Periodic solutions of the Navier-Stokes equations with inhomogeneous boundary conditions

Takahiro Okabe

Mathematical Institute, Tohoku University, Sendai 980-8578, Japan

e-mail: sa6m08@math.tohoku.ac.jp

Abstract

We show the existence of time periodic solutions of the Navier-Stokes equations in bounded domains of $\mathbb{R}^3$ with inhomogeneous boundary conditions in the strong and weak sense. In particular, for weak solutions, we deal with more generalized conditions on the boundary data for Leray's problem.

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1 Introduction.

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. In this paper, we consider the Navier-Stokes equations with inhomogeneous boundary data of Dirichlet type:

$$\text{(N-S)}\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = f, & \text{in } \Omega \times (t_0, t_1), \\ \text{div } u = 0, & \text{in } \Omega \times (t_0, t_1), \\ u|_{\partial\Omega} = \beta, & \text{on } \partial\Omega \times (t_0, t_1), \\ u(t_0) = \tilde{a}, & \text{in } \Omega, \quad \text{(if necessary)} \end{cases}$$

where $-\infty \le t_0 < t_1 \le \infty$, $u = u(x,t) = (u^1(x,t), u^2(x,t), u^3(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and pressure of the fluid at $(x,t) \in \Omega \times (t_0, t_1)$, respectively, while $f = f(x,t) = (f^1(x,t), f^2(x,t), f^3(x,t))$ is the given external force at $(x,t) \in \Omega \times (t_0,t_1)$, $\beta = \beta(x) = (\beta^1(x), \beta^2(x), \beta^3(x))$ is the given boundary data and $\tilde{a}(x) = (\tilde{a}^1(x), \tilde{a}^2(x), \tilde{a}^3(x))$ is the given initial data at $x \in \partial\Omega$.

The purpose of this paper is to prove that if the external force f is periodic in time, then there exists a solution of (N-S) which has the same period as f.

Kaniel-Shinbrot [5] considered the reproductive property, and showed the existence of periodic solutions with small periodic forces f. For the two-dimensional case, Takeshita [11] got the same result as Kaniel-Shinbrot [5] without assuming the smallness of f. Miyakawa-Teramoto [10] showed the periodic weak solution on a bounded domain whose boundary moves periodically in time. On the other hand, Kozono-Nakao [6] and Yamazaki [12] obtained the existence of strong periodic solutions with homogeneous boundary condition in unbounded domains when f is small.

As for the stationary problem, Leray proposed to solve the Navier-Stokes equations with inhomogeneous boundary data in $H^{1/2}(\partial\Omega)$. Up to now, we are not yet successful to give a complete answer to this problem. However, based on the Helmholtz-Weyl decomposition, Kozono-Yanagisawa [8] recently solved this problem under a more generalized condition on the boundary data in $H^{1/2}(\Omega)$.

In the present paper, we shall show the existence of strong and weak periodic solutions of (N-S) with the inhomogeneous boundary condition. To prove the existence of strong periodic solutions, we first consider the boundary value problem of the steady Navier-Stokes equations:

$$\text{(S)} \qquad \begin{cases} -\Delta v + v\cdot\nabla v + \nabla\pi = 0, & \text{in } \Omega,\\ \operatorname{div} v = 0, & \text{in } \Omega,\\ v = \beta, & \text{on } \partial\Omega. \end{cases}$$

With a solution v of (S) we can reduce the problem (N-S) to the following equations:

$$\text{(N-S}'\text{)} \qquad \begin{cases} \dfrac{\partial w}{\partial t} - \Delta w + v\cdot\nabla w + w\cdot\nabla v + w\cdot\nabla w + \nabla p' = f, & \text{in } \Omega\times\mathbb{R},\\ \operatorname{div} w = 0, & \text{in } \Omega\times\mathbb{R},\\ w|_{\partial\Omega} = 0, & \text{on } \partial\Omega\times\mathbb{R}. \end{cases}$$

To prove the existence of time periodic solutions to (N-S$'$), we need to introduce an operator $\mathcal{L}$ defined by

$$\mathcal{L}w = Aw + P(v\cdot\nabla w + w\cdot\nabla v),$$

where A denotes the usual Stokes operator and P is the Helmholtz projection. It is important to show that $-\mathcal{L}$ generates a bounded analytic semigroup $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ in L^r_σ as well as the L^q-L^r estimates. In particular, the asymptotic behavior $\|e^{-t\mathcal{L}}a\|_r = O(e^{-\beta t})$ as $t\to\infty$ for some $\beta > 0$ plays an essential role in constructing time periodic strong solutions.

Concerning the weak solutions, we establish a reproductive property of (N-S). To this end, similarly to (N-S$'$), we introduce the perturbed equations such as:

$$\text{(N-S}^*\text{)} \qquad \begin{cases} \dfrac{\partial w}{\partial t} - \Delta w + b\cdot\nabla w + w\cdot\nabla b + w\cdot\nabla w + \nabla p' = F, & \text{in } \Omega\times(0,T),\\ \operatorname{div} w = 0, & \text{in } \Omega\times(0,T),\\ w|_{\partial\Omega} = 0, & \text{on } \partial\Omega\times(0,T),\\ w(0) = a, & \text{in } \Omega, \end{cases}$$

where the coefficient b may satisfy div $b = 0$ and $b|_{\partial\Omega} = \beta$ and $F = f + \Delta b - b \cdot \nabla b$. Under some restriction on b, for an arbitrary large F, we prove the reproductive property which may be regarded as generalization of periodicity. In particular, in the two-dimensional case, our weak solution is actually the periodic solution of (N-S*) with the same period as f.

2 Results.

Before stating our results, we impose the following assumption on the domain Ω and the boundary value β. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^\infty$, and assume that

$$\partial\Omega = \bigcup_{j=0}^{L} \Gamma_j,$$

where

(i) $\Gamma_0, \ldots, \Gamma_L$ are C^∞-surfaces,

(ii) $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$,

(iii) $\Gamma_1, \ldots, \Gamma_L$ are inside of Γ_0, and outside of one another.

Throughout this paper, we impose the *general flux condition* (G.F.) on the boundary data β, i.e.,

$$\text{(G.F.)} \qquad \sum_{j=0}^{L} \int_{\Gamma_j} \beta \cdot \nu \, dS = 0,$$

where ν denotes the unit outer normal to $\partial\Omega$.

We shall next introduce some notations and function spaces. The space $C_{0,\sigma}^\infty(\Omega)$ denotes the set of all C^∞-real vector fields ϕ with compact support in Ω such that div $\phi = 0$. The space $L_\sigma^r(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ with respect to the L^r-norm $\|\cdot\|_r$; $(\cdot, \cdot)$ is the L^r-$L^{r'}$ pairing with $1/r + 1/r' = 1$. Here, $L^r(\Omega)$ stands for the usual (vector-valued) L^r-space in Ω, $1 \leq r \leq \infty$ and $H_{0,\sigma}^{m,r}(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ with respect to the usual $W^{m,r}$-norm $\|\cdot\|_{W^{m,r}}$. When X is a Banach space, we denote by $\|\cdot\|_X$ the norm on X and $\boldsymbol{B}(X)$ denotes the set of all bounded operators on X. Furthermore, $C^m([t_1, t_2]\,; X)$, $BC([t_1, t_2]\,; X)$ and $L^r(t_1, t_2\,; X)$ are the usual Banach spaces of X-valued functions on $[t_1, t_2]$, where $m = 0, 1, \ldots,$ and t_1 and t_2 are real numbers such that $t_1 < t_2$.

Let us define the Stokes operator A_r in $L_\sigma^r(\Omega)$. We have the following Helmholtz decomposition:

$$L^r(\Omega) = L_\sigma^r(\Omega) \oplus G^r(\Omega), \quad 1 < r < \infty,$$

where $G^r(\Omega) = \{\nabla p \in L^r(\Omega)\,; p \in W^{1,r}(\Omega)\}$. P denotes the projection operator from $L^r(\Omega)$ onto $L_\sigma^r(\Omega)$. The Stokes operator A_r is defined by $A_r = -P\Delta$ with the domain $D(A_r) = W^{2,r}(\Omega) \cap H^{1,r}_{0,\sigma}(\Omega)$. It is known that the adjoint operator A_r^* of A_r is $A_{r'}$ with $1/r + 1/r' = 1$. We abbreviate A_r to A, if we have no confusion. Moreover, we have the embedding estimate:

$$\|u\|_q \le C\|Au\|_r, \quad \text{for } u \in D(A_r), \quad \frac{1}{q} = \frac{1}{r} - \frac{2}{3},$$

where $C = C(q, r, \Omega) > 0$.

Definition 2.1. *Let $1 < r < \infty$ and $v \in W^{1,3/2}(\Omega)$ with $\operatorname{div} v = 0$. Then we define the operator $\mathcal{L}_v$ on L^r_σ with $D(\mathcal{L}_v) = W^{2,r}(\Omega) \cap H^{1,r}_{0,\sigma}(\Omega)$ associated with v by:*

$$\mathcal{L}_v w := Aw + B_v w \quad \text{for } w \in D(\mathcal{L}_v), \tag{2.1}$$

where B_v is defined by

$$B_v w := P(v \cdot \nabla w + w \cdot \nabla v).$$

Similarly to A, we abbreviate $\mathcal{L}_v$ and B_v to $\mathcal{L}$ and B, respectively, when the vector field v is known from the context.

Proposition 2.1. *For every $3/2 < r < \infty$, there exists $\eta = \eta(r) > 0$ such that if $v \in W^{1,3/2}(\Omega)$ with $\operatorname{div} v = 0$ satisfies $\|v\|_3 < \eta$, then $-\mathcal{L}_v$ generates a bounded analytic semigroup $\{e^{-t\mathcal{L}_v}\}_{t\ge 0}$ on $L^r_\sigma(\Omega)$.*

For the proof, see Lemma 3.2.

Making use of the operator $\mathcal{L}_v$ and its semigroup $\{e^{-t\mathcal{L}}\}_{t\ge 0}$, we introduce an abstract evolution equation (N-S$''$) and the integral equation (I.E.) related to (N-S$'$):

$$\frac{dw}{dt} + \mathcal{L}w + P[w \cdot \nabla w] = Pf, \tag{N-S$''$}$$

$$w(t) = \int_{-\infty}^t e^{-(t-s)\mathcal{L}} Pf(s)\, ds - \int_{-\infty}^t e^{-(t-s)\mathcal{L}} P[w \cdot \nabla w](s)\, ds, \tag{I.E.}$$

for all $t \in \mathbb{R}$.

We first show the existence of strong solutions to the stationary problem (S). Indeed, we have:

Theorem 2.1. *Let $1 < p < 3/2$, $1/p^* = 1/p - 1/3$, and let $\beta \in W^{2-1/p,p}(\partial\Omega)$ satisfying (G.F.). For every $\varepsilon > 0$ there exists $\gamma = \gamma(\varepsilon, p) > 0$ such that if $\|\beta\|_{W^{1-1/p^*,p^*}(\partial\Omega)} < \gamma$, then there is a solution $v \in W^{2,p}(\Omega)$ of (S) with $\|v\|_3 \le \varepsilon$.*

We next consider the existence of time periodic solutions to (I.E.).

