}S^\kappa)\Big)\, dx \\
&\leq C\int_\Omega \psi^*(\varepsilon^\kappa, S^\kappa, S^\kappa_x)\, dx\,.
\end{aligned}$$

Thus we can choose μ sufficiently small to obtain

$$\int_\Omega \psi^*(\varepsilon^\kappa, S^\kappa, S^\kappa_x)\, dx \leq C\left(1 + \int_0^t \int_\Omega \psi^*(\varepsilon^\kappa, S^\kappa, S^\kappa_x)\, dx d\tau\right). \tag{2.25}$$

Applying Gronwall's inequality in the integral form we conclude from this inequality that there is C_{T_e} such that for every $t \in [0, T_e]$

$$\int_0^t \int_\Omega \psi^*(\varepsilon^\kappa, S^\kappa, S^\kappa_x)\, dx d\tau \leq C_{T_e}\,. \tag{2.26}$$

Since (1.14) and (1.7) imply

$$|S|^2 + \frac{\nu}{2}|S_x|^2 \leq (M+1)(\psi^*(\varepsilon, S, S_x) + 1),$$

(2.25) becomes

$$\|S^\kappa_x(t)\|^2 + \|S^\kappa(t)\|^2 + \|\varepsilon^\kappa(t)\|^2 \leq C.$$

The proof of the lemma is complete.

Lemma 2.2 *There holds for any* $t \in [0, T_e]$

$$\|S_x^\kappa(t)\|^2 + \nu \int_0^t \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx d\tau \quad \leq \quad C. \tag{2.27}$$

Proof. Observe first that $S_{tx}^\kappa \in L^2(Q_{T_e})$, by Theorem 2.1, which yields that for almost all t

$$\frac{1}{2}\frac{d}{dt}\|S_x^\kappa(t)\|^2 = \int_\Omega S_x^\kappa(t) S_{xt}^\kappa(t) dx.$$

Using this relation, (2.15) and (2.14) we obtain by multiplication of (2.4) by $-S_{xx}^\kappa$ and integration by parts with respect to x, where we take the boundary condition (2.6) into account, that for almost all t

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|S_x^\kappa\|^2 + \nu \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx = c \int_\Omega \left(\hat{\psi}'(S^\kappa) - T^\kappa \cdot \overline{\varepsilon}\right) |S_x^\kappa|_\kappa S_{xx}^\kappa dx \\
&\leq C \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa| dx = C \int_\Omega |S_x^\kappa|_\kappa^{\frac{1}{2}} |S_x^\kappa|_\kappa^{\frac{1}{2}} |S_{xx}^\kappa| dx \\
&\leq \frac{\nu}{2} \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx + \frac{2C^2}{\nu} \int_\Omega (|S_x^\kappa|_\kappa)^2 dx.
\end{aligned} \tag{2.28}$$

We subtract the term $\frac{\nu}{2}\int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx$ on both sides of this inequality and use Gronwall's Lemma to derive (2.27) from the resulting estimate, noting also (2.11). This completes the proof.

From this estimate we obtain more estimates which we collect now.

Corollary 2.1 *There holds for any* $t \in [0, T_e]$

$$\int_0^t \int_\Omega (|S_x^\kappa|_\kappa |S_{xx}^\kappa|)^{\frac{4}{3}} \, dx d\tau \quad \leq \quad C, \tag{2.29}$$

$$\int_0^t \int_\Omega (|S_x^\kappa| \, |S_{xx}^\kappa|)^{\frac{4}{3}} \, dx d\tau \quad \leq \quad C, \tag{2.30}$$

$$\int_0^t \left\| \int_0^{S_x^\kappa} |y|_\kappa dy \right\|_{W^{1,\frac{4}{3}}(\Omega)}^{\frac{4}{3}} d\tau \quad \leq \quad C, \tag{2.31}$$

$$\int_0^t \left\| \int_0^{S_x^\kappa} |y|_\kappa dy \right\|_{L^\infty(\Omega)}^{\frac{4}{3}} d\tau \quad \leq \quad C, \tag{2.32}$$

$$\| \, |S_x^\kappa| S_x^\kappa \|_{L^{\frac{4}{3}}(0,T_e;L^\infty(\Omega))} \quad \leq \quad C, \tag{2.33}$$

$$\int_0^t \|S_x^\kappa\|_{L^\infty(\Omega)}^{\frac{8}{3}} \, d\tau \quad \leq \quad C. \tag{2.34}$$

