

EXISTENCE AND UNIQUENESS FOR RATE-INDEPENDENT INFINITESIMAL GRADIENT PLASTICITY WITH ISOTROPIC HARDENING AND PLASTIC SPIN

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ABSTRACT. Existence and uniqueness for infinitesimal dislocation based rate-independent gradient plasticity with linear isotropic hardening and plastic spin is shown using convex analysis and variational inequality methods. The dissipation potential is extended non-uniquely from symmetric plastic rates to non-symmetric plastic rates and three qualitatively different formats for the dissipation potential are distinguished.

Key words: Plasticity, gradient plasticity, dislocations, plastic spin, quasistatic evolution, rate-independent process, variational inequality.

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1. INTRODUCTION

We study the existence of solutions of quasistatic initial-boundary value problems arising in gradient plasticity with isotropic hardening. The models we study use constitutive equations with internal variables to describe the deformation behaviour of metals at small

strain [2, 14, 27, 3]. While gradient plasticity is of high current interest especially for describing novel effects at very small length scales [16, 11, 13, 12, 1, 28, 4], mathematical studies of the time continuous higher gradient plasticity problem are still rather scarce. Recently Reddy [26] (see also Ebobisse [8]) treats a geometrically linear model introduced by Gurtin [11], essentially different from the model we consider here. We mention also the very successful energetic approach of Mielke [19, 18] which led to existence results in both small strain and finite strain models.

Our model has been introduced in Neff et al. [22]. The model features a non-symmetric plastic distortion $p \in \mathfrak{sl}(3)$ and second spatial gradients of the plastic distortion D^2p acting as dislocation based kinematical backstresses which are directly related to the geometrically necessary dislocations (GND) which induce a long-range interaction. Furthermore, local kinematical hardening is incorporated as well. Uniqueness of classical solutions for rate-independent and rate-dependent formulations of this model is shown in [21]. The more difficult existence question for the rate-independent model in terms of a weak reformulation is addressed in [22]. First numerical results for a simplified rate-independent irrotational formulation (no plastic spin, i.e., symmetric plastic distortion p) are presented in [23]. A distinguishing feature of this model with kinematical hardening is that, similar to classical approaches, only the symmetric part $\varepsilon_p := \text{sym } p$ of the plastic distortion appears in the local part of the Eshelby stress tensor $\sigma - h^+ \text{sym } p$, while the higher order stresses are the only source of non-symmetry. Here, σ is the symmetric Cauchy stress tensor and h^+ is the local kinematical hardening modulus. For more on the basic invariance questions related to this issue, see [29, 20].

Here, we modify the previous model. On one hand we skip the local kinematical hardening i.e. we put $h^+ = 0$. On the other hand we augment the model with classical isotropic hardening based on the equivalent plastic strain concept which we understand to be related to statistically stored dislocations (SSD).¹ Thus both different physical mechanisms related to GND's and SSD's contribute in their specific way to the hardening regime.

The related viscoplastic formulation of dislocation based gradient plasticity with kinematical hardening is treated in [24].

Note that equivalent (or accumulated) plastic strain, classically means that a measure of the total plastic strain, i.e. $\gamma(t) = \int_0^t \sqrt{\frac{2}{3}} |\dot{\varepsilon}_p| dt$ is driving the increase of the yield limit $\sigma_y = \sigma_y^0 + k \gamma$ and thus leads to isotropic hardening. This is tantamount to prescribe the plastic dissipation to be related to $|\dot{\varepsilon}_p| = \sqrt{|\dot{\varepsilon}_p|^2}$. In our case, where the plastic distortion

¹The isotropic increase of the yield surface will depend only on local measures of equivalent plastic strain γ . Nonlocal extensions have been discussed in [1, 9, 5, 6, 7].

p is not necessarily symmetric, the generalization of the former is not unique. Therefore, we introduce a parameter $0 \leq \alpha \leq \infty$ and weight the contribution to the dissipation according to $\sqrt{|\text{sym } \dot{p}|^2 + \alpha |\text{skew } \dot{p}|^2}$.

