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# WELL-POSEDNESS FOR DISLOCATION BASED GRADIENT VISCOPLASTICITY I: SUBDIFFERENTIAL CASE*

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**Abstract.** Well-posedness for infinitesimal dislocation based gradient viscoplasticity with linear kinematic hardening is shown using a time-discretization technique for the rate-dependent model and methods of convex analysis.

**Key words.** plasticity, gradient plasticity, viscoplasticity, dislocations, plastic spin, Rothe's time-discretization method, rate-dependent models

**AMS subject classifications.** 35B65, 35D10, 74C10, 74D10, 35J25, 34G20, 34G25, 47H04, 47H05

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## 1. Introduction.

We study the existence of solutions of quasi-static initial-boundary value problems arising in gradient viscoplasticity. The models we study use rate-dependent constitutive equations with internal variables to describe the deformation behavior of metals at small strain. While gradient plasticity is of high current interest [13, 14, 10, 9], mathematical studies of the time-continuous higher gradient plasticity problem are still rather scarce. Reddy, Ebobisse, and McBride [20] treats a geometrically linear model of Gurtin and Anand [8], different from the model we consider.

Our model has been derived in [15]. Contrary to more classical approaches, the model features a nonsymmetric plastic distortion field $p \in \mathcal{M}^3$, a dislocation based energy storage based on $|\operatorname{Curl} p|$ and second gradients of the plastic distortion in the form of $\operatorname{Curl}\operatorname{Curl} p$ acting as dislocation based kinematical backstresses. Uniqueness of classical solutions for rate-independent and rate-dependent formulations is shown in [19]. The existence question for the rate-independent model in terms of a weak reformulation is addressed in [15]. The rate-independent model with isotropic hardening is treated in [6]. First numerical results for a simplified rate-independent irrotational formulation (no plastic spin, symmetric plastic distortion $p$) are presented in [17]. While $p$ is nonsymmetric, a distinguishing feature of our model is that similar to classical approaches, only the symmetric part $\varepsilon_p := \operatorname{sym} p$ of the plastic distortion appears in the local Cauchy stress $\sigma$, while the higher order stresses are nonsymmetric. For more on the basic invariance questions related to this issue dictating this type of behavior, see [25, 18].

It is well known that classical viscoplasticity (without gradient effects) gives rise to a well-posed problem. We extend this result to our formulation of rate-dependent

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gradient plasticity. We allow our model both to include ($C_1 > 0$) or not to include ($C_1 = 0$) classical linear kinematic hardening. The presence of the nonlocal gradient term is always related to $C_2 > 0$.

**Setting of the problem.** Let $\Omega \subset \mathbb{R}^3$ be an open bounded set, the set of material points of the solid body, with a $C^1$-boundary. By $T_e$ we denote a positive number (time of existence), which can be chosen arbitrarily large, and for $0 < t \leq T_e$

$$\Omega_t = \Omega \times (0, t).$$

The sets $\mathcal{M}^3$ and $\mathcal{S}^3$ denote the sets of all $3 \times 3$ matrices and of all symmetric $3 \times 3$ matrices, respectively. Unknown in our small strain formulation are the displacement $u(x,t) \in \mathbb{R}^3$ of the material point $x$ at time $t$ and the nonsymmetric infinitesimal plastic distortion $p(x,t) \in \mathcal{M}^3$.

The model equations of the problem are

$$-\operatorname{div}_x \sigma(x,t) = b(x,t), \tag{1}$$

$$\sigma(x,t) = \mathbb{C}(\operatorname{sym}(\nabla_x u(x,t) - p(x,t))), \tag{2}$$

$$\partial_t p(x,t) \in g\big(\Sigma^{\text{lin}}(x,t)\big), \quad \Sigma^{\text{lin}} = \Sigma^{\text{lin}}_e + \Sigma^{\text{lin}}_{\text{sh}} + \Sigma^{\text{lin}}_{\text{curl}}, \tag{3}$$
$$\Sigma^{\text{lin}}_{\text{e}} = \sigma, \ \Sigma^{\text{lin}}_{\text{sh}} = -C_1 \operatorname{dev}\operatorname{sym} p, \quad \Sigma^{\text{lin}}_{\text{curl}} = -C_2 \operatorname{Curl}\operatorname{Curl} p,$$

which must be satisfied in $\Omega \times [0, T_e)$. Here, $C_1, C_2 \geq 0$ are given material constants and $\Sigma^{\text{lin}}$ is the infinitesimal Eshelby stress tensor driving the evolution of the plastic distortion $p$. The initial condition and Dirichlet boundary condition are

$$p(x,0) = p^{(0)}(x), \qquad x \in \Omega, \tag{4}$$

$$\operatorname{Curl} p(x,t) \times \tau(x) = 0, \qquad (x,t) \in \partial\Omega \times [0, T_e), \tag{5}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0, T_e), \tag{6}$$

where $\tau$ is a normal vector on the boundary $\partial\Omega$. For simplicity we consider only the homogeneous boundary condition. The elasticity tensor $\mathbb{C} : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, positive definite mapping. Classical linear kinematic hardening is included for $C_1 > 0$. Here, the nonlocal backstress contribution is given by the dislocation density motivated term $\Sigma^{\text{lin}}_{\text{curl}} = -C_2 \operatorname{Curl}\operatorname{Curl} p$ together with corresponding Neumann conditions.

For the model we require that the nonlinear constitutive mapping $g : \mathcal{M}^3 \to 2^{\mathcal{M}^3}$ is monotone,[1] i.e., it satisfies

$$0 \in g(0), \tag{7}$$

$$0 \leq (v_1 - v_2) \cdot (v_1^* - v_2^*) \tag{8}$$

for all $v_i \in \mathbb{R}^{3\times 3}$, $v_i^* \in g(v_i)$, $i = 1, 2$. Later on we restrict our attention to the subdifferential case $g = \partial\phi$. The case of monotone functions $g$, not necessarily having the subdifferential structure, will be treated separately in [16]. Given are the volume force $b(x,t) \in \mathbb{R}^3$ and the initial datum $p^{(0)}(x) \in \mathcal{M}^3$.

[1] Here $2^{\mathcal{M}^3}$ denotes the power set of $\mathcal{M}^3$.