Proof. By Hölder's inequality, we have for some $2 > p \geq 1, q = \frac{2}{p}$ and $\frac{1}{q} + \frac{1}{q'} = 1$ that

$$\begin{aligned}
& \int_0^t \int_\Omega (|S_x^\kappa|_\kappa |S_{xx}^\kappa|)^p \, dx d\tau \\
= & \int_0^t \int_\Omega (|S_x^\kappa|_\kappa)^{\frac{p}{2}} \left((|S_x^\kappa|_\kappa)^{\frac{p}{2}} |S_{xx}^\kappa|^p \right) dx d\tau \\
\leq & \left(\int_0^t \int_\Omega (|S_x^\kappa|_\kappa)^{\frac{pq'}{2}} \, dx d\tau \right)^{\frac{1}{q'}} \left(\int_0^t \int_\Omega (|S_x^\kappa|_\kappa)^{\frac{pq}{2}} |S_{xx}^\kappa|^{pq} dx d\tau \right)^{\frac{1}{q}} \\
\leq & \left(\int_0^t \int_\Omega (|S_x^\kappa|_\kappa)^{\frac{p}{2-p}} \, dx d\tau \right)^{\frac{2-p}{2}} \left(\int_0^t \int_\Omega |S_x^\kappa|_\kappa |S_{xx}^\kappa|^2 dx d\tau \right)^{\frac{p}{2}}. \qquad (2.35)
\end{aligned}$$

Inequality (2.27) implies for $\frac{p}{2-p} \leq 2$, i.e. $p \leq \frac{4}{3}$, that the right hand side of (2.35) is bounded. This yields the estimate (2.29). Writing

$$|S_x^\kappa|_\kappa S_{xx}^\kappa = \left(\int_0^{S_x^\kappa} |y|_\kappa dy \right)_x, \qquad (2.36)$$

and invoking the boundary condition $S_x^\kappa|_{x \in [0,T_e] \times \partial\Omega} = 0$, we show easily that (2.29) implies (2.31) by the Poincaré inequality, and one has

$$\int_0^t \left\| \int_0^{S_x^\kappa} |y|_\kappa dy \right\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} \leq C,$$

thence we get $\int_0^{S_x^\kappa} |y|_\kappa dy \in L^{\frac{4}{3}}(0, T_e; W^{1,\frac{4}{3}}(\Omega))$. Making use of the Sobolev embedding theorem, we get (2.32).

Sine $\frac{1}{2}(|y|y)' = |y|$ we know that (2.34) is equivalent to (2.33). To prove (2.34), we rewrite $\int_0^{S_x^\kappa} |y|_\kappa dy$ as

$$\begin{aligned}
\int_0^{S_x^\kappa} |y|_\kappa dy &= \int_0^{S_x^\kappa} |y| dy + \int_0^{S_x^\kappa} (|y|_\kappa - |y|) dy \\
&= \frac{1}{2}|y|y \Big|_0^{S_x^\kappa} + \int_0^{S_x^\kappa} \frac{\kappa^2}{|y|_\kappa + |y|} dy \\
&= \frac{1}{2}|S_x^\kappa| S_x^\kappa + \int_0^{S_x^\kappa} \frac{\kappa^2}{|y|_\kappa + |y|} dy. \qquad (2.37)
\end{aligned}$$

Thus

$$\frac{1}{2}(|S_x^\kappa| S_x^\kappa)_x = \left(\int_0^{S_x^\kappa} |y| dy \right)_x = \left(\int_0^{S_x^\kappa} |y|_\kappa dy \right)_x - \frac{\kappa^2 S_{xx}^\kappa}{|S_x^\kappa|_\kappa + |S_x^\kappa|}. \qquad (2.38)$$

From $|y|_\kappa + |y| \geq \kappa$ it is easy to see that

$$\frac{\kappa^2}{|y|_\kappa + |y|} \leq \frac{\kappa^2}{\kappa} = \kappa. \qquad (2.39)$$

By the Young inequality we obtain from (2.27) and the assumption that $k \leq 1$ that

$$\|\kappa S_{xx}^\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})} \leq \left(\int_{Q_{T_e}} \left(\kappa^2 + \kappa |S_{xx}^{\kappa\,2}| \right) d(\tau, x) \right)^{\frac{3}{4}} \leq C.$$

Combination with (2.32), (2.38) and (2.39) yields

$$\|(|S_x^\kappa|S_x^\kappa)_x\|_{L^{\frac{4}{3}}(Q_{T_e})} \le C\left(\left\|\left(\int_0^{S_x^\kappa}|y|_\kappa dy\right)_x\right\|_{L^{\frac{4}{3}}(Q_{T_e})} + \|\kappa S_{xx}^\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})}\right) \le C.$$

