L^q-almost Solvability of Viscoplastic Models of Monotone Type

Sergiy Nesenenko*

December 10, 2009

Abstract

It is well-known that the loss of the coerciveness for governing monotone nonlinearities in evolution equations/inclusions can cost the non-solvability of the problem for a given data and the rule how to choose an appropriate data has to be prescribed. Allowing constitutive functions in the evolution relations for elasto/visco-plastic models of monotone type to be non-coercive we fisrt give a new (relaxed) meaning to the solvability of the systems of equations under consideration and then we define a criteria for choosing "right" data, which guarantees the solvability in the defined sense. Realizing this strategy a slight extension of the well-developed monotone operator method to our needs is performed. The theory is applied to some particular well-known models in elasto/visco-plasticity. The relations between the standard notion of the solvability and defined one are investigated.

Key words: existence, L^q-almost solvability, non-coercive problem, plasticity, viscoplasticity, monotone operator method, degenerate equations, general duality principle.

AMS 2000 subject classification: 74C05, 74C10, 74D10, 35J25, 34G20, 34G25, 47H04, 47H05, 47H14, 35F50, 47J40, 47J05, 47J20, 47J35

1 Introduction

Phenomenologically the inelastic rate-dependent or rate-independent respond of the solid materials on the deformation at small strains is usually modeled by the balance law for the linear momentum (the equilibrium equations), by the finite number of the evolution laws for the internal variables and by the constitutive relations, which connect stresses with the displacement gradient and the internal variables. More precisely, in the quasi-static case the following system of linear elliptic partial differential equations coupled with ordinary differential equations/inclusions governed by strong nonlinearities is studied.

*Sergiy Nesenenko, Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, 64289 Darmstadt, Germany, email: nesenenko@mathematik.tu-darmstadt.de, Tel.: +49-6151-16-2788 and Laboratoire Jacques-Louis Lions, Universite Pierre et Marie Curie, Boite courrier 187, 75252 Paris Cedex 05, France, email: nesenenko@ann.jussieu.fr, Tel.: +33-(0)144-2785-16

Setting of the problem. Let $\Omega \subset \mathbb{R}^3$ denote an open bounded set, the set of material points of the body, with C^1-boundary. $\mathcal{S}^3$ denotes the space of symmetric 3×3-matrices. The space $\mathcal{S}^3$ can be isomorphically identified with the space $\mathbb{R}^6$ (see [1, p. 31]). Therefore we can define a linear mapping $B : \mathbb{R}^N \to \mathcal{S}^3$ as a composition of a projector from $\mathbb{R}^N$ onto $\mathbb{R}^6$ and the isomorphism between $\mathbb{R}^6$ and $\mathcal{S}^3$. The transpose $B^T : \mathcal{S}^3 \to \mathbb{R}^N$ is given then by

$$B^T \tau = (\hat{z}, 0)^T$$

for $\tau \in \mathcal{S}^3$ and $z = (\hat{z}, \tilde{z})^T \in \mathbb{R}^N$, $\tilde{z} \in \mathbb{R}^{N-6}$. T_e denotes a positive number (time of existence) and for $0 < t < T_e$

$$\Omega_t = \Omega \times (0, t).$$

If a nonlinear function $g : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ satisfying the inequality $v^* \cdot v \geq 0$ for all $v^* \in g(v)$ is given, then one looks for the unknown displacement $u(x,t) \in \mathbb{R}^3$, the Cauchy stress tensor $T(x,t) \in \mathcal{S}^3$ and the vector of internal variables $z(x,t) = (\varepsilon_p(x,t), \tilde{z}(x,t)) \in \mathbb{R}^6 \times \mathbb{R}^{N-6}$ of the following model equations

$$-\operatorname{div}_x T(x,t) = b(x,t), \tag{1}$$

$$T(x,t) = \mathcal{D}(\varepsilon(u(x,t)) - Bz(x,t)), \tag{2}$$

$$\begin{aligned} \partial_t z(x,t) &\in g\Big(-\nabla_z \psi(\varepsilon(u(x,t)), z(x,t))\Big) \\ &= g\Big(B^T T(x,t) - Lz(x,t)\Big), \end{aligned} \tag{3}$$

with the initial condition

$$z(x,0) = 0, \tag{4}$$

and with the Dirichlet boundary condition

$$u(x,t) = \gamma(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty). \tag{5}$$

The term $\varepsilon(u(x,t))$ in the equations denotes the symmetrized gradient of u, the strain tensor. The linear mapping B assigns to the vector $z(x,t)$ the plastic strain tensor $\varepsilon_p(x,t) = Bz(x,t) \in \mathcal{S}^3$. We denote by $\mathcal{D} : \mathcal{S}^3 \to \mathcal{S}^3$ a linear, symmetric, positive definite mapping, the elasticity tensor. The function $b : \Omega \times [0,\infty) \to \mathbb{R}^3$ is the volume force and $\gamma : \partial\Omega \times [0,\infty) \to \mathbb{R}^3$ is the boundary data. The positive semi-definite quadratic form in (3)

$$\psi(\varepsilon, z) = \frac{1}{2}\mathcal{D}\Big(\varepsilon - Bz\Big) \cdot \Big(\varepsilon - Bz\Big) + \frac{1}{2}\Big(Lz\Big) \cdot z$$

represents the free energy. The linear mapping $L : \mathbb{R}^N \to \mathbb{R}^N$ above is symmetric and positive semi-definite.

Definition 1.1. *The system of equations (1) - (5) with the mappings B and L introduced above is called a problem/model of monotone type iff the symmetric $N \times N-$matrix $M := L + B^T \mathcal{D} B$ is positive definite and the nonlinear function $g : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ satisfying the inequality $v^* \cdot v \geq 0$ for all $v^* \in g(v)$ is monotone.*

To simplify computations in this work we also assume that the linear mappings B und L satisfy the following condition

$$\dim(\ker B) = N - 6, \quad \dim(\ker L) = 6, \quad \ker B + \ker L = \mathbb{R}^N. \tag{6}$$

It was pointed out in [3] that condition (6) being the consequence of the physical structure of the equations (1) - (5) is always fulfilled, and, therefore, condition (6) can be added to the definition of the models of monotone type without loss of generality.

The class of problems of monotone type was introduced by Alber in [1] and generalized naturally the class of generalized standard materials defined by Halphen and Nguyen Quoc Son in [24]. The function g in (3) for generalized standard materials is a subdifferential of a convex function and, since the subdifferential of a convex function is monotone, the class of generalized standard materials is a sub-class of problems of monotone type. Typical application of such models is elasto/visco-plasticity with or without hardening effects at small strains. Such classical models of Prandtl-Reuss and Norton-Hoff belong to this class and are examples from rate-independent and rate-dependent sub-classes of monotone problems, respectively. It is worth to mention here that the initial boundary value problem (1) - (5) is written in the most general form and, describing thermodynamically admissible processes, includes all elasto/visco-plastic models at small strains used in engineering (see [1]), yet the function g is not monotone quite often. In the rate independent case, i.e. when $g = \partial I_K$ for a closed convex set K, an alternative approach for such models was proposed by Mielke and Theil in [34], a so-called energetic formulation. In the setting of Mielke and Theil the effects like damage, fracture and hysteretic behavior in ferroelectic materials at finite strains can be also analyzed.

Here we are not going to give a complete survey of the relevant literature (we refer to [1, 9, 12, 16, 17, 20, 33, 32, 35, 39, 46, 49, 55]) and mention only those publications which actually motivated this work. The mathematical analysis of the classical models, which were later included into the monotone class, started with works by Moreau, Duvaut and Lions and Johnson [19, 29, 36] and followed by the further investigations in [30, 37]. In these publications the stress for the Prandtl-Reuss model was properly characterized while the existence of the appropriate displacement remained unclear. Suquet [50] and Temam [51, 52] carried out a complete analysis of this problem and showed that the displacements in general belong to the space of bounded deformations, only (see the monographs [21, 25] for a historical survey). The extension of the ideas of Suquet and Temam to other models in elasto/visco-plasticity is performed in [12]. With the introduction of the problems of monotone type the positivity of the free energy (like the case of linear kinematic hardening) started to play an important role in the existence issues. In [1, 2] the autors proved that the problem (1) - (5) with positive definite free energy has a (unique) solution under very mild assumptions on the function g, it must be maximal monotone. The case of positive semi-definite free energy, which contains the mentioned ealier models of Prandtl-Reuss and Norton-Hoff as well as models with linear isotropic hardening [13, 16, 25, 30], turned out to be quite challenging comparing to the case of positive definite free energy even for a maximal monotone g. In [3, 13, 14, 22, 42, 43] an approach for the derivation of the existence of solutions to the problem (1) - (5) initiated in [1, 2] was continued and extended to particular models of monotone type with a positive semi-definite energy. Using a time-discretization technique and convex analysis the existence for the nonlinear Maxwell model of viscoelasticity was proved in [56]. A common feature of all just mentioned works investigating the rate-dependent case is that the function g is coercive in an appropriate functional space (see Section 3 for the definition) and the usage of either the monotone operator method[1] or the convex analysis or both. We note that the monotone operator method and the convex analysis are called for the treetment of the rate-independent problems as well. In the present paper we are trying to drop the coercivity property of the function g to be able to handle more general rate-dependent problems of monotone type and to develop such an existence theory which could be simultaneously applied to rate-independent problems. However, it is well-known that the absence of the coercivity infers the non-solvability of a problem for a fixed data and the rule how to choose an "admissible" data has to be prescribed. Following this strategy we have slightly extended some techniques of the monotone operator method to our needs in order to be able to treet maximal monotone functions g, which can blow up at some point, for example, (Theorem 4.1 in Section 4). The other observation we made is that, if the domain of the Nemitsckyi operator generated by a function g has an empty interior, like the case for Prandtl-Reuss model, then one can not expect the solvability in $L^q(W^{1,q})-$spaces in general and the notion of solutions has to be relaxed to be a unified substance for all models, for those which have solutions in $L^q(W^{1,q})-$spaces for some $q \in (1, \infty)$ and for those which not. The last led us to the introduction of the L^q-almost solability in Section 5. In Section 6 we apply the theory constructed in Section 5 to the models of Norton-Hoff and Prandtl-Reuss and to the model of nonlinear kinematic hardening and show that the model of Prandtl-Reuss can be only L^q-almost solvable and not sovable in $L^q(W^{1,q})-$setting. This agrees completely with the existence theory derived for this model ealier. In contrast to the Prandtl-Reuss model, the models of Norton-Hoff and nonlinear kinematic hardening possess solutions in $L^q(W^{1,q})-$spaces.

Functional spaces and notation. We denote the Banach space of Lebesgue integrable with the power p together with their weak derivatives up to the order m functions by $W^{m,p}(\Omega, \mathbb{R}^n)$. The norm in $W^{m,p}(\Omega, \mathbb{R}^n)$ is denoted by $\|\cdot\|_{m,p,\Omega}$ ($\|\cdot\|_{p,\Omega} := \|\cdot\|_{0,p,\Omega}$). By $W_0^{m,p}(\Omega, \mathbb{R}^n)$ we denote the closure of $C_0^\infty(\Omega, \mathbb{R}^n)$ of all infinitely differentiable functions with compact support in Ω with respect to $\|\cdot\|_{m,p,\Omega}$. The spaces $W^{-m,q}(\Omega, \mathbb{R}^n)$ are the dual spaces for $W_0^{m,p}(\Omega, \mathbb{R}^n)$. If m is not integer, then the corresponding Sobolev-Slobodeckij space is denoted by $W^{m,p}(\Omega, \mathbb{R}^n)$. For p and q satisfying $1 < p, q < \infty$ and $1/p + 1/q = 1$ one can define a bilinear form on the product space $L^p(\Omega, \mathbb{R}^n) \times L^q(\Omega, \mathbb{R}^n)$ by

$$(\xi, \zeta)_\Omega = \int_\Omega \xi(x) \cdot \zeta(x) dx.$$

[1]The monotone operator method was basically developed in 60ies (see the two-volume monograph [27, 28] for a historical survey on the subject) and from that time constantly improved and generalized. During its time of existence it has penetrated into many different fields of mathematics and has been applied to the variety of problems in differential and evolution inclusions, control and optimization theories, mathematical economics, game theory and calculus of variations. Some recent applications of the monotone operator method in continuum (thermo-) mechanics of solids and fluids, electrically (semi-) conductive media, modelling of biological systems, or in mechanical engineering can be found in [47].