**Notation.** Throughout the work we choose the numbers $q, q^*$ satisfying the conditions

$$1 < q, q^* < \infty \quad \text{and} \quad 1/q + 1/q^* = 1,$$

and $|\cdot|$ denotes a norm in $\mathbb{R}^k$, $k \in \mathbb{N}$. Moreover, the following notation is used in this work. The space $W^{m,q}(\Omega, \mathbb{R}^k)$ with $q \in [1, \infty]$ consists of all functions in $L^q(\Omega, \mathbb{R}^k)$ with weak derivatives in $L^q(\Omega, \mathbb{R}^k)$ up to order $m$. If $m$ is not integer, then $W^{m,q}(\Omega, \mathbb{R}^k)$ denotes the corresponding Sobolev–Slobodecki space. We set $H^m(\Omega, \mathbb{R}^k) = W^{m,2}(\Omega, \mathbb{R}^k)$. The norm in $W^{m,q}(\Omega, \mathbb{R}^k)$ is denoted by $\|\cdot\|_{m,q,\Omega}$ ($\|\cdot\|_q := \|\cdot\|_{0,q,\Omega}$). The operator $\Gamma_0$ defined by

$$\Gamma_0 : W^{1,q}(\Omega, \mathbb{R}^k) \to W^{1-1/q,q}(\partial\Omega, \mathbb{R}^k)$$

denotes the usual trace operator. The space $W_0^{m,q}(\Omega, \mathbb{R}^k)$ with $q \in [1, \infty]$ consists of all functions $v$ in $W^{m,q}(\Omega, \mathbb{R}^k)$ with $\Gamma_0 v = 0$. One can define the bilinear form on the product space $L^q(\Omega, \mathcal{M}^3) \times L^{q^*}(\Omega, \mathcal{M}^3)$ by

$$(\xi, \zeta)_\Omega = \int_\Omega \xi(x) \cdot \zeta(x) dx.$$

On $L^q(\Omega, \mathcal{S}^3) \times L^{q^*}(\Omega, \mathcal{S}^3)$ we define another bilinear form involving the elasticity tensor $\mathbb{C}$ by

$$[\xi, \zeta]_\Omega = (\mathbb{C}\xi, \zeta)_\Omega.$$

The space

$$L^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3) = \{v \in L^q(\Omega, \mathcal{M}^3) \mid \operatorname{Curl} v \in L^q(\Omega, \mathcal{M}^3)\}$$

is a Banach space with respect to the norm

$$\|v\|_{q,\mathrm{Curl}} = \|v\|_q + \|\operatorname{Curl} v\|_q.$$

The well known result on the generalized trace operator can be easily adapted to the functions with values in $\mathcal{M}^3$ (see [24, section II.1.2]). Then, according to this result, there is a bounded operator $\Gamma_\tau$ on $L^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$

$$\Gamma_\tau : L^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3) \to \left(W^{1-1/q^*,q^*}(\partial\Omega, \mathcal{M}^3)\right)^*$$

with

$$\Gamma_\tau v = v \times \tau\big|_{\partial\Omega} \text{ if } v \in C^1(\bar{\Omega}, \mathcal{M}^3),$$

where $X^*$ denotes the dual of a Banach space $X$. Moreover, analogically to the derivation of the generalized Green formula in [24, section II.1.2] one can obtain that the generalized Stokes formula

$$(9) \qquad \big(v, \operatorname{Curl} \phi\big)_\Omega = \big(\operatorname{Curl} v, \phi\big)_\Omega + \big(\Gamma_\tau v, \Gamma_0 \phi\big)_{\partial\Omega}$$

holds for all $v \in L^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$ and $\phi \in W^{1,q^*}(\Omega, \mathcal{M}^3)$. Here $(\cdot,\cdot)_{\partial\Omega}$ denotes the duality pairing between $W^{1-1/q^*,q^*}(\partial\Omega, \mathcal{M}^3)$ and $\left(W^{1-1/q^*,q^*}(\partial\Omega, \mathcal{M}^3)\right)^*$. Next,

$$L^q_{\mathrm{Curl},0}(\Omega, \mathcal{M}^3) = \{w \in L^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3) \mid \Gamma_\tau(w) = 0\}.$$

Let us define a space $V^q(\Omega, \mathbb{R}^k)$ by

$$V^q(\Omega, \mathcal{M}^3) = \{v \in L^q(\Omega, \mathcal{M}^3) \mid \operatorname{div} v, \operatorname{Curl} v \in L^q(\Omega, \mathcal{M}^3), \Gamma_\tau v = 0\},$$

which is a Banach space with respect to the norm

$$\|v\|_{V^q} = \|v\|_q + \|\operatorname{Curl} v\|_q + \|\operatorname{div} v\|_q.$$

According to [12, Theorem 2][2] the space $V^q(\Omega, \mathcal{M}^3)$ is continuously embedded into $W^{1,q}(\Omega, \mathcal{M}^3)$. In the case of a bounded domain $\Omega$ with a connected smooth boundary the embedding holds with the estimate [12, Theorem 2]

$$\|v\|_q + \|\nabla v\|_q \le C(\|\operatorname{Curl} v\|_q + \|\operatorname{div} v\|_q), \tag{10}$$

where $C = C(\Omega, q)$ (see the appendix). We also define the space $V^q_{\operatorname{Curl}}(\Omega, \mathcal{M}^3)$ by

$$V^q_{\operatorname{Curl}}(\Omega, \mathcal{M}^3) = \{v \in L^q(\Omega, \mathcal{M}^3) \mid \operatorname{Curl} v \in V^q(\Omega, \mathcal{M}^3)\},$$

which is a Banach space with respect to the norm

$$\|v\|_{V^q_{\operatorname{Curl}}} = \|v\|_{V^q} + \|\operatorname{Curl}\operatorname{Curl} v\|_q.$$

For functions $v$ defined on $\Omega \times [0, \infty)$ we denote by $v(t)$ the mapping $x \mapsto v(x,t)$, which is defined on $\Omega$. The space $L^q(0, T_e; X)$ denotes the Banach space of all Bochner-measurable functions $u : [0, T_e) \to X$ such that $t \mapsto \|u(t)\|^q_X$ is integrable on $[0, T_e)$. Finally, we frequently use the spaces $W^{m,q}(0, T_e; X)$, which consist of Bochner measurable functions having $q$-integrable weak derivatives up to order $m$.

**Main result.** In this work we restrict our attention to monotone functions $g$ which are given by the subdifferential of a proper convex lower semicontinuous function, i.e.,

$$g = \partial\phi,$$

where $\phi : \mathcal{M}^3 \to \bar{\mathbb{R}}$ ($\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$) is a proper convex lower semicontinuous function. (See section 3 for basics on convex analysis.) We also require that the function $\phi$ satisfy the two-sided estimate

$$a_0|v|^q - b_0 \le \phi(v) \le a_1|v|^q + b_1 \tag{11}$$

for positive $a_0$ and $a_1$, some $b_0$ and $b_1$, and any $v \in \mathcal{M}^3$. The last condition on $\phi$ implies that

$$\phi^*(v) \ge c|v|^{q^*} - d$$

for positive $c$, some $d$, and any $v \in \mathcal{M}^3$, where $\phi^*$ is the Legendre–Fenchel conjugate of $\phi$. Note that $q = 1$ is excluded in assumption (11); it would correspond to a rate-independent model.