Using the boundary condition $S_x^\kappa|_{x\in[0,T_e]\times\partial\Omega} = 0$, and applying again the Poincaré inequality we obtain

$$\||S_x^\kappa|S_x^\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})} \le C,$$

hence

$$\||S_x^\kappa|S_x^\kappa\|_{L^{\frac{4}{3}}(0,T_e;W^{1,\frac{4}{3}}(\Omega))} \le C.$$

Therefore one concludes by the Sobolev embedding theorem that

$$\||S_x^\kappa|S_x^\kappa\|_{L^{\frac{4}{3}}(0,T_e;L^\infty(\Omega))} \le C,$$

that is

$$\|S_x^\kappa\|_{L^{\frac{8}{3}}(0,T_e;L^\infty(\Omega))} \le C.$$

This proves the corollary.

Lemma 2.3 *The function S_t^κ belongs to $L^{\frac{4}{3}}(Q_{T_e})$ and we have the estimates*

$$\|S_t^\kappa\|_{L^{4/3}(Q_{T_e})} \le C\,, \tag{2.40}$$

$$\|(|S_x^\kappa|S_x^\kappa)_t\|_{L^1(0,T_e;H^{-2}(\Omega))} \le C\,. \tag{2.41}$$

Proof. From the equation (2.4) and the estimates (2.29), and (2.27) we immediately see that $S_t^\kappa \in L^{\frac{4}{3}}(Q_{T_e})$ and that (2.40) holds. Therefore we only need to prove the second estimate.

To verify (2.41) we must show that there exists a constant C, which is independent of κ, such that

$$\left|((|S_x^\kappa|S_x^\kappa)_t\,,\varphi)_{Q_{T_e}}\right| \le C\|\varphi\|_{L^\infty(0,T_e;H^2(\Omega))} \tag{2.42}$$

for all functions $\varphi \in L^\infty(0,T_e;H_0^2(\Omega))$. To prove (2.42), we first prove that for any $1 \ge \delta > 0$ there holds

$$\left|\left(\left(\int_0^{S_x^\kappa}|y|_\delta dy\right)_t,\varphi\right)_{Q_{T_e}}\right| \le C\|\varphi\|_{L^\infty(0,T_e;H^2(\Omega))} \tag{2.43}$$

for all functions $\varphi \in L^\infty(0,T_e;H_0^2(\Omega))$. Here δ is independent of κ. Inequality (2.42) is obtained from this estimate as follows: From $S_x^\kappa \in L^\infty(0,T_e,L^2(\Omega)) \subset L^2(Q_{T_e})$, $S_{xt}^\kappa \in L^2(Q_{T_e})$ and $|\,|y|_\delta - |y|\,| \le \delta \to 0$ as $\delta \to 0$ we infer that $\|\,|S_x^\kappa|_\delta - |S_x^\kappa|\,\|_{L^\infty(Q_{T_e})} \to 0$. A straightforward computation yields that

$$\left(\int_0^{S_x^\kappa}|y|_\delta dy\right)_t = |S_x^\kappa|_\delta S_{xt}^\kappa\,. \tag{2.44}$$

Therefore, $\left(\int_0^{S_x^\kappa} |y|_\delta dy\right)_t = |S_x^\kappa|_\delta S_{xt}^\kappa \to |S_x^\kappa|\, S_{xt}^\kappa$ strongly in $L^2(Q_{T_e})$. Whence, as $\delta \to 0$,

$$\left(\left(\int_0^{S_x^\kappa} |y|_\delta dy\right)_t, \varphi\right) \to \frac{1}{2}\left((|S_x^\kappa| S_x^\kappa)_t\,, \varphi\right)_{Q_{T_e}}$$

for all $\varphi \in L^\infty(0, T_e; H_0^2(\Omega)) \subset L^\infty(Q_{T_e})$. This relation together with (2.43) implies (2.42).

Thus it suffices to prove (2.43). To simply the notations we define

$$\mathcal{R}_\kappa := c\left(\nu S_{xx}^\kappa + T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa)\right)|S_x^\kappa|_\kappa\,. \tag{2.45}$$