Theorem 2.2. *For every $3/2 < r < 3$, $2 < q < 3$ with $1/r + 1/3 < 2/q$ and $3/2 < l < \infty$, there is a constant $\delta = \delta(r, q, l)$ with the following properties. Suppose* *that* $f \in BC(\mathbb{R}; L^l(\Omega))$ *is periodic in time with the period* T_**, i.e.,* $f(t) = f(t + T_*)$ *for all* $t \in \mathbb{R}$ *and that* $v \in W^{1,3/2}(\Omega)$ *with* $\operatorname{div} v = 0$*. If* f *and* v *satisfy*

$$\|v\|_3 + \sup_{s\in\mathbb{R}} \|f(s)\|_l < \delta, \tag{2.2}$$

then there exists a solution w *of (I.E.) with the property* $w(t) = w(t + T_*)$ *for all* $t \in \mathbb{R}$ *in the class* $w \in BC(\mathbb{R}; L^r_\sigma(\Omega))$ *with* $\nabla w \in BC(\mathbb{R}; L^q(\Omega))$*. Moreover, such a solution* w *is unique within this class provided* $\sup_{s\in\mathbb{R}} \|w(s)\|_r + \sup_{s\in\mathbb{R}} \|\nabla w(s)\|_q$ *is sufficiently small.*

The solution w given by Theorem 2.2 is actually a solution of (N-S$''$) provided f is regular in time.

Theorem 2.3. *In addition to the hypotheses of Theorem 2.2, let us assume that* f *is a Hölder continuous function on* $\mathbb{R}$ *in* $L^3(\Omega)$*. Then the periodic solution* w *given by Theorem 2.2 has the following additional properties:*

(i) $w \in BC(\mathbb{R}; L^3_\sigma(\Omega)) \cap C^1(\mathbb{R}; L^3_\sigma(\Omega))$,

(ii) $w(t) \in D(\mathcal{L})$ *for* $t \in \mathbb{R}$ *and* $\mathcal{L}w \in C(\mathbb{R}; L^3_\sigma(\Omega))$,

(iii)

$$\text{(N-S}''\text{)} \qquad \frac{dw}{dt} + \mathcal{L}w + P[w \cdot \nabla w] = Pf \quad \text{in } L^3_\sigma(\Omega) \quad \text{for all } t \in \mathbb{R}.$$

Now we conclude that we obtain time periodic solutions of (N-S) from Theorem 2.1, Theorem 2.2 and Theorem 2.3, i.e., the corollary stated below immediately follows.

Corollary 2.1. *For* $3/2 < l < \infty$, $1 < p < 3/2$*, there exists a constant* $\delta = \delta(l, p) > 0$ *with the following property. Suppose that* $\beta \in W^{2-1/p,p}(\partial\Omega)$*. Assume that* $f \in BC(\mathbb{R}; L^l(\Omega))$ *is periodic with the period* T_**, i.e.,* $f(t) = f(t + T_*)$ *for all* $t \in \mathbb{R}$*, and is Hölder continuous on* $\mathbb{R}$ *with values in* $L^3(\Omega)$*. If* β *and* f *satisfy*

$$\|\beta\|_{W^{1-1/p^*,p^*}(\partial\Omega)} + \sup_{s\in\mathbb{R}} \|f(s)\|_l < \delta, \quad \text{with} \quad p^* = \frac{3p}{3-p},$$

then there is a solution $u \in BC(\mathbb{R}; L^3(\Omega))$ *of (N-S) with* $u(t) = u(t + T_*)$ *for all* $t \in \mathbb{R}$.

Remark 2.1. (1) By Theorem 2.2, we obtain a time periodic strong solution w of (N-S$'$) if the solenoidal function v and the external force f is small. However, our aim is to obtain a time periodic solution of (N-S). To this end, we take v in Theorem 2.2 as the steady solution of (S). On the other hand, Theorem 2.1 ensures that if the boundary data β is small, the steady solution v is small. Consequently, we obtain a time periodic solution of (N-S).

(2) To construct time periodic strong solutions, we need the smallness of the boundary data β and the external force f. It seems to be hard to obtain time periodic solutions without these smallness assumption on β and f. For this reason, we introduce the reproductive property which may be regarded as generalization of time periodicity.

Next we define weak solutions of (N-S*).

Definition 2.2. *Let $b \in H^1(\Omega)$ with* div $b = 0$, *$a \in L^2_\sigma(\Omega)$ and $f \in L^2(0, T; L^2(\Omega))$. A measurable function w on $\Omega \times (0, T)$ is called a weak solution of the initial-boundary value problem (N-S*) on $(0, T)$ if*

(i) $w \in L^\infty(0, T; L^2_\sigma(\Omega))$,

(ii) $w \in L^2(0, T'; H^1_{0,\sigma}(\Omega))$ *for any* $T' \in (0, T)$,

(iii)

$$\int_0^T \{-(w, \partial_t \Phi) + (\nabla w, \nabla \Phi) + (b \cdot \nabla w + w \cdot \nabla b, \Phi) + (w \cdot \nabla w, \Phi)\} d\tau$$
$$= \int_0^T \langle F, \Phi \rangle d\tau + (a, \Phi(0)),$$

for all $\Phi \in H^1(0, T; H^1_{0,\sigma}(\Omega))$ such that, for some T_0, $\Phi(t) = 0$ on $t \in (T_0, T)$, where $\langle F, \Phi \rangle := (f, \Phi) - (\nabla b, \nabla \Phi) - (b \cdot \nabla b, \Phi)$.

To show the existence of a weak solution w of (N-S*), we need to introduce the harmonic vector fields $V_{har}(\Omega)$ on Ω defined by

$$V_{har}(\Omega) := \{h \in C^\infty(\bar{\Omega}) \,;\, \mathrm{div}\, h = 0,\ \mathrm{rot}\, h = 0 \text{ in } \Omega,\ h \times \nu = 0 \text{ on } \partial\Omega\}$$

It is shown by Kozono-Yanagisawa [8] that $\dim V_{har} = L$ and that

$$V_{har}(\Omega) = \mathrm{span}\,\{h_1, \dots, h_L\} \quad \text{with } h_j = \nabla q_j,$$

where $\{q_j\}_{j=1}^L$ are harmonic functions on Ω such that

$$\Delta q_j = 0 \quad \text{in } \Omega, \quad q_j|_{\Gamma_0} = 0, \quad q_j|_{\Gamma_i} = \delta_{ij}, \quad i, j = 1, \dots L.$$

Instead of $\{h_j\}_{j=1}^L$, it is useful to take an orthonormal basis $\{\psi_1, \dots, \psi_L\}$ in L^2-sense. More precisely, there exists a regular $L \times L$ matrix $(\alpha_{jk})_{j,k=1}^L$ depending only on Ω such that

$$\psi_j(x) = \sum_{k=1}^L \alpha_{jk} h_k(x), \quad j = 1, \dots, L, \tag{2.3}$$

and such that

$$(\psi_i, \psi_j) = \delta_{ij}. \tag{2.4}$$

Now our existence result on weak solutions of (N-S*) reads:

Theorem 2.4. *Let $a \in L^2_\sigma$ and $f \in L^2_{loc}([0,\infty); L^2)$. Suppose that $\beta \in H^{1/2}(\partial\Omega)$ satisfies (G.F.) with the restriction*

$$\Big\| \sum_{j,k=1}^{L} \alpha_{jk} \Big(\int_{\Gamma_k} \beta \cdot \nu dS \Big) \psi_j \Big\|_3 < \frac{1}{4C_s}, \tag{2.5}$$

where $C_s = 3^{-1/2} 2^{2/3} \pi^{-2/3}$ is the best constant of the Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$. Then there exist $b \in H^1(\Omega)$ with $\operatorname{div} b = 0$ and $b|_{\partial\Omega} = \beta$, and a function $w \in L^\infty_{loc}([0,\infty); L^2_\sigma) \cap L^2_{loc}([0,\infty); H^1_{0,\sigma})$ such that w gives a weak solution of (N-S) on $(0,T)$ for all $0 < T < \infty$.*

As an application of Theorem 2.4, we show a reproductive property of the weak solution to (N-S*). This implies that for an arbitrary prescribed time interval $[0, T_*]$, we can construct an initial data a and a weak solution $w(t)$ which behaves at $t = T_*$ in the same way as the initial state a.

Theorem 2.5. *Let f and β satisfy the hypothesis of Theorem 2.4. Then for every $0 < T_* < \infty$, there exists an initial value $a \in L^2_\sigma(\Omega)$ and a weak solution w of (N-S*) on $(0,\infty)$ having the property $w(T_*) = w(0) = a$ in $L^2_\sigma(\Omega)$.*

Remark 2.2. In the two dimensional case, by uniqueness of weak solutions for the initial value problem, we remark that the reproductive property necessarily yields the time periodicity of weak solutions provided f is time periodic. Hence, Theorem 2.5 shows that the existence of time reproductive solutions in two dimensional multi-connected domains without any smallness assumption on β.

3 L^q-L^r estimates for the semigroup $e^{-t\mathcal{L}}$.

In this section, we show that $-\mathcal{L}_v$ generates a bounded analytic semigroup $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ in $L^r_\sigma(\Omega)$ and satisfies an L^q-L^r estimate.

In what follows, we denote by C various constants.

Let us introduce the following operator.

Definition 3.1. *Let $1 < r < 3$ and $v \in W^{1,3/2}(\Omega)$ with $\operatorname{div} v = 0$. We define the operator $B' = B'_v$ and $\mathcal{L}' = \mathcal{L}'_v$ on $L^r_\sigma(\Omega)$ with $D(\mathcal{L}') = W^{2,r}(\Omega) \cap H^{1,r}_{0,\sigma}(\Omega)$ associated with v by:*

$$\begin{aligned} B'_v w &:= -P\Big[v \cdot \nabla w + \sum_{j=1}^{3} v^j \nabla w^j \Big], \\ \mathcal{L}'_v w &:= Aw + B'_v w, \end{aligned} \tag{3.1}$$

for all $w \in D(\mathcal{L}'_v)$.

Remark 3.1. Since div $v = 0$, it is easy to see that the adjoint operator $(\mathcal{L}_v)^*$ of $\mathcal{L}_v$ on $L^r_\sigma(\Omega)$ is the operator $\mathcal{L}'_v$ on $L^{r'}_\sigma(\Omega)$.

To begin with, we investigate the adjoint operator $\mathcal{L}'_v$ instead of $\mathcal{L}_v$.

Lemma 3.1. *For every $1 < r < 3$, there exists $\tilde{\eta} = \tilde{\eta}(r) > 0$ such that if $v \in W^{1,3/2}(\Omega)$ with* div $v = 0$ *satisfies $\|v\|_3 < \tilde{\eta}$, then $-\mathcal{L}'_v$ generates a bounded analytic semigroup on $L^r_\sigma(\Omega)$.*

Proof. Since $-A$ generates a bounded analytic semigroup on $L^r_\sigma(\Omega)$, there exists a sector

$$\Sigma^{\mu,\delta}_{-A} := \Big\{\lambda \in \mathbb{C}\,;\, |\arg(\lambda - \mu)| < \delta + \frac{\pi}{2}\Big\} \subset \rho(-A), \tag{3.2}$$

for some $\mu < 0$ where $\rho(-A)$ is the resolvent set of $-A$. Moreover, there exists $M_r \geq 1$ such that for each $\lambda \in \Sigma^{\mu,\delta}_{-A}$ the resolvent estimate

$$\|(\lambda + A)^{-1}\|_{\boldsymbol{B}(L^r_\sigma)} \leq \frac{M_r}{1 + |\lambda|}, \tag{3.3}$$

holds, see e.g. [3].

We show the existence of the resolvent $(\lambda + \mathcal{L}')^{-1}$ for any $\lambda \in \Sigma^{\mu,\delta}_{-A}$.