Next, we present some intuitive ideas which lead to the definition of weak solutions for the initial boundary value problem (1)–(6). Let us assume that the functions

[2]This theorem has to be applied to each row of a function with values in $\mathcal{M}^3$ to obtain the desired result.

$(u,\sigma,p)$ have the regularities

$$(u,\sigma)\in W^{1,1}(0,T_e;H_0^1(\Omega,\mathbb{R}^3)\times L^2(\Omega,\mathcal{S}^3)),$$

$$p\in W^{1,1}(0,T_e;V^2_{\mathrm{Curl}}(\Omega,\mathcal{M}^3)),\quad \Sigma^{\mathrm{lin}}\in L^q(\Omega_{T_e},\mathcal{M}^3);$$

for every $t\in[0,T_e]$ the function $(u(t),\sigma(t))$ is a weak solution of the boundary value problem formed by (1)–(2) and (6) with the given $\operatorname{sym} p(t)$, and the equations (3)–(5) are satisfied for a.e. $(x,t)$, and $b$ is smooth enough. Then, since the inclusion (3) holds pointwise, from the Young–Fenchel inequality (17), which is just equality in our case, we have that

$$\phi^*(\partial_t p(x,t))+\phi(\Sigma^{\mathrm{lin}}(x,t))=\Sigma^{\mathrm{lin}}(x,t)\cdot\partial_t p(x,t).$$

Integrating the last relation over $\Omega$ and taking into account (2) then yield for all $t$

$$\int_\Omega\big(\phi^*(\partial_t p(x,t))+\phi(\Sigma^{\mathrm{lin}}(x,t))\big)dx=\int_\Omega\Big(\sigma\cdot\big(\partial_t\operatorname{sym}(\nabla_x u)-\partial_t\mathbb{C}^{-1}\sigma\big)$$

$$-\partial_t p\cdot\big(C_1\operatorname{dev}\operatorname{sym}p+C_2\operatorname{Curl}\operatorname{Curl}p\big)\Big)dx. \tag{12}$$

If we now test (1) with the function $\partial_t u$, we obtain that for all $t$

$$\big(\sigma(t),\partial_t\operatorname{sym}(\nabla_x u(t))\big)_\Omega=(b(t),\partial_t u(t))_\Omega\,.$$

Then inserting the last identity into (12), using the boundary condition (5), and integrating the result over $(0,t)$ with $t\in(0,T_e)$ gives the inequality

$$\begin{aligned}&\frac12\int_\Omega\mathbb{C}^{-1}\sigma(x,t)\cdot\sigma(x,t)dx+C_1\|\operatorname{dev}\operatorname{sym}p(t)\|_2^2+C_2\|\operatorname{Curl}p(t)\|_2^2\\&+\int_0^t\int_\Omega\big(\phi^*(\partial_s p(x,s))+\phi(\Sigma^{\mathrm{lin}}(x,s))\big)\,dxds\le\int_0^t(b(s),\partial_s u(s))_\Omega\,ds\\&+\frac12\int_\Omega\mathbb{C}^{-1}\sigma^{(0)}(x)\cdot\sigma^{(0)}(x)dx+C_1\|\operatorname{dev}\operatorname{sym}p^{(0)}\|_2^2+C_2\|\operatorname{Curl}p^{(0)}\|_2^2,\end{aligned}\tag{13}$$

which is satisfied for all $t\in(0,T_e)$. Here, the function $\sigma^{(0)}\in L^2(\Omega,\mathcal{S}^3)$ is determined by (1)–(2) and (6) for given $p^{(0)}$ and $b(0)$. Obviously, $\sigma^{(0)}=\sigma(0)$.

Conversely, it can be shown that if the functions $(u,\sigma,p)$, which enjoy the inequality (13) with appropriately chosen $\sigma^{(0)}$, are such that the function $(u(t),\sigma(t))$ is a weak solution of the boundary value problem (14)–(16) for every $t\in[0,T_e]$ with $\hat\varepsilon_p=\operatorname{sym}p(t)$, $\hat b=b(t)$, and have the above regularities and the initial value of $p$ is equal to $p^{(0)}$, then these functions are a solution of the initial boundary value problem (1)–(6). Indeed, if $\sigma^{(0)}$ is taken as the solution of the elliptic problem formed by (1)–(2) and (6) for given $p^{(0)}$ and $b(0)$, we get that $\sigma^{(0)}=\sigma(0)$ and the following identities hold:

$$\big(\mathbb{C}^{-1}\sigma(t),\sigma(t)\big)_\Omega-\big(\mathbb{C}^{-1}\sigma^{(0)},\sigma^{(0)}\big)_\Omega=\int_{\Omega_t}\frac{\partial}{\partial s}\big(\mathbb{C}^{-1}\sigma(x,s)\cdot\sigma(x,s)\big)\,dsdx,$$

$$\big\|\operatorname{dev}\operatorname{sym}p(t)\big\|_2^2-\big\|\operatorname{dev}\operatorname{sym}p^{(0)}\big\|_2^2=\int_0^t\frac{\partial}{\partial s}\|\operatorname{dev}\operatorname{sym}p(s)\|_2^2ds,$$

and

$$\|\operatorname{Curl} p(t)\|_2^2 - \|\operatorname{Curl} p^{(0)}\|_2^2 = \int_0^t \frac{\partial}{\partial s} \|\operatorname{Curl} p(s)\|_2^2 ds.$$

Then, the inequality (13) can be rewritten as follows:

$$\int_{\Omega_t} \Big( \mathbb{C}^{-1} \partial_s \sigma \cdot \sigma + \operatorname{dev} \operatorname{sym} p \cdot \partial_s p + C_2 (\operatorname{Curl} \partial_s p \cdot \operatorname{Curl} p) \\ + \phi^*(\partial_s p(x,s)) + \phi(\Sigma^{\text{lin}}(x,s)) \Big) ds dx \le \int_{\Omega_t} b \cdot (\partial_s u) ds dx.$$

Handling (1)–(2) as above we obtain that the last inequality takes the form

$$\int_{\Omega_t} \Big( \operatorname{dev} \operatorname{sym} p \cdot \partial_s p + C_2 (\operatorname{Curl} \operatorname{Curl} p \cdot \partial_s p) + \phi^*(\partial_s p(x,s)) + \ \phi(\Sigma^{\text{lin}}(x,s)) \Big) ds dx \\ \le \int_{\Omega_t} \sigma(x,s) \cdot \partial_s p(x,s) ds dx$$

or, equivalently,

$$\int_{\Omega_t} \Big( \phi^*(\partial_s p(x,s)) + \phi(\Sigma^{\text{lin}}(x,s)) \Big) ds dx \le \int_{\Omega_t} \Sigma^{\text{lin}}(x,s) \cdot \partial_s p(x,s) ds dx.$$

Therefore, by the equivalence result for convex integrands in section 3 we may conclude that the inclusion (3) is satisfied pointwise. The pointwise meaning of (4) and (5) follows from the assumed regularity of $(u, \sigma, p)$.