Multiplying equation (2.4) by $\left(|S_x^\kappa|_\delta \varphi\right)_x$, integrating the resulting equation with respect to (t, x) over Q_{T_e}, using integration by parts for the term with the time derivative and noting (2.44), we obtain

$$\begin{aligned} 0 &= \left(S_t^\kappa - \mathcal{R}_\kappa, (|S_x^\kappa|_\delta \varphi)_x\right)_{Q_{T_e}} \\ &= -\left(S_{xt}^\kappa, |S_x^\kappa|_\delta \varphi\right)_{Q_{T_e}} - \left(\mathcal{R}_\kappa, (|S_x^\kappa|_\delta \varphi)_x\right)_{Q_{T_e}} \\ &= -\left(\left(\int_0^{S_x^\kappa} |y|_\delta dy\right)_t, \varphi\right)_{Q_{T_e}} - \left(\mathcal{R}_\kappa, (|y|_\delta)'\big|_{y=S_x^\kappa} S_{xx}^\kappa \varphi\right) - \left(\mathcal{R}_\kappa, |S_x^\kappa|_\delta \varphi_x\right). \end{aligned} \tag{2.46}$$

Remembering that $S_{xt}^\kappa \in L^2(Q_{T_e})$ for any fixed κ, we see that the first term in the second equality of (2.46) is properly defined. To estimate the last two terms on the right hand side of this inequality we note that there holds

$$\left|(|y|_\delta)'\right| = \left|\frac{y}{|y|_\delta}\right| \le 1 \quad \text{and} \quad |y|_\delta \le |y| + 1,$$

which yields the estimates

$$\begin{aligned} &\left|\left(\mathcal{R}_\kappa, (|y|_\delta)'\big|_{y=S_x^\kappa} S_{xx}^\kappa \varphi\right)_{Q_{T_e}}\right| \le \left|\left(|\mathcal{R}_\kappa|, |S_{xx}^\kappa \varphi|\right)_{Q_{T_e}}\right| \\ \le\ & \left|\left(|S_x^\kappa|_\kappa |S_{xx}^\kappa|^2, |\varphi|\right)_{Q_{T_e}}\right| + \left|\left(|T^\kappa \cdot \varepsilon - \hat{\psi}_S(S^\kappa)|, |S_x^\kappa|_\kappa |S_{xx}^\kappa \varphi|\right)_{Q_{T_e}}\right| \\ \le\ & C\|\varphi\|_{L^\infty(Q_{T_e})} \le C\|\varphi\|_{L^\infty(0,T_e;H^2(\Omega))} + I, \end{aligned} \tag{2.47}$$

and

$$\begin{aligned} \left|\left(\mathcal{R}_\kappa, |S_x^\kappa|_\delta \varphi_x\right)_{Q_{T_e}}\right| &\le C\int_{Q_{T_e}} (|S_x^\kappa| + 1)\, |S_x^\kappa|_\kappa |S_{xx}^\kappa \varphi_x| d(\tau, x) \\ &\quad + C\int_{Q_{T_e}} \left|(T^\kappa \cdot \varepsilon - \hat{\psi}_S(S^\kappa))\right| |S_x^\kappa|_\kappa (|S_x^\kappa| + 1)\, |\varphi_x|\, d(\tau, x) \\ &= I_1 + I_2. \end{aligned} \tag{2.48}$$

We estimate I first. Recalling that $|T^\kappa\cdot\varepsilon-\hat\psi_S(S^\kappa)|\le C$, one has

$$
\begin{aligned}
I &\le C\int_{Q_{T_e}} |S_x^\kappa|_\kappa^{\frac12}|S_x^\kappa|_\kappa^{\frac12}|S_{xx}^\kappa|\,|\varphi|\,d(t,x)\\
&\le C\int_0^{T_e}\|\,|S_x^\kappa|_\kappa^{\frac12}\|_{L^4(\Omega)}\|\,|S_x^\kappa|_\kappa^{\frac12}S_{xx}^\kappa\|\,\|\varphi\|_{L^4(\Omega)}d\tau\\
&\le C\left(\int_0^{T_e}\|\,|S_x^\kappa|_\kappa^{\frac12}S_{xx}^\kappa\|^2d\tau\right)^{\frac12}\left(\int_0^{T_e}\|\varphi\|_{L^4(\Omega)}^2d\tau\right)^{\frac12}\\
&\le C\|\varphi\|_{L^2(0,T_e;L^4(\Omega))}.
\end{aligned}\tag{2.49}
$$