If we had chosen, instead, to use the concept of equivalent plastic work, i.e. $\tilde{\gamma}(t) = \int_0^t \sqrt{\frac{2}{3}} \langle \sigma, \dot{\varepsilon}_p \rangle dt$ as a measure of plastic work is driving the increase of the yield limit σ_y , then the generalization to nonsymmetric plastic distortions is unique since $\tilde{\gamma}(t) = \int_0^t \sqrt{\frac{2}{3}} \langle \sigma, \dot{p} \rangle dt = \int_0^t \sqrt{\frac{2}{3}} \langle \sigma, \text{sym } \dot{p} + \alpha \text{skew } \dot{p} \rangle dt = \int_0^t \sqrt{\frac{2}{3}} \langle \sigma, \text{sym } \dot{p} \rangle dt$ because the Cauchy stress tensor σ is symmetric. Therefore, from a modelling point of view, in order to have both concepts of equivalent plastic strain and equivalent plastic work being based on the same primitive variable, the parameter value $\alpha = 0$ would suggest itself. However, we are presently unable to include a full analysis for $\alpha = 0$.

The outline of this contribution is as follows: first, we shortly motivate the model and derive the strong form of equations based on a Cahn-Allen type energetic approach. Then we reformulate the model in a weak sense as a variational inequality. Existence and uniqueness is shown in a suitable Hilbert-space.

2. STRONG FORMULATION

2.1. The balance equation. The conventional macroscopic force balance leads to the equation of equilibrium

$$(2.1) \quad \text{div } \sigma + f = \mathbf{0}$$

in which σ is the infinitesimal symmetric Cauchy stress tensor and f is the body force.

2.2. Constitutive relations. The constitutive equations are obtained from a free energy imbalance together with a flow law that characterizes plastic behaviour. Since the model under study involves plastic spin, we consider an additive decomposition of the displacement gradient ∇u into elastic and plastic distortions e and p , so that

$$(2.2) \quad \nabla u = e + p, \quad (\nabla u)_{ij} = \frac{\partial u_i}{\partial x_j},$$

with the nonsymmetric plastic distortion p incapable of sustaining volumetric changes; that is,

$$(2.3) \quad \text{tr } p = 0 \Leftrightarrow p \in \mathfrak{sl}(3).$$

We define $\varepsilon_e := \text{sym } e = \text{sym}(\nabla u - p)$ as the infinitesimal elastic strain and $\varepsilon_p := \text{sym } p$ is the plastic strain while $\varepsilon := \text{sym } \nabla u = (\nabla u + \nabla u^T)/2$ is the total strain. Equation (2.2)

induces the well known additive decomposition of strains as follows

$$(2.4) \quad \varepsilon = \varepsilon_e + \varepsilon_p .$$

We consider here a quadratic free energy given in the additively decoupled format

$$(2.5) \quad \Psi(\nabla u, p, \text{Curl } p, \gamma) := \underbrace{\Psi_e^{\text{lin}}(\varepsilon_e)}_{\text{elastic energy}} + \underbrace{\Psi_{\text{curl}}^{\text{lin}}(\text{Curl } p)}_{\text{defect energy (GND)}} + \underbrace{\Psi_{\text{iso}}(\gamma)}_{\text{hardening energy (SSD)}} ,$$

where

$$\Psi_e^{\text{lin}}(\varepsilon_e) := \frac{1}{2} \langle \varepsilon_e, \mathbb{C} \cdot \varepsilon_e \rangle, \quad \Psi_{\text{curl}}^{\text{lin}}(\text{Curl } p) = \frac{\mu L_c^2}{2} |\text{Curl } p|^2, \quad \Psi_{\text{iso}}(\gamma) = \frac{1}{2} k |\gamma|^2 .$$

L_c is an energetic intrinsic length scale and k is the positive isotropic hardening constant. The defect energy is related to geometrically necessary dislocations (GND's) present through the dislocation density tensor $\text{Curl } p$ [25, 15, 10], where the operator Curl is acting on the rows of any second order tensor X such that $\text{Curl } \nabla u = 0$. The isotropic hardening energy is related to statistically stored dislocations (SSD's) phenomenologically through a measure of equivalent plastic strain γ .

For isotropic media the fourth order elasticity tensor \mathbb{C} is given by

$$(2.6) \quad \mathbb{C} \cdot X = 2\mu X + \lambda \text{tr}(X) I$$

for any symmetric second-order tensor X , where λ and μ are the Lamé moduli supposed to satisfy

$$\mu > 0 \quad \text{and} \quad 3\lambda + 2\mu > 0 .$$

These conditions suffice for pointwise ellipticity of the elasticity tensor in the sense that there exists a constant $m_0 > 0$ such that for any symmetric second order tensor we have

$$(2.7) \quad \langle X, \mathbb{C} \cdot X \rangle \geq m_0 |X|^2 ,$$

the magnitude of any second-order tensor X being given by the Frobenius-matrix norm $|X| = \langle X, X \rangle^{1/2}$. The anisotropic case is included by allowing a general \mathbb{C} to satisfy (2.7).²