We also define another bilinear form on $L^p(\Omega,\mathbb{R}^n)\times L^q(\Omega,\mathbb{R}^n)$ by

$$[\xi,\zeta]_\Omega = (\mathcal{D}\xi,\zeta)_\Omega.$$

Spaces of functions of bounded deformation. We recall that a Radon measure, denoted by μ, is a linear continuous functional on the space $C_0(\Omega,\mathbb{R})$, the space of continuous functions with compact support in Ω. The space of Radon measures, which are bounded with respect to the norm

$$\|\mu\|_{\mathcal{M}} := \sup\left\{\frac{|\langle\mu,\phi\rangle|}{\|\phi\|_\infty} \mid \phi\in C_0(\Omega,\mathbb{R})\right\},$$

where $\|\cdot\|_\infty$ denotes the supremum norm, is donoted by $\mathcal{M}(\Omega,\mathbb{R})$. It is a Banach space when equipped with the above norm. The space of functions of bounded deformation is defined by

$$BD(\Omega,\mathbb{R}^n) = \left\{v\in L^1(\Omega,\mathbb{R}^n) \mid \varepsilon_{ij}(v)\in\mathcal{M}(\Omega,\mathbb{R}),\ i,j=1,...,n\right\}.$$

The space $BD(\Omega,\mathbb{R}^n)$ is a Banach space with norm $\|\cdot\|_{BD}$ defined by

$$\|v\|_{BD} = \|v\|_{1,\Omega} + \sum_{i,j=1}^{n}\|\varepsilon_{ij}(v)\|_{\mathcal{M}}.$$

The embedding properties of spaces $BD(\Omega,\mathbb{R}^n)$ can be found in [51, 53].

Spaces of Bochner-measurable functions. If (X,H,X^*) is an evolution triple (known as "Gelfand triple") and $1<p,q<\infty$, $1/p+1/q=1$, then

$$W_{p,q}(0,T_e;X) := \left\{u\in L^p(0,T_e;X) \mid \dot{u}\in L^q(0,T_e;X^*)\right\}$$

are separable reflexive Banach spaces when furnished with the norm

$$\|u\|^2_{W_{p,q}} = \|u\|^2_{L^p(0,T_e;X)} + \|\dot{u}\|^2_{L^q(0,T_e;X^*)},$$

where the time derivative of $u(\cdot)$ is understood in the sense of vector-valued distributions. The space $L^p(0,T_e;X)$ in the definition of $W_{p,q}(0,T_e;X)$ denotes the Banach space of all Bochner-measurable functions $u:[0,T_e)\to X$ such that $t\mapsto\|u(t)\|^p_X$ is integrable on $[0,T_e)$. We recall that the embedding $W_{p,q}(0,T_e;X)\subset C([0,T_e],H)$ is continuous ([28, p. 4], for instance).

Finally, we frequently use the spaces $W^{k,p}(0,T_e;X)$, which consist of Bochner measurable functions having a p-integrable weak derivatives up to order k.

2 The Helmholtz projection on tensor fields

The construction of the solutions for the initial boundary value problem (1)–(5) is based on the existence theory for the evolution inclusions in a reflexive Banach space derived in Section 4. The construction procedure requires the introduction of projection operators to spaces of tensor fields, which are symmetric gradients and to spaces of tensor fields with vanishing divergence. All material for this section is taken from [2, 3], where more details and proofs of stated hier results can be found.

We recall ([54, Theorem 4.2]) that a Dirichlet boundary value problem from the linear elasticity theory formed by equations

$$-\mathrm{div}_x T(x) = \hat{b}(x), \qquad x \in \Omega, \tag{7}$$

$$T(x) = \mathcal{D}(\varepsilon(u(x)) - \hat{\varepsilon}_p(x)), \qquad x \in \Omega, \tag{8}$$

$$u(x) = \hat{\gamma}(x), \qquad x \in \partial\Omega, \tag{9}$$

to given $\hat{b} \in W^{-1,p}(\Omega, \mathbb{R}^3)$, $\hat{\varepsilon}_p \in L^p(\Omega, \mathcal{S}^3)$ and $\hat{\gamma} \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$ has a unique weak solution $(u, T) \in W^{1,p}(\Omega, \mathbb{R}^3) \times L^p(\Omega, \mathcal{S}^3)$ with $1 < p < \infty$ and $1/p + 1/q = 1$. For $\hat{b} = \hat{\gamma} = 0$ the solution of (7) - (9) satisfies the inequality

$$\|\varepsilon(u)\|_{p,\Omega} \le C \|\hat{\varepsilon}_p\|_{p,\Omega}$$

with some positive constant C depending on p and Ω.

Definition 2.1. *For every $\hat{\varepsilon}_p \in L^p(\Omega, \mathcal{S}^3)$ we define a linear operator $P_p : L^p(\Omega, \mathcal{S}^3) \to L^p(\Omega, \mathcal{S}^3)$ by*

$$P_p \hat{\varepsilon}_p = \varepsilon(u),$$

where $u \in W_0^{1,p}(\Omega, \mathbb{R}^3)$ is a unique weak solution of (7) - (9) to the given function $\hat{\varepsilon}_p$ and $\hat{b} = \hat{\gamma} = 0$.

A subset $\mathcal{G}^p$ of $L^p(\Omega, \mathcal{S}^3)$ is defined by

$$\mathcal{G}^p = \{\varepsilon(u) \mid u \in W_0^{1,p}(\Omega, \mathbb{R}^3)\}.$$

The main properties of P_p are stated in the following lemma.

Lemma 2.1. *For every $1 < p < \infty$ the operator P_p is a bounded projector onto the subset $\mathcal{G}^p$ of $L^p(\Omega, \mathcal{S}^3)$. The projector $(P_p)^*$ adjoint with respect to the bilinear form $[\xi, \zeta]_\Omega$ on $L^p(\Omega, \mathcal{S}^3) \times L^q(\Omega, \mathcal{S}^3)$ satisfy*

$$(P_p)^* = P_q, \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

This implies $\ker(P_p) = H^p_{sol}$ *with*

$$H^p_{sol} = \{\xi \in L^p(\Omega, \mathcal{S}^3) \mid [\xi, \zeta]_\Omega = 0 \quad \text{for all } \zeta \in \mathcal{G}^q\}.$$

From the last lemma it follows that the projection operator

$$Q_p = (I - P_p) : L^p(\Omega, \mathcal{S}^3) \to L^p(\Omega, \mathcal{S}^3)$$

with $Q_p(L^p(\Omega, \mathcal{S}^3)) = H^p_{sol}$ is a generalization of the classical Helmholtz projection.

Corollary 2.0.1. *Let $(B^T \mathcal{D} Q_p B)^*$ denote the adjoint operator to $B^T \mathcal{D} Q_p B : L^p(\Omega, \mathbb{R}^N) \to L^p(\Omega, \mathbb{R}^N)$ with respect to the bilinear form $(\xi, \zeta)_\Omega$ on the product space $L^p(\Omega, \mathbb{R}^N) \times L^q(\Omega, \mathbb{R}^N)$. Then*

$$(B^T \mathcal{D} Q_p B)^* = B^T \mathcal{D} Q_q B : L^q(\Omega, \mathbb{R}^N) \to L^q(\Omega, \mathbb{R}^N).$$

Moreover, the operator $B^T \mathcal{D} Q_2 B$ is non-negative and self-adjoint.

Proof. This result is shown in [2, 3]. For the reader's convenience we give here another proof of the non-negativity of $B^T \mathcal{D} Q_2 B$, which is based only on the definition of P_2. To this end, we note first that, if $\hat{u}$ is a solution of (7) - (9) to the given $\hat{\varepsilon}_p = Bv$ and $\hat{b} = \hat{\gamma} = 0$ for an arbitrary chosen $v \in L^2(\Omega, \mathrm{I\!R}^{\mathrm{N}})$, then using it as a test function in (7) - (9) we obtain

$$\int_\Omega \mathcal{D}\big(Bv(x) - \varepsilon(\hat{u}(x))\big) \cdot \varepsilon(\hat{u}(x)) dx = 0.$$

Therefore, for any $v \in L^2(\Omega, \mathrm{I\!R}^{\mathrm{N}})$,

$$\int_\Omega B^T \mathcal{D} Q_2 B v(x) \cdot v(x) dx = \int_\Omega B^T \mathcal{D} B v(x) \cdot v(x) dx$$

$$- \int_\Omega B^T \mathcal{D} \varepsilon(\hat{u}(x)) \cdot v(x) dx = \int_\Omega \mathcal{D}(Bv(x) - \varepsilon(\hat{u}(x))) \cdot Bv(x) dx$$

$$= \int_\Omega \mathcal{D}\big(Bv(x) - \varepsilon(\hat{u}(x))\big) \cdot \big(Bv(x) - \varepsilon(\hat{u}(x))\big) dx \geq 0.$$

This completes the proof of the corollary. □

3 Sum of two operators

In this section we present some results on the sum of two operators, which will be used for the construction of the solutions for models of monotone type with positive semi-definite free energy.

Let V be a reflexive Banach space with the norm $\|\cdot\|$, V^* be its dual space with the norm $\|\cdot\|_*$. The brackets $\langle\cdot,\cdot\rangle$ denotes the dual pairing between V and V^*. Under V we shall always mean a reflexive Banach space throughout this section. For a multivalued mapping $A : V \to 2^{V^*}$ the sets

$$D(A) = \{v \in V \mid Av \neq \emptyset\}$$

and

$$GrA = \{[v, v^*] \in V \times V^* \mid v \in D(A),\ v^* \in Av\}$$

are called the *effective domain* and the *graph* of A, respectively.

Quite often it is important to know when the sets

$$R(A + B) = \cup_{v\in V}(A + B)v \quad \text{and} \quad R(A) + R(B) = \cup_{v\in V, u\in V}(Av + Bu)$$

are almost equel[2].

Remark 3.1. *The set $R(A) + R(B)$ is larger than $R(A + B)$. For example, we can take on $V = \mathbb{R}^N$ the rotations on $\pi/2$ and $-\pi/2$ as the operators A and B, respectively. Then $R(A) + R(B) = \mathbb{R}^N$, but $R(A + B) = \{0\}$. Therefore, one needs to look for conditions, which guarantee the almost equivalence of the sets $R(A) + R(B)$ and $R(A + B)$.*

[2]We say that two sets U_1 and U_2 of V are *almost equel* and write $U_1 \simeq U_2$ provided that

$$\overline{U}_1 = \overline{U}_2 \quad \text{and} \quad \operatorname{int} U_1 = \operatorname{int} U_2.$$

We introduce some notions.

Definition 3.1. *A mapping* $A : V \to 2^{V^*}$ *on a real Banach space* V *is* n-monotone *iff the condition*

$$\langle u_1^*, u_1 - u_2\rangle + \langle u_2^*, u_2 - u_3\rangle + ... + \langle u_n^*, u_n - u_{n+1}\rangle \geq 0 \tag{10}$$

holds for all $[u_i, u_i^*] \in GrA$, $i = 1, ..., n$, *for fixed* $n \geq 2$, *where we set* $u_{n+1} = u_1$. *A mapping* $A : V \to 2^{V^*}$ *is called* cyclic monotone *iff the condition (10) holds for any* $n = 2, 3, ...$ *A mapping* $A : V \to 2^{V^*}$ *is called* monotone *if (10) holds for* $n = 2$.

A mapping $A : V \to 2^{V^*}$ *is called* $3-\sigma-$monotone *iff* A *is monotone and there is a number* $\sigma > 0$ *such that*

$$\langle v^* - u^*, w - v\rangle \leq \sigma \langle u^* - w^*, u - w\rangle$$

holds for all $[w, w^*], [u, u^*], [v, v^*] \in GrA$.

A mapping $A : V \to 2^{V^*}$ *is called* 3^*-monotone *iff* A *is monotone and*

$$\sup_{[v,v^*]\in GrA} \langle v^* - u^*, w - v\rangle < \infty \tag{11}$$

holds for all $w \in D(A), u^* \in R(A)$.

A mapping $A : V \to 2^{V^*}$ *is called* strongly coercive *iff either* $D(A)$ *is bounded or* $D(A)$ *is unbounded and the condition*

$$\frac{\langle v^*, v - w\rangle}{\|v\|} \to +\infty \quad \textit{as } \|v\| \to \infty, \quad [v, v^*] \in GrA,$$

is satisfied for each $w \in D(A)$.

Remark 3.2. *Some basic facts on maximal monotone and generalized pseudo-monotone operators the reader can find in Appendix 7.*

From the last definition it is seen that any 3-monotone mapping (in particular, cyclic monotone) is $3-\sigma-$monotone with $\sigma = 1$ and that a $3-\sigma-$monotone mapping satisfies the condition (11). Less obvious consequence is that if A is monotone and strongly coercive or monotone and has the bounded range $R(A)$, then A is 3^*-monotone (see [57, Proposition 32.41], [40, Chapter V.3.2]). In particular, the Yosida approximation $A_\lambda = (A^{-1} + \lambda J^{-1})^{-1}$ of an operator A, where J is the duality mapping, gives an example of the 3^*-monotone map. We note as well that the inverse of a 3^*-monotone operator is 3^*-monotone.

The condition (11) plays a decisive role in solving the problem when the range $R(A + B)$ of the sum operator $A + B$ is almost equal to $R(A) + R(B)$, i.e.

$$\operatorname{int} R(A + B) = \operatorname{int}\big(R(A) + R(B)\big) \ \text{ and } \ \overline{R(A + B)} = \overline{R(A) + R(B)}.$$

The particular answer, when $R(A + B) \simeq R(A) + R(B)$ holds, is given by the following theorem.