We note that the inequality (13) has a pointwise meaning under less regularity assumptions on functions $(u, \sigma, p)$ as in the above computations, where we have shown the equivalence between the pointwise inclusion (3) and the inequality (13) for a certain class of functions. Therefore, the above computations suggest the following notion of weak solutions for the initial boundary value problem (1)–(6).

Definition 1.1. *A function* $(u, \sigma, p)$ *such that*

$$(u, \sigma) \in W^{1,q^*}(0, T_e; W_0^{1,q^*}(\Omega, \mathbb{R}^3) \times L^{q^*}(\Omega, \mathcal{S}^3)), \quad \Sigma^{\text{lin}} \in L^q(\Omega_{T_e}, \mathcal{M}^3), \\ p \in W^{1,q^*}(0, T_e; L^{q^*}(\Omega, \mathcal{M}^3)) \cap L^{q^*}(0, T_e; V^{q^*}_{\text{Curl}}(\Omega, \mathcal{M}^3))$$

*with*

$$(\sigma, \operatorname{dev} \operatorname{sym} p, \operatorname{Curl} p) \in L^\infty(0, T_e; L^2(\Omega, \mathcal{S}^3 \times \mathcal{M}^3 \times \mathcal{M}^3))$$

*is called a* weak solution *of the initial boundary value problem* (1)–(6) *if for every* $t \in [0, T_e]$ *the function* $(u(t), \sigma(t))$ *is a weak solution of the boundary value problem* (14)–(16) *with* $\hat{\varepsilon}_p = \operatorname{sym} p(t)$ *and* $\hat{b} = b(t)$*, the initial condition* (4) *is satisfied, and the inequality* (13) *holds with the function* $\sigma^{(0)} \in L^2(\Omega, \mathcal{S}^3)$ *determined by* (14)–(16) *for* $\hat{\varepsilon}_p = \operatorname{sym} p^{(0)}$ *and* $\hat{b} = b(0)$.

Next, we state the main result of this work.

Theorem 1.2. *Suppose that* $1 < q^* \le 2 \le q < \infty$*. Assume that* $\Omega \subset \mathbb{R}^3$ *is an open bounded set with a* $C^1$*-boundary. Let the functions* $b \in W^{1,q}(0, T_e; L^q(\Omega, \mathbb{R}^3))$ *and* $p^{(0)} \in L^2_{\text{Curl}}(\Omega, \mathcal{M}^3)$ *be given. Assume that the function* $\phi : \mathcal{M}^3 \to \bar{\mathbb{R}}$ *satisfies condition* (11)*. Then there exists a weak solution* $(u, \sigma, p)$ *of the initial boundary value problem* (1)–(6).

*Remark* 1.3. Viscoplasticity is typically included in the former conditions by choosing the function $g$ to be in Norton–Hoff form, i.e.,

$$g(\Sigma) = [|\Sigma| - \sigma_{\mathrm{y}}]_+^r \frac{\Sigma}{|\Sigma|}, \quad \Sigma \in \mathcal{M}^3,$$

where $\sigma_{\mathrm{y}}$ is the flow stress and $r$ is some parameter together with $[x]_+ := \max(x, 0)$. If $g : \mathcal{M}^3 \to \mathcal{S}^3$, then the flow is called irrotational (no plastic spin).

**2. The Helmholtz projection on tensor fields.** In this section we present some auxiliary material without proofs concerning projection operators to spaces of tensor fields, which are symmetric gradients, and to spaces of tensor fields with vanishing divergence. For details see [2].

In the linear elasticity theory it is well known (see [7, Theorem 10.15]) that a Dirichlet boundary value problem formed by the equations

$$-\operatorname{div}_x \sigma(x) = \hat{b}(x), \qquad x \in \Omega, \tag{14}$$

$$\sigma(x) = \mathbb{C}(\operatorname{sym}(\nabla_x u(x)) - \hat{\varepsilon}_p(x)), \qquad x \in \Omega, \tag{15}$$

$$u(x) = 0, \qquad x \in \partial\Omega, \tag{16}$$

to given $\hat{b} \in W^{-1,q}(\Omega, \mathbb{R}^3)$ and $\hat{\varepsilon}_p \in L^q(\Omega, \mathcal{S}^3)$ has a unique weak solution $(u, \sigma) \in W_0^{1,q}(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathcal{S}^3)$ provided the open set $\Omega$ has a $C^1$-boundary. Here the number $q$ satisfies $1 < q < \infty$. For $\hat{b} = 0$ the solution of (14)–(16) satisfies the inequality

$$\|\operatorname{sym}(\nabla_x u)\|_q \le C \|\hat{\varepsilon}_p\|_q$$

with some positive constant $C$.

DEFINITION 2.1. *For every $\hat{\varepsilon}_p \in L^q(\Omega, \mathcal{S}^3)$ we define a linear operator $P_q : L^q(\Omega, \mathcal{S}^3) \to L^q(\Omega, \mathcal{S}^3)$ by*

$$P_q \hat{\varepsilon}_p := \operatorname{sym}(\nabla_x u),$$

*where $u \in W_0^{1,q}(\Omega, \mathbb{R}^3)$ is the unique weak solution of* (14)–(16) *to the given function $\hat{\varepsilon}_p$ and $\hat{b} = 0$.*

Next, a subset $\mathcal{G}^q$ of $L^q(\Omega, \mathcal{S}^3)$ is defined by

$$\mathcal{G}^q = \{\operatorname{sym}(\nabla_x u) \mid u \in W_0^{1,q}(\Omega, \mathbb{R}^3)\}.$$

The main properties of $P_q$ are stated in the following lemma.