For every $\lambda \in \Sigma^{\mu,\delta}_{-A}$, we have

$$\begin{aligned}(\lambda + \mathcal{L}') &= (\lambda + A + B') \\ &= (1 + B'(\lambda + A)^{-1})(\lambda + A).\end{aligned} \tag{3.4}$$

Hence, using a Neumann series, it is suffices to show that

$$\|B'(\lambda + A)^{-1}\|_{\boldsymbol{B}(L^r_\sigma)} < 1. \tag{3.5}$$

By the definition (3.1), we have

$$\begin{aligned}\|B'(\lambda + A)^{-1}w\|_r &\leq \Big\|-P\Big[v \cdot \nabla(\lambda + A)^{-1}w + \sum_{j=1}^{3} v^j \nabla[(\lambda + A)^{-1}w]^j\Big]\Big\|_r \\ &\leq C\Big(\|v \cdot \nabla(\lambda + A)^{-1}w\|_r + \Big\|\sum_{j=1}^{3} v^j \nabla[(\lambda + A)^{-1}w]^j\Big\|_r\Big) \\ &=: I_1 + I_2.\end{aligned} \tag{3.6}$$

By (3.3) and the Sobolev inequality, we have

$$\begin{aligned}I_1 &= \|v \cdot \nabla(\lambda + A)^{-1}w\|_r \\ &\leq \|v\|_3 \|\nabla(\lambda + A)^{-1}w\|_{3r/(3-r)} \\ &\leq C\|v\|_3 \|\nabla^2(\lambda + A)^{-1}w\|_r \\ &\leq C\|v\|_3 \|w\|_r,\end{aligned} \tag{3.7}$$

and by the same way as (3.7),

$$I_2 = \Big\| \sum_{j=1}^{3} v^j \nabla [(\lambda + A)^{-1}]^j w \Big\|_r \leq C \|v\|_3 \|w\|_r. \tag{3.8}$$

Hence (3.7) and (3.8) yield that

$$\|B'(\lambda + A)^{-1}\|_{\boldsymbol{B}(L^r_\sigma)} \leq C_1 \|v\|_3, \tag{3.9}$$

where the constant C_1 is independent of λ. To obtain (3.5), it suffices to take $\tilde{\eta}$ in Lemma 3.1 so that $C_1\|v\|_3 < 1$. Then we obtain the estimate of the resolvent $(\lambda + \mathcal{L}')^{-1}$:

$$\|(\lambda + \mathcal{L}')^{-1}\|_{\boldsymbol{B}(L^r_\sigma)} \leq \frac{M_r}{1 + |\lambda|} \frac{1}{1 - C_1\|v\|_3}. \tag{3.10}$$

Hence (3.10) guarantees the generation of the bounded analytic semigroup $\{e^{-t\mathcal{L}'}\}_{t \geq 0}$ on $L^r_\sigma(\Omega)$. This completes the proof of Lemma 3.1. □

With aid of $\mathcal{L}'_v$ and its resolvent estimate (3.10), we show that $-\mathcal{L}_v$ generates a bounded analytic semigroup on $L^r_\sigma(\Omega)$.

Lemma 3.2. *For every $3/2 < r < \infty$, there exists a constant $\eta = \eta(r) > 0$ such that if $v \in W^{1,3/2}(\Omega)$ with* div $v = 0$ *satisfies $\|v\|_3 < \eta$, then $-\mathcal{L}_v$ generates a bounded analytic semigroup on $L^r_\sigma(\Omega)$.*

Proof. For each $\lambda \in \Sigma^{\mu,\delta}_{-A}$, we show the existence of $(\lambda + \mathcal{L})^{-1}$. By the definition (2.1),

$$\begin{aligned} (\lambda + \mathcal{L}) &= (\lambda + A + B) \\ &= (\lambda + A)^{1/2} \left(1 + (\lambda + A)^{-1/2} B (\lambda + A)^{-1/2}\right) (\lambda + A)^{1/2}. \end{aligned} \tag{3.11}$$

Hence it suffices to show that

$$\|(\lambda + A)^{-1/2} B (\lambda + A)^{-1/2}\|_{\boldsymbol{B}(L^r_\sigma)} < 1. \tag{3.12}$$

By duality, we have

$$\begin{aligned} |\langle(\lambda + A)^{-1/2} B (\lambda + A)^{-1/2} w, \phi\rangle| &= |\langle w, (\lambda + A)^{-1/2} B' (\lambda + A)^{-1/2} \phi\rangle| \\ &\leq \|w\|_r \|(\lambda + A)^{-1/2} B' (\lambda + A)^{-1/2} \phi\|_{r'}, \end{aligned} \tag{3.13}$$

and by the Sobolev inequality,

$$\begin{aligned} &\|(\lambda + A)^{-1/2} B' (\lambda + A)^{-1/2} \phi\|_{r'} \\ &\leq C \|B' (\lambda + A)^{-1/2} \phi\|_{3r'/(3+r')} \\ &\leq C \Big(\|v \cdot \nabla (\lambda + A)^{-1/2} \phi\|_{3r'/(3+r')} + \Big\| \sum_{j=1}^{3} v^j \nabla [(\lambda + A)^{-1/2} \phi]^j \Big\|_{3r'/(3+r')} \Big) \\ &\leq C \|v\|_3 \|\nabla (\lambda + A)^{-1/2} \phi\|_{r'} \\ &\leq C \|v\|_3 \|\phi\|_{r'}, \end{aligned} \tag{3.14}$$

for all $\phi \in C^\infty_{0,\sigma}(\Omega)$. Hence (3.13) and (3.14) imply that

$$\|(\lambda + A)^{-1/2} B (\lambda + A)^{-1/2}\|_{\boldsymbol{B}(L^r_\sigma)} \le C_2 \|v\|_3, \tag{3.15}$$

where the constant C_2 is independent of λ. It suffices to take η in Lemma 3.2 so that $C_2\|v\|_3 < 1$. Then it remains to estimate $(\lambda + \mathcal{L})^{-1}$. By the adjoint operator $(\lambda + \mathcal{L}')^{-1}$ and its estimate (3.10), we obtain that

$$\begin{aligned} |\langle (\lambda + \mathcal{L})^{-1} w, \phi\rangle| &\le |\langle w, (\lambda + \mathcal{L}')^{-1}\phi\rangle| \\ &\le \|w\|_r \|(\lambda + \mathcal{L}')^{-1}\|_{\boldsymbol{B}(L^{r'}_\sigma)} \|\phi\|_{r'} \\ &\le C\frac{M_{r'}}{1+|\lambda|}\|w\|_r\|\phi\|_{r'}, \end{aligned} \tag{3.16}$$

for all $w \in L^r_\sigma(\Omega)$ and $\phi \in C^\infty_{0,\sigma}(\Omega)$, where the constant C in (3.16) is independent of λ. This completes the proof of Lemma 3.2. □

Next we show the L^q-L^r estimate for $e^{-t\mathcal{L}'}$ and $e^{-t\mathcal{L}}$.

Lemma 3.3. *Let $1 < r < 3$, $0 < \alpha < 1$ and $v \in W^{1,3/2}(\Omega)$ with* div $v = 0$. *Suppose that $\|v\|_3 < \tilde{\eta}$, where $\tilde{\eta} = \tilde{\eta}(r)$ is the same as in Lemma 3.1. Then there exists a constant $M = M(r, \alpha) > 0$ independent of λ such that*

$$\|A^\alpha (\lambda + \mathcal{L}')^{-1} w\|_r \le M(1 + |\lambda|)^{\alpha - 1}\|w\|_r, \tag{3.17}$$

for all $w \in L^r_\sigma(\Omega)$ and for all $\lambda \in \Sigma^{\mu,\delta}_{-A}$.

Proof. By the proof of Lemma 3.1, we see that

$$(\lambda + \mathcal{L}')^{-1} = (\lambda + A)^{-1}(1 + B'(\lambda + A)^{-1})^{-1}, \tag{3.18}$$

for all $\lambda \in \Sigma^{\mu,\delta}_{-A}$, and we see that $\|(1 + B'(\lambda + A)^{-1})^{-1}\|_{\boldsymbol{B}(L^r_\sigma)}$ is estimated independently of λ. By the moment inequality for A, i.e., the interpolation inequality of the operator with respect to the fractional powers, we have

$$\begin{aligned} &\|A^\alpha(\lambda + \mathcal{L}')^{-1}\|_{\boldsymbol{B}(L^r_\sigma)} \\ &\le C\|(\lambda + \mathcal{L}')^{-1}\|^{1-\alpha}_{\boldsymbol{B}(L^r_\sigma)} \|A(\lambda + A)^{-1}(1 + B'(\lambda + A)^{-1})^{-1}\|^\alpha_{\boldsymbol{B}(L^r_\sigma)} \\ &\le M(1 + |\lambda|)^{\alpha - 1}. \end{aligned} \tag{3.19}$$

Hence (3.19) yields the estimate (3.17). □

Lemma 3.4. *Let $3/2 < r < \infty$ and $v \in W^{1,3/2}(\Omega)$ with* div $v = 0$. *Suppose that $\|v\|_3 < \eta$, where $\eta = \eta(r)$ is the same as in Lemma 3.2. Then there exist constants $\mu' < 0$ and $M = M(r)$ such that*

$$\|A^{1/2}(\lambda + \mathcal{L})^{-1} w\|_r \le M(1 + |\lambda|)^{-1/2}\|w\|_r, \tag{3.20}$$

for all $w \in L^r_\sigma(\Omega)$ and for all $\lambda \in \Sigma^{\mu',\delta}_{-A}$.

Proof. Firstly we mention an important fact about the constant which appears in moment inequalities. Let $\mathcal{A}$ be a general operator on a Banach space X so that the resolvent estimate:

$$\|(z-\mathcal{A})^{-1}\|_{\boldsymbol{B}(X)} \le \frac{M'}{1+|z|}, \quad z<0, \tag{3.21}$$

holds. Then the constant of the moment inequality depends only on the exponents and M' as in (3.21). We see that there exists $\mu'<0$ such that the resolvent estimate:

$$\|(z-(\lambda+A))^{-1}\|_{\boldsymbol{B}(L^r_\sigma)} \le \frac{\tilde{M}}{1+|z|}, \quad \text{for } z<0,$$

holds for all $\lambda \in \Sigma^{\mu',\delta}_{-A}$ where the constant $\tilde{M}$ is independent of λ.

Since $\|\big(1+(\lambda+A)^{-1/2}B(\lambda+A)^{-1/2}\big)^{-1}\|_{\boldsymbol{B}(L^r_\sigma)}$ is estimated independently of λ in the proof of Lemma 3.2, by the moment inequality for $(\lambda+A)$ we have

$$\begin{aligned}
&\|A^{1/2}(\lambda+\mathcal{L})^{-1}w\|_r \\
&\le \|A^{1/2}(\lambda+A)^{-1/2}\big(1+(\lambda+A)^{-1/2}B(\lambda+A)^{-1/2}\big)^{-1}(\lambda+A)^{-1/2}w\|_r \\
&\le C\|\big(1+(\lambda+A)^{-1/2}B(\lambda+A)^{-1/2}\big)^{-1}(\lambda+A)^{-1/2}w\|_r \\
&\le C\|\big(1+(\lambda+A)^{-1/2}B(\lambda+A)^{-1/2}\big)^{-1}\|_{\boldsymbol{B}(L^r_\sigma)}\|(\lambda+A)^{1/2}(\lambda+A)^{-1}w\|_r \\
&\le C\|(\lambda+A)^{-1}w\|_r^{1/2}\|w\|_r^{1/2} \\
&\le M(1+|\lambda|)^{-1/2}\|w\|_r,
\end{aligned}$$

for all $w \in L^r_\sigma(\Omega)$ and all $\lambda \in \Sigma^{\mu',\delta}_{-A}$, where the constant M is independent of λ. $\square$

By Lemma 3.3 and Lemma 3.4, the Dunford integrals of $e^{-t\mathcal{L}'}$ and $e^{-t\mathcal{L}}$ immediately yield the following estimates (3.22) and (3.23).