Theorem 3.1. (Brezis & Haraux, '76)
Let V *be a reflexive Banach space and let* A *and* B *be two monotone operators in* V *such that* $A + B$ *is maximal monotone. If either*

i) both A *and* B *satisfy (11),*

or

ii) B *satisfies (11) and* $D(A) \subset D(B)$,

then $R(A + B) \simeq R(A) + R(B)$.

Theorem 3.1 was firstly proved in a Hilbert space by Brezis and Haraux [7], and then generalized to a reflexive Banach space by Reich [44]. For general Banach spaces we refer the reader to [45]. The proof of Theorem 3.1 in a Hilbert space can be found in [40, Theorem V.3.] and in [57, Corollary 32.46] as well. One of the most important example of 3^*-monotone operator, which appears very often in applications, is the subdiferential of a proper functional on a real Banach space V. The subdiferential satisfies (11), since it is cyclic monotone, what is an easy consequence of the definition of the subdiferential. Further examples of cyclic monotone operators (hence 3^*-monotone), which also appear frequently in applications, the reader can find in [5, Chapter II.2.3], for example.

For further discussion we also recall some basic facts from convex analysis. The Legendre-Fenchel transformation of a convex lower semi-continuous function $\phi : V \to (-\infty, +\infty]$ is the function ϕ^* defined for each $v^* \in V^*$ by

$$\phi^*(v^*) = \sup_{v \in V} \{\langle v^*, v\rangle - \phi(v)\}.$$

The indicator funktion of a convex set K is the convex function I_K defined by

$$I_K(v) = \begin{cases} 0 & \text{if } v \in K \\ +\infty & \text{otherwise} \end{cases}.$$

Then $I_K^* = \sigma_K$ holds, where σ_K is the support function of K given by

$$\sigma_K(v^*) = \sup_{v \in K} \langle v^*, v\rangle \, .$$

Remark 3.3. *If two closed and convex sets K_1 and K_2 in V are given, then it is well-known that $K_1 \subset K_2$ if and only if, for any $v^* \in V^*$, $\sigma_{K_1}(v^*) \leq \sigma_{K_1}(v^*)$ (see [15, Proposition II.1.3]).*

The following result of Brezis and Nirenberg [8, Proposition II.5, Proposition II.6] is of great importance for our further investigations.

Proposition 3.1. (Brezis & Nirenberg, '78)
Let N be a closed subspace of a reflexive Banach space V and B be a non-linear map from V into V^*. The following statements are equivalent

$$I^*_{R(B)}(v) \geq \langle f, v\rangle \, , \quad \forall v \in N,$$

$$f \in \overline{N^\perp + \mathrm{conv} R(B)}.$$

Remark 3.4. *We note that Theorem 3.1 in [8] is actually proved in a Hilbert space, but it can be easily generalized to a reflexive Banach space.*

If the operator B in Proposition 3.1 is maximal monotone, then based on results in Section 7 the following corollary holds true.

Corollary 3.1.1. *If, additionally to the assumptions in Proposition 3.1, the operator B is maximal monotone, then the statements are equivalent*

$$I^*_{R(B)}(v) \geq \langle f, v\rangle \, , \quad \forall v \in N,$$

$$f \in \overline{N^\perp + R(B)}.$$

The next lemma, which is due to Browder, will be important in Section 4.

Lemma 3.1. *Let V be a Banach space, $\{u_n\}$ a sequence in V, and $\{\alpha_n\}$ a sequence of positive numbers with $\alpha_n \to 0$ as $n \to \infty$. Fix $r > 0$ and assume that for every $h \in V^*$ with $\|h\| \leq r$ there exists a constant C_h such that $\langle h, u_n \rangle \leq \alpha_n \|u_n\| + C_h$, for all n. Then the sequence $\{u_n\}$ is bounded.*

Proof. See [11, Lemma 1]. □

4 Some results on the sum of two operators

Case of a maximal monotone operator B. The first part of this section is devoted to the proof of the following result on the sum of two maximal monotone operators in a reflexive Banach space.

Theorem 4.1. *Let V be a reflexive Banach space, $A : D(A) \subset V \to V^*$ be a linear maximal monotone operator and $B : D(B) \subset V \to 2^{V^*}$ be a maximal monotone operator with $[0, 0] \in GrB$. Suppose that the following conditions are satisfied:*

i) the operator B is 3^*-monotone,

ii) the operator B is locally bounded at every $v \in D(B)$.

Then

$$R(A + B) \simeq R(A) + R(B).$$

Remark 4.1. *Theorem 7.2 guarantees that the operator A in Theorem 4.1 is a densely defined closed operator.*

Remark 4.2. *Since B is locally bounded at every $v \in D(B)$, by Lemma 7.1 the domain $D(B)$ of B is open.*

Proof. Since $0 \in D(B)$ and $D(B)$ is open, by Theorem 7.3, the sum $A + B$ is maximal monotone. Hence, by Theorem 7.1, the operator $A + B + \lambda J$ is surjective for all $\lambda > 0$, i.e. $R(A + B + \lambda J) = V^*$, where J denotes the duality mapping.
Next, we claim that $R(A_\lambda + B) \simeq R(A) + R(B)$, where A_λ is the Yosida approximation of A. Indeed, A_λ is 3^*-monotone and maximal monotone, then, by Theorem 3.1, we obtain that

$$R(A_\lambda + B) \simeq R(A_\lambda) + R(B) = R(A) + R(B).$$

In the last equality we used the property of the Yosida appriximation of single-valued maps, namely that the relation $A_\lambda u = A(j_\lambda^A u)$ holds for any $u \in V$, where $j_\lambda^A : V \to D(A)$ is the resolvent of A.
Now we are going to establish two inclusions

$$R(A_\lambda + B) \subset \overline{R(A + B)} \quad \text{and} \quad \operatorname{int} R(A_\lambda + B) \subset R(A + B), \tag{12}$$

which will complete the proof of the theorem. To this end, we adopt the proof of Theorem 2.1 in [23] to our situation.
Since $R(A + B + (1/n)J) = V^*$, the equation

$$Au + Bu + (1/n)Ju \ni s$$

has a sulution u_n for any $s \in R(A_\lambda + B)$, i.e. for $s = A_\lambda v + v^*$ with $[v, v^*] \in GrB$. If $\|u_n\|$ is bounded, then trivially one has $s \in \overline{R(A+B)}$. Assume that $\|u_n\|$ is unbounded and choose $\lambda > 0$ such that

$$u_s = j_\lambda^A v \in D(B), \quad \text{for} \quad v \in D(B).$$

In the last line we are able to choose $u_s \in D(B)$ provided $\lambda > 0$ is small enough, since $j_\lambda^A v$ converges strongly in V to v for all $v \in \overline{D(A)}$ as λ tends to zero (see Proposition 7.1). In our case $\overline{D(A)} = V$. Then, for some $u_n^* \in B(u_n)$, we obtain

$$\left\langle \frac{1}{n} Ju_n, u_n - u_s \right\rangle = - \langle Au_n + u_n^* - A_\lambda v - v^*, u_n - u_s \rangle$$

$$= - \left\langle Au_n - A_\lambda v, u_n - j_\lambda^A v \right\rangle + \left\langle u_n^* - v^*, j_\lambda^A v - u_n \right\rangle$$
$$\leq \left\langle u_n^* - v^*, j_\lambda^A v - u_n \right\rangle \leq k(v^*, \lambda),$$

which implies

$$\frac{1}{n} \|Ju_n\|_* \leq \frac{1}{n} \|u_s\| + k(v^*, \lambda)/\|u_n\|.$$

The last inequality says that $(1/n)Ju_n$ converges to 0 as $n \to \infty$ provided $\|u_n\|$ is sufficiently large. It implies that $s \in \overline{R(A+B)}$.

Next, we shall show that the second inclusion in (12) holds. Indeed, choose any $s \in \operatorname{int} R(A_\lambda + B)$ and consider any $h \in \overline{B(0,r)}$ for some $r > 0$ such that $s + h \in R(A_\lambda + B)$. We claim that the solution u_n of $Au + Bu + (1/n)Ju \ni s$ is bounded. Otherwise, we could assume that $\|u_n\| \to \infty$ as $n \to \infty$. Then

$$\langle h, u_n - u_{s+h} \rangle = - \langle Au_n + u_n^* - (s+h), u_n - u_{s+h} \rangle - \frac{1}{n} \langle Ju_n, u_n - u_{s+h} \rangle,$$

where u_{s+h} is a fixed element in V depending on $s + h$ and $u_n^* \in B(u_n)$. The last equality implies that

$$\langle h, u_n \rangle \leq \langle h, u_{s+h} \rangle + \frac{k(s+h, u_{s+h})}{\|u_n\|} \|u_n\| - \frac{1}{n} \|Ju_n\| \left(\|u_n\| - \|u_{s+h}\| \right)$$

$$\leq \langle h, u_{s+h} \rangle + \frac{k(s+h, u_{s+h})}{\|u_n\|} \|u_n\|$$

for large enough n. But then, by Lemma 3.1, the sequence $\{u_n\}$ has to be bounded, what is a contradiction to the assumption. Since $\|u_n\|$ is bounded, we obtain (up to extracting a subsequence) that

$$Au_n + u_n^* = -\frac{1}{n} Ju_n + s \to s \quad \text{and} \quad u_n \rightharpoonup u_0,$$

and hence, by maximal monotonicity of $A + B$, that $s \in R(A+B)$. The last inclusion gives $\operatorname{int} R(A_\lambda + B) \subset R(A+B)$, what completes the proof of Theorem 4.1. □

Remark 4.3. *According to assumptions in Theorem 4.1, the sum $A+B$ is maximal monotone, since A is linear and $0 \in \operatorname{int} D(B)$. But none of the conditions i) or ii) of Theorem 3.1 is satisfied*[3] *and, therefore, this theorem can not be*

[3] Example 2.21 in [41] shows that a linear maximal monotone map can not be 3^*−monotone.

applied directly to an operator B with the domain not equal to the whole space (see an example in the next section). Moreover, the condition ii) can not be dropped in Theorem 3.1. Take, for example, on $V = \mathbb{R}^2$ the maps $A =$ rotation by $\pi/2$ and $B = \partial I_K$, where I_K is the idicator function of $K = \mathbb{R} \times \{0\}$. Then $R(A) + R(B) = \mathbb{R}^2$ while $R(A + B) = \{0\} \times \mathbb{R}$. This example also explains why we require in Theorem 4.1 the openness of the domain of B.

Combining Proposition 3.1 with Theorem 4.1 we get the following corollary.

Corollary 4.1.1. *Suppose that all conditions of Theorem 4.1 are satisfied and assume that*

$$I^*_{R(B)}(v) \geq \langle f, v \rangle \,, \quad \forall v \in \ker A^*$$

holds for $f \in V^*$, then the problem

$$Av + Bv \ni f, \quad v \in V \tag{13}$$

is almost solvable, i.e. $f \in \overline{R(A + B)}$.

Case of the generalized pseudomonotone operator B. If the operator B is generalized pseudomonotone, then we can get the almost solvability of (13) applying a result on the range of non-linear operarors in [23]. To state the mentioned result on the range of the sum of two operarors in [23] we need the following definitions.

Definition 4.1. *The mapping $A : V \to 2^W$, with topological spaces V and W, is called* upper semicontinuous *at $v_0 \in D(A)$ iff for any given open set $\tilde{W} \subset W$ such that $Av_0 \subset \tilde{W}$ there exists a neighborhood $\tilde{V}$ of v_0 such that $A(\tilde{V}) \subset \tilde{W}$.*
The mapping $A : V \to 2^{V^*}$ is called finitely continuous *iff A is upper semicontinuous from each finite-dimensional subspace F of V to the weak topology of V^*.*
The mapping $A : V \to 2^{V^*}$ is called quasibounded *iff for each $M > 0$ there exists $K(M) > 0$ such that for $v \in V$ and $v^* \in Av$ with $\|v\| \leq M$ and $\langle v^*, v\rangle \leq M\|v\|$ we have that $\|v^*\| \leq K(M)$.*

Remark 4.4. *A maximal monotone operator $A : V \to 2^{V^*}$ is upper semicontinuous from* $\operatorname{int} D(A)$ *into V^* furnished with the weak* topology ([10, Theorem 3.18], [27, Theorem III.1.28]).*

We denote by Γ the set of all functions $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\beta(r) \to 0$ as $r \to \infty$. The main theorem in [23] reads as follows.

Theorem 4.2. (Guan & Kartsatos & Skrypnik, '03)
Let V be a reflexive Banach space, $A : V \to 2^{V^*}$ maximal monotone, and $B : V \to 2^{V^*}$ quasibounded, finitely continuous and generalized pseudomonotone. Let $V' \subset D(B)$, where V' is a dense subspace of V with $V' \cap D(A) \neq \emptyset$. Let S be a subset of V^* such that for every $s \in S$, there exist $v_s \in V$ and $\beta = \beta_s \in \Gamma$ such that

$$\langle u^* + v^* - s, v - v_s\rangle \geq -\beta(\|v\|)\|v\| \tag{14}$$

for all $v \in D(A) \cap D(B)$ with $\|v\|$ sufficiently large, and all $u^* \in Av, v^* \in Bv$. Then

$$S \subset \overline{R(A + B)} \quad \textit{and} \quad \operatorname{int} S \subset \operatorname{int} R(A + B).$$

Combining now Proposition 3.1 with Theorem 4.2 for $S = R(A) + R(B)$ we get the following result.