LEMMA 2.2. *The operator $P_q$ is a bounded projector onto the subset $\mathcal{G}^q$ of $L^q(\Omega, \mathcal{S}^3)$. The projector $(P_q)^*$ adjoint with respect to the bilinear form $[\xi, \zeta]_\Omega := (\mathbb{C}\xi, \zeta)_\Omega$ on $L^q(\Omega, \mathcal{S}^3) \times L^{q^*}(\Omega, \mathcal{S}^3)$ satisfies*

$$(P_q)^* = P_{q^*}.$$

*This implies* $\ker(P_q) = H^q_{sol}$ *with*

$$H^q_{sol} = \{\xi \in L^q(\Omega, \mathcal{S}^3) \mid [\xi, \zeta]_\Omega = 0 \quad \textit{for all } \zeta \in \mathcal{G}^{q^*}\}.$$

Since $\mathbb{C}$ is symmetric, the relation $[\xi, \zeta]_\Omega = 0$ holds for all $\zeta \in \mathcal{G}^{q^*}$ iff

$$(\mathbb{C}\xi, \nabla_x v)_\Omega = (\mathbb{C}\xi, \operatorname{sym}(\nabla_x v))_\Omega = [\xi, \operatorname{sym}(\nabla_x v)]_\Omega = 0$$

for all $v \in W_0^{1,q^*}(\Omega, \mathbb{R}^3)$. Consequently

$$H^q_{sol} = \{\xi \in L^q(\Omega, \mathcal{S}^3) \mid \operatorname{div}(\mathbb{C}\xi) = 0\}.$$

Therefore, the projection operator

$$Q_q = (I - P_q) : L^p(\Omega, \mathcal{S}^3) \to L^p(\Omega, \mathcal{S}^3)$$

with $Q_q(L^q(\Omega, \mathcal{S}^3)) = H^q_{sol}$ is a generalization of the classical Helmholtz projection.

Let $L : \mathcal{S}^3 \to \mathcal{S}^3$ be the linear, positive semidefinite mapping given by

$$Lv = C_1 \operatorname{dev} v.$$

The next result is needed for the subsequent analysis.

COROLLARY 2.3. *Let $(\mathbb{C}P_q + L)^*$ be the operator adjoint to $\mathbb{C}P_q + L : L^q(\Omega, \mathcal{S}^3) \to L^q(\Omega, \mathcal{S}^3)$ with respect to the bilinear form $(\xi, \zeta)_\Omega$ on the product space $L^q(\Omega, \mathcal{S}^3) \times L^{q^*}(\Omega, \mathcal{S}^3)$. Then $(\mathbb{C}P_q + L)^* = \mathbb{C}P_{q^*} + L$. Moreover, the operator $\mathbb{C}Q_2 + L$ is nonnegative and self-adjoint.* For the proof of this result see [1].

**3. Basic facts from convex analysis.** In this section we briefly recall some basic facts about convex functions, their subdifferentials, and the surjectivity results for them.

Let $V$ be a reflexive Banach space with the norm $\|\cdot\|$ and let $V^*$ be its dual space with the norm $\|\cdot\|_*$. The brackets $\langle\cdot,\cdot\rangle$ denote the duality pairing between $V$ and $V^*$. Throughout this section, by $V$ we mean a reflexive Banach space.

For a function $\phi : V \to \bar{\mathbb{R}}$ the sets

$$\operatorname{dom}(\phi) = \{v \in V \mid \phi(v) < \infty\}, \quad \operatorname{epi}(\phi) = \{(v,t) \in V \times \mathbb{R} \mid \phi(v) \le t\}$$

are called the *effective domain* and the *epigraph* of $\phi$, respectively. One says that the function $\phi$ is *proper* if $\operatorname{dom}(\phi) \neq \emptyset$ and $\phi(v) > -\infty$ for every $v \in V$. The epigraph is a nonempty closed convex set iff $\phi$ is a proper lower semicontinuous convex function or, equivalently, iff $\phi$ is a proper weakly lower semicontinuous convex function (see [26, Theorem 2.2.1]).

The Legendre–Fenchel conjugate of a proper convex lower semicontinuous function $\phi : V \to \bar{\mathbb{R}}$ is the function $\phi^*$ defined for each $v^* \in V^*$ by

$$\phi^*(v^*) = \sup_{v \in V} \{\langle v^*, v\rangle - \phi(v)\}.$$

The Legendre–Fenchel conjugate $\phi^*$ is convex, lower semicontinuous, and proper on the dual space $V^*$. Moreover, the *Young–Fenchel inequality* holds

$$\forall v \in V,\ \forall v^* \in V^* : \quad \phi^*(v^*) + \phi(v) \ge \langle v^*, v\rangle\,, \tag{17}$$

and the inequality $\phi \le \psi$ implies $\psi^* \le \phi^*$ for any two proper convex lower semicontinuous functions $\psi, \phi : V \to \bar{\mathbb{R}}$ (see [26, Theorem 2.3.1]).

Due to Proposition II.2.5 in [3] a proper convex lower semicontinuous function $\phi$ satisfies the identity

$$\operatorname{int}\operatorname{dom}(\phi) = \operatorname{int}\operatorname{dom}(\partial\phi), \tag{18}$$

where $\partial\phi : V \to 2^{V^*}$ denotes the subdifferential of the function $\phi$. We note that the equality in (17) holds iff $v^* \in \partial\phi(v)$.

*Remark* 3.1. We recall that the subdifferential of a lower semicontinuous proper and convex function is maximal monotone[3] (see [3, Theorem II.2.1]).

The next surjectivity result on subdifferentials of convex functions is one of the key tools in the proof of our main existence result.

THEOREM 3.2. *Let $A := \partial\phi$, where $\phi : V \to \bar{\mathbb{R}}$ is a proper convex lower semicontinuous function on $V$. Then the following conditions are equivalent:*[4]

$$\lim_{\|v\|\to\infty} \frac{\phi(v)}{\|v\|} = \infty; \tag{19}$$

$$R(A) = V^* \text{ and } A^{-1} \text{ is bounded.} \tag{20}$$

*Proof.* See [3, Theorem II.2.6], for example. ∎

To state our next result, we recall that the relation

$$\partial\phi + \partial\psi = \partial(\phi + \psi)$$

holds for any two convex functions $\psi$ and $\phi$ if there exists a point in $\mathrm{dom}(\phi)\cap\mathrm{dom}(\psi)$, where $\phi$ is continuous (see [23, Proposition II.7.7]). Then, since a proper convex lower semicontinuous function is continuous on the interior of its domain [3, Proposition II.2.2], we get the following important result.

PROPOSITION 3.3. *Let $\phi$ be a proper convex lower semicontinuous function and $\psi$ be convex. Suppose that*

$$\big(\mathrm{int}\,\mathrm{dom}(\phi)\big) \cap \mathrm{dom}(\psi) \neq \emptyset. \tag{21}$$

*Then*

$$\partial\phi + \partial\psi = \partial(\phi + \psi).$$

Now we turn to some examples of convex functions which will show up in the next section.