Next, we consider I_1, I_2. The term I_1 can be handled as

$$
\begin{aligned}
I_1 &\le C\int_0^t\|\,|S_x^\kappa|_\kappa^{\frac12}\|_{L^\infty(\Omega)}\|\varphi_x\|_{L^\infty(\Omega)}\left(\int_\Omega(1+|S_x^\kappa|^2)dx\right)^{\frac12}\left(\int_\Omega|S_x^\kappa|_\kappa|S_{xx}^\kappa|^2dx\right)^{\frac12}d\tau\\
&\le C\|\varphi_x\|_{L^\infty(0,T_e;H^1(\Omega))}\int_0^t(\|S_x^\kappa\|_{L^\infty(\Omega)}^{\frac12}+1)\left(\int_\Omega|S_x^\kappa|_\kappa|S_{xx}^\kappa|^2dx\right)^{\frac12}d\tau\\
&\le C\|\varphi\|_{L^\infty(0,T_e;H^2(\Omega))}\left(\int_0^t(\|S_x^\kappa\|_{L^\infty(\Omega)}^{\frac12}+1)^2d\tau\right)^{\frac12}\left(\int_0^t\|\,|S_x^\kappa|_\kappa^{\frac12}S_{xx}^\kappa\|^2d\tau\right)^{\frac12}\\
&\le C\|\varphi\|_{L^\infty(0,T_e;H^2(\Omega))}.
\end{aligned}\tag{2.50}
$$

Here we used the estimates in Lemma 2.2 and Lemma 2.3. The term I_2 is easier to estimate. Using again the inequallity $|T^\kappa\cdot\varepsilon-\hat\psi_S(S^\kappa)|\le C$, we obtain that

$$
\begin{aligned}
I_2 &\le C\int_0^t\int_\Omega|S_x^\kappa|_\kappa(1+|S_x^\kappa|)\,|\varphi_x|dxd\tau\le\int_0^t\int_\Omega(|S_x^\kappa|+1)^2\,|\varphi_x|dxd\tau\\
&\le C\|\varphi_x\|_{L^\infty(Q_{T_e})}\int_0^t\left(\|S_x^\kappa\|^2+1\right)d\tau\\
&\le C\|\varphi\|_{L^\infty(0,T_e;H^2(\Omega))}.
\end{aligned}\tag{2.51}
$$

Combination of (2.46) – (2.51) yields

$$
\begin{aligned}
\left|\left(\left(\int_0^{S_x^\kappa}|y|_\delta dy\right)_t,\varphi\right)_{Q_{T_e}}\right| &\le C(\|\varphi\|_{L^\infty(0,T_e;H_0^2(\Omega))}+\|\varphi\|_{L^2(0,T_e;L^4(\Omega))})\\
&\le C\|\varphi\|_{L^\infty(0,T_e;H_0^2(\Omega))}\,,
\end{aligned}\tag{2.52}
$$

which implies (2.43) and completes the proof.

3 Existence of solutions to the phase field model

In this section we use the a priori estimates established in the previous section to study the convergence of $(u^\kappa, T^\kappa, S^\kappa)$ as $\kappa\to 0$. We shall show that there is a subsequence, which converges to a weak solution of the initial-boundary value problem (1.1) – (1.6), thereby proving Theorem 1.1.

Note first that the estimates (2.13), (2.34), (2.40), the fact that Ω is bounded, and Poincaré's inequality imply

$$\|S^\kappa\|_{W^{1,4/3}(Q_{T_e})}\le C\,,\tag{3.1}$$

for a constant C independent of κ. Hence, we can select a sequence $\kappa_n \to 0$ and a function $S \in W^{1,4/3}(Q_{T_e})$, such that the sequence S^{κ_n}, which we again denote by S^κ, satisfies

$$\|S^\kappa - S\|_{L^{4/3}(Q_{T_e})} \to 0, \qquad S^\kappa_x \rightharpoonup S_x\,, \qquad S^\kappa_t \rightharpoonup S_t\,, \tag{3.2}$$

where the weak convergence is in $L^{4/3}(Q_{T_e})$.

As usual, since equation (2.4) is nonlinear, the weak convergence of S^κ_x is not enough to prove that the limit function solves this equation. In the following lemma we therefore show that S^κ_x converges pointwise almost everywhere:

Lemma 3.1 *There exists a subsequence of S^κ_x, we still denote it by S^κ_x, such that*

$$S^\kappa_x \to S_x, \qquad a.e. \ \ in \ \ Q_{T_e}, \tag{3.3}$$
$$|S^\kappa_x|_\kappa \to |S_x|, \qquad a.e. \ \ in \ \ Q_{T_e}, \tag{3.4}$$
$$|S^\kappa_x|_\kappa \rightharpoonup |S_x|, \qquad weakly \ in \ L^{\frac{4}{3}}(Q_{T_e}), \tag{3.5}$$
$$\int_0^{S^\kappa_x} |y|dy \to \frac{1}{2}S_x|S_x|, \qquad strongly \ \ in \ \ L^{\frac{4}{3}}(0,T_e;L^2(\Omega)), \tag{3.6}$$
$$\int_0^{S^\kappa_x} |y|_\kappa dy \to \frac{1}{2}S_x|S_x|, \qquad strongly \ \ in \ \ L^{\frac{4}{3}}(0,T_e;L^2(\Omega)), \tag{3.7}$$

as $\kappa \to 0$.