The local free-energy imbalance states that

$$(2.8) \quad \dot{\Psi} - \langle \sigma, \nabla u_t \rangle \leq 0 \Leftrightarrow \dot{\Psi} - \langle \sigma, \dot{e} + \dot{p} \rangle \leq 0 \Leftrightarrow \dot{\Psi} - \langle \sigma, \dot{\varepsilon}_e \rangle - \langle \sigma, \dot{p} \rangle \leq 0 .$$

Now we expand $\dot{\Psi}$, substitute (2.5) and get

$$(2.9) \quad \langle \mathbb{C} \cdot \varepsilon_e - \sigma, \dot{\varepsilon}_e \rangle - \langle \sigma, \dot{p} \rangle + \mu L_c^2 \langle \text{Curl } p, \text{Curl } \dot{p} \rangle + k \gamma \dot{\gamma} \leq 0 ,$$

²The defect energy may also be more general, as long as an estimate $\Psi_{\text{curl}}^{\text{lin}}(\text{Curl } p) \geq c^+ |\text{Curl } p|^2$ is satisfied.

which, using arguments from thermodynamics,³ implies on the one hand the elasticity relation

$$(2.10) \quad \sigma = \mathbb{C} \cdot \varepsilon_e = 2\mu \operatorname{sym}(\nabla u - p) + \lambda \operatorname{tr}(\nabla u - p) I$$

and the reduced dissipation inequality

$$(2.11) \quad -\langle \sigma, \dot{p} \rangle + \mu L_c^2 \langle \operatorname{Curl} p, \operatorname{Curl} \dot{p} \rangle + k\gamma \dot{\gamma} \leq 0.$$

Now we integrate (2.11) over Ω and get

$$\begin{aligned} 0 &\geq \int_{\Omega} \left[-\langle \sigma, \dot{p} \rangle + \mu L_c^2 \langle \operatorname{Curl} p, \operatorname{Curl} \dot{p} \rangle + k\gamma \dot{\gamma} \right] dV \\ &= - \int_{\Omega} \left[\langle \sigma, \dot{p} \rangle + \mu L_c^2 \langle \operatorname{Curl} \operatorname{Curl} p, \dot{p} \rangle + \sum_{i=1}^3 \operatorname{div} \left(\mu L_c^2 \frac{d}{dt} p^i \times (\operatorname{Curl} p)^i \right) + k\gamma \dot{\gamma} \right] dV. \end{aligned}$$

Using the divergence theorem we obtain

$$(2.12) \quad \int_{\Omega} \left[\langle -\sigma + \mu L_c^2 \operatorname{Curl} \operatorname{Curl} p, \dot{p} \rangle + k\gamma \dot{\gamma} \right] dV + \sum_{i=1}^3 \int_{\partial\Omega} \mu L_c^2 \left\langle \frac{d}{dt} p^i \times (\operatorname{Curl} p)^i, \vec{n} \right\rangle dS \leq 0,$$

where \vec{n} is the unit outward normal on $\partial\Omega$.

In order to obtain a dissipation inequality in the spirit of classical plasticity, we assume that the infinitesimal plastic distortion p satisfies the so-called *linearized insulation condition*

$$(2.13) \quad \sum_{i=1}^3 \int_{\partial\Omega} \mu L_c^2 \left\langle \frac{d}{dt} p^i \times (\operatorname{Curl} p)^i, \vec{n} \right\rangle dS = 0$$

In order for the insulation condition to be satisfied we will impose the following boundary conditions on the plastic distortion:

$$(2.14) \quad p(x, t) \cdot \tau = p(x, 0) \cdot \tau \text{ on } \Gamma_D \quad \text{and} \quad \operatorname{Curl} p \cdot \tau = 0 \text{ on } \partial\Omega \setminus \Gamma_D,$$

where $\Gamma_D \subset \partial\Omega$ is that part of the boundary where zero Dirichlet conditions on the displacements u are prescribed.

Under (2.13), we then obtain the dissipation inequality

$$(2.15) \quad \int_{\Omega} [\langle \sigma + \Sigma_{\operatorname{curl}}^{\operatorname{lin}}, \dot{p} \rangle + g\dot{\gamma}] dV \geq 0,$$

where

$$(2.16) \quad \Sigma_{\operatorname{curl}}^{\operatorname{lin}} := -\mu L_c^2 \operatorname{Curl} \operatorname{Curl} p \quad \text{and} \quad g = -k\gamma.$$

³The rate $\dot{\varepsilon}_e$ can be chosen arbitrarily.