**Convex integrands.** For a proper convex lower semicontinuous function $\phi : \mathbb{R}^k \to \bar{\mathbb{R}}$ we define a functional $I_\phi$ on $L^q(\Omega, \mathbb{R}^k)$ by

$$I_\phi(v) = \begin{cases} \int_\Omega \phi(v(x))dx, & \phi(v) \in L^1(\Omega, \mathbb{R}^k), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with some $N \in \mathbb{N}$. Due to Proposition II.8.1 in [23] the functional $I_\phi$ is proper, convex, lower semicontinuous, and $v^* \in \partial I_\phi(v)$ iff

$$v^* \in L^{q^*}(\Omega, \mathbb{R}^k), \quad v \in L^q(\Omega, \mathbb{R}^k), \quad \text{and} \quad v^*(x) \in \partial\phi(v(x)) \text{ a.e.}$$

Due to the result of Rockafellar in [21, Theorem 2] the Legendre–Fenchel conjugate of $I_\phi$ is equal to $I_{\phi^*}$, i.e.,

$$\big(I_\phi\big)^* = I_{\phi^*},$$

where $\phi^*$ is the Legendre–Fenchel conjugate of $\phi$.

[3]A monotone mapping $A : V \to 2^{V^*}$ is called maximal monotone iff the inequality

$$\langle v^* - u^*, v - u\rangle \geq 0 \quad \forall\, u \in V \text{ and } u^* \in A(u)$$

implies $v^* \in A(v)$.

[4]Here $R(A)$ denotes the range of the operator $A$.

**Boundary value problems.** Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a Lipschitz boundary. For every $v \in L^2(\Omega, \mathcal{M}^3)$ we define a functional $\Psi$ on $L^2(\Omega, \mathcal{M}^3)$ by

$$\Psi(v) = \begin{cases} \frac{1}{2}\int_\Omega |\operatorname{Curl} v(x)|^2 dx, & v \in L^2_{\operatorname{Curl}}(\Omega, \mathcal{M}^3), \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to check that $\Psi$ is proper, convex, and lower semicontinuous. The next lemma gives a precise description of the subdifferential $\partial\Psi$.

Lemma 3.4. *We have that* $\partial\Psi = \operatorname{Curl}\operatorname{Curl}$ *with*

$$\operatorname{dom}(\partial\Psi) = V^2_{\operatorname{Curl}}(\Omega, \mathcal{M}^3).$$

*Proof.* Let $A : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3)$ be the linear operator defined by

$$Av = \operatorname{Curl}\operatorname{Curl} v$$

and $\operatorname{dom}(A) = V^2_{\operatorname{Curl}}(\Omega, \mathcal{M}^3)$. Due to Lemma 4.1, the identity

$$\int_\Omega \operatorname{Curl}\operatorname{Curl} v(x) \cdot w(x) dx = \int_\Omega \operatorname{Curl} v(x) \cdot \operatorname{Curl} w(x) dx \tag{22}$$

holds for any $v, w \in V^2_{\operatorname{Curl}}(\Omega, \mathcal{M}^3)$. Therefore, using (22) we obtain

$$\int_\Omega \operatorname{Curl}\operatorname{Curl} v \cdot (w - v) dx = \int_\Omega \operatorname{Curl} v \cdot (\operatorname{Curl} w - \operatorname{Curl} v) dx \leq \Psi(w) - \Psi(v)$$

for every $v, w \in \operatorname{dom}(A)$. This shows that $A \subset \partial\Psi$. Since $A$ is maximal monotone (see Corollary 4.3) we conclude that $A = \partial\Psi$. □

**4. Some properties of the Curl Curl-operator.** In this section we collect some properties of the Curl Curl-operator, which are relevant to further investigations.

Lemma 4.1 (self-adjointness of Curl Curl). *Let* $\Omega \subset \mathbb{R}^3$ *be an open bounded set with a Lipschitz boundary and* $A : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3)$ *be the linear operator defined by*

$$Av = \operatorname{Curl}\operatorname{Curl} v$$

*with* $\operatorname{dom}(A) = L^2_{\operatorname{Curl}}(\Omega, \mathcal{M}^3)$. *The operator* $A$ *is self-adjoint and nonnegative.*

*Proof.* Indeed, let us consider first the following linear closed operator $S : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3)$ defined by

$$Sv = \operatorname{Curl} v, \qquad v \in \operatorname{dom}(S) = L^2_{\operatorname{Curl}}(\Omega, \mathcal{M}^3).$$

It is easily seen that its adjoint is given by

$$S^*v = \operatorname{Curl} v, \qquad v \in \operatorname{dom}(S^*) = L^2_{\operatorname{Curl},0}(\Omega, \mathcal{M}^3).$$

Then, by Theorem V.3.24 in [11], the operator $S^*S$ with

$$\operatorname{dom}(S^*S) = \{v \in \operatorname{dom}(S) \mid Sv \in \operatorname{dom}(S^*)\},$$

which is exactly the operator $A$, is self-adjoint in $L^2(\Omega, \mathcal{M}^3)$. The nonnegativity of $A$ follows from its representation by the operator $S$, i.e., $A = S^*S$, and the identity

$$(Av, u)_\Omega = (S^*Sv, u)_\Omega = (Sv, Su)_\Omega,$$

which holds for all $v \in \operatorname{dom}(A)$ and $u \in \operatorname{dom}(S)$. □

*Remark* 4.2. We note that the nonnegativity of $A$ defined in Lemma 4.1 can be shown directly by using the generalized Stokes formula (9) and the density of the space $C^\infty(\bar{\Omega}, \mathcal{M}^3)$ in $L^2_{\text{Curl}}(\Omega, \mathcal{M}^3)$ (see [5, Lemma VII.4.1]).

COROLLARY 4.3. *The operator* $A : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3)$ *defined in Lemma* 4.1 *is maximal monotone.*

*Proof.* According to the result of Brezis (see [4, Theorem 1]), a linear monotone operator $A$ is maximal monotone if it is a densely defined closed operator such that its adjoint $A^*$ is monotone. The statement of the corollary then follows directly from Lemma 4.1 and the result of Brezis. □

In the following lemma we prove that the Curl Curl-operator is closed as an operator acting from $L^q(\Omega, \mathcal{M}^3)$ into $L^q(\Omega, \mathcal{M}^3)$. This result is irrelevant to further investigations and is stated here only for completeness. In contrast to Lemma 4.1, where the closedness of the Curl Curl-operator as an operator from $L^2(\Omega, \mathcal{M}^3)$ into $L^2(\Omega, \mathcal{M}^3)$ follows directly from its self-adjointness, to show the similar result for the Curl Curl-operator as an operator from $L^q(\Omega, \mathcal{M}^3)$ into $L^q(\Omega, \mathcal{M}^3)$ we require some additional regularity of the open bounded set $\Omega$ in order to apply the results in [12, Theorem 2].