The proof is based on the following two results:

Theorem 3.1 *Let B_0 be a normed linear space imbedded compactly into another normed linear space B which is continuously imbedded into a Hausdorff locally convex space B_1. Assume that $1 \le p < +\infty$, that $v, v_i \in L^p(0,T_e;B_0)$ for all $i \in \mathbb{N}$, that the sequence $\{v_i\}_{i\in\mathbb{N}}$ converges weakly to v in $L^p(0,T_e;B_0)$ and that $\{\frac{\partial v_i}{\partial t}\}_{i\in\mathbb{N}}$ is bounded in $L^1(0,T_e;B_1)$. Then v_i converges to v strongly in $L^p(0,T_e;B)$.*

Lemma 3.2 *Let $(0,T_e)\times\Omega$ be an open set in $\mathbb{R}^+\times\mathbb{R}^n$ and assume that $1 < q < \infty$. Suppose that the functions $g_n, g \in L^q((0,T_e)\times\Omega)$ satisfy*

$$\|g_n\|_{L^q((0,T_e)\times\Omega)} \le C, \quad g_n \to g \ \textit{almost everywhere in} \ (0,T_e)\times\Omega.$$

Then g_n converges to g weakly in $L^q((0,T_e)\times\Omega)$.

Theorem 3.1 is a general version of Aubin-Lions lemma valid under the weak assumption $\partial_t v_i \in L^1(0,T_e;B_1)$. This version, which we need here, is proved in [18] and in [17]. A proof of Lemma 3.2 can be found in [16, p.12].

Proof of Lemma 3.1: We choose $p = \frac{4}{3}$ and

$$B_0 = W^{1,\frac{4}{3}}(\Omega), \quad B = L^2(\Omega), \quad B_1 = H^{-2}(\Omega).$$

These spaces satisfy the assumptions of the theorem. Since the estimates (2.29), (2.31) and (2.41) imply that the sequence $\int_0^{S^\kappa_x}|y|dy$ is uniformly bounded in $L^p(0,T_e;B_0)$ for $\kappa \to 0$ and $\left(\int_0^{S^\kappa_x}|y|dy\right)_t$ is uniformly bounded in $L^1(0,T_e;B_1)$, it follows from Theorem 3.1 that there is a subsequence, still denoted by $\int_0^{S^\kappa_x}|y|dy$, which converges strongly in $L^p(0,T_e;B) = L^{\frac{4}{3}}(0,T_e;L^2(\Omega))$ to a limit function $G \in L^{\frac{4}{3}}(0,T_e;L^2(\Omega))$. Consequently, from this sequence we can select another subsequence, denoted in the same way, which converges almost everywhere in Q_{T_e}. Using that the mapping $y \mapsto f(y) := \int_0^y |\xi| d\xi = \frac{1}{2} y|y|$ has a continuous inverse $f^{-1}: \mathbb{R} \to \mathbb{R}$, we infer that also the sequence $S_x^\kappa = f^{-1}\left(\int_0^{S_x^\kappa} |y| dy\right)$ converges pointwise almost everywhere to $f^{-1}(G)$ in Q_{T_e}. From the uniqueness of the weak limit we conclude that $f^{-1}(G) = S_x$ almost everywhere in Q_{T_e}.

For the proof of (3.7) we write

$$\int_0^{S_x^\kappa} |y|_\kappa dy = \int_0^{S_x^\kappa} |y| dy + \int_0^{S_x^\kappa} (|y|_\kappa - |y|) dy = I_1 + I_2.$$

It is easy to estimate I_2 as $\|I_2\|_{L^2(Q_{T_e})} \le \|\kappa S_x^\kappa\|_{L^2(Q_{T_e})} \le C\kappa \|S_x^\kappa\|_{L^\infty(0,T_e;L^2(\Omega))} \le C\kappa \to 0$. Therefore, $\int_0^{S_x^\kappa} |y|_\kappa dy \to \lim_{\kappa\to 0} I_1 = \frac{1}{2}|S_x|S_x$ strongly in $L^{\frac{4}{3}}(0,T_e;L^2(\Omega)$. This is (3.7).

To prove (3.5) we note that the estimate $|S_x^\kappa|_\kappa \le |S_x^\kappa| + \kappa$ and the inequality (3.1) together imply that the sequence $|S_x^\kappa|_\kappa$ is uniformly bounded in $L^{\frac{4}{3}}(Q_{T_e})$. Thus, (3.5) is a consequence of (3.4) and Lemma 3.2.