For further use we define the non-symmetric stress tensor

$$(2.17) \quad \Sigma_E := \sigma + \Sigma_{\text{curl}}^{\text{lin}},$$

the non-symmetry being only related to the nonlocal term $\Sigma_{\text{curl}}^{\text{lin}}$.

2.3. The plastic flow law: dual formulation. The classical yield function for isotropic hardening von Mises plasticity is given by

$$\phi_{\infty}(\Sigma_{\infty}^p) := |\text{dev } \sigma| + g - \sigma_y$$

for the generalized stress $\Sigma_{\infty}^p = (\sigma, g)$.

Here, given a parameter $\alpha \in [0, +\infty]$, we consider a yield function $\phi_{1/\alpha}$ defined for every generalized stress $\Sigma^p = (\Sigma_E, g)$ by

$$(2.18) \quad \phi_{1/\alpha}(\Sigma^p) := \begin{cases} \sqrt{|\text{dev sym } \Sigma_E|^2 + \frac{1}{\alpha} |\text{dev skew } \Sigma_E|^2} + g - \sigma_y^0 & \text{if } \alpha > 0, \\ \begin{cases} |\text{dev } \Sigma_E| + g - \sigma_y^0 & \text{if } \Sigma_E \in M_{\text{sym}}^{3 \times 3}, \\ +\infty & \text{otherwise} \end{cases} & \text{if } \alpha = 0, \\ |\text{dev sym } \Sigma_E| + g - \sigma_y^0 & \text{if } \alpha = +\infty. \end{cases}$$

Here, σ_y^0 is the initial yield stress of the material. So the set of admissible (elastic) generalized stresses is

$$(2.19) \quad \mathcal{K}_{\alpha} := \{\Sigma^p = (\Sigma_E, g) : \phi_{1/\alpha}(\Sigma^p) \leq 0\}.$$

If we let $\Gamma^p = (p, \gamma)$ then the maximum dissipation principle gives the normality law

$$(2.20) \quad \dot{\Gamma}^p \in N_{\mathcal{K}_{\alpha}}(\Sigma^p),$$

where $N_{\mathcal{K}_{\alpha}}(\Sigma^p)$ denotes the normal cone to \mathcal{K}_{α} at Σ^p , which is the set of generalised strain rates $\dot{\Gamma}^p$ that satisfy

$$(2.21) \quad \langle \bar{\Sigma} - \Sigma^p, \dot{\Gamma}^p \rangle \leq 0 \quad \text{for all } \bar{\Sigma} \in \mathcal{K}_{\alpha}.$$

2.4. The plastic flow law: primal formulation. Using convex analysis (Legendre transformation) the relation (2.20) can be inverted and we find that

$$(2.22) \quad \dot{\Gamma}^p \in N_{\mathcal{K}_{\alpha}}(\Sigma^p) \Leftrightarrow \Sigma^p \in \partial \mathcal{D}_{\alpha}(\dot{\Gamma}^p),$$

where \mathcal{D}_{α} is the one-homogeneous dissipation function for rate-independent processes which in this case is defined by

$$(2.23) \quad \mathcal{D}_{\alpha}(q, \beta) = \sup \{ \langle \Sigma_E, q \rangle + g\beta \mid \phi_{1/\alpha}(\Sigma_E, g) \leq 0, g \leq 0 \}.$$

$\alpha \in [0, \infty]$	$\mathcal{D}_\alpha(q, \beta) =$
$\alpha \in (0, \infty)$	$\begin{cases} \sigma_y \sqrt{ \text{sym } q ^2 + \alpha \text{skew } q ^2} & \text{if } \sqrt{ \text{sym } q ^2 + \alpha \text{skew } q ^2} \leq \beta, \\ \infty & \text{otherwise} \end{cases}$
$\alpha = 0$	$\begin{cases} \sigma_y \text{sym } q & \text{if } \text{sym } q \leq \beta, \\ \infty & \text{otherwise} \end{cases}$
$\alpha = \infty$	$\begin{cases} \sigma_y q & \text{if } q \in M_{\text{sym}}^{3 \times 3} \text{ and } q \leq \beta, \\ \infty & \text{otherwise} \end{cases}$

TABLE 1. The dissipation function $\mathcal{D}_\alpha(q, \beta)$ for $\alpha \in [0, \infty]$

By easy calculations, we get the expressions of the dissipation function \mathcal{D}_α given in Table 1 below.

Such a case distinction for the dissipation function has been introduced in Mielke [17] for finite plasticity.