LEMMA 4.4 (closedness of Curl Curl). *Let* $\Omega \subset \mathbb{R}^3$ *be a domain with a smooth connected boundary. Then the operator* $A : \text{dom}(A) \subset L^q(\Omega, \mathcal{M}^3) \to L^q(\Omega, \mathcal{M}^3)$ *given by*

$$Av(x) := \operatorname{Curl}\operatorname{Curl} v(x) \quad \textit{with} \quad \text{dom}(A) = V^q_{\text{Curl}}(\Omega, \mathcal{M}^3)$$

*is closed.*

*Proof.* Let $v_n \in D(A)$ be a sequence such that $v_n$ and $Av_n$ converge strongly in $L^q(\Omega, \mathcal{M}^3)$ to $v_0$ and $y_0$, respectively. The function $w_n := \operatorname{Curl} v_n \in V^q(\Omega, \mathcal{M}^3)$ solves the problem

$$\begin{aligned} \operatorname{Curl} w_n(x) &= f_n(x), && x \in \Omega, \\ \operatorname{div} w_n(x) &= 0, && x \in \Omega, \\ \Gamma_\tau w_n &= 0, \end{aligned}$$

where $f_n := Av_n$. Since the boundary $\partial\Omega$ is connected, by inequality (10) the function $w_n \in V^q(\Omega, \mathcal{M}^3)$ satisfies the inequality

$$\|w_n\|_q + \|\nabla w_n\|_q \le C(\|\operatorname{Curl} w_n\|_q + \|\operatorname{div} w_n\|_q) = C\|f_n\|_q \tag{23}$$

with a constant $C$ independent of $w_n$. Inequality (23) yields that $w_n$ is a Cauchy sequence in $W^{1,q}(\Omega, \mathcal{M}^3)$ with the limit denoted by $w_0$. Therefore, $w_0$ belongs to $V^q(\Omega, \mathcal{M}^3)$ and solves the following problem:

$$\operatorname{Curl} w_0(x) = y_0(x), \qquad x \in \Omega, \tag{24}$$

$$\operatorname{div} w_0(x) = 0, \qquad x \in \Omega, \tag{25}$$

$$\Gamma_\tau w_0 = 0. \tag{26}$$

Since $v_n$ and $w_n$ converge strongly in $L^q(\Omega, \mathcal{M}^3)$ to $v_0$ and $w_0$, respectively, and $L^q_{\text{Curl}}(\Omega, \mathcal{M}^3)$ is a Banach space with respect to the norm $\|\cdot\|_{q,\text{Curl}}$, we conclude that

$$w_0 = \operatorname{Curl} v_0.$$

Equations (24)–(26) yield that $v_0 \in V^q_{\text{Curl}}(\Omega, \mathcal{M}^3)$. The proof of Lemma 4.4 is complete. □

**5. Existence of weak solutions.** In this section we prove the main existence result for (1)–(6). To show the existence of weak solutions a time-discretization method is used in this work. In the first step, we prove the existence of the solutions of the time-discretized problem in appropriate Hilbert spaces based on the Helmholtz projection in $L^2(\Omega, \mathcal{S}^3)$ (section 2) and the methods of convex analysis (section 3). In the second step, we derive the uniform a priori estimates for the solutions of the time-discretized problem using growth condition (11) and then we pass to the weak limit in the equivalent formulation of the time-discretized problem employing the weak lower semicontinuity of lower semicontinuous convex functions. We note that due to the uniform a priori estimates obtained here based on growth condition (11), some regularity of the solutions of the time-discretized problem is lost in the passage from discrete to continuous time.

*Proof of Theorem* 1.2. Let us recall that $1 < q^* \leq 2 \leq q < \infty$. We show the existence of weak solutions using the Rothe method (a time-discretization method; see [22] for details). In order to introduce a time-discretized problem, let us fix any $m \in \mathbb{N}$ and set

$$h := \frac{T_e}{2^m},\ p_m^0 := p^{(0)},\ b_m^n := \frac{1}{h}\int_{(n-1)h}^{nh} b(s)ds \in L^q(\Omega, \mathbb{R}^3), \quad n = 1, \ldots, 2^m.$$

Then we are looking for functions $u_m^n \in H^1(\Omega, \mathbb{R}^3)$, $\sigma_m^n \in L^2(\Omega, \mathcal{S}^3)$, and $p_m^n \in V^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$ with

$$\Sigma^{\mathrm{lin}}_{n,m} := \sigma_m^n - C_1 \operatorname{dev}\operatorname{sym} p_m^n - \frac{1}{m} p_m^n - C_2 \operatorname{Curl}\operatorname{Curl} p_m^n \in L^q(\Omega, \mathcal{M}^3)$$

solving the problem

$$-\operatorname{div}_x \sigma_m^n(x) = b_m^n(x), \tag{27}$$

$$\sigma_m^n(x) = \mathbb{C}(\mathrm{sym}(\nabla_x u_m^n(x) - p_m^n(x))), \tag{28}$$

$$\frac{p_m^n(x) - p_m^{n-1}(x)}{h} \in \partial\phi\big(\Sigma^{\mathrm{lin}}_{n,m}(x)\big), \tag{29}$$

together with the boundary conditions

$$\operatorname{Curl} p_m^n(x) \times \tau(x) = 0, \qquad x \in \partial\Omega, \tag{30}$$

$$u_m^n(x) = 0, \qquad x \in \partial\Omega. \tag{31}$$

Next, we adopt the reduction technique proposed in [1] to the above equations. Let $(u_m^n, \sigma_m^n, p_m^n)$ be a solution of the boundary value problem (27)–(31). The equations (27)–(28), (31) form a boundary value problem for the solution $(u_m^n, \sigma_m^n)$ of the problem of linear elasticity. Due to linearity of this problem we can write these components of the solution in the form

$$(u_m^n, \sigma_m^n) = (\tilde{u}_m^n, \tilde{\sigma}_m^n) + (\hat{v}_m^n, \hat{\sigma}_m^n)$$

with the solution $(\hat{v}_m^n, \hat{\sigma}_m^n)$ of the Dirichlet boundary value problem (14)–(16) to the data $\hat{b} = b_m^n$, $\hat{\varepsilon}_p = 0$, and with the solution $(\tilde{u}_m^n, \tilde{\sigma}_m^n)$ of the problem (14)–(16) to the data $\hat{b} = 0$, $\hat{\varepsilon}_p = \mathrm{sym}(p_m^n)$. We thus obtain

$$\mathrm{sym}(\nabla_x u_m^n) - \mathrm{sym}(p_m^n) = (P_2 - I)\mathrm{sym}(p_m^n) + \mathrm{sym}(\nabla_x \hat{v}_m^n).$$