Lemma 3.5. *Let $1<r<3$, $0<\alpha<1$ and $v \in W^{1,3/2}(\Omega)$ with $\operatorname{div} v = 0$. Suppose that $\|v\|_3 < \tilde{\eta}$ with $\tilde{\eta} = \tilde{\eta}(r)$ as in Lemma 3.1. Then there are constants $C = C(r,\alpha) > 0$ and $\beta = \beta(r) > 0$ such that for all $t>0$*

$$\|A^\alpha e^{-t\mathcal{L}'}\|_{\boldsymbol{B}(L^r_\sigma)} \le Ce^{-\beta t}t^{-\alpha}. \tag{3.22}$$

Lemma 3.6. *Let $3/2<r<\infty$ and $v \in W^{1,3/2}(\Omega)$ with $\operatorname{div} v = 0$. Suppose that $\|v\|_3 < \eta$ with $\eta = \eta(r)$ as in Lemma 3.2. Then there are constants $C = C(r) > 0$ and $\beta = \beta(r) > 0$ such that for all $t>0$*

$$\|A^{1/2} e^{-t\mathcal{L}}\|_{\boldsymbol{B}(L^r_\sigma)} \le Ce^{-\beta t}t^{-1/2}. \tag{3.23}$$

By Lemma 3.5 and by Lemma 3.6, we obtain the L^q-L^r estimates for $e^{-t\mathcal{L}'}$ and $e^{-t\mathcal{L}}$.

Lemma 3.7. *Let $1 < q \le r < 3$ and $v \in W^{1,3/2}(\Omega)$ with* div $v = 0$. *Suppose that $\|v\|_3 < \min\{\tilde\eta(r), \tilde\eta(q)\}$, where $\tilde\eta(r)$ and $\tilde\eta(q)$ are the same as in Lemma 3.1. Then there exist constants $C = C(r,q) > 0$ and $\beta = \beta(r,q) > 0$ such that there hold*

$$\begin{aligned} \|e^{-t\mathcal{L}'}w\|_r &\le Ce^{-\beta t}t^{-3(1/q-1/r)/2}\|w\|_q, \\ \|\nabla e^{-t\mathcal{L}'}w\|_r &\le Ce^{-\beta t}t^{-3(1/q-1/r)/2-1/2}\|w\|_q, \end{aligned} \tag{3.24}$$

for all $w \in L^q_\sigma(\Omega)$ and $t > 0$.

Proof. Since $3(1/q - 1/r)/2 < 1$, by Lemma 3.5 we have

$$\begin{aligned} \|e^{-t\mathcal{L}'}w\|_r &\le \|A^{3(1/q-1/r)/2}e^{-t\mathcal{L}'}w\|_q \\ &\le Ce^{-\beta t}t^{-3(1/q-1/r)/2}\|w\|_q, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \|\nabla e^{-t\mathcal{L}'}w\|_r &\le C\|A^{1/2}e^{-t\mathcal{L}'}w\|_r \\ &\le Ce^{-\beta t}t^{-1/2}\|e^{-\frac{t}{2}\mathcal{L}'}w\|_r \\ &\le Ce^{-\beta t}t^{-3(1/q-1/r)/2-1/2}\|w\|_q, \end{aligned} \tag{3.26}$$

for all $w \in L^q_\sigma(\Omega)$. $\square$

Lemma 3.8. *Let $3/2 < q \le r < \infty$ and $v \in W^{1,3/2}(\Omega)$ with* div $v = 0$. *Suppose that $\|v\|_3 < \min\{\eta(r), \eta(q), \tilde\eta(r'), \tilde\eta(q')\}$, where $\eta(r)$, $\eta(q)$, $\tilde\eta(r')$ and $\tilde\eta(q')$ are the same as in Lemma 3.1 and Lemma 3.2. Then there exist constants $C = C(r,q) > 0$ and $\beta = \beta(r,q) > 0$ such that*

$$\begin{aligned} \|e^{-t\mathcal{L}}w\|_r &\le Ce^{-\beta t}t^{-3(1/q-1/r)/2}\|w\|_q, \\ \|\nabla e^{-t\mathcal{L}}w\|_r &\le Ce^{-\beta t}t^{-3(1/q-1/r)/2-1/2}\|w\|_q, \end{aligned} \tag{3.27}$$

for all $w \in L^q_\sigma(\Omega)$ and $t > 0$.

Proof. Since $3/2 < q \le r < \infty$, we have $1 < r' \le q' < 3$. Then by Lemma 3.7 duality yields

$$\begin{aligned} \|e^{-t\mathcal{L}}\|_{\boldsymbol{B}(L^q_\sigma, L^r_\sigma)} = \|e^{-t\mathcal{L}'}\|_{\boldsymbol{B}(L^{r'}_\sigma, L^{q'}_\sigma)} &\le Ce^{-\beta t}t^{-3(1/r'-1/q')/2} \\ &\le Ce^{-\beta t}t^{-3(1/q-1/r)/2}, \end{aligned}$$

for all $t > 0$, where $\boldsymbol{B}(L^q_\sigma, L^r_\sigma)$ is the set of all bounded linear operators from $L^q_\sigma(\Omega)$ to $L^r_\sigma(\Omega)$. By Lemma 3.6, we have

$$\begin{aligned} \|\nabla e^{-t\mathcal{L}}w\|_r &\le C\|A^{1/2}e^{-t\mathcal{L}}w\|_r \\ &\le Ce^{-\beta t}t^{-1/2}\|e^{-\frac{t}{2}\mathcal{L}}w\|_r \\ &\le Ce^{-\beta t}t^{-3(1/q-1/r)/2-1/2}\|w\|_q, \end{aligned}$$

for all $w \in L^q_\sigma(\Omega)$. $\square$

4 Existence of steady solutions of (S); Proof of Theorem 2.1.

4.1 Preliminary.

We consider the following Dirichlet problem of the steady Navier-Stokes equations:

$$(\mathrm{S}) \qquad \begin{cases} -\Delta v + v\cdot\nabla v + \nabla\pi = 0, & \text{in } \Omega,\\ \operatorname{div} v = 0, & \text{in } \Omega,\\ v = \beta, & \text{on } \partial\Omega. \end{cases}$$

Let $b \in W^{1,3/2}(\Omega)$ satisfy $\operatorname{div} b = 0$ and $b|_{\partial\Omega} = \beta$. In order to show the existence of a solution for (S), we set $u = v - b$ and reduce the problem to the following equations:

$$(\mathrm{S}') \qquad \begin{cases} -\Delta u + b\cdot\nabla u + u\cdot\nabla b + u\cdot\nabla u + \nabla\pi = F, & \text{in } \Omega,\\ \operatorname{div} u = 0, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $F := \Delta b - b\cdot\nabla b$.

Lemma 4.1. *Let $b \in W^{1,3/2}(\Omega)\cap W^{1,q}(\Omega)$ with $\operatorname{div} b = 0$ for some $3/2 \le q < \infty$. Further assume that $F \in W^{-1,q}(\Omega)$. Then for each $\varepsilon > 0$ there exists $\tilde\gamma = \tilde\gamma(\varepsilon, q) > 0$ such that if $\|b\|_{W^{1,3/2}} < \tilde\gamma$, we have a weak solution $u \in W_0^{1,3/2}(\Omega)\cap W_0^{1,q}(\Omega)$ of (S′) which satisfies*

$$(4.1) \qquad \|u\|_3 < \varepsilon.$$

Proof. Firstly, we construct a solution u on $\{u \in L^3_\sigma(\Omega)\,;\, \nabla u \in L^{3/2}(\Omega)\}$ by the iteration method:

$$(4.2) \qquad \begin{cases} u_0 := A^{-1}PF,\\ u_{j+1} := -A^{-1}P[b\cdot\nabla u_j + u_j\cdot\nabla b + u_j\cdot\nabla u_j] + u_0. \end{cases}$$

To begin with, we estimate $\|u_0\|_3$. By duality and the Sobolev inequality,

$$(4.3) \qquad \begin{aligned} |\langle u_0, \phi\rangle| &= |\langle F, A^{-1}\phi\rangle|\\ &= |\langle \Delta b - b\cdot\nabla b, A^{-1}\phi\rangle|\\ &\le \|\nabla b\|_{3/2}\|\nabla A^{-1}\phi\|_3 + \|b\|_3^2\|\nabla A^{-1}\phi\|_3\\ &\le C(\|\nabla b\|_{3/2} + \|b\|_3^2)\|\phi\|_{3/2}, \end{aligned}$$

for all $\phi \in C^\infty_{0,\sigma}(\Omega)$, from which it follows that

$$(4.4) \qquad \|u_0\|_3 \le C(\|\nabla b\|_{3/2} + \|b\|_3^2).$$

Next we have

$$\|u_{j+1}\|_3 \le \|u_0\|_3 + \|A^{-1}P[b\cdot\nabla u_j]\|_3 + \|A^{-1}P[u_j\cdot\nabla b]\|_3 + \|A^{-1}P[u_j\cdot\nabla u_j]\|_3.$$

By duality, the Sobolev inequality and since $\|\nabla A^{-1}\phi\|_3 \le C\|\phi\|_{3/2}$, we have

$$\begin{aligned} |\langle A^{-1}P[b\cdot\nabla u_j],\phi\rangle| &\le |\langle b\cdot\nabla u_j, A^{-1}\phi\rangle| \\ &\le |\langle b\cdot\nabla A^{-1}\phi, u_j\rangle| \\ &\le C\|b\|_3\|u_j\|_3\|\phi\|_{3/2}, \end{aligned} \tag{4.5}$$

for all $\phi \in C^\infty_{0,\sigma}(\Omega)$. Hence we obtain that

$$\|A^{-1}P[b\cdot\nabla u_j]\|_3 \le C\|b\|_3\|u_j\|_3, \tag{4.6}$$

similarly,

$$\begin{aligned} \|A^{-1}P[u_j\cdot\nabla b]\|_3 \le C\|b\|_3\|u_j\|_3, \\ \|A^{-1}P[u_j\cdot\nabla u_j]\|_3 \le C\|u_j\|_3^2. \end{aligned} \tag{4.7}$$

Hence (4.6) and (4.7) yield

$$\|u_{j+1}\|_3 \le \|u_0\|_3 + C_3\|b\|_3\|u_j\|_3 + C_3\|u_j\|_3^2, \tag{4.8}$$

for all $j = 0,1,2,\ldots$. We assume that

$$\|b\|_3 + \|u_0\|_3 < \frac{1}{4C_3}, \tag{4.9}$$

where the constant C_3 is the same constant as in (4.8). Then there holds

$$\begin{aligned} \|u_j\|_3 &\le \frac{(1-C_3\|b\|_3) - \sqrt{(1-C_3\|b\|_3)^2 - 4C_3\|u_0\|_3}}{2C_3} \\ &=: K_b \le \frac{1}{2C_3}, \end{aligned} \tag{4.10}$$

for all $j = 0,1,2,\ldots$. for all $j = 0,1,2,\ldots$. We note that K_b in (4.10) tends to 0 as $\|b\|_3$ and $\|\nabla b\|_{3/2}$ goes to 0.