In (2.22), $\partial \mathcal{D}_\alpha(\dot{\Gamma}^p)$ denotes the subdifferential of \mathcal{D}_α evaluated at $\dot{\Gamma}^p$. That is, $\Sigma^p \in \partial \mathcal{D}_\alpha(\dot{\Gamma}^p)$ means that

$$(2.24) \quad \mathcal{D}_\alpha(M) \geq \mathcal{D}_\alpha(\dot{\Gamma}^p) + \langle \Sigma^p, M - \dot{\Gamma}^p \rangle \quad \text{for any } M.$$

That is,

$$(2.25) \quad \mathcal{D}_\alpha(q, \beta) \geq \mathcal{D}_\alpha(\dot{p}, \dot{\gamma}) + \langle \sigma + \Sigma_{\text{curl}}^{\text{lin}}, q - \dot{p} \rangle + g(\beta - \dot{\gamma}) \quad \text{for any } (q, \beta).$$

We refer to $\alpha = \infty$ as irrotational (p is symmetric), $\alpha = 1$ as equal spin and $\alpha = 0$ as free spin case.

2.5. Strong formulation of the model. To summarize, we have obtained the following strong formulation for the model of infinitesimal gradient plasticity with isotropic hardening and plastic spin: Find

- (i) the displacement $u \in H^1(0, T; H_0^1(\Omega, \Gamma_D, \mathbb{R}^3))$ satisfying zero Dirichlet conditions on Γ_D ,
- (ii) the infinitesimal plastic distortion $p \in H^1(0, T; L^2(\Omega, \mathfrak{sl}(3)))$ for the case $\alpha > 0$ (while only $\text{sym } p \in H^1(0, T; L^2(\Omega, \mathfrak{sl}(3)))$ for the case $\alpha = 0$ and p symmetric

<i>Equilibrium equation:</i>	$\operatorname{div} \sigma + f = 0$ in $\Omega \times [0, T]$
<i>Elasticity relation:</i>	$\sigma = \mathbb{C} \cdot \operatorname{sym}(\nabla u - p)$
<i>Constitutive relation:</i>	$(\sigma + \Sigma_{\operatorname{curl}}^{\operatorname{lin}}, g) \in \partial \mathcal{D}_\alpha(\dot{p}, \dot{\gamma})$ where $\Sigma_{\operatorname{curl}}^{\operatorname{lin}} = -\mu L_c^2 \operatorname{Curl} \operatorname{Curl} p$ and \mathcal{D}_α is defined in (2.23)
<i>Boundary conditions:</i>	$\begin{cases} u(x, t) = 0 \text{ on } \Gamma_D, \\ p(x, t) \cdot \tau = p(x, 0) \cdot \tau \text{ on } \Gamma_D, \\ \operatorname{Curl} p \cdot \tau = 0 \text{ on } \partial\Omega \setminus \Gamma_D \end{cases}$
<i>Initial conditions:</i>	$\begin{cases} u(x, 0) = u_0(x), \\ p(x, 0) = p_0(x), \\ \gamma(x, 0) = \gamma_0(x). \end{cases}$

TABLE 2. Strong primal formulation of the model

and $p \in H^1(0, T; L^2(\Omega, \mathfrak{sl}(3)))$ for $\alpha = +\infty$) with $\operatorname{Curl} p(t) \in L^2(\Omega, M^{3 \times 3})$ and $\operatorname{Curl} \operatorname{Curl} p(t) \in L^2(\Omega, M^{3 \times 3})$,

(iii) the isotropic hardening variable (the equivalent plastic strain) $\gamma \in H^1(0, T; L^2(\Omega))$ such that

3. WEAK FORMULATION OF THE MODEL

Assume that the problem in Section 2.5 has a solution (u, p, γ) . Let $v \in H^1(\Omega, \mathbb{R}^3)$ with $v|_{\Gamma_D} = 0$ in the sense of traces. Multiply the equilibrium equation with $v - \dot{u}$ and integrate in space to get

$$(3.1) \quad \int_{\Omega} \langle \sigma, \nabla v - \nabla \dot{u} \rangle dV = \int_{\Omega} f(v - \dot{u}) dV.$$

Using the symmetry of σ and the elasticity relation we get

$$(3.2) \quad \int_{\Omega} \langle \mathbb{C} \cdot \operatorname{sym}(\nabla u - p), \operatorname{sym}(\nabla v - \nabla \dot{u}) \rangle dV = \int_{\Omega} f(v - \dot{u}) dV.$$