Proof of Theorem 1.1: Define the functions u and T by

$$u(t,x) = u^* \left(\int_a^x S(t,y) dy - \frac{x-a}{d-a} \int_a^d S(t,y) dy \right) + w(t,x), \tag{3.8}$$

$$T(t,x) = D(\varepsilon^* - \bar{\varepsilon}) S - D\varepsilon^* \frac{1}{d-a} \int_a^d S(t,y) dy + \sigma(t,x), \tag{3.9}$$

where for S we insert the limit function of the sequence S^κ given in (3.2), and where $u^* \in \mathbb{R}^3$, $\varepsilon^* \in \mathcal{S}^3$ and (w,σ) are the same constants and functions as in (2.8) and (2.9). We prove that (u,T,S) is a weak solution of problem (1.1) – (1.6).

Remember first that by Lemma 2.1 we have $S \in L^\infty(Q_{T_e})$. From this relation, from the above definition of u and T and from $(w,\sigma) \in C^{2,1}(\bar{Q}_{T_e}) \times C^{1,1}(\bar{Q}_{T_e})$ we immediately see that u and T satisfy (1.10) and (1.11). Observe next that $\|S^\kappa\|_{L^\infty(0,T_e;H_0^1(\Omega))} \le C$, by Lemma 2.1 and Sobolev's embedding theorem. This implies $S \in L^\infty(0,T_e;H_0^1(\Omega))$, since we can select a subsequence of S^κ which converges weakly to S in this space. Thus, S satisfies (1.12).

It is shown in [4] that the functions u and T defined in this way satisfy the equations (1.1), (1.2) and (1.5). We remarked this previously. It therefore suffices to show that the equations (1.3) and (1.6) are fulfilled in the weak sense. By definition, these equations are satisfied in the weak sense if the relation (1.13) holds. To verify (1.13) we use that by construction (T^κ, S^κ) solves (2.4), (2.6) and (2.7). If we multiply equation (2.4) by a test function $\varphi \in C_0^\infty((-\infty,T_e) \times \Omega)$ and integrate the resulting equation over Q_{T_e} we obtain

$$\begin{aligned} 0 &= (S_t^\kappa, \varphi)_{Q_{T_e}} + \left(-c\nu |S_x^\kappa|_\kappa S_{xx}^\kappa - c\left(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa)\right) |S_x^\kappa|_\kappa, \varphi \right)_{Q_T} \\ &= -(S_0^\kappa, \varphi(0))_\Omega - (S^\kappa, \varphi_t)_{Q_{T_e}} + \left(c\nu \int_0^{S_x^\kappa} |y|_\kappa dy, \varphi_x \right)_{Q_{T_e}} \\ &\quad - \left(c\left(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa)\right) |S_x^\kappa|_\kappa, \varphi \right)_{Q_{T_e}}. \end{aligned}$$

Equation (1.13) follows from this relation if we show that

$$(S_0^\kappa, \varphi(0))_\Omega \quad \to \quad (S_0, \varphi(0))_\Omega, \tag{3.10}$$

$$(S^\kappa, \varphi_t)_{Q_{T_e}} \quad \to \quad (S, \varphi_t)_{Q_{T_e}}, \tag{3.11}$$

$$\left(\int_0^{S_x^\kappa} |y|_\kappa dy, \varphi_x\right)_{Q_{T_e}} \quad \to \quad \left(\frac{1}{2}|S_x|S_x, \varphi_x\right)_{Q_{T_e}}, \tag{3.12}$$

$$\left(\left(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa)\right)|S_x^\kappa|_\kappa, \varphi\right)_{Q_{T_e}} \quad \to \quad \left(\left(T \cdot \bar{\varepsilon} - \hat{\psi}'(S)\right)|S_x|, \varphi\right)_{Q_{T_e}}, \tag{3.13}$$

for $\kappa \to 0$. Now, the relation (3.10) follows from (2.11), and the relation (3.11) is a consequence of (3.2). To verify (3.13) we note that (2.9) and (3.9) yield

$$\begin{aligned} & T^\kappa(t,x) - T(t,x) \\ = \; & D(\varepsilon^* - \bar{\varepsilon})(\chi_\kappa * S^\kappa - S)(t,x) - \frac{D\varepsilon^*}{d-a}\int_a^d (\chi_\kappa * S^\kappa - S)(t,y)dy. \end{aligned} \tag{3.14}$$