Now we assume (4.9), setting $w_{j+1} = u_{j+1} - u_j$ $(u_{-1} \equiv 0)$, we have

$$w_{j+1} = -A^{-1}P\left[b\cdot\nabla w_j + w_j\cdot\nabla b + w_j\cdot\nabla u_j + u_{j-1}\cdot\nabla w_j\right], \tag{4.11}$$

and, by (4.10) and by the same way as (4.8) we have

$$\begin{aligned} \|w_{j+1}\|_3 &\le C_3\Big(\|b\|_3\|w_j\|_3 + \big(\|u_{j-1}\|_3 + \|u_j\|_3\big)\|w_j\|_3\Big) \\ &\le C_3\left(\|b\|_3 + 2K_b\right)\|w_j\|_3 \\ &\le \cdots \\ &\le \left[C_3\left(\|b\|_3 + 2K_b\right)\right]^{j+1}\|u_0\|_3, \end{aligned} \tag{4.12}$$

for all $j = 0, 1, 2, \dots$ where the constant C_3 is the same as (4.8). Then an elementary calculation shows $C_3(\|b\|_3 + 2K_b) < 1$. Since $u_j = \sum_{k=0}^{j} w_k$ we see that there exists a function $u \in L^3_\sigma(\Omega)$ such that

$$u_j \to u \quad \text{in } L^3_\sigma(\Omega) \quad \text{as } j \to \infty. \tag{4.13}$$

We shall next show that $\nabla u \in L^{3/2}(\Omega)$. Since

$$\|\nabla A^{-1} P[b \cdot \nabla u_j]\|_{3/2} \le C \|A^{-1/2} P[b \cdot \nabla u_j]\|_{3/2}, \tag{4.14}$$

by duality and the Sobolev inequality,

$$\begin{aligned} |\langle A^{-1/2} P[b \cdot \nabla u_j], \phi\rangle| &\le |\langle b \cdot \nabla u_j, A^{-1/2}\phi\rangle| \\ &\le |\langle b \cdot \nabla A^{-1/2}\phi, u_j\rangle| \\ &\le C_4 \|b\|_3 \|u_j\|_3 \|\phi\|_3, \end{aligned} \tag{4.15}$$

for all $\phi \in C^\infty_{0,\sigma}(\Omega)$. Hence we have

$$\|\nabla A^{-1} P[b \cdot \nabla u_j]\|_{3/2} \le C_4 \|b\|_3 \|u_j\|_3, \tag{4.16}$$

and similarly,

$$\begin{aligned} \|\nabla A^{-1} P[u_j \cdot \nabla b]\|_{3/2} &\le C_4 \|b\|_3 \|u_j\|_3, \\ \|\nabla A^{-1} P[u_j \cdot \nabla u_j]\|_{3/2} &\le C_4 \|u_j\|_3^2. \end{aligned} \tag{4.17}$$

Hence we obtain a uniform bound of $\|\nabla u_j\|_{3/2}$ by

$$\begin{aligned} \|\nabla u_j\|_{3/2} &\le \|\nabla u_0\|_{3/2} + 2C_4 \|b\|_3 \|u_j\|_3 + C_4 \|u_j\|_3^2 \\ &\le \|\nabla u_0\|_{3/2} + 2C_4 \|b\|_3 K_b + C_4 K_b^2 < \infty, \end{aligned} \tag{4.18}$$

for all $j = 0, 1, \dots$. From (4.18), it is easy to see that $\nabla u \in L^{3/2}(\Omega)$. We note that the estimate of $\|\nabla u_0\|_{3/2}$ is stated below for the case $q = 3/2$.

By the same argument as in (4.12), we have

$$\begin{aligned} &A^{-1} P[b \cdot \nabla u_j] \to A^{-1} P[b \cdot \nabla u], \\ &A^{-1} P[u_j \cdot \nabla b] \to A^{-1} P[u \cdot \nabla b], \\ &A^{-1} P[u_j \cdot \nabla u_j] \to A^{-1} P[u \cdot \nabla u], \quad \text{in } L^3_\sigma(\Omega) \quad \text{as } j \to \infty. \end{aligned} \tag{4.19}$$

Hence letting $j \to \infty$ in (4.2), we see by (4.13) and (4.19) that u satisfy (4.2) in $L^3_\sigma(\Omega)$, from which follows that $\nabla u_j \to \nabla u$ in $L^{3/2}(\Omega)$ as $j \to \infty$. Then u is a desired solution in $W^{1,3/2}_0(\Omega)$.

It remains to show that $u \in W^{1,q}_0(\Omega)$. Since $F \in W^{-1,q}(\Omega)$, we have $\|\nabla u_0\|_q < C\|F\|_{W^{-1,q}}$. By the Sobolev inequality with $1/q = 1/r - 1/3$ and by the Hölder inequality, we have

$$
\begin{aligned}
&\|\nabla A^{-1}P[b\cdot\nabla u_j+u_j\cdot\nabla b+u_j\cdot\nabla u_j]\|_q\\
&\le C\|A^{-1/2}P[b\cdot\nabla u_j+u_j\cdot\nabla b+u_j\cdot\nabla u_j]\|_q\\
&\le C\|b\cdot\nabla u_j+u_j\cdot\nabla b+u_j\cdot\nabla u_j\|_r\\
&\le C\left(\|b\|_3\|\nabla u_j\|_q+\|u_j\|_3\|\nabla b\|_q+\|u_j\|_3\|\nabla u_j\|_q\right)\\
&\le C\Big(K_b\|\nabla b\|_q+(\|b\|_3+K_b)\|\nabla u_j\|_q\Big).
\end{aligned}
\tag{4.20}
$$

Hence by (4.20) we obtain

$$
\|\nabla u_{j+1}\|_q\le\|\nabla u_0\|_q+C_5K_b\|\nabla b\|_q+C_5(\|b\|_3+K_b)\|\nabla u_j\|_q, \tag{4.21}
$$

which shows a *linear recurrence*. Hence if

$$
\|b\|_3+K_b<\frac{1}{C_5}, \tag{4.22}
$$

then we have a uniform bound of $\|\nabla u_j\|_q$ as

$$
\|\nabla u_j\|_q\le\frac{\|\nabla u_0\|_q+C_5K_b\|\nabla b\|_q}{1-C_5(\|b\|_3+K_b)}\quad\text{for all } j=0,1,2,\ldots. \tag{4.23}
$$

Hence (4.23) implies $u_j\to u$ strongly in $W_0^{1,q}(\Omega)$. By (4.4), we can take the constant $\tilde{\gamma}$ in Lemma 4.1 so that the conditions (4.9) and (4.22) are satisfied. This completes the proof of Lemma 4.1. □

4.2 Proof of Theorem 2.1.

Let $1/p^*=1/p-1/3$. Since $\beta\in W^{2-1/p,p}(\partial\Omega)\subset W^{1-1/p^*,p^*}(\partial\Omega)$ with (G.F.), by the trace theorem and Bogovskiĭ [1] and Borchers-Sohr [2], there exists $b\in W^{2,p}(\Omega)\subset W^{1,p^*}(\Omega)$ such that div $b=0$, $b|_{\partial\Omega}=\beta$ and

$$
\|b\|_{W^{1,p^*}(\Omega)}\le M\|\beta\|_{W^{1-1/p^*,p^*}(\partial\Omega)}. \tag{4.24}
$$

Take $\gamma=\gamma(\varepsilon,p)$ so that if $\|\beta\|_{W^{1-1/p^*,p^*}(\partial\Omega)}<\gamma$ there holds

$$
\|b\|_{W^{1,3/2}}\le\min\{\tilde{\gamma}(\varepsilon/2,p^*),\,\varepsilon/2\}, \tag{4.25}
$$

where $\tilde{\gamma}$ is the same constant as in Lemma 4.1. Then we have by Lemma 4.1 that there exists a solution $u\in W_0^{1,3/2}(\Omega)\cap W_0^{1,p^*}(\Omega)$ of (S′) with $\|u\|_3<\varepsilon/2$. Let $v=u+b$. Then v is a desired solution of (S) with

$$
\|v\|_3\le\|u\|_3+\|b\|_3\le\varepsilon. \tag{4.26}
$$

On the other hand, since $\beta\in W^{2-1/p,p}(\partial\Omega)$, it is easy to see that $v\cdot\nabla v\in L^p$. Hence the regularity criterion for the steady Stokes equations ensures that $v\in W^{2,p}(\Omega)$. This completes the proof of Theorem 2.1. □

5 Existence of periodic solutions of (I.E.); Proof of Theorem 2.2.

5.1 Preliminary.

Now we consider the existence of solutions of (I.E.).

Lemma 5.1. *Let $3/2 < r < 3$ and $2 < q < 3$ with $1/r + 1/3 < 2/q$. Define a function space X and a bilinear operator $G[\cdot, \cdot]$ on X by*

$$X := \{u \in BC(\mathbb{R}\,;\, L^r_\sigma(\Omega))\,;\, \nabla u \in BC(\mathbb{R}\,;\, L^q(\Omega))\}, \tag{5.1}$$

$$G[u, v](t) := -\int_{-\infty}^{t} e^{-(t-s)\mathcal{L}} P[u \cdot \nabla v](s)\, ds, \tag{5.2}$$

for $u, v \in X$ respectively. Then we have $G[u, v] \in X$. Moreover, if $u(s) \in D(A_r)$ for $s \in \mathbb{R}$, we obtain $G[u, v] \in X$ with

$$\sup_{s\in\mathbb{R}} \|G[u, v](s)\|_r \le C_{r,q} \sup_{s\in\mathbb{R}} \|\nabla u(s)\|_q \sup_{s\in\mathbb{R}} \|\nabla v(s)\|_q, \tag{5.3}$$

$$\sup_{s\in\mathbb{R}} \|\nabla G[u, v](s)\|_q \le C_{r,q} \sup_{s\in\mathbb{R}} \|\nabla u(s)\|_q \sup_{s\in\mathbb{R}} \|\nabla v(s)\|_q, \tag{5.4}$$

for all for $u, v \in X$ and $u(s) \in D(A_r)$ for $s \in \mathbb{R}$.

Proof. Firstly, we assume $u(s) \in D(A_r)$ for $s \in \mathbb{R}$. In this case, we note that $u(s) \in W^{1,r}_0(\Omega) \cap W^{1,q}(\Omega) \subset W^{1,q}_0(\Omega)$ for $s \in \mathbb{R}$. By Lemma 3.8 and since $3/q - 3/2r - 1/2 < 1$, we have

$$\begin{aligned}
\|G[u, v](t)\|_r &\le \int_{-\infty}^{t} \|e^{-(t-s)\mathcal{L}} P[u \cdot \nabla v](s)\|_r\, ds \\
&\le C \int_{-\infty}^{t} e^{-\beta(t-s)} (t-s)^{-3(2/q-1/3-1/r)/2} \|u \cdot \nabla v(s)\|_{3q/(6-q)}\, ds \\
&\le C \int_{-\infty}^{t} e^{-\beta(t-s)} (t-s)^{-(3/q-3/2r-1/2)} \|u(s)\|_{3q/(3-q)} \|\nabla v(s)\|_q\, ds \\
&\le C \sup_{s\in\mathbb{R}} \|\nabla u(s)\|_q \sup_{s\in\mathbb{R}} \|\nabla v(s)\|_q,
\end{aligned} \tag{5.5}$$

for all $t \in \mathbb{R}$. Similarly, we have

$$\begin{aligned}
\|\nabla G[u, v]\|_q &\le \int_{-\infty}^{t} \|\nabla e^{-(t-s)\mathcal{L}} P[u \cdot \nabla v](s)\|_q\, ds \\
&\le C \int_{-\infty}^{t} e^{-\beta(t-s)} (t-s)^{-3(2/q-1/3-1/q)/2-1/2} \|u \cdot \nabla v(s)\|_{\frac{3q}{6-q}}\, ds \\
&\le C \int_{-\infty}^{t} e^{-\beta(t-s)} (t-s)^{-3/2q} \|u(s)\|_{\frac{3q}{3-q}} \|\nabla v(s)\|_q\, ds \\
&\le C \sup_{s\in\mathbb{R}} \|\nabla u(s)\|_q \sup_{s\in\mathbb{R}} \|\nabla v(s)\|_q.
\end{aligned} \tag{5.6}$$

for all $t \in \mathbb{R}$.