Now, for any $q \in C^\infty(\overline{\Omega}, \mathfrak{sl}(3))$ such that $q \cdot \tau = 0$ on Γ_D and any $\beta \in L^2(\Omega)$, we integrate (2.25) over Ω , integrate by parts the term with $\operatorname{Curl} \operatorname{Curl}$ using the boundary conditions

$$(q - \dot{p}) \cdot \tau = 0 \text{ on } \Gamma_D, \quad \operatorname{Curl} p \cdot \tau = 0 \text{ on } \partial\Omega \setminus \Gamma_D$$

and get

$$\begin{aligned}
(3.3) \quad \int_{\Omega} \mathcal{D}_{\alpha}(q, \beta) dV &\geq \int_{\Omega} \mathcal{D}_{\alpha}(\dot{p}, \dot{\gamma}) dV + \int_{\Omega} [(\langle \sigma + \Sigma_{\text{curl}}^{\text{lin}}, q - \dot{p} \rangle + g(\beta - \dot{\gamma}))] dV \\
&\geq \int_{\Omega} \mathcal{D}_{\alpha}(\dot{p}, \dot{\gamma}) dV + \int_{\Omega} \langle \mathbb{C} \cdot \text{sym}(\nabla u - p), \text{sym}(q - \dot{p}) \rangle dV \\
&\quad - \mu L_c^2 \int_{\Omega} \langle \text{Curl Curl } p, q - \dot{p} \rangle dV - \int_{\Omega} k\gamma(\beta - \dot{\gamma}) dV \\
&\geq \int_{\Omega} \mathcal{D}_{\alpha}(\dot{p}, \dot{\gamma}) dV + \int_{\Omega} \langle \mathbb{C} \cdot \text{sym}(\nabla u - p), \text{sym}(q - \dot{p}) \rangle dV \\
&\quad - \mu L_c^2 \int_{\Omega} \langle \text{Curl } p, \text{Curl}(q - \dot{p}) \rangle dV - \int_{\Omega} k\gamma(\beta - \dot{\gamma}) dV.
\end{aligned}$$

Now adding up (3.2) and (3.3) we arrive at the following weak formulation of the problem in Section 2.5 in the form of a variational inequality:

$$\begin{aligned}
(3.4) \quad &\int_{\Omega} \left[\langle \mathbb{C} \cdot \text{sym}(\nabla u - p), \text{sym}(\nabla v - q) - \text{sym}(\nabla \dot{u} - \dot{p}) \rangle + \mu L_c^2 \langle \text{Curl } p, \text{Curl}(q - \dot{p}) \rangle \right. \\
&\left. + k\gamma(\beta - \dot{\gamma}) \right] dV + \int_{\Omega} \mathcal{D}_{\alpha}(q, \beta) dV - \int_{\Omega} \mathcal{D}_{\alpha}(\dot{p}, \dot{\gamma}) dV \geq \int_{\Omega} f(v - \dot{u}) dV
\end{aligned}$$

for all (v, q, β) .

4. EXISTENCE AND UNIQUENESS

4.1. The case $\alpha > 0$ (with spin). To prove the existence and uniqueness of a solution of the variational inequality (3.4) for $\alpha > 0$, we introduce the following function spaces:

$$(4.1) \quad V := H_0^1(\Omega, \Gamma_D, \mathbb{R}^3) = \{v \in H^1(\Omega, \mathbb{R}^3) : v|_{\Gamma_D} = 0\},$$

$$(4.2) \quad Q_{\alpha} := \{q : \Omega \rightarrow \mathfrak{sl}(3) : q \in H_{\text{curl}}(\Omega, M^{3 \times 3}), q|_{\Gamma_D} \cdot \tau = 0\},$$

$$(4.3) \quad \Lambda := L^2(\Omega),$$

$$(4.4) \quad Z_{\alpha} := V \times Q_{\alpha} \times \Lambda,$$

$$(4.5) \quad W_{\alpha} := \{z = (v, q, \beta) \in Z_{\alpha} : \sqrt{|\text{sym } q|^2 + \alpha |\text{skew } q|^2} \leq \beta \text{ a.e. in } \Omega\}$$

equipped with the norms

$$(4.6) \quad \|v\|_V := \|\nabla v\|_{L^2}, \quad \|q\|_{Q_{\alpha}}^2 := \|q\|_{L^2}^2 + \|\text{Curl } q\|_{L^2}^2,$$

$$(4.7) \quad \|z\|_{Z_{\alpha}}^2 := \|v\|_V^2 + \|q\|_{Q_{\alpha}}^2 + \|\beta\|_{L^2}^2 \quad \text{for } z = (v, q, \beta) \in Z_{\alpha},$$

and the functionals:

$$(4.8) \quad a(w, z) := \int_{\Omega} \left[\langle \mathbb{C} \cdot \text{sym}(\nabla u - p), \text{sym}(\nabla v - q) \rangle + \mu L_c^2 \langle \text{Curl } p, \text{Curl } q \rangle + k\gamma\beta \right] dV$$

$$(4.9) \quad j_{\alpha}(z) := \begin{cases} \int_{\Omega} \mathcal{D}_{\alpha}(q, \beta) dV & \text{if } \sqrt{|\text{sym } q|^2 + \alpha |\text{skew } q|^2} \leq \beta \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise} \end{cases}$$

$$(4.10) \quad \langle \ell, z \rangle := \int_{\Omega} f v dV,$$

for $w = (u, p, \gamma)$ and $z = (v, q, \beta)$.