Corollary 4.2.1. *Suppose that all conditions of Theorem 4.2 are satisfied and the operator A is linear and single-valued. Additionally, assume that*

$$I^*_{R(B)}(v) \geq \langle f, v \rangle , \quad \forall v \in \ker A^*$$

holds for $f \in V^*$. Then the problem (13) is almost solvable, i.e. $f \in \overline{R(A+B)}$.

In the next section we apply the constructed theory to the problems of monotone type with the positive semi-definite free energy.

5 Existence for models of monotone type

As we pointed out in Introduction, the loss of the coerciveness for the fubction g can cost the non-solvability of the problem (1) - (5) for a fixed data and the rule how to choose an appropriate data has to be prescribed. In this section we define a new notion of the solvability of the systems of equations (1) - (5) for a non-coercive g, L^q-almost solvability, and then we give a criteria for choosing "right" data, which guarantees the L^q-almost solvability of the problem. In the end of the section the relations between the standard notion of the solvability and L^q-almost solvability are investigated and sufficient conditions for the solvability of (1) - (5), if it is L^q-almost solvable, are presented.

We suppose from now on that the numbers p and q satisfy the relations

$$1 \leq q \leq 2 \leq p \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Next we give the definition of the strong solutions of the system (1) - (5).

Definition 5.1. *Let functions b and γ such that $b \in L^p(0, T_e; W^{-1,p}(\Omega, \mathbb{R}^3))$, $\gamma \in L^p(0, T_e; W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3))$ be given. A function (u, T, z) such that*

$$(u, T) \in L^q(0, T_e; W^{1,q}(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathcal{S}^3)),$$
$$z \in W^{1,q}(0, T_e; L^q(\Omega, \mathbb{R}^N)), \quad B^T T - Lz \in L^p(\Omega_{T_e}, \mathbb{R}^N)$$

is called a strong solution *of the intial boundary value problem (1) - (5), if for almost every $t \in [0, T_e]$ the function $(u(t), T(t))$ is a weak solution of the boundary value problem (7) - (9) with $\hat{\varepsilon}_p = Bz(t)$, $\hat{b} = b(t)$ und $\hat{\gamma} = \gamma(t)$ and the equations (3) and (4) is satisfied for almost every (x, t).*
The set of functions having the above regularity we shall denote by $\mathcal{F}$. For any given $\nu \in L^p(0, T_e; W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3))$, by $\mathcal{F}_\nu$ the following set of functions will be denoted

$$\mathcal{F}_\nu = \{(u, T, z) \in \mathcal{F} \mid \Gamma_{\partial\Omega}(u(t)) = \nu(t), \quad z(x, 0) = 0, \text{ a.e.}\},$$

where $\Gamma_{\partial\Omega} : W^{1,q}(\Omega, \mathbb{R}^3) \to W^{1-1/q,q}(\partial\Omega, \mathbb{R}^3)$ is the usual trace operator.

Before we give the definition of the L^p-almost solvability and state the main result of this section, let us define the follwing operator $G : L^p(\Omega_{T_e}, \mathbb{R}^N) \mapsto 2^{L^q(\Omega_{T_e}, \mathbb{R}^N)}$ by

$$G(\xi) := \{\zeta \in L^q(\Omega_{T_e}, \mathbb{R}^N) \mid \exists \xi \in L^p(\Omega_{T_e}, \mathbb{R}^N) \ : \ \zeta(x,t) \in g(\xi(x,t)) \text{ a.e.}\} \quad (15)$$

with

$$D(G) = \{\xi \in L^p(\Omega_{T_e}, \mathbb{R}^N) \mid G(\xi) \neq \emptyset\}.$$

The operator G is well defined, since $0 \in D(G)$ holds. Therefore, if we assume that the function g is maximal monotone, by Proposition 2.13 in [18], the operator G must be maximal monotone as well. If, instead of maximality, g is 3^*-monotone, then the fact that G is 3^*-monotone follows from the definition.

Definition 5.2. *Let functions b and γ such that $b \in L^p(0, T_e; W^{-1,p}(\Omega, \mathbb{R}^3))$, $\gamma \in L^p(0, T_e; W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3))$ be given. The initial boundary value problem (1) - (5) is called L^q-*almost solvable*, iff*

$$\inf_{(u,T,z)\in\mathcal{F}_\gamma} \big\{\| \operatorname{div}_x T + b\|_{L^p(0,T_e;W^{-1,q}(\Omega,\mathbb{R}^3))} + \|T - \mathcal{D}(\varepsilon(u) - Bz)\|_{q,\Omega_{T_e}}$$

$$+|G^{-1}(\partial_t z) - (B^T T - Lz)|_{p,\Omega_{T_e}}\big\} = 0,$$

where

$$|G^{-1}(\hat{z})|_{p,\Omega_{T_e}} := \inf\{\|\hat{z}^*\|_{p,\Omega_{T_e}} \mid \hat{z}^* \in G^{-1}(\hat{z})\}.$$

Remark 5.1. *Obviously the solvability implies the L^q-almost solvability.*

Remark 5.2. *At this point we would like to mention that we can only treet the case of a single-valued function g in Theorem 5.1. Nevertheless, since the realization of the multi-valued situation seems to us just a technical issue, we shall study the Prandtl-Reuss model in the next section assuming that Theorem 5.1 holds for multi-valued function g as well. The rigerous verification of the multi-valued case we leave for future work.*

We can state the main result of this section.

Theorem 5.1. *Assume that the functions b and γ are given with the regularity as in Definition 5.2, the problem (1) - (5) is of monotone type (see Definition 1.1) and that the conditions (6) hold. Let for a.e. $t \in [0, T_e]$ the function*

$$(v, \sigma) \in L^p(0, T_e, W^{1,p}(\Omega, \mathbb{R}^3)) \times L^p(\Omega_{T_e}, \mathcal{S}^3)$$

be a solution of the problem (7) - (9) with $\hat{\varepsilon}_p = 0$, $\hat{b} = b(t)$ and $\hat{\gamma} = \gamma(t)$. Suppose that the mapping $g : \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following conditions:

(a) the function g is 3^*-monotone and maximal monotone;

(b) the inverse of G is locally bounded on its domain.

If we assume additionally that for any $w \in L^q(0, T_e; \mathcal{G}^q)$ there exists $z \in \overline{D(G)}$ such that the inequality

$$\int_0^{T_e}\int_\Omega \big(Bz(x,t), w(x,t)\big)dxdt \geq \int_0^{T_e}\int_\Omega \big(\sigma(x,t), w(x,t)\big)dxdt \tag{16}$$

holds, then the problem (1) - (5) is L^q-almost solvable.

We note that condition (b) in Theorem 5.1 implies that g^{-1} has a polynomical growth with the rate q/p, i.e. the following inequality for g^{-1}

$$|g^{-1}(z)| \leq C(1 + \|z\|^{q/p}_{\mathbb{R}^N})$$

holds, where

$$|g^{-1}(z)| := \inf\{\|z^*\|_{\mathbb{R}^N} \mid z^* \in g^{-1}(z)\}$$

and C is some positive constant, and, as a consequence of that, the domain of G^{-1} is the whole space $L^q(\Omega_{T_e}, \mathbb{R}^N)$. This can be seen from the following lemma, which we prove for the sake of simplicity only for single-valued functions and for $p = 2$. The proof of the lemma is easily generalizable to the multi-valued case and any $p \in (1, \infty)$.

Lemma 5.1. *Let $g : \mathbb{R}^N \to \mathbb{R}^N$ be maximal monotone and such that $g(0) = 0$. If the operator G defined by (15) for $p = 2$ is locally bounded at 0, then g has a linear growth and, as a consequence of that, the domain of G is the whole space $L^2(\Omega_{T_e}, \mathbb{R}^N)$.*

Proof. Indeed, let $\delta > 0$ be a fixed number such that

$$\|G(z)\|_{\Omega} \le C, \quad \text{for any } z \in B_\delta(0),$$

where $B_\delta(0)$ denotes an open ball in $L^2(\Omega_{T_e}, \mathbb{R}^N)$. The fact that the whole ball of the radius δ belongs to the domain of G follows from the local boundness of G at 0 (see Lemma 7.1). Next, we choose a vector $z_1 \in \mathbb{R}^N$ with $\|z_1\|_{\mathbb{R}^N} = 1$ and define a sequence

$$z_m(x,t) = \begin{cases} 0, & \text{if } x \in \Omega \setminus \Omega_m \\ k_m z_1, & \text{if } x \in \Omega_m \end{cases},$$

where $\{k_m\}_{m \in \mathbb{N}}$ is a sequence of positive real numbers such that $k_m \to \infty$ as $m \to \infty$ and the set Ω_m is a subset of Ω with the measure $\mu(\Omega_m) = \delta^2/(4k_m^2)$. Obviously, $z_m \in B_\delta(0)$. Since G is locally bounded at 0, we get that

$$\|G(z_m)\|_{\Omega_{T_e}} = T_e^{1/2} \mu(\Omega_m)^{1/2} \|G(k_m z_1)\|_{\mathbb{R}^N} \le C,$$

and, therefore,

$$\|g(k_m z_1)\|_{\mathbb{R}^N} \le C k_m / T_e^{1/2} \delta = \frac{C}{T_e^{1/2} \delta} \|k_m z_1\|_{\mathbb{R}^N},$$

what implies the linear growth for g (the last inequality holds for any $z \in \mathbb{R}^N$). Thus, due to the linear growth of the function g, the domain of G is the whole space $L^2(\Omega_{T_e}, \mathbb{R}^N)$. □

Remark 5.3. *We note that condition (16) allows a nice geometrical interpretation: roughly speaking condition (16) means that the projection of σ on every vector w from $L^q(0, T_e; \mathcal{G}^q)$ must be less than the largest projection among all projections of vectors from the set $\overline{D(G)}$ on the vector w.*

Remark 5.4. *If we strengthen the condition (16) by imposing that it should hold for every w from $L^q(\Omega_{T_e}, \mathcal{S}^3)$, then, by Remark 3.3, (16) is equivalent to following inclusion*

$$B^T \sigma \in \overline{D(G)}.$$

The last condition is much more easier to verify in practice than (16) and, therefore, can be used in applications instead of (16).

Remark 5.5. *Since the closure of the domain $\overline{D(G)}$ of the maximal monotone operator G is convex (see Appendix 7) and $0 \in D(G)$, condition (16) give restrictions on the possible choice of functions b and γ, but not on the domain of the maximal monotone operator G. This is due to the fact that $\sigma \in L^p(\Omega_{T_e}, \mathcal{S}^3)$, as the solution of linear elasticity problem (7) - (9) for $\hat{\varepsilon}_p = 0$, $\hat{b} = b(t)$ und $\hat{\gamma} = \gamma(t)$, is controlled by the given functions b and γ (see [54], if it is needed). Therefore, by choosing b and γ small enough we can always make σ verify condition (16). This observation suggests that condition (16) always holds for any maximal monotone operator G with $0 \in D(G)$ provided b and γ are chosen appropriately.*

Proof. Before we start the proof of Theorem 5.1, let us introduce the following notations

$$X = L^p(\Omega, \mathbb{R}^N), \quad \mathcal{X} = L^p(0, T_e, X), \quad H = L^2(\Omega, \mathbb{R}^N), \quad \mathcal{H} = L^2(0, T_e, H),$$

and

$$M_p = B^T \mathcal{D} Q_p B + L : X \to X, \quad M_2 = B^T \mathcal{D} Q_2 B + L : H \to H.$$

We note that the operator M_2 is non-negative by Corollary 2.0.1. Since $(\mathcal{X}, \mathcal{H}, \mathcal{X}^*)$ forms an evolution triple, we are able to define a linear maximal monotone operator $\mathcal{L} : \mathcal{X} \to \mathcal{X}^*$ by

$$\mathcal{L} z = \partial_t z \;\text{ with } D(\mathcal{L}) = \{z \in W_{p,q}(0, T_e, \mathcal{X}) \mid z(0) = 0\}.$$

The idea of the proof of Theorem 5.1 is to show the almost solvability of the abstract equation (17) in a reflexive Banach space $\mathcal{X}^*$ applying Theorem 4.1 and then, based on this result, to construct solutions for the initial boundary value problem (1) - (5). We note that the idea of the proof is strongly connected to the general duality principle for the sum of two operators obtained in [4].

Let us consider now the following inclusion in $\mathcal{X}^*$

$$\mathcal{L}^{-1} M_q v + G^{-1} v \ni B^T \sigma, \quad v \in \mathcal{X}^*. \tag{17}$$

The next lemma proves that the operator $\mathcal{L}^{-1} M_q$ in (17) is maximal monotone.