In the case that $u \in X$, $\|\nabla u(s)\|_q$ as in (5.5) and (5.6) may be replaced by $\|u(s)\|_{W^{1,q}}$. This completes the proof of Lemma 5.1. $\square$

Lemma 5.2. *Let $3/2 < r < 3$ and $2 < q < 3$ with $1/r + 1/3 < 2/q$, and let $f \in BC(\mathbb{R}; L^l(\Omega))$ with $3/2 < l < \infty$. Define*

$$F(t) := \int_{-\infty}^{t} e^{-(t-s)\mathcal{L}} Pf(s)\, ds, \quad t \in \mathbb{R}. \tag{5.7}$$

Then $F \in X$ and the following estimates hold:

$$\begin{aligned} \sup_{s\in\mathbb{R}} \|F(s)\|_r &\le C_{r,q,l} \sup_{s\in\mathbb{R}} \|Pf(s)\|_l, \\ \sup_{s\in\mathbb{R}} \|\nabla F(s)\|_q &\le C_{r,q,l} \sup_{s\in\mathbb{R}} \|Pf(s)\|_l. \end{aligned} \tag{5.8}$$

Proof. Firstly we take $3/2 < l' < l$ so that $3/2 < l' < \min\{r, q\}$. Since $\Omega \subset \mathbb{R}^3$ is a bounded domain, we note that $\|Pf\|_{l'} < C\|Pf\|_l$. Then we obtain

$$\begin{aligned} \|F(t)\|_r &\le \int_{-\infty}^{t} \|e^{-(t-s)\mathcal{L}} Pf(s)\|_r\, ds \\ &\le C \int_{-\infty}^{t} e^{-\beta(t-s)} (t-s)^{-3(1/l'-1/r)/2} \|Pf(s)\|_{l'}\, ds \\ &\le C \sup_{s\in\mathbb{R}} \|Pf(s)\|_l. \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} \|\nabla F(t)\|_q &\le \int_{-\infty}^{t} \|\nabla e^{-(t-s)\mathcal{L}} Pf(s)\|_q\, ds \\ &\le C \int_{-\infty}^{t} e^{-\beta(t-s)} (t-s)^{-3(1/l'-1/q)/2-1/2} \|Pf(s)\|_{l'}\, ds \\ &\le C \sup_{s\in\mathbb{R}} \|Pf(s)\|_l, \end{aligned} \tag{5.10}$$

for all $t \in \mathbb{R}$. The proof of Lemma 5.2 completes. $\square$

5.2 Proof of Theorem 2.2.

According to Kozono and Nakao [6], we shall prove the existence and uniqueness of solutions to the integral equation (I.E.) by successive approximation with the aid of Lemma 5.1 and Lemma 5.2. Recall the function space X and the bilinear operator G on X introduced in Lemma 5.1. Equipped with the norm $\|\cdot\|_X$ defined by

$$\|u\|_X := \sup_{s\in\mathbb{R}} \|u(s)\|_r + \sup_{s\in\mathbb{R}} \|\nabla u(s)\|_q, \tag{5.11}$$

X is a Banach space. We construct a periodic solution of (I.E.) with the iteration method:

$$
\begin{aligned}
w_0(t) &:= \int_{-\infty}^{t} e^{-(t-s)\mathcal{L}} Pf(s)\, ds, \\
w_{m+1}(t) &:= w_0(t) + G[w_m, w_m](t), \quad m = 0, 1, \dots.
\end{aligned} \tag{5.12}
$$

By Lemma 5.2, we have $w_0 \in X$ with

$$
\|w_0\|_X \le C \sup_{s\in\mathbb{R}} \|Pf(s)\|_l. \tag{5.13}
$$

Since f is a periodic function with period T_*, we can easily verify that w_0 is also periodic with the same period T_*. By induction and Lemma 5.1, so is w_m for $m = 0, 1, \dots$. Moreover, it follows by Lemma 5.1 that

$$
\|w_{m+1}\|_X \le \|w_0\|_X + \|G(w_m, w_m)\|_X \le \|w_0\|_X + C_6\|w_m\|_X^2, \quad m = 0, 1, \dots, \tag{5.14}
$$

where the constant C_6 is in Lemma 5.1. Hence if

$$
\|w_0\|_X \le \frac{1}{4C_6}, \tag{5.15}
$$

then there holds

$$
\|w_m\|_X \le \frac{1-\sqrt{1-4C_6\|w_0\|_X}}{2C_6} =: K < \frac{1}{2C_6}, \quad \text{for all } m = 0, 1, \dots. \tag{5.16}
$$

By (5.13), we should take the constant δ in Theorem 2.2 so that the condition (5.15) is satisfied.

Now we assume (5.13). Setting $u_m := w_m - w_{m-1}$, $u_{-1} \equiv 0$, we have

$$
\begin{aligned}
u_{m+1} &= G[w_m, w_m](t) - G[w_{m-1}, w_{m-1}](t) \\
&= G[u_m, w_m](t) - G[w_{m-1}, u_m](t).
\end{aligned} \tag{5.17}
$$

By Lemma 5.1 and (5.16), we have

$$
\begin{aligned}
\|u_{m+1}\|_X &\le \|G[u_m, w_m]\|_X + \|G[w_{m-1}, u_m]\|_X \\
&\le C_6(\|u_m\|_X\|w_m\|_X + \|w_{m-1}\|_X\|u_m\|_X) \\
&\le 2C_6K\|u_m\|_X \\
&\le \dots \\
&\le (2C_6K)^{m+1}\|w_0\|_X,
\end{aligned} \tag{5.18}
$$

for all $m = 0, 1, \dots$. Since $w_m(t) = \sum_{j=0}^{m} u_j(t)$, by (5.16) and (5.18), we see that there exists a function $w \in X$ such that

$$
w_m \to w \quad \text{in } X \quad \text{as } m \to \infty. \tag{5.19}
$$

This limit function $w \in X$ is periodic in t with the same period as f. By the same argument as (5.18), we see that

$$\begin{aligned} \|G[w_m, w_m] - G[w, w]\|_X &\leq \|G[w_m - w, w_m]\|_X + \|G[w, w_m - w]\|_X \\ &\leq C_6(\|w_m - w\|_X \|w_m\|_X + \|w\|_X \|w_m - w\|_X) \\ &\leq 2C_6 K \|w_m - w\|_X, \end{aligned} \tag{5.20}$$

hence

$$G[w_m, w_m] \to G[w, w] \quad \text{in } X \quad \text{as } m \to \infty. \tag{5.21}$$

Now letting $m \to \infty$ in (5.12), we conclude by (5.19) and (5.21) that w is a desired periodic solution of (I.E.). Next we show the uniqueness. Suppose that $w' \in X$ is another solution of (I.E.) with $\|w'\|_X \leq K$, where K is the same constant as in (5.16). Then we have

$$\begin{aligned} \|w - w'\|_X &\leq C_6(\|w - w'\|_X \|w\|_X + \|w'\|_X \|w - w'\|_X) \\ &\leq 2C_6 K \|w - w'\|_X. \end{aligned} \tag{5.22}$$

Since $2C_6 K < 1$ there holds $w = w'$. This proves Theorem 2.2. □

6 Regularity of mild solutions; Proof of Theorem 2.3.

We shall show that the periodic solution w obtained in Theorem 2.2 is actually a solution of the differential equation (N-S$''$) if the external force f is regular. To this end, we need the local existence of strong solutions to the initial-boundary value problem for (N-S$''$). We follow the argument of Kozono and Nakao [6].

Let us define strong solutions of the initial value problem for (N-S$''$).

Definition 6.1. *Let $a \in L^3_\sigma(\Omega)$ and let $Pf \in C((t_0, t_1); L^3_\sigma(\Omega))$, where $t_0 < t_1$. Then a measurable function $\tilde{w}$ on $\Omega \times (t_0, t_1)$ is called a strong solution of (I N-S$''$) on (t_0, t_1) with the initial data a at $t = t_0$ if*

(i) $\tilde{w} \in BC([t_0, t_1); L^3_\sigma(\Omega)) \cap C^1((t_0, t_1); L^3_\sigma(\Omega))$,

(ii) $\tilde{w}(t) \in D(\mathcal{L})$ *for* $t_0 < t < t_1$ *and* $\mathcal{L}\tilde{w} \in C((t_0, t_1); L^3_\sigma(\Omega))$,

(iii)

$$\text{(I N-S}'') \qquad \begin{cases} \dfrac{d\tilde{w}}{dt} + \mathcal{L}\tilde{w} + P[\tilde{w} \cdot \nabla \tilde{w}] = Pf \quad \text{in } L^3_\sigma(\Omega) \quad \text{for } t_0 < t < t_1, \\ \tilde{w}(t_0) = a. \end{cases}$$

Lemma 6.1. *Let $2 < q < 3$ and let $3/2 < l < \infty$. Assume that $a \in L^3_\sigma(\Omega)$ and $L^{q^*}_\sigma(\Omega)$ with $1/q^* = 1/q - 1/3$, $f \in BC(\mathbb{R}; L^l)$ and that Pf is a Hölder continuous function on $\mathbb{R}$ with values in $L^3_\sigma(\Omega)$. Then there exists $T > 0$ such that for every $t_0 \in \mathbb{R}$, we have a unique strong solution $\tilde{w}$ of (I N-S$''$) on (t_0, t_0+T) with initial data a at t_0. Moreover, $\tilde{w}$ has the additional property $\tilde{w} \in BC([t_0, t_0+T); L^{q^*}_\sigma(\Omega))$ with*

$$\sup_{t_0 \le t < t_0+T} \|\tilde{w}(t)\|_{q^*} \le C_7, \tag{6.1}$$

where $C_7 = C_7(\|a\|_{q^*}, \sup_{s\in\mathbb{R}} \|Pf(s)\|_l)$ is independent of t_0. Here T is estimated as

$$T = C_8\Big(\|a\|_{q^*} + \sup_{s\in\mathbb{R}} \|Pf(s)\|_l\Big)^{-2q^*/(q^*-3)}, \tag{6.2}$$

with $C_8 = C_8(q, l)$ independent of a, f and t_0.

Due to Lemma 3.8, we can prove Lemma 6.1 in the same way as [6], see Lemma 4.1 in [6].