Thus assuming for instance that the model is initially unstressed and undeformed and this corresponds to $f(x, 0) = 0$ and $u_0 = 0$, $p_0 = 0$ and $\gamma_0 = 0$, the weak formulation of the model in the case $\alpha > 0$ reads as follows: find $w = (u, p, \gamma) \in H^1(0, T; Z_{\alpha})$ such that $w(0) = 0$, $\dot{w}(t) \in W_{\alpha}$ for a.e. $t \in [0, T]$ and the variational inequality

$$(4.11) \quad a(\dot{w}, z - w) + j_{\alpha}(z) - j_{\alpha}(\dot{w}) \geq \langle \ell, z - \dot{w} \rangle \text{ for every } z \in W_{\alpha} \text{ and for a.e. } t \in [0, T]$$

The existence and uniqueness of the solution of (4.11) is obtained from the following abstract result by Han-Reddy [14].

Theorem 4.1. *Let H be a Hilbert space; $K \subset H$ a nonempty, closed, convex cone; $a: H \times H \rightarrow \mathbb{R}$ a bilinear form that is symmetric, bounded on H and coercive on K ; $\ell \in H^1(0, T; H')$ with $\ell(0) = 0$; and $j: H \rightarrow \mathbb{R}$ nonnegative, convex, positively homogeneous, and Lipschitz continuous on K . Then there exists a unique solution $w \in H^1(0, T; H)$ of the problem*

$$(4.12) \quad a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t)) \geq \langle \ell(t), z - \dot{w}(t) \rangle \quad \forall z \in K,$$

satisfying $w(0) = 0$ and $\dot{w}(t) \in K$.

While it is easy to see that the set W_{α} , the functionals j_{α} and ℓ satisfy the assumptions of Theorem 4.1 and that the bilinear form a is continuous on Z_{α} , let us show that a is

coercive on W_α . Let therefore $z = (v, q, \beta) \in W_\alpha$. Then

$$\begin{aligned}
a(z, z) &\geq m_0 \|\text{sym}(\nabla v) - \text{sym } q\|_2^2 + \mu L_c^2 \|\text{Curl } q\|_2^2 + k \|\beta\|_2^2 \quad (m_0 > 0 \text{ is from (2.7)}) \\
&= m_0 \left[\|\text{sym}(\nabla v)\|_2^2 + \|\text{sym } q\|_2^2 - 2\langle \text{sym}(\nabla v), \text{sym } q \rangle \right] + \mu L_c^2 \|\text{Curl } q\|_2^2 + k \|\beta\|_2^2 \\
&\geq m_0 \left[\|\text{sym}(\nabla v)\|_2^2 + \|\text{sym } q\|_2^2 - \theta \|\text{sym}(\nabla v)\|_2^2 - \frac{1}{\theta} \|\text{sym } q\|_2^2 \right] \\
&\quad + \mu L_c^2 \|\text{Curl } q\|_2^2 + \frac{1}{2}k \|\text{sym } q\|_2^2 + \frac{\alpha}{2}k \|\text{skew } q\|_2^2 + \frac{1}{2}k \|\beta\|_2^2 \\
&\quad \text{(using Young's inequality and } |\text{sym } q|^2 + \alpha |\text{skew } q|^2 \leq \beta^2 \text{ from } W_\alpha) \\
&= m_0(1 - \theta) \|\text{sym}(\nabla v)\|_2^2 + \left[m_0(1 - \frac{1}{\theta}) + \frac{1}{2}k \right] \|\text{sym } q\|_2^2 + \frac{\alpha}{2}k \|\text{skew } q\|_2^2 \\
&\quad + \mu L_c^2 \|\text{Curl } q\|_2^2 + \frac{1}{2}k \|\beta\|_2^2.
\end{aligned}$$