Lemma 5.2. *The operator $\mathcal{L}^{-1} M_q : D(\mathcal{L}^{-1} M_q) \subset \mathcal{X}^* \to \mathcal{X}$ is linear maximal monotone.*

Proof. According to Theorem 7.2, the operator $\mathcal{L}^{-1} M_q$ with $D(\mathcal{L}^{-1} M_q) = \{v \in \mathcal{X}^* \mid M_q v \in D(\mathcal{L}^{-1})\}$ is maximal monotone, if it is a densely defined closed monotone operator such that its adjoint $(\mathcal{L}^{-1} M_q)^*$ is monotone.

We note that the operator $\mathcal{L}^{-1} M_q$ is the closure in $\mathcal{X}^* \times \mathcal{X}$ of the operator $\mathcal{L}_0$ given by

$$\mathcal{L}_0 v := \mathcal{L}^{-1} M_q v, \quad v \in D(\mathcal{L}_0) = \{v \in \mathcal{X}^* \mid \mathcal{L}^{-1} v \in \mathcal{X}\}.$$

The last operator is monotone, what can be shown using the generalized integration by parts formula and the following identity

$$\mathcal{L}^{-1} M_q v = M_p \mathcal{L}^{-1} v, \quad v \in D(\mathcal{L}_0). \tag{18}$$

The identity (18) is proved in the end of this work. Therefore, the operator $\mathcal{L}^{-1}M_q$ is monotone as the closure in $\mathcal{X}^* \times \mathcal{X}$ of the monotone operator $\mathcal{L}_0$. Since the operator $(\mathcal{L}^{-1}M_q)$ is the closure of $\mathcal{L}_0$, thier adjoint operators coincide. The adjoint of $\mathcal{L}_0$ is easy to compute and is equal to $(\mathcal{L}^{-1})^*M_q$, by a well-known result from the functional analysis. Therefore, by arguing in the same way as above, we obtain that the adjoint $(\mathcal{L}^{-1}M_q)^*$ (we recall that $(\mathcal{L}^{-1}M_q)^* = (\mathcal{L}^{-1})^*M_q$) is monotone. Thus, since $\mathcal{L}^{-1}M_q$ verifies all assumptions of Theorem 7.2, it is a maximal monotone operator. The proof of Lemma 5.2 is complete. □

In order to apply Theorem 4.1 we set

$$V = \mathcal{X}^*, \quad A = \mathcal{L}^{-1}M_q, \quad B = G^{-1}, \quad f = B^T\sigma. \tag{19}$$

So we have that V is reflexive Banach space and that the operators A and B are maximal monotone. As the inverse of the 3^*-monotone operator G, the operator B is 3^*-monotone. Therefore, in order to apply Theorem 4.1 it is left to verify that the domain $D(B)$ is open.

Since the inverse of G is locally bounded on its domain, i.e. on $R(G)$, Lemma 7.1 implies that the domain of B is open.

Thus, Theorem 4.1 applied to the chosen space V and operators A, B in (19) yields that

$$R(A+B) \simeq R(A) + R(B).$$

In particular, the equality

$$\overline{R(A+B)} = \overline{R(A)+R(B)}. \tag{20}$$

holds. Next, we show that the given function f in (19) belongs to $\overline{R(A+B)}$. To this end, it is enough to prove that f belongs to $\overline{R(A)+R(B)}$. Since then, by (20), this gives the desired result.

In virtue of Theorem 3.1, the inclusion $f \in \overline{R(A)+R(B)}$ holds iff the following inequality

$$I^*_{R(B)}(v) \geq \langle f, v\rangle\,, \quad \forall v \in \ker A^* \tag{21}$$

is valid. Therefore, it is left to check whether the inquality (21) is satisfied. We show below that this inequality is a direct consequence of the condition (16). To see this, let us compute $\ker A^* = \ker(\mathcal{L}^{-1}M_q)^*$. First, we note that

$$\ker A^* = \ker(\mathcal{L}^{-1}M_q)^* = \ker\left((\mathcal{L}^{-1})^*M_q\right) = L^q(0, T_e, \ker M_q).$$

Therefore, it left to compute the kernal of $B^T\mathcal{D}Q_qB + L$ only. Obviously,

$$\ker\left(B^T\mathcal{D}Q_qB + L\right) = \ker(B^T\mathcal{D}Q_qB) \cap \ker L,$$

where $\ker L$ and $\ker L$ denote the sets

$$\ker L := \left\{v \in X^* \mid Lv(x) = 0 \text{ a.e.}\right\} \text{ and } \ker B := \left\{v \in X^* \mid Bv(x) = 0 \text{ a.e.}\right\}^4.$$

[4] In the brackets L and B denote the linear mappings from the first section, i.e. $L : \mathbb{R}^N \to \mathbb{R}^N$ and $B : \mathbb{R}^N \to \mathcal{S}^3$

Due to the assumption (6) we have that $\ker L \cap \ker B = \emptyset$. Hence,

$$\begin{aligned}\ker\big(B^T\mathcal{D}Q_qB+L\big) &= \ker(\mathcal{D}Q_qB)\cap\ker L = \ker(Q_qB)\cap\ker L\\ &= \big(B^T\big(\ker Q_q\cap R(B)\big)\cup\ker B\big)\cap\ker L\\ &\subset \big(B^T(\ker Q_q)\cup\ker B\big)\cap\ker L\\ &= \big(B^T(\ker Q_q)\cap\ker L\big)\cup\big(\ker B\cap\ker L\big)\\ &\subset B^T(\ker Q_q) = B^T(R(P_q)) = B^T(\mathcal{G}^q).\end{aligned}$$

The last computations obviously yield that

$$\ker M_q \subset B^T(\mathcal{G}^q).$$

Now it is easily seen that the inequality (16) implies (21).

Thus, we have shown that $f \in \overline{R(A+B)}$. The last inclusion means that there exists a sequence $f_n \in \mathcal{X}^*$ such that f_n converges strongly to f and $f_n \in R(A+B)$. Therefore, there exists a sequence $v_n \in D(A)\cap D(B)$ such that $f_n = Av_n + Bv_n$. Returning back to the old notations we get

$$\mathcal{L}^{-1}M_qv_n + G^{-1}v_n \ni f_n, \quad v_n \in L^q(\Omega_{T_e},\mathbb{R}^N). \tag{22}$$

Denoting by $\tau_n = \mathcal{L}^{-1}M_qv_n$ we obtain from (22) that τ_n solves the problem

$$\mathcal{L}\tau_n = M_qG(-\tau_n + f_n), \quad \tau_n \in L^p(\Omega_{T_e},\mathbb{R}^N). \tag{23}$$

Using the last result, the construction of the "solutions" of the problem (1) - (5) can be now performed as in [3]:

Let $\tau_n \in L^p(\Omega_{T_e},\mathbb{R}^N)$ be a solution of (23). With the function τ_n let $z_n \in W^{1,q}(0,T_e,L^q(\Omega,\mathcal{S}^3))$ be the solution of

$$\begin{aligned}\partial_t z_n(t) &= g\big(-\tau_n(t)+f_n(t)\big), \quad \text{a.e. } (0,T_e) &(24)\\ z_n(0) &= 0. &(25)\end{aligned}$$

By the linear ellipticity theory, we can now obtain the existence of the solution $(\tilde{u}_n(t),\tilde{T}_n(t))$ for the problem (7) - (9) to the following data $\hat{b}=\hat{\gamma}=0$, $\hat{\varepsilon}_p = Bz_n(t)$. Next, we apply the operator M_q to (24) - (25) and obtain

$$\partial_t(M_qz_n(t)) = M_qG\big(-\tau_n(t)+f_n(t)\big) = \partial_t\tau_n(t), \quad M_qz_n(0)=\tau_n(0)=0.$$

The last implies that $M_qz_n = \tau_n$. Thus, recalling definitions of $\mathcal{L}$, G and M_q we arrive at the problem

$$\begin{aligned}\partial_t z_n(x,t) &= g\Big(B^TT_n(x,t) - Lz_n(x,t) + f_n(x,t) - B^T\sigma(x,t)\Big), &(26)\\ z_n(x,0) &= 0, &(27)\end{aligned}$$

with $(u_n,T_n) = (\tilde{u}_n + v, \tilde{T}_n + \sigma)$ solving the following boundary value problem

$$\begin{aligned}-\mathrm{div}_x T_n(x,t) &= b(x,t), &(28)\\ T_n(x,t) &= \mathcal{D}(\varepsilon(u_n(x,t)) - Bz_n(x,t)), &(29)\\ u_n(x,t) &= \gamma(x,t), \quad (x,t)\in\partial\Omega\times[0,\infty). &(30)\end{aligned}$$

Therefore, we can conclude that the triple (u_n,T_n,z_n) belongs to $\mathcal{F}_\gamma$ and satisfy the equations (26) - (30). The last conclusions imply that the problem (1) - (5) is L^q-almost solvable. This completes the proof of Theorem 5.1. □

Remark 5.6. *In the proof of Theorem 5.1 we used Theorem 4.1 to study the sum of two maximal monotone operators* $A = \mathcal{L}^{-1}M_q$ *and* $B = G^{-1}$*. Since the domain of* B *is the whole space* $\mathcal{X}^*$*, Theorem 3.1 or Theorem 4.2 could be applied instead of Theorem 4.1 as well. Here we give an example of an operator arising in the existence theory for models of ferroelectric material behavior in [31], which has an open domain strictly contained in the whole space*[5]*.*
Let a function $g : \mathbb{R}^n \to \mathbb{R}^n$ *with the domain* $D(g) = B_\rho(0)$*, an open ball in* $\mathbb{R}^n$ *with a radius* $\rho > 0$*, and with the property* $g(0) = 0$ *be given. Let* $\psi : \mathbb{R}^n \to \mathbb{R}_+$ *be a standard mollifier. We define the operator* $\hat{G} : L^2(\Omega, \mathbb{R}^n) \to L^2(\Omega, \mathbb{R}^n)$ *by*

$$\hat{G}(z) := \psi * g(\psi * z)$$

with

$$D(\hat{G}) = \{z \in L^2(\Omega, \mathbb{R}^n) \mid R(\psi * \tilde{z}(x)) \in D(g), \text{ a.e. } x \in \mathbb{R}^n\},$$

where $\tilde{z}$ *denotes the extension of* z *outside of* Ω *by 0. Suppose that* z_0 *belongs to the domain* $D(\hat{G})$*. Next, choose* $\delta > 0$ *such that* $\max_x \|\psi * \tilde{z}_0(x)\|_{\mathbb{R}^n} = \rho - \delta$ *and* $C = \mu(\Omega)^{1/2} \max_x \psi(x)$*. We note that such* $\delta > 0$ *exists, since* $\psi * \tilde{z}_0$ *has a compact support in* $\mathbb{R}^n$*, the image of which contains in the open set* $D(g) \subset \mathbb{R}^n$*. If we now choose* $\epsilon > 0$ *such that* $\epsilon < \delta/C$*, the following inequality*

$$\begin{aligned} \|\psi * \tilde{z}(x)\|_{\mathbb{R}^n} &\le \| \int_{\mathbb{R}^n} \psi(x-y)\big(\tilde{z}(y) - \tilde{z}_0(y)\big)dy\|_{\mathbb{R}^n} + \|\psi * \tilde{z}_0(x)\|_{\mathbb{R}^n} \\ &\le C\|z_0 - z\|_\Omega + \rho - \delta \le C\epsilon + \rho - \delta < \rho \end{aligned}$$

holds for any $z \in B_\epsilon(z_0) := \{v \in L^2(\Omega, \mathbb{R}^n) \mid \|z_0 - v\|_\Omega < \epsilon\}$*. This shows that* $D(\hat{G})$ *is open. The fact that* $D(\hat{G})$ *does not coincide with the whole* $L^2(\Omega, \mathbb{R}^n)$ *is easily seen from the definition.*

The next corollaries give immediate sufficient conditions for the the solvability of the initial boundary value problem (1) - (5).

Corollary 5.1.1. *Let instead of (16) in Theorem 5.1 the following condition*

$$B^T\sigma \in \operatorname{int} D(G), \tag{31}$$

where $(v(t), \sigma(t))$ *is a solution of the Dirichlet boundary value problem (7) - (9) to the data* $\hat{b} = b(t)$*,* $\hat{\gamma} = \gamma(t)$*,* $\hat{\varepsilon}_p = 0$*, be satisfied. Then the initial boundary value problem (1) - (5) has a strong (not necessary unique) solution.*

Proof. Theorem 5.1 guarantees that

$$R(A + B) \simeq R(A) + R(B)$$

for operators A and B defined in (19). In particular, the equality

$$\operatorname{int} R(A + B) = \operatorname{int}(R(A) + R(B))$$

holds. But, since $R(B) := R(G^{-1}) = D(G)$, in virtue of the assumtion (31) of the corollary, we obtain that

$$\operatorname{int} R(A + B) = \operatorname{int}(R(A) + D(G)) \ni f.$$

[5]The example is proposed by Nataliya Kraynyukova (TU-Darmstadt).