6.1 Completion of the proof of Theorem 2.3.

Let w be the periodic solution of the integral equation (I.E.) given by Theorem 2.2. Since $w \in X$, we have by the Sobolev inequality that $w \in BC(\mathbb{R}; L^3_\sigma(\Omega) \cap L^{q^*}_\sigma(\Omega))$, where $q^* = 3q/(3-q)$. By Lemma 6.1, for every $t_0 \in \mathbb{R}$ there exists a unique strong solution $\tilde{w}$ of (I N-S$''$) with the initial data $w(t_0)$ on (t_0, t_0+T) with

$$T = C_8\Big(\|w(t_0)\|_{q^*} + \sup_{s\in\mathbb{R}} \|Pf(s)\|_l\Big)^{-2q^*/(q^*-3)},$$

where C_8 is the same constant as in (6.2). By (5.16) and (6.1), we have

$$\sup_{t_0 < s < t_0+T} \|\tilde{w}(s)\|_{q^*} + \sup_{t_0 < s < t_0+T} \|\nabla w(s)\|_q \le C_7 + K =: C_9 \tag{6.3}$$

where C_9 is independent of t_0. Now consider the integral equation of (I N-S$''$) with a replaced by $w(t_0)$;

$$\tilde{w}(t) = e^{-t\mathcal{L}} w(t_0) + \int_{t_0}^t e^{-(t-s)\mathcal{L}} Pf(s)\,ds + \int_{t_0}^t e^{-(t-s)\mathcal{L}} P[\tilde{w}\cdot\nabla\tilde{w}](s)\,ds,$$

for $t_0 \le t < t_0 + T$. It is easy to show that

$$\begin{aligned} &w(t) - \tilde{w}(t) \\ &= -\int_{t_0}^t e^{-(t-s)\mathcal{L}} P[(w-\tilde{w})\cdot\nabla w](s)\,ds - \int_{t_0}^t e^{-(t-s)\mathcal{L}} P[\tilde{w}\cdot\nabla(w-\tilde{w})](s)\,ds \\ &=: J_1(t) + J_2(t), \quad t_0 < t < t_0 + T. \end{aligned} \tag{6.4}$$

By Lemma 3.8, we have

$$
\begin{aligned}
\|J_1(t)\|_3 &\le C\int_{t_0}^{t}(t-s)^{-3(1/3+1/q-1/3)/2}\|w(s)-\tilde{w}(s)\|_3\|\nabla w(s)\|_q ds \\
&\le C\sup_{s\in R}\|\nabla w(s)\|_q \sup_{t_0<s<t}\|w(s)-\tilde{w}(s)\|_3(t-t_0)^{1-3/2q},
\end{aligned} \tag{6.5}
$$

for all $t_0 < t < t_0+T$, where the constant $C = C(q)$ is independent t_0. By integration by parts we have

$$
\begin{aligned}
|\langle J_2(t),\phi\rangle| &\le \left|\int_{t_0}^{t}\langle \tilde{w}(s)\cdot\nabla e^{-(t-s)\mathcal{L}'}\phi, w(s)-\tilde{w}(s)\rangle\, ds\right| \\
&\le C\int_{t_0}^{t}\|\tilde{w}(s)\|_{q^*}\|\nabla e^{-(t-s)\mathcal{L}'}\phi\|_{q'}\|w(s)-\tilde{w}(s)\|_3\, ds \\
&\le C\sup_{t_0<s<t_0+T}\|\tilde{w}(s)\|_{q^*}\sup_{t_0<s<t}\|w(s)-\tilde{w}(s)\|_3 \\
&\quad\times\int_{t_0}^{t}(t-s)^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{q'})-\frac{1}{2}}ds\|\phi\|_{3/2} \\
&\le C\sup_{t_0<s<t_0+T}\|\tilde{w}(s)\|_{q^*}\sup_{t_0<s<t}\|w(s)-\tilde{w}(s)\|_3(t-t_0)^{1-3/2q}\|\phi\|_{3/2},
\end{aligned}
$$

for all $\phi \in C_{0,\sigma}^{\infty}(\Omega)$ and all $t_0 < t < t_0+T$, where the constant $C = C(q)$ is independent of t_0. Hence by duality, we obtain

$$
\|J_2\|_3 \le C\sup_{t_0<s<t_0+T}\|\tilde{w}(s)\|_{q^*}\sup_{t_0<s<t}\|w(s)-\tilde{w}(s)\|_3(t-t_0)^{1-3/2q}, \tag{6.6}
$$

for all $t_0 < t < t_0+T$. Now it follows from (6.3) to (6.6) that

$$
\|w(t)-\tilde{w}(t)\|_3 \le C_{10}\sup_{t_0<s<t}\|w(s)-\tilde{w}(s)\|_3(t-t_0)^{1-3/2q}, \quad t_0<t<t_0+T,
$$

where C_{10} is independent of t_0. Defining $\tau := \min\{(1/2C_{10})^{2q/(2q-3)}, T\}$, we obtain from the above estimate that

$$
\|w(t)-\tilde{w}(t)\|_3 \le \frac{1}{2}\sup_{t_0<s<t}\|w(s)-\tilde{w}(s)\|_3,
$$

for all $t_0 < t < t_0+\tau$, which yields

$$
w \equiv \tilde{w} \quad \text{on } [t_0, t_0+\tau].
$$

Since τ can be taken independently of t_0, we conclude that

$$
w \equiv \tilde{w} \quad \text{on } [t_0, t_0+T).
$$

Then since t_0 is arbitrary, if follows from Lemma 6.1 that w has the desired properties (i), (ii) and (iii) in Theorem 2.3. □

7 Existence of weak solutions; Proof of Theorem 2.4.

To construct weak solutions of (N-S*), we introduce the following lemma as an immediate corollary of Kozono-Yanagisawa [8].

Lemma 7.1. *Let $\beta \in H^{1/2}(\partial\Omega)$ satisfy (G.F.). If*

$$\left\| \sum_{j,k=1}^{L} \alpha_{jk} \Big(\int_{\Gamma_k} \beta \cdot \nu dS \Big) \psi_j \right\|_3 < \frac{1}{4C_s}, \tag{7.1}$$

then there exists $0 < \varepsilon_0 < 1/4$ and $b_{\varepsilon_0} \in H^1(\Omega)$ with div $b_{\varepsilon_0} = 0$ *and $b_{\varepsilon_0}|_{\partial\Omega} = \beta$ such that*

$$|(u \cdot \nabla b_{\varepsilon_0}, u)| \le \varepsilon_0 \|\nabla u\|_2^2, \tag{7.2}$$

for all $u \in H^1_{0,\sigma}$. Here $C_s = 3^{-1/2}2^{2/3}\pi^{-2/3}$ is the best constant of the Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$.

Hence we may take the solenoidal extension b of the boundary data β so that (7.2) is fulfilled.

According to Masuda [9], we first construct approximate solutions of (N-S$'$) by the Galerkin method, in the Hilbert space $H^1_{0,\sigma}(\Omega)$. At first, we introduce the following lemma which plays an important role for the convergence of the nonlinear term.

Lemma 7.2 (Masuda [9]). *For any $\varepsilon > 0$ and $\Phi \in C([0,T]; H^1_{0,\sigma}(\Omega))$, there exist a constant $M > 0$ and an integer N, and functions $\phi_j \in L^2_\sigma(\Omega)$, $(j = 1, \dots, N)$ such that*

$$\begin{aligned}\int_s^t |(u \cdot \nabla v, \Phi)|\, d\tau \le \varepsilon \int_s^t (\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u\|_2 \|\nabla v\|_2)\, d\tau \\ + M \sum_{i=1}^{N} \int_s^t |(u, \phi_i)|^2 d\tau,\end{aligned} \tag{7.3}$$

holds for all $u \in L^2(0,T; H^1_{0,\sigma}(\Omega))$, $v \in L^2(0,T; H^1(\Omega))$ and $0 \le s < t \le T$.

For the proof, see Masuda [9], Lemma 2.5, p.632.

Remark 7.1. Although Masuda [9] proved (7.3) for $v \in L^2(0,T; H^1_{0,\sigma}(\Omega))$, it is easy to see that the same inequality holds for all $v \in L^2(0,T; H^1(\Omega))$.

7.1 Proof of Theorem 2.4.

Let $\{\varphi_k\}_{k=1}^{\infty} \subset C_{0,\sigma}^{\infty}(\Omega)$ be a complete orthonormal system in $L^2_\sigma(\Omega)$ and dense in $H^1_{0,\sigma}(\Omega)$. Using $\{\varphi_k\}_{k=1}^{\infty}$, we construct approximate solutions $w_m = w_m(x,t)$, $m \in \mathbb{N}$, of (N-S*) which have the form;

$$w_m(x,t) = \sum_{l=1}^{m} g_{m,l}(t)\varphi_l(x). \tag{7.4}$$

Here the coefficient $g_{m.j}(t)$, $(j = 1, \ldots, m)$ is determined by the following system of ordinary differential equations;

$$\begin{cases} (\partial_t w_m, \varphi_j) + (\nabla w_m, \nabla \varphi_j) + (b \cdot \nabla w_m, \varphi_j) + (w_m \cdot \nabla b, \varphi_j) \\ \qquad\qquad\qquad\qquad\qquad + (w_m \cdot \nabla w_m, \varphi_j) = \langle F, \varphi_j \rangle, \\ \displaystyle\sum_{l=1}^{m} (w_m(0), \varphi_l)\varphi_l = \sum_{l=1}^{m} (a, \varphi_l)\varphi_l := a_m, \quad j = 1, \ldots, m. \end{cases} \tag{7.5}$$

Let $\lambda_j(t) \in H^1((0,T))$, $(j = 1, \ldots, m)$. We multiply the first equation of (7.5) by $\lambda_j(t)$ and integrate over $(s,\, t)$, to get

$$\begin{aligned} &\int_s^t \{-(w_m, \partial_t \Phi_j) + (\nabla w_m, \nabla \Phi_j) \\ &\qquad + (b \cdot \nabla w_m, \Phi_j) + (w_m \cdot \nabla b, \Phi_j) + (w_m \cdot \nabla w_m, \Phi_j)\}\, d\tau \\ &\qquad\qquad = \int_s^t \langle F, \Phi_j \rangle\, d\tau - (w_m(t), \Phi_j(t)) + (w_m(s), \Phi_j(s)), \end{aligned} \tag{7.6}$$

where $\Phi_j = \lambda_j(t)\varphi_j(x)$. Putting $\lambda_j(t) = g_{m,j}(t)$ in (7.6), and taking summation with respect to j, we obtain with $s = 0$,

$$\|w_m(t)\|_2^2 + 2\int_0^t \|\nabla w_m\|_2^2\, d\tau + 2\int_0^t (w_m \cdot \nabla b, w_m)\, d\tau = 2\int_0^t \langle F, w_m \rangle\, d\tau + \|a_m\|_2^2. \tag{7.7}$$

On the other hand, by (7.2), we estimate

$$\int_s^t |(w_m \cdot \nabla b, w_m)|\, d\tau \le \varepsilon_0 \int_s^t \|\nabla w_m\|_2^2\, d\tau, \tag{7.8}$$

and by the Poincaré inequality and the Young inequality, we estimate

$$\begin{aligned} |\langle F, w_m \rangle| &\le |(f, w_m)| + |(\nabla b, \nabla w_m)| + |(b \cdot \nabla b, w_m)| \\ &\le C\|f\|_2 \|\nabla w_m\|_2 + \|\nabla b\|_2 \|\nabla w_m\|_2 + \|b\|_4^2 \|\nabla w_m\|_2 \\ &\le 3\varepsilon_0 \|\nabla w_m\|_2^2 + C(\|f\|_2^2 + \|\nabla b\|_2^2 + \|b\|_4^4). \end{aligned} \tag{7.9}$$

Hence (7.8) and (7.9) yield

$$
\begin{aligned}
(7.10)\quad \|w_m(t)\|_2^2 + 2(1-4\varepsilon_0)\int_0^t \|\nabla w_m\|_2^2\,d\tau \\
\le C\int_0^T (\|f\|_2^2 + \|\nabla b\|_2^2 + \|b\|_4^4)\,d\tau + \|a\|_2^2 =: K(T),
\end{aligned}
$$

for $0 \le t < T$. The a priori estimate (7.10) guarantees the global existence of solutions of (7.5). Moreover we obtain the following lemma.