So, choosing θ such that $\frac{1}{1 + \frac{k}{2m_0}} \leq \theta < 1$, and using Korn's first inequality, there exist two positive constants $C_1(m_0, \mu, k, L_c, \Omega) > 0$, $C_2(m_0, \mu, k, L_c, \alpha, \Omega) > 0$ such that

$$\begin{aligned}
(4.13) \quad a(z, z) &\geq C_1 \left[\|v\|_V^2 + \|\text{sym } q\|_2^2 + \alpha \|\text{skew } q\|_2^2 + \|\text{Curl } q\|_2^2 + \|\beta\|_2^2 \right] \\
&\geq C_2 \left[\|v\|_V^2 + \|q\|_2^2 + \|\text{Curl } q\|_2^2 + \|\beta\|_2^2 \right] = C_2 \|z\|_{Z_\alpha}^2 \quad \forall z = (v, q, \beta) \in W_\alpha.
\end{aligned}$$

Remark that the pointwise control of the skewsymmetric part of q is coming from $\alpha > 0$ here. In addition, for $\alpha > 0$ we could as well treat the completely "microfree case", i.e., the boundary condition for the plastic distortion $\text{Curl } p \cdot \tau = 0$ on $\partial\Omega$.

4.2. The case $\alpha = \infty$ (irrotational, no spin). To prove the existence and uniqueness of the solution of the variational inequality (3.4) for $\alpha = \infty$ is, in fact, quite easy. We only have to remark that necessarily the plastic distortions p remain symmetric throughout and have to adapt the Hilbert-space framework accordingly. An irrotational model with linear kinematic hardening in the spirit of the present approach has been dealt with in [22, 23]. For another irrotational model see [26] and [8], where the full gradient of the plastic strain is involved in the primal formulation of the flow law, leading to the space $H^1(\Omega, M_{\text{sym}}^{3 \times 3})$ as the appropriate space for infinitesimal plastic strains. Another irrotational gradient plasticity model has been considered in [6, 7], where gradient effects appear only through a dependence of the yield limit on spatial gradients of the equivalent plastic strain γ , i.e. $\sigma_y = \sigma_y^0 + k \gamma - k_3 \Delta \gamma$. This amounts to use $\Psi_{\text{iso}}(\gamma, \nabla \gamma) = \frac{1}{2}k|\gamma|^2 + \frac{1}{2}k_3|\nabla \gamma|^2$ and to set $L_c = 0$.

5. OPEN PROBLEM: THE CASE $\alpha = 0$ (FREE SPIN)

We have seen that for $\alpha > 0$ the precise form of the defect energy did not really matter. We could also replace the defect energy by a full gradient term $\Psi_{\nabla}^{\text{lin}}(\nabla p) = \frac{\mu L_{\varepsilon}^2}{2} \|\nabla p\|^2$ without affecting the mathematical development at all.

This changes completely for the free spin case $\alpha = 0$, in which the form of the defect energy matters. The interplay between, on one hand, a pointwise control of the symmetric part of the plastic distortion $\text{sym } p$ and on the other hand only a nonlocal control of the skewsymmetric part $\text{skew } p$ by the $\text{Curl } p$ -contribution provides for challenging questions.

In order to prove the existence and uniqueness of the solution of the variational inequality (3.4) for $\alpha = 0$, the first task is to provide a suitable Hilbert space environment where one is able to apply Theorem 4.1. For $\alpha = 0$ we infer from (4.13) the estimate

$$(5.1) \quad a(z, z) \geq C_1 [\|v\|_V^2 + \|\text{sym } q\|_2^2 + \|\text{Curl } q\|_2^2 + \|\beta\|_2^2] .$$

Thus, taking into account the structure of the bilinear form a and the set W_0 , a natural attempt for the space of infinitesimal plastic distortions, is to consider the completion $H_{\text{curl}}^{\text{sym}}(\Omega, \Gamma_D; M^{3 \times 3})$ of the linear space

$$\{q \in C^\infty(\bar{\Omega}, M^{3 \times 3}): \text{tr } q = 0, \quad q \cdot \tau = 0 \text{ on } \Gamma_D\}$$

with respect to the norm

$$(5.2) \quad \|q\|_{\text{sym, curl}}^2 := \|\text{sym } q\|_2^2 + \|\text{Curl } q\|_2^2 .$$

Despite appearance, this quadratic form indeed defines a norm as shown in [22]. Thus Theorem 4.1 carries over by replacing Q_α with $H_{\text{curl}}^{\text{sym}}(\Omega, \Gamma_D; M^{3 \times 3})$ and using the norm (5.2). However, in this space it is not immediately obvious how to define a linear bounded tangential trace operator. We will investigate the detailed properties of this Hilbert space and the regularity of the obtained solution in a future contribution.

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