The last means that the equition (17) is solvable. Thus, the problem (23) has a solution as well. The construction of solutions of the problem (1) - (5) coincides completely with the one performed in the end of the proof of Theorem 5.1. This completes the proof of Corollary 5.1.1. □

Remark 5.7. *The condition (31) in Corollary 5.1.1 is fulfilled if the function* $g : \mathbb{R}^N \to \mathbb{R}^N$ *has the polynomical growth, i.e. g satisfies the inequality*

$$\|g(z)\| \leq C(1 + \|z\|^{p/q}).$$

If the operator G^{-1} is coercive, then the condition (c) and condition (16) in Theorem 5.1 can be dropped.

Corollary 5.1.2. *If instead of the conditions* (c) *and (16) in Theorem 5.1 we assume that the operator* G^{-1} *is coercive, then the initial boundary value problem (1) - (5) has a strong (not necessary unique) solution.*

Proof. Following the proof of Theorem 5.1 we come to the question of the solvability of equation (17), which can be solved now by using the standard existence theory for monotone operator. Indeed, the operators A and B in (17) are maximal monotone. By Theorem 7.3, the sum $A + B$ is maximal monotone, since A is linear and $0 \in \operatorname{int} D(B)$. Moreover, the sum $A + B$ is a coercive operator as the following line shows:

$$\frac{\langle Av + v^*, v\rangle}{\|v\|} \geq \frac{\langle v^*, v\rangle}{\|v\|} \to +\infty \quad \text{as } \|v\| \to \infty,$$

for $v^* \in B(v)$. The last holds due to the coecivity of B. In virtue of Theorem 7.4, the maximal monotone and coercive operator $A + B$ is surjective. Therefore, equation (17) has a solution. The rest of the proof is as in Corollary 5.1.1. □

Remark 5.8. *As it is easily seen, the operator* G^{-1} *is coercive if the function* g^{-1} *satisfies the inequality*

$$C\|z\|^q_{\mathbb{R}^N} - a(x,t) \leq (z^*, z)_{\mathbb{R}^N}, \quad z^* \in g^{-1}(z), \ a \in L^q(\Omega_{T_e}, \mathbb{R}).$$

Corollary 5.1.2 is sucsessfully applied in [22].

6 Application

In this section we apply the constructed theory from the previous section to the models of Norton-Hoff and Prandtl-Reuss and to the model of nonlinear kinematic hardening. All these models are L^q-almost solvable in the sense of Definition 5.2. In contrast to the model of Prandtl-Reuss, which is only L^q-almost solvable and not strongly solvable in the sense of Definition 5.1, the model of Norton-Hoff and nonlinear kinematic hardening possess strong solutions. As we shall show below none of the sufficient conditions, which implies the solvability (see Corollary 5.1.1 and Corollary 5.1.2), is satisfied and, therefore, one can not expect that the model of Prandtl-Reuss does have strong solutions. This explains the need of the so-called "safe-load condition"[6], which actually guarantees that this model has its solutions in BD, the space of bounded deformations

[6] For the definition of the "safe-load condition" we refer the reader to [12, 20, 30, 51, 52], for example.

(see [51, 52]). Therefore, the Prandtl-Reuss model is an example of a problem of monotone type (1) - (5) (see below), which only L^q-almost solvability but does not possess strong solutions. Due to technical reasons in all models presented here we drop the requirement that $\varepsilon_p \in P_0\mathcal{S}^3$ in equations describing the plastic flow process[7]. The treetment of this requirement is left for the future work.

Model of Prandtl-Reuss. The model equations are ([1, 20])

$$
\begin{aligned}
-\operatorname{div}_x T(x,t) &= b(x,t), && (32)\\
T(x,t) &= \mathcal{D}(\varepsilon(u(x,t)) - \varepsilon_p(x,t)), && (33)\\
\partial_t \varepsilon_p(x,t) &\in \partial I_K(T(x,t)), && (34)\\
\varepsilon_p(x,0) &= 0, && (35)\\
u(x,t) &= \gamma(x,t), \quad x \in \partial\Omega, && (36)
\end{aligned}
$$

where I_K is the indicator function of some closed convex set $K \subset \mathcal{S}^3$, which is specified by a yield criterion. For the von Mises yield criterion the set K has the form

$$K = \{\sigma \in \mathcal{S}^3 \mid \sigma \cdot \sigma \leq C\},$$

where $C > 0$ is some given constant. In the case of the von Mises yield criterion the equation (34) reads (see [20] if needed)

$$\partial_t \varepsilon_p \in g(T) := \begin{cases} 0, & \text{if } T \cdot T < C, \\ \lambda T/|T|, & \text{otherwise,} \end{cases} \tag{37}$$

where $\lambda \geq 0$ is the so-called plastic multiplier. From inclusion (37) it is easily seen that for the von Mises yield criterion equations (32) - (36) can be written in the form (1) - (5). We note as well that the free energy for the model of perfect plasticity given by

$$\psi(\varepsilon, \varepsilon_p) = \frac{1}{2}\mathcal{D}(\varepsilon - \varepsilon_p) \cdot (\varepsilon - \varepsilon_p) \tag{38}$$

is positive semi-definite. The vector of internal veriables z consists of ε_p only, i.e. $z = \varepsilon_p$.

The L^q-almost solvability for equations (32) - (36) is easy to establish. Indeed, the mapping ∂I_K is maximal monotone and 3^*-monotone as a subdifferential of a proper convex lower semi-continous function. The inverse of G generated by g in (37) is bounded, what can be seen from the formular:

$$g^{-1}(v) := \begin{cases} \{\xi \mid \xi \cdot \xi < C\}, & \text{if } v = 0, \\ Cv/|v|, & v \neq 0. \end{cases} \tag{39}$$

Choosing $\sigma \in L^p(\Omega_{T_e}, \mathcal{S}^3)$, the solution of elasticity problem (7) - (9) to $\hat{\varepsilon}_p = 0$, $\hat{b} = b(t)$ und $\hat{\gamma} = \gamma(t)$, in a way that the condition (16) is satisfied[8], we can

[7] The linear mapping $P_0 : \mathcal{S}^3 \to \mathcal{S}^3$ is the orthogonal projection onto the subspace

$$P_0\mathcal{S}^3 := \{\tau \in \mathcal{S}^3 \mid \operatorname{trace}(\tau) = 0\},$$

called the deviator space. The fact that $\varepsilon_p \in P_0\mathcal{S}^3$ is due to plastic incompressibility, i.e. that there is no change in volume accompanying plastic deformation.

[8] We recall that this can be always done, since σ as a solution of the linear elliptic problem is controlled by the given functions b and γ.

conclude that the problem (32) - (36) is L^q-almost solvable in the sense of Definition 5.2. Since the operator G^{-1} generated by g^{-1} is not coercive and, since the domain of G, i.e. the set $D(G) = \{v \in L^p(\Omega_{T_e}, \mathcal{S}^3) \mid v(x,t) \in K \text{ a.e.}\}$, is not open, Corollary 5.1.1 and Corollary 5.1.2 are not applicable to this model. Therefore, one can not expect the solvability of the problem (32) - (36) in the sence of Definition 5.1. But it is already known ([52, 51]) that the solution u of (32) - (36) together with its time derivative belong to the following space $L^\infty(0, T_e; BD(\Omega, \mathbb{R}^3))$ provided that the "safe-load condition" holds, and not to the space $L^q(0, T_e; W^{1,q}(\Omega, \mathbb{R}^3))$ as it is required in Definition 5.1. Therefore, the Prandtl-Reuss model is an example of the L^q-almost solvable problem, which does not have strong solutions.

Model of Norton-Hoff. The equations of the Norton-Hoff model are ([1])

$$\begin{aligned} -\operatorname{div}_x T(x,t) &= b(x,t), && (40)\\ T(x,t) &= \mathcal{D}(\varepsilon(u(x,t)) - \varepsilon_p(x,t)), && (41)\\ \partial_t \varepsilon_p(x,t) &= C|T(x,t)|^r \frac{T(x,t)}{|T(x,t)|}, && (42)\\ \varepsilon_p(x,0) &= 0, && (43)\\ u(x,t) &= \gamma(x,t), \quad x \in \partial\Omega, && (44) \end{aligned}$$

where $r > 1$ and C is some material constant. The free energy for this model has the form

$$\psi(\varepsilon, \varepsilon_p) = \frac{1}{2}\mathcal{D}(\varepsilon - \varepsilon_p) \cdot (\varepsilon - \varepsilon_p).$$

Obviously, it is positive semi-definite. If we define the function $g : \mathcal{S}^3 \to \mathcal{S}^3$ as follows

$$g(w) := C|w|^r \frac{w}{|w|},$$

then equations (40) - (44) fit into the framework of the problem (1) - (5). The function g is maximal monotone and 3^*-monotone (see Remark 7.1) as a gradient of the continuous convex function

$$\phi(w) := \frac{C}{r+1} C|w|^{r+1}.$$

The condition (b) of Theorem 5.1 is satisfied, since the inverse of g given by

$$g^{-1}(\zeta) = 1/C|\zeta|^{1/r} \frac{\zeta}{|\zeta|}$$

has the polynomical growth, what infers that the inverse of the operator G generated by g is locally bounded on its domain (the whole space). The condition (16) is easily fulfilled, since the domain of G is the whole space. Therefore, the conditions of Theorem 5.1 are satisfied, and, thus, the problem (40) - (44) is L^q-almost solvable. Moreover, since the domain of G is the whole space, by Corollary 5.1.1, the problem (40) - (44) has a stong solution.

Nonlinear kinematic hardening. This model we study in details. The model of nonlinear kinematic hardening consists of the equations

$$
\begin{aligned}
-\operatorname{div}_x T(x,t) &= b(x,t), && (45)\\
T(x,t) &= \mathcal{D}\big(\varepsilon(u(x,t)) - \varepsilon_p(x,t)\big), && (46)\\
\partial_t \varepsilon_p(x,t) &= c_1 |T(x,t) - Y(x,t)|^r \frac{T(x,t) - Y(x,t)}{|T(x,t) - Y(x,t)|}, && (47)\\
\partial_t \varepsilon_n(x,t) &= c_2 |Y(x,t)|^m \frac{Y(x,t)}{|Y(x,t)|}, && (48)\\
Y(x,t) &= k(\varepsilon_p(x,t) - \varepsilon_n(x,t)), && (49)\\
Y(x,0) &= \varepsilon_p(x,0) = 0, && (50)\\
u(x,t) &= \gamma(x,t), \quad x \in \partial\Omega. && (51)
\end{aligned}
$$

The variable $Y \in \mathcal{S}^3$ in equations (47) - (50) is called backstress, a variable of kinematic hardening. c_1,c_2 and k are material constants. The typical values for m and r in practice are taken between 5 and 7.

Next, we show that the model (45) - (51) can be written in the general form (1) - (5). To this end, we choose for the vector of the internal variable $z = (\varepsilon_p, \varepsilon_n) \in \mathcal{S}^3 \times \mathcal{S}^3 \cong \mathbb{R}^{12}$, and define the mapping $B : \mathcal{S}^3 \times \mathcal{S}^3 \to \mathcal{S}^3$ by $Bw = B(\hat{w}, \tilde{w}) = \hat{w} \in \mathcal{S}^3$. The transpose B^T is given by $B^T\hat{w} = (\hat{w}, 0)$, thus $B^T\mathcal{D}Bw = (\mathcal{D}\hat{w}, 0)$ and $B^T\mathcal{D}\varepsilon = (\mathcal{D}\varepsilon, 0)$. Let us define a linear symmetric mappings $M : \mathcal{S}^3 \times \mathcal{S}^3 \to \mathcal{S}^3 \times \mathcal{S}^3$ by

$$
Mw = M(\hat{w}, \tilde{w}) = \big(\mathcal{D}\hat{w} + k(\hat{w} - \tilde{w}), -k(\hat{w} - \tilde{w})\big).
$$

Obviously, M is positive definite. But the linear mapping $L := M - B^T\mathcal{D}B$ is only positive semi-definite, because for $w \neq 0$

$$
Lw \cdot w = Mw \cdot w - (\mathcal{D}\hat{w}, 0) \cdot (\hat{w}, \tilde{w}) = k|\hat{w} - \tilde{w}|^2 \geq 0.
$$

Let us define the mappings $\hat{g} : \mathcal{S}^3 \to \mathcal{S}^3$ and $\tilde{g} : \mathcal{S}^3 \to \mathcal{S}^3$ by

$$
\hat{g}(\hat{w}) = c_1 |\hat{w}|^r \frac{\hat{w}}{|\hat{w}|} \quad \text{and} \quad \tilde{g}(\tilde{w}) = c_2 |\tilde{w}|^m \frac{\tilde{w}}{|\tilde{w}|}
$$

for $w = (\hat{w}, \tilde{w}) \in \mathcal{S}^3 \times \mathcal{S}^3$. Finally, if we define $g : \mathcal{S}^3 \times \mathcal{S}^3 \to \mathcal{S}^3 \times \mathcal{S}^3$ by

$$
g(w) = g(\hat{w}, \tilde{w}) = (\hat{g}(\hat{w}), \tilde{g}(\tilde{w})),
$$

we obtain that the equations (45) - (51) can be written in the form (1) - (5). The free energy for the model of the nonlinear kinematic hardening (45) - (51) is given by

$$
\psi(\varepsilon, z) = \frac{1}{2}\mathcal{D}(\varepsilon - Bz) \cdot (\varepsilon - Bz) + \frac{1}{2}Lz \cdot z = \frac{1}{2}\mathcal{D}(\varepsilon - \varepsilon_p) \cdot (\varepsilon - \varepsilon_p) + \frac{1}{2}|\tilde{z}|^2.
$$