From (2.1) and (3.2) we conclude that

$$\begin{aligned} \|\chi_\kappa * S^\kappa - S\|_{L^{\frac{4}{3}}(Q_{T_e})} & \le \|\chi_\kappa * (S^\kappa - S)\|_{L^{\frac{4}{3}}(Q_{T_e})} + \|(S - \chi_\kappa * S)\|_{L^{\frac{4}{3}}(Q_{T_e})} \\ & \le \|(S - \chi_\kappa * S)\|_{L^{\frac{4}{3}}(Q_{T_e})} + \|S^\kappa - S\|_{L^{\frac{4}{3}}(Q_{T_e})} \to 0, \end{aligned}$$

for $\kappa \to 0$. Since ε^* and $\bar{\varepsilon}$ are constants, we infer from this relation and from (3.14) that

$$\|T - T^\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})} \to 0,$$

for $\kappa \to 0$. Thus, after selecting a subsequence we have $T^\kappa \to T$ a.e in Q_{T_e}. Together with (3.3) and (3.4) we see that $(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa))|S_x^\kappa|_\kappa$ tends to $(T \cdot \bar{\varepsilon} - \hat{\psi}'(S))|S_x|$, almost everywhere in Q_{T_e}. Since (2.27) and (2.15) imply that $(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa))|S_x^\kappa|_\kappa$ is uniformly bounded in $L^2(Q_{T_e})$, we deduce from Lemma 3.2 that

$$(T^\kappa \cdot \bar{\varepsilon} - \hat{\psi}'(S^\kappa))|S_x^\kappa|_\kappa \rightharpoonup (T \cdot \bar{\varepsilon} - \hat{\psi}'(S))|S_x|,$$

weakly in $L^2(Q_{T_e})$, which implies (3.13). Consequently (1.13) holds.

It remains to prove that the solution has the regularity properties stated in (1.15) and (1.16). The relation $S_t \in L^{\frac{4}{3}}(Q_{T_e})$ is implied by (3.2). To verify the second assertion in (1.15), we use estimate (2.34) to get

$$\int_0^{T_e} \|S_x^\kappa\|_{L^\infty(\Omega)}^{\frac{8}{3}} dt \le C.$$

This inequality and $S_x^\kappa \rightharpoonup S_x$ in $L^{\frac{8}{3}}(0,T_e;L^\infty(\Omega))$ imply $S_x \in L^{\frac{8}{3}}(0,T_e;L^\infty(\Omega))$.

To prove (1.16), we recall that $\int_0^{S_x^\kappa} |y|_\kappa dy$ converges to $|S_x|S_x$ strongly in the space $L^{\frac{4}{3}}(0,T_e;L^2(\Omega)) \subset L^{\frac{4}{3}}(Q_{T_e})$ and that $\left(\int_0^{S_x^\kappa} |y|_\kappa dy\right)_x$ is uniformly bounded in $L^{\frac{4}{3}}(Q_{T_e})$ for $\kappa \to 0$, by (2.29). This together implies that $(|S_x|S_x)_x \in L^{\frac{4}{3}}(Q_{T_e})$. Finally, to prove the second assertion of (1.16) we choose a test function $\varphi \in L^4(0,T_e,W_0^{1,4}(\Omega))$, multiply equation (2.4) by $-\varphi_x$ and integrate the resulting equation over Q_{T_e} to obtain

$$0 = (S_t^\kappa - \mathcal{R}_\kappa, -\varphi_x)_{Q_{T_e}} = (S_{xt}^\kappa, \varphi)_{Q_{T_e}} + (\mathcal{R}_\kappa, \varphi_x)_{Q_{T_e}}, \tag{3.15}$$

with $\mathcal{R}_\kappa$ defined in (2.45). Invoking the estimates (2.29), (2.15) and (2.27) we deduce that

$$\|\mathcal{R}_\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})} \le C,$$

hence (3.15) yields

$$(S^\kappa_{xt}, \varphi)_{Q_{T_e}} \le \|\mathcal{R}_\kappa\|_{L^{\frac{4}{3}}(Q_{T_e})} \|\varphi_x\|_{L^4(Q_{T_e})} \le C\|\varphi\|_{L^4(0,T_e;W^{1,4}_0(\Omega))},$$

and this means that S^κ_{xt} is uniformly bounded in $L^{\frac{4}{3}}(0, T_e; W^{-1,\frac{4}{3}}(\Omega))$. From this estimate and from $S^\kappa_t \rightharpoonup S_t$ in $L^{\frac{4}{3}}(Q_{T_e})$ we deduce easily that S_{xt} belongs to the dual space of $L^4(0, T_e; W^{1,4}_0(\Omega))$, which is $L^{\frac{4}{3}}(0, T_e; W^{-1,\frac{4}{3}}(\Omega))$.

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