Lemma 7.3. *For each fixed j, the family $\{(w_m(t), \varphi_j)\}_{m=1}^{\infty}$ forms a uniformly bounded and equicontinuous set of continuous functions on $[0, T]$.*

Proof. The uniform boundedness is an immediate consequence of (7.10). By (7.5), we have

$$
\begin{aligned}
(w_m(t),\varphi_j) - (w_m(s),\varphi_j) &= \int_s^t (\partial_t w_m(\tau),\varphi_j)\,d\tau \\
&= -\int_s^t (\nabla w_m, \nabla\varphi_j)\,d\tau - \int_s^t (b\cdot\nabla w_m, \varphi_j)\,d\tau \\
&\quad - \int_s^t (w_m\cdot\nabla b, \varphi_j)\,d\tau - \int_s^t (w_m\cdot\nabla w_m, \varphi_j)\,d\tau \qquad (7.11)\\
&\quad + \int_s^t \langle F, \varphi_j\rangle\,d\tau \\
&=: -I_1 - I_2 - I_3 - I_4 + I_5.
\end{aligned}
$$

So we estimate I_k, $(k = 1, 2, 3, 4, 5)$. By the Schwarz inequality, the Sobolev inequality and (7.10), we have

$$
\begin{aligned}
|I_1| &\le C\|\nabla\varphi_j\|_2 K(T)^{1/2}\,|t-s|^{1/2}, \\
|I_2| &\le C\|\nabla\varphi_j\|_2 K(T)^{1/2}\|b\|_3\,|t-s|^{1/2}, \\
|I_3| &\le \int_s^t \|w_m\|_6\|\nabla b\|_2\|\varphi_j\|_3\,d\tau \\
&\le CK(T)^{1/2}\|\nabla\varphi_j\|_2\|\nabla b\|_2\,|t-s|^{1/2},
\end{aligned}
$$

where constant C does not depend on t, s nor m. By Lemma 7.2, for each $\varepsilon > 0$, there exist $M > 0$, $N \in \mathbb{N}$ and $\phi_i \in L^2_\sigma$, $(i = 1, \dots, N)$ such that

$$
|I_4| \le \varepsilon\frac{5}{2}K(T) + \frac{\varepsilon}{2}K(T)\,|t-s| + MNK(T)\max_{i=1,\dots,N}\|\phi_i\|_2\,|t-s|,
$$

and

$$
|I_5| \le \Big(\int_0^\infty \|f\|_2^2\,d\tau\Big)^{1/2}|t-s|^{1/2} + \|\nabla b\|_2\|\nabla\varphi_j\|_2\,|t-s| + \|b\|_4^2\,\|\nabla\varphi_j\|_2\,|t-s|.
$$

Thus for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|(w_m(t), \varphi_j) - (w_m(s), \varphi_j)| \le \varepsilon, \qquad \text{if } |t-s| < \delta. \tag{7.12}$$

This completes the proof of Lemma 7.3. $\square$

7.2 Completion of the proof of Theorem 2.4.

By the Ascoli-Arzelà theorem, and the diagonal argument, it follows from (7.10) and Lemma 7.3 that there exists a subsequence $\{w_{m_k}(t)\}_{k=1}^{\infty} \subset \{w_m(t)\}$ converging to some $w(t)$, uniformly in $t \in [0, T]$, in the weak topology of $L^2_\sigma(\Omega)$ and the uniform limit $w(t)$ is weakly continuous in $L^2_\sigma(\Omega)$. On the other hand, since $\{w_m\}$ is bounded in $L^2(0, T; H^1_{0,\sigma}(\Omega))$ by (7.10), we obtain that $\{w_{m_j}(t)\} \subset \{w_{m_k}(t)\}$ converging to $w(t)$ weakly in $L^2(0, T; H^1_{0,\sigma}(\Omega))$. We may assume that the original sequence $\{w_m(t)\}$ converges to $\{w(t)\}$, for simplification of notation. Next we show that the limit function $w(t)$ is a desired solution. In order to prove this, we show that

$$\int_s^t (w_m \cdot \nabla w_m, \Phi)\, d\tau \to \int_s^t (w \cdot \nabla w, \Phi)\, d\tau, \tag{7.13}$$

$$\int_s^t (b \cdot \nabla w_m, \Phi)\, d\tau \to \int_s^t (b \cdot \nabla w, \Phi)\, d\tau, \tag{7.14}$$

$$\int_s^t (w_m \cdot \nabla b, \Phi)\, d\tau \to \int_s^t (w \cdot \nabla b, \Phi)\, d\tau, \tag{7.15}$$

as $m \to \infty$ for all Φ of the form :

$$\Phi = \sum_{\text{finite}} \lambda_l(\tau)\varphi_l(x), \quad \lambda_l \in C^1([s,t]). \tag{7.16}$$

We first show (7.13). In fact,

$$\begin{aligned} &\int_s^t (w_m \cdot \nabla w_m, \Phi)\, d\tau - \int_s^t (w \cdot \nabla w, \Phi)\, d\tau \\ &= \int_s^t ((w_m - w) \cdot \nabla w_m, \Phi)\, d\tau + \int_s^t (w \cdot \nabla (w_m - w), \Phi)\, d\tau \\ &=: J_1 + J_2. \end{aligned} \tag{7.17}$$

Now we estimate J_1. By Lemma 7.2 and the a priori estimate (7.10), we obtain that for each $\varepsilon > 0$ there exist a constant $M_\varepsilon > 0$, $N_\varepsilon \in \mathbb{N}$ and functions $\phi_1, \ldots, \phi_{N_\varepsilon}$ in $L^2_\sigma(\Omega)$ such that

$$|J_1| \le \varepsilon C K(T) + M_\varepsilon \sum_{j=1}^{N_\varepsilon} \int_s^t |(w_m - w, \phi_i)|^2 d\tau, \tag{7.18}$$

where the constant C is independent of ε, m. Since $w_m(t) \rightharpoonup w(t)$ uniformly in t weakly in $L^2_\sigma(\Omega)$, letting $m \to \infty$, we have

$$\limsup_{m\to\infty} |J_1| \leq \varepsilon C K(T). \tag{7.19}$$

Since for arbitrariness of $\varepsilon > 0$, it follows that $J_1 \to 0$ as $m \to \infty$. We next show $J_2 \to 0$. We set $u^i(x,\tau) := w^i(x,\tau)\Phi(x,\tau)$, $1 \leq i \leq 3$, where w^i stands for the i-th component of w, then $u^i \in L^2(\Omega \times (s,t))$. Hence there is a sequence $\{u^i_k\}_{k=1}^\infty \subset C_0^\infty(\Omega \times (s,t))$ with $u^i_k \to u^i$ in $L^2(\Omega \times (s,t))$ as $k \to \infty$. For the u^i_k, we have by integration by part,

$$\begin{aligned} |J_2| \leq& \sum_{i=1}^3 \int_s^t |(w_m - w, \partial_i u^i_k)|\, d\tau \\ &+ \sum_{i=1}^3 \Big(\int_s^t \|\nabla w_m - \nabla w\|^2\, d\tau \Big)^{1/2} \Big(\int_s^t \|u^i_k - u^i\|^2 \Big)^{1/2}, \end{aligned} \tag{7.20}$$

where $\partial_i = \partial/\partial x_i$. Letting $m \to \infty$ and then $k \to \infty$ in (7.20), we have by the a priori estimate (7.10), $J_2 \to 0$. Hence we obtain (7.13). The same argument as J_1 and J_2 yields (7.14) and (7.15).

In (7.6) taking finite sum with respect to l and letting $m \to \infty$, we obtain

$$\begin{aligned} \int_s^t \{ &-(w, \partial_t \Phi) + (\nabla w, \nabla \Phi) \\ &+ (b \cdot \nabla w, \Phi) + (w \cdot \nabla b, \Phi) + (w \cdot \nabla w, \Phi)\}\, d\tau \\ &= \int_s^t \langle F, \Phi \rangle\, d\tau - (w(t), \Phi(t)) + (w(s), \Phi(s)), \end{aligned} \tag{7.21}$$

for all Φ with the form (7.16). Since the set of all Φ with the form (7.16) is dense in $H^1(s,t\,; H^1_{0,\sigma}(\Omega))$ (see Masuda [9], Lemma 2.2), w is our desired weak solution of (N-S*). Once we obtain a weak solution of (N-S*) on $(0,T)$ by Galerkin method, then we can construct a global weak solution of (N-S*) by the diagonal argument. This proves Theorem 2.4. □

8 Existence of reproductive solutions; Proof of Theorem 2.5.

We show the existence of reproductive solutions. So we consider the approximate solutions constructed in the previous section. By (7.2), (7.5) and (7.9) we have

$$\frac{d}{dt}\|w_m\|_2^2 + 2(1 - 4\varepsilon_0)\|\nabla w_m\|_2^2 \leq C(\|f\|_2^2 + \|\nabla b\|_2^2 + \|b\|_4^4) =: K(t). \tag{8.1}$$

Then, by the Poincaré inequality, we obtain with a positive constant α independent of $m \in \mathbb{N}$

$$\frac{d}{dt}\|w_m\|_2^2 + \alpha\|w_m\|_2^2 \le K(t), \tag{8.2}$$

so that by integration on $(0, T_*)$

$$e^{\alpha T_*}\|w_m(T_*)\|_2^2 \le \|w_m(0)\|_2^2 + \int_0^{T_*} e^{\alpha t} K(t)\, dt. \tag{8.3}$$

If we choose $R > 0$, so that

$$R^2(1 - e^{-\alpha T_*}) \ge \int_0^{T_*} e^{-\alpha(T_* - t)} K(t)\, dt, \tag{8.4}$$

then it follows from (8.3) that

$$\|w_m(T_*)\|_2 \le R, \quad \text{if } \|w_m(0)\|_2 \le R. \tag{8.5}$$

On the other hand, we note that the map $w_m(0) \mapsto w_m(T_*)$ is continuous. Then the Brouwer fixed point theorem ensures the existence of an approximate solution w_m such that $w_m(0) = w_m(T_*)$ and $\|w_m(0)\|_2 = \|w_m(T_*)\|_2 \le R$ in the finite dimensional linear span of $\{\varphi_1, \ldots, \varphi_m\}$. Since R is independent of m, for each m there exists $a^{(m)} \in L^2_\sigma(\Omega)$ such that $a^{(m)} = w_m(0) = w_m(T_*)$. This sequence $\{w_m\}_{m=1}^\infty$ is bounded in $L^\infty(0, T_*; L^2_\sigma(\Omega)) \cap L^2(0, T_*; H^1_{0,\sigma}(\Omega))$. Then there is a function $w \in L^\infty(0, T_*; L^2_\sigma(\Omega)) \cap L^2(0, T_*; H^1_{0,\sigma}(\Omega))$ and $a \in L^2_\sigma(\Omega)$ such that

(i) $w_m \rightharpoonup w$ weakly$*$ in $L^\infty(0, T_*; L^2_\sigma(\Omega))$,

(ii) $w_m \rightharpoonup w$ weakly in $L^2(0, T_*; H^1_{0,\sigma}(\Omega))$,

(iii) $a^{(m)} \rightharpoonup a$ weakly in $L^2_\sigma(\Omega)$.

It is easy to see that w is a weak solution of (N-S*) with $w(0) = a$ and weakly continuous on $[0, T_*]$ in $L^2_\sigma(\Omega)$. Then $w(0) = w(T_*) = a$. This completes the proof of Theorem 2.5.

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