It is easily seen that the free energy is only positive semi-definite. Now we verify the conditions of Theorem 5.1:

By definition of the function g we have that $0 \in g(0)$. The function g is also monotone as the gradient of the convex function $\phi := \hat{\phi} + \tilde{\phi}$, where

$$
\hat{g} = \nabla\hat{\phi}, \quad \text{and} \quad \tilde{g} = \nabla\tilde{\phi}.
$$

Since g is the gradient of ϕ, it is also 3^*-monotone. Moreover, again by definition of g, one sees that the function ϕ is continuous, therefore g must be maximal monotone due to Remark 7.1. Hence, it is left to check that the inverse of the operator G generated by g is locally bounded on its domain. To this end, we show that g^{-1} has a polynomical growth. This will imply the local boundness of G^{-1}. The inverse of g is easy to compute. Indeed, the function g^{-1} has the form

$$g^{-1}(\zeta) = g^{-1}(\hat{\zeta}, \tilde{\zeta}) = (\hat{g}^{-1}(\hat{\zeta}), \tilde{g}^{-1}(\tilde{\zeta}))$$

with $\hat{g}^{-1}$ and $\tilde{g}^{-1}$ given by

$$\hat{g}^{-1}(\hat{\zeta}) = c_3|\hat{\zeta}|^{1/r}\frac{\hat{\zeta}}{|\hat{\zeta}|} \quad \text{and} \quad \tilde{g}^{-1}(\tilde{\zeta}) = c_4|\tilde{\zeta}|^{1/m}\frac{\tilde{\zeta}}{|\tilde{\zeta}|},$$

where $c_3 = c_1^{-1/r}$, $c_4 = c_2^{-1/m}$.

Consider the case when $r \geq m$. In this case we set $p = 1 + m$. Then the conjugate to p number $q = p/(p-1)$ is equal to $1 + 1/m$, i.e. $q = 1 + 1/m$. Thus, by Young's inequality with $s = r/m$, we obtain that

$$\big|\tilde{g}^{-1}(\tilde{\zeta})\big|^q \leq c_3^q\Big(\frac{1}{s}|\tilde{\zeta}|^{qs} + \frac{1}{s'}1^{s'}\Big) \leq c_5\big(|\tilde{\zeta}|^p + 1\big)$$

and

$$\big|\hat{g}^{-1}(\hat{\zeta})\big|^q = c_4^q|\hat{\zeta}|^p$$

for some constant c_5. Combining the last two relations we get that the function g^{-1} enjoys the inequality

$$\big|g^{-1}(\zeta)\big|^q \leq c_6\big(|\zeta|^p + 1\big)$$

with some constant c_6. This means that g^{-1} has the polynomical growth with the rate p/q and, therefore, the operator G^{-1} is locally bounded on its domain. Similarly, in the case $r \leq m$ we obtain that the function g^{-1} satisfies the inequality

$$\big|g^{-1}(\zeta)\big|^q \leq c_7\big(|\zeta|^p + 1\big)$$

with $p = 1 + r$ and $q = 1 + 1/r$ and some constant c_7. Thus G^{-1} is locally bounded for $r \leq m$ also.

The condition (16) is easily fulfilled, since the domain of G is the whole space. Hence, all assumptions of Theorem 5.1 are satisfied, and therefore the problem (45) - (51) is L^q-almost solvable. Moreover, since the domain of G is the whole space, by Corollary 5.1.1 the problem (40) - (44) has a stong solution.

Remark 6.1. *In [3] the existence of strong solutions to the model of nonlinear kinematic hardening is shown under the condition that m and r satisfy the inequality $m > r$. Based on Corollary 5.1.2 this condition is removed in [22] (here we use Corollary 5.1.1). Using the theory of Orlic spaces and the monotone operator method similar results are obtained in [43] without imposing any restrictions on m and r.*

7 Appendix I: Locally bounded operators

For the reader's convenience we collect here some basic facts about maximal monotone operators in reflexive Banach spaces used in this work.

Definition 7.1. *The mapping* $A : V \to 2^{V^*}$ *is called* locally bounded *at a point* $v_0 \in V$ *if there exists a neighbourhood* U *of* v_0 *such that the set*

$$A(U) = \{Av \mid v \in D(A) \cap U \}$$

is bounded in V^*.

In particular, A is locally bounded at any $v \notin \overline{D(A)}$. Indeed, because for such a point v there is a neighborhood U that does not have common points with $D(A)$ and therefore has $A(U) = \emptyset$. (The empty set is, of course, regarded as bounded.)

Definition 7.2. *A mapping* $A : V \to 2^{V^*}$ *is called* maximal monotone *iff the inequality*

$$\langle v^* - u^*, v - u\rangle \geq 0 \qquad (\forall)\ [u, u^*] \in GrA$$

implies $[v, v^*] \in GrA$.
A mapping $A : V \to 2^{V^*}$ *is called* generalized pseudomonotone *iff the set* Av *is closed, convex and bounded for all* $v \in D(A)$ *and for every pair of sequences* $\{v_n\}$ *and* $\{v_n^*\}$ *such that* $v_n^* \in Av_n$, $v_n \rightharpoonup v_0$, $v_n^* \rightharpoonup v_0^* \in V^*$ *and*

$$\limsup_{n\to\infty} \langle v_n^*, v_n - v_0\rangle \leq 0,$$

we have that $[v_0, v_0^*] \in GrA$ *and* $\langle v_n^*, v_n\rangle \to \langle v_0^*, v_0\rangle$.

It is well known ([40, p. 105]) that if A is a maximal monotone operator, then for any $v \in D(A)$ the image Av is closed convex subset of V^* and the graph GrA is demiclosed[9]. A maximal monotone operator is also generalized pseudomonotone (see [5, 27, 40], for example).

Remark 7.1. We recall that the subdifferential of a lower semi-continuous and convex function is maximal monotone (see [41, Theorem 2.25]).

Definition 7.3. *The* duality mapping $J : V \to 2^{V^*}$ *is defined by*

$$J(v) = \{v^* \in V^* \mid \langle v^*, v\rangle = \|v\|^2 = \|v^*\|_*^2 \}$$

for all $v \in V$.

For maximal monotone operators we have the following characterization in reflexive Banach spaces.

Theorem 7.1. *Let* $A : V \to 2^{V^*}$ *be a monotone mapping. Then* A *is maximal monotone iff for any* $\lambda > 0$ *the following surjectivity result holds*

$$R(A + \lambda J) = V^*.$$

Proof. See [5, Theorem II.1.2]. □

[9]A set $A \in V \times V^*$ is demiclosed if v_n converges strongly to v_0 in V and v_n^* converges weakly to v_0^* in V^* (or v_n converges weakly to v_0 in V and v_n^* converges strongly to v_0^* in V^*) and $[v_n, v_n^*] \in GrA$, then $[v, v^*] \in GrA$

Without loss of generality (due to Asplund's theorem) we can assume that both V and V^* are strictly convex, i.e. if the unit ball in the corresponding space is strictly convex. In virtue of Theorem 7.1, the equation

$$J(v_\lambda - v) + \lambda A v_\lambda \ni 0$$

has a solution $v_\lambda \in D(A)$ for every $v \in V$ and $\lambda > 0$ if A is maximal monotone. The solution is unique (see [5, p. 41]).

Definition 7.4. *Setting*

$$v_\lambda = j_\lambda^A v \quad \textit{and} \quad A_\lambda v = -\lambda^{-1} J(v_\lambda - v)$$

we define two single valued operators: the Yosida approximation $A_\lambda : V \to V^*$ *and the* resolvent $j_\lambda^A : V \to D(A)$ *with* $D(A_\lambda) = D(j_\lambda^A) = V$.

By the definition, one immediately sees that $A_\lambda v \in A\big(j_\lambda^A v\big)$. For the main properties of the Yosida approximation we refer to [5, 27, 40] and mention only that it is a bounded maximal monotone operator.

Proposition 7.1. *If* $v \in \overline{\mathrm{conv} D(A)}$*, then* $j_\lambda^A v \to v$ *in* V *as* $\lambda \to 0$.

Proof. See [5, Proposition II.1.1] or [40, Proposition III.3.1]. □

If the operator A is linear and single valued, then the following result holds.

Theorem 7.2. *The following assertions are equivalent:*

(a) $A : V \to V^*$ *is maximal monotone;*

(b) A *is a densely defined closed operator such that its adjoint* A^* *is monotone;*

(c) A *is a densely defined closed operator such that* A^* *is maximal monotone.*

Proof. See Theorem 1 [6]. □

It turns out that every monotne operator is local bounded at the interior points of its domain ([40, Theorem 2.2]). It means that the image Av is a bounded subset of V^* for any $v \in \mathrm{int}\, D(A)$. But it can happen that A is not local bounded at any point of the boundary of $D(A)$ (see an example in [40, p. 158]).

The next lemma is one of the main tools in the construction of the existence theory for models of monotone type with positive semi-definite free energy. For the completeness of work we give the proof of this result here (see also [26, Lemma III.24]).

Lemma 7.1. *Let* $A : V \to 2^{V^*}$ *be a maximal monotone mapping. Assume that* A^{-1} *is locally bounded at a point* $v_0^* \in R(A)$*. Then the point* v_0^* *is the interior point of* $R(A)$.

Proof. Since A^{-1} is locally bounded at a point $v_0^* \in R(A)$ ($[v_0, v_0^*] \in GrA$), there exists $r > 0$ such that A^{-1} is bounded on

$$B(v_0^*, r) := \{ w^* \in R(A) \mid \|w^* - v_0^*\|_* \le r \}.$$

Let $v^* \in V^*$ be any element satisfying the inequality $\|v^* - v_0^*\|_* \le r/2$ (i.e. $v^* \in B(v_0^*, r/2)$), then, by Theorem 7.1, the equation

$$A v_\lambda + \lambda J(v_\lambda - v_0) \ni v^*$$

has a solution $v_\lambda \in D(A)$ for every $\lambda > 0$. In virtue of the monotonocity of A we have

$$\langle v_\lambda^* - v_0^*, v_\lambda - v_0 \rangle \ge 0, \qquad v_\lambda^* = v^* - \lambda J(v_\lambda - v_0).$$

This inequality implies that

$$\lambda \|v_\lambda - v_0\| \le \|v^* - v_0^*\|_* \le r/2.$$

Therefore

$$\|v_\lambda^* - v_0^*\|_* \le r.$$

Since A^{-1} is bounded on $B(v_0^*, r)$, the set of the solutions v_λ remains bounded and

$$\|v_\lambda^* - v^*\|_* = \lambda \|v_\lambda - v_0\| \to 0, \quad \text{as } \lambda \to 0.$$

Hence, the demicontinuity of A yields that $[v, v^*] \in GrA$, where v is a weak limit of v_λ. Thus we have proved that $B(v_0^*, r/2) \subset R(A)$, what completes the proof of Lemma 7.1. □

A particular answer on the maximality of the sum of two maximal monotone operators is given by the following result.

Theorem 7.3. *Let V be a reflexive Banach space, and let A and B be maximal. Suppose that the condition*

$$D(A) \cap \operatorname{int} D(B) \neq \emptyset$$

is fulfilled. Then the sum $A + B$ is a maximal monotone operator.

Proof. See [40, Theorem III.3.6] or [5, Theorem II.1.7]). □

For deeper results on the maximality of the sum of two maximal monotone operators we refer the reader to the book [48]. The next surjectivity result plays an important role in the existence theory for monotone operators.

Theorem 7.4. *If V is a (strictly convex) reflexive Banach space and $A : V \to 2^{V^*}$ is maximal monotone and coercive, then A is surjective.*

Proof. See [40, Theorem III.2.10]. □

Among other interesting properties of maximal monotone operators in reflexive Banach spaces we also mention that both $\overline{D(A)}$ and $\overline{R(A)}$ are convex (see [5, Theorem II.1.5]). Proposition 2.34 in [27, p. 328] gives a condition on a maximal monotone A under which $\operatorname{int} D(A) \neq \emptyset$; in particular, this is the case if the convex hull of $D(A)$ is assumed to have nonempty interior. Under this hypothesis, the interior of $D(A)$ is convex, and for any point $v \in D(A) \setminus \operatorname{int} D(A)$, the set Av is unbounded (see [41, p. 30]). For further reading on maximal monotone operators we refer the reader to [5, 10, 40, 41, 27] or [57].

8 Appendix II: Proof of Lemma 5.2

Here we prove the identity (18) used in Lemma 5.2.

Lemma 8.1. *The following identity*

$$M_p \mathcal{L}^{-1} v = \mathcal{L}^{-1} M_q v \tag{52}$$

holds for all $v \in D(M_p \mathcal{L}^{-1}) = D(\mathcal{L}^{-1}) := \{z \in \mathcal{X}^* \mid \int_0^t z(s) ds \in \mathcal{X}\}$.

Proof. Note first that the identity (52) follows easily from

$$P_p \mathcal{L}^{-1} v = \mathcal{L}^{-1} P_q v, \tag{53}$$

which holds for $v \in D(\mathcal{L}^{-1})$. (53) can be proved as follows:
Let $v \in D(\mathcal{L}^{-1})$. Then, according to the definition of P_p, the boundary value problem

$$\begin{aligned} -\operatorname{div} \mathcal{D}\varepsilon(u(x,t)) &= -\operatorname{div} \mathcal{D}v(x,t), \quad x \in \Omega &(54)\\ u(x,t) &= 0, \qquad x \in \partial\Omega, &(55)\end{aligned}$$

has a unique solution $u(t) \in W_0^{1,q}(\Omega, \mathbb{R}^3)$, i.e. the function u satisfies the equation

$$(\mathcal{D}\varepsilon(u(t)), \varepsilon(\phi))_\Omega = (\mathcal{D}v(t), \varepsilon(\phi))_\Omega$$

for all $\phi \in W_0^{1,p}(\Omega, \mathbb{R}^3)$. Similary, we obtain that the problem

$$\begin{aligned} -\operatorname{div} \mathcal{D}\varepsilon(w(x,t)) &= -\operatorname{div} \mathcal{D}\Big(\int_0^t v(x,s)ds\Big), \quad x \in \Omega &(56)\\ w(x,t) &= 0, \qquad x \in \partial\Omega, &(57)\end{aligned}$$

has a unique solution $w(t) \in W_0^{1,p}(\Omega, \mathbb{R}^3)$. Integrating (54) we get that the identity

$$\Big(\mathcal{D}\varepsilon\Big(\int_0^t u(s)ds\Big), \varepsilon(\phi)\Big)_\Omega = \Big(\mathcal{D}\Big(\int_0^t v(s)ds\Big), \varepsilon(\phi)\Big)_\Omega$$

holds for all $\phi \in W_0^{1,p}(\Omega, \mathbb{R}^3)$. Thus, by the definition of P_p, we have

$$w(t) = \int_0^t u(s)ds.$$

Since

$$\nabla\Big(\int_0^t u ds\Big) = \int_0^t (\nabla u) ds, \qquad u \in L^q(0,T; W_0^{1,q}(\Omega, \mathbb{R}^3)),$$

it must be

$$\nabla w(t) = \int_0^t (\nabla u(s)) ds,$$

and hereby (53) is proved. The proof is complete. □

Acknowledgement. The author would like to thank H.-D. Alber, A. Damlamian, N. Kraynyukova and F. Murat for helpful suggestions.

The author acknowledges the financial support by the Alexander von Humboldt Stiftung within Feodor Lynen Fellowship Programme.

References

[1] H.-D. Alber. *Materials with Memory. Initial-Boundary Value Problems for Constitutive Equations with Internal Variables*, volume 1682 of *Lecture Notes in Mathematics*. Springer, Berlin, 1998.

[2] H.-D. Alber and K. Chełmiński. Quasistatic problems in viscoplasticity theory I: Models with linear hardening. In I. Gohberg, A.F. dos Santos, F.-O. Speck, F.S. Teixeira, and W. Wendland, editors, *Theoretical Methods and Applications to Mathematical Physics.*, volume 147 of *Operator Theory. Advances and Applications*, pages 105–129. Birkhäuser, Basel, 2004.

[3] H.-D. Alber and K. Chełmiński. Quasistatic problems in viscoplasticity theory. II. Models with nonlinear hardening. *Math. Models Meth. Appl. Sci.*, 17(2):189–213, 2007.

[4] H. Attouch and M. Thera. A general duality principle for the sum of two operators. *J. Convex Anal.*, 3(1):1–24, 1996.

[5] V. Barbu. *Nonlinear Semigroups and Differential Equations in Banach Spaces.* Editura Academiei, Bucharest, 1976.

[6] H. Brézis. On some degenerate nonlinear parabolic equations. In *Nonlinear Functional Analysis*, volume XVIII, Part 1 of *Proceedings of Symposia in Pure Mathematics, Chicago, 1968*, pages 28–39. AMS, Providence, 1970.

[7] H. Brezis and A. Haraux. Image d'une somme d'operateurs monotones et applications. *Israel J. Math.*, 23(2):165–186, 1976.

[8] H. Brezis and L. Nirenberg. Characterizations of ranges of some nonlinear operators and applications to boundary value problems. *Annali Scu. norm. Sup. Pisa*, 15:225–325, 1978.

[9] M. Brokate, C. Carstensen, and J. Valdman. A quasi-static boundary value problem in multi-surface elastoplasticity. I. Analysis. *Math. Methods Appl. Sci.*, 27(14):1697–1710, 2004.

[10] F. E. Browder. Nonlinear operators and nonlinear equations of evolution in Banach spaces. In *Nonlinear Functional Analysis*, volume XVIII, Part 2 of *Proceedings of Symposia in Pure Mathematics, Chicago, 1968*, pages 1–308. AMS, Providence, 1976.

[11] F. E. Browder. On a principle of brezis and its applications. *J. Funct. Anal.*, 25:356–365, 1977.

[12] K. Chełmiński. Global existence of weak-type solutions for models of monotone type in the theory of inelastic deformations. *Math. Methods Appl. Sci.*, 25:1195–1230, 2002.

[13] K. Chełmiński and P. Gwiazda. Nonhomogenious initial boundary value problems for coercive and self-controlling models of monotone type. *Continuum Mech. Thermodyn.*, 12:217–234, 2000.

[14] K. Chełmiński and Z. Naniewicz. Coercive limits for constitutive equations of monotone-gradient type. *Nonlin. Anal.*, 48:1197–1214, 2002.

[15] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R Wolenski. *Nonsmooth analysis and control theory*, volume 178 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1998.

[16] G. Dal Maso, A. De Simone, and M. G. Mora. Quasistatic evolution problems for linearly elastic - perfectly plastic materials. *Arch. Ration. Mech. Anal.*, 180:237–291, 2006.

[17] G. Dal Maso, A. De Simone, M. G. Mora, and M. Morini. A vanishing viscosity approach to quasistatic evolution in plasticity with softening. *Arch. Ration. Mech. Anal.*, 189(3):469–544, 2008.

[18] A. Damlamian, N. Meunier, and J. Van Schaftingen. Periodic homogenization of monotone multivalued operators. *Nonlinear Anal., Theory Methods Appl.*, 67:3217–3239, 2007.

[19] G. Duvaut and J.L. Lions. *Les inequations en mecanique et en physique.* Dunod, 1972.

[20] F. Ebobisse and B.D. Reddy. Some mathematical problems in perfect plasticity. *Comput. Methods Appl. Mech. Eng.*, 193:5071 – 5094, 2004.

[21] M. Fuchs and G. Seregin. *Variational Methods for Problems from Plasticity Theory and for Generalized Newtonian Fluids*, volume 1749 of *Lecture Notes in Mathematics.* Springer, Berlin, 2000.

[22] P. Gruber, D. Knees, S. Nesenenko, and M. Thomas. Analytical and numerical aspects of time-dependent models with internal variables. *To appear in Z. Angew. Math. Mech.*, 2010.

[23] Z. Guan, A.G. Kartsatos, and I.V. Skrypnik. Ranges of densely defined generalized pseudo-monotone peturbations of maximal monotone operators. *J. Diff. Equa.*, 118:332–351, 2003.

[24] B. Halphen and Nguyen Quoc Son. Sur les materaux standards generalises. *J. Mec.*, 14:39–63, 1975.

[25] W. Han and B. D. Reddy. *Plasticity, Mathematical Theory and Numerical Analysis.* Springer Verlag Inc., New York, 1999.

[26] A. Haraux. *Nonlinear Evolution Equation - Global Behavior of Solutions*, volume 841 of *Lecture Notes in Mathematics*. Springer, Berlin, 1981.

[27] Sh. Hu and N. S. Papageorgiou. *Handbook of Multivalued Analysis. Volume I: Theory.* Mathematics and its Applications. Kluwer, Dordrecht, 1997.

[28] Sh. Hu and N. S. Papageorgiou. *Handbook of Multivalued Analysis. Volume II: Applications.* Mathematics and its Applications. Kluwer, Dordrecht, 2000.

[29] C. Johnson. Existence theorems for plasticity problems. *J. Math. Pures Appl., IX. Sér*, 55:431–444, 1976.

[30] C. Johnson. On plasticity with hardening. *J. Math. Anal. Appl.*, 62(2):325–336, 1978.

[31] N. Kraynyukova. *Existence for mathematical models of ferroelectric material behavior.* Dissertation, Darmstadt University of Technology, 2009.

[32] A. Mainik and A. Mielke. Existence results for energetic models for rate-independent systems. *Calc. Var. Partial Differ. Equ.*, 22:73–99, 2005.

[33] A. Mielke. Existence of minimizers in incremental elasto-plasticity with finite strains. *SIAM J. Math. Analysis*, 36:384–404, 2004.

[34] A. Mielke and F. Theil. On rate–independent hysteresis models. *Nonl. Diff. Eqns. Appl. (NoDEA)*, 11:151–189, 2004.

[35] A. Mielke and A.M. Timofte. An energetic material model for time-dependent ferroelectric behaviour: Existence and uniqueness. *Math. Meth. Appl. Sci.*, 29:1393–1410, 2006.

[36] J.-J. Moreau. Sur l'evolution d'un systeme elasto-visco-plastique. *C. R. Acad. Sci. Paris Ser. A-B*, 273:118–121, 1971.

[37] J.J. Moreau. Application of convex analysis to the treatment of elastoplastic systems. Appl. Methods Funct. Anal. Probl. Mech., IUTAM/IMU-Symp. Marseille 1975, Lect. Notes Math. 503, 56-89 (1976), 1976.

[38] F. Murat. Private communications. 2009.

[39] P. Neff, K. Chelminski, and H.-D. Alber. Notes on strain gradient plasticity: finite strain covariant modelling and global existence in the infinitesimal rate-independent case. *Math. Models Methods Appl. Sci.*, 19(2):307–346, 2009.

[40] D. Pascali and S. Sburlan. *Nonlinear Mappings of Monotone Type.* Editura Academiei, Bucharest, 1978.

[41] R. R. Phelps. *Convex Functions, Monotone Operators and Differentiability*, volume 1364 of *Lecture Notes in Mathematics.* Springer, Berlin, 1993.

[42] W. Pompe. *Existence Theorems in the Viscoplasticity Theory.* Dissertation, Darmstadt University of Technology, 2003.

[43] W. Pompe. Linear functionals on nonlinear spaces and applications to problems from viscoplasticity theory. *Math. Methods Appl. Sci.*, 31(7):775–792, 2008.

[44] S. Reich. The range of sums of accretive and monotone operators. *J. Math. Anal. Appl.*, 68:310–317, 1979.

[45] H. Riahi. On the range of the sum of monotone operators in general banach spaces. *Proc. Amer. Math. Soc.*, 124(11):3333–3338, 1996.

[46] T. Roubicek. Rate-independent processes in viscous solids at small strains. *Math. Methods Appl. Sci.*, 32(7):825–862, 2009.

[47] T. Roubiček. *Nonlinear Partial Differential Equations with Applications*, volume 153 of *International Series of Numerical Mathematics.* Birkhäuser, Basel, 2005.

[48] S. Simons. *Minimax and Monotonicity*, volume 1693 of *Lecture Notes in Mathematics.* Springer, Berlin, 1998.

[49] B. Sören and T. Roubicek. Thermoviscoplasticity at small strains. *Z. Angew. Math. Mech.*, 88(9):735–754, 2008.

[50] P.-M. Suquet. Evolution problems for a class of dissipative materials. *Quart. Appl. Math.*, 38(4):391–414, 1980/81.

[51] R. Temam. *Mathematical Problems in Plasticity.* Bordas, Paris, 1983.

[52] R. Temam. A generelized Norton-Hoff model and the Prandtl-Reuss law of plasticity. *Arch. Ration. Mech. Anal.*, 95(2):137–183, 1986.

[53] R. Temam and G. Strang. Functions of bounded deformation. *Arch. Ration. Mech. Anal.*, 75(1):7–21, 1980.

[54] T. Valent. *Boundary Value Problems of Finite Elasticity.* Springer, Berlin, 1988.

[55] A. Visintin. Electromagnetic processes in doubly-nonlinear composites. *Comm. Partial Differential Equations*, 33(4-6):808–841, 2008.

[56] A. Visintin. Homogenization of the nonlinear Maxwell model of viscoelasticity and of the Prandtl-Reuss model of elastoplasticity. *Proc. Roy. Soc. Edinburgh Sec. A*, 138(6):1363–1401, 2008.

[57] E. Zeidler. *Nonlinear Functional Analysis and its Applications*, volume II/B. Springer, Berlin, 1990.