We insert this equation into (28) and get that (29) can be rewritten in the form

$$\frac{p_m^n - p_m^{n-1}}{h} \in \partial I_\phi\big( - M_m p_m^n - C_2 \operatorname{Curl}\operatorname{Curl} p_m^n + \hat{\sigma}_m^n\big), \tag{32}$$

$$\operatorname{Curl} p_m^n(x) \times \tau(x) = 0, \qquad x \in \partial\Omega, \tag{33}$$

where

$$M_m := (\mathbb{C}Q_2 + L)\,\mathrm{sym} + \frac{1}{m} I : L^2(\Omega, \mathcal{M}^3) \to L^2(\Omega, \mathcal{M}^3)$$

with the Helmholtz projection $Q_2$ and $I_\phi$ is given by

$$I_\phi(v) = \begin{cases} \int_\Omega \phi(v(x))dx, & \phi(v) \in L^1(\Omega, \mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases}$$

We recall that the functional $I_\phi$ is proper, convex, and lower semicontinuous (see section 3). Since $(I_\phi)^* = I_{\phi^*}$, the problem (32)–(33) reads

$$\partial\Psi(p_m^n) \ni \hat{\sigma}_m^n, \tag{34}$$

where

$$\Psi(v) = I_{\phi^*}\left(\frac{v - p_m^{n-1}}{h}\right) + \Phi_1(v) + \Phi_2(v).$$

Here, the functional $\Phi_1 : L^2(\Omega, \mathcal{M}^3) \to \bar{\mathbb{R}}$ and the functional $\Phi_2 : L^2(\Omega, \mathcal{M}^3) \to \bar{\mathbb{R}}$ are given by

$$\Phi_1(v) := \frac{1}{2}\|M_m^{1/2} v\|_2^2 \quad \text{and} \quad \Phi_2(v) := \begin{cases} \frac{1}{2}\int_\Omega |\operatorname{Curl} v(x)|^2 dx, & v \in L^2_{\operatorname{Curl}}(\Omega, \mathcal{M}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

respectively. The facts that $\Phi_2$ is a proper convex lower semicontinuous functional and that $\operatorname{Curl}\operatorname{Curl} = \partial\Phi_2$ are proved in section 3. To see that $\Phi_1$ has similar properties as well, we note first that $M_m$ is a bounded and positive definite operator (see Corollary 2.3 and the definition of $M_m$). Thus, it is maximal monotone by Theorem II.1.3 in [3]. Since by Corollary 2.3 the operator $M_m$ is also self-adjoint, one has that $M_m = \partial\Phi_1$ by Proposition II.2.7 in [3]. All other properties of $\Phi_1$ follow from its definition. The last thing we have to verify is whether the relation

$$\partial\Psi = \partial I_{\phi^*} + \partial\Phi_1 + \partial\Phi_2$$

holds. By condition (11) and the definition of $\Phi_1$, we conclude that the domains of $I_{\phi^*}$ and $\Phi_1$ are equal to the whole space $L^2(\Omega, \mathcal{M}^3)$. Therefore, condition (21) is fulfilled and, since all functionals are proper, convex, and lower semicontinuous, Proposition 3.3 gives the desired result. With the relation (18) in hand, the last observation implies that

$$\operatorname{dom}(\partial\Psi) = \operatorname{dom}(\partial\Phi_2) := V^2_{\operatorname{Curl}}(\Omega, \mathcal{M}^3).$$

Since $\Phi_1$ is coercive in $L^2(\Omega, \mathcal{M}^3)$, which obviously yields the coercivity of $\Psi$, the operator $A = \partial\Psi$ is surjective by Theorem 3.2. Thus, we conclude that (34) as well as the problem (32)–(33) have the solutions with the required regularity, i.e., $p_m^n \in$

$V^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$. By the constructions this implies that the boundary value problem (27)–(31) is solvable as well (for more detail see [1]).

**Rothe approximation functions.** For any family $\{\xi^n_m\}_{n=0,\ldots,m}$ of functions in a reflexive Banach space $X$, we define *the piecewise affine interpolant* $\xi_m \in C([0, T_e], X)$ by

$$\xi_m(t) := \left(\frac{t}{h} - (n-1)\right)\xi^n_m + \left(n - \frac{t}{h}\right)\xi^{n-1}_m \quad \text{for } (n-1)h \le t \le nh \tag{35}$$

and *the piecewise constant interpolant* $\bar{\xi}_m \in L^\infty(0, T_e; X)$ by

$$\bar{\xi}_m(t) := \xi^n_m \text{ for } (n-1)h < t \le nh,\ n = 1, \ldots, 2^m, \text{ and } \bar{\xi}_m(0) := \xi^0_m. \tag{36}$$

For further analysis we recall the following property of $\bar{\xi}_m$ and $\xi_m$:

$$\|\xi_m\|_{L^q(0,T_e;X)} \le \|\bar{\xi}_m\|_{L^q(-h,T_e;X)} \le \Big(h\|\xi^0_m\|^q_X + \|\bar{\xi}_m\|^q_{L^q(0,T_e;X)}\Big)^{1/q}, \tag{37}$$

where $\bar{\xi}_m$ is formally extended to $t \le 0$ by $\xi^0_m$ and $1 \le q \le \infty$ (see [22]).

**A priori estimates.** Multiplying (27) by $(u^n_m - u^{n-1}_m)/h$ and then integrating over $\Omega$ we get

$$\big(\sigma^n_m, \mathrm{sym}(\nabla_x(u^n_m - u^{n-1}_m)/h)\big)_\Omega = \big(b^n_m, (u^n_m - u^{n-1}_m)/h\big)_\Omega.$$

The Fenchel property and (28), (29) yield for a.e. $x \in \Omega$

$$\begin{aligned}
&\sigma^n_m \cdot \Big(\mathrm{sym}(\nabla_x(u^n_m - u^{n-1}_m)/h) - \mathbb{C}^{-1}(\sigma^n_m - \sigma^{n-1}_m)/h\Big)\\
&\quad - \frac{p^n_m - p^{n-1}_m}{h} \cdot \Big(C_1 \,\mathrm{dev}\,\mathrm{sym}\, p^n_m + \frac{1}{m}p^n_m + C_2 \,\mathrm{Curl}\,\mathrm{Curl}\, p^n_m\Big)\\
&= \phi^*\left(\frac{p^n_m - p^{n-1}_m}{h}\right) + \phi\left(\Sigma^{\mathrm{lin}}_{n,m}\right).
\end{aligned}$$

After integrating the last identity over $\Omega$, the above computations imply

$$\begin{aligned}
&\frac{1}{h}\Big(\mathbb{C}^{-1}\sigma^n_m, \sigma^n_m - \sigma^{n-1}_m\Big)_\Omega + C_1\frac{1}{h}\Big(\mathrm{dev}\,\mathrm{sym}(p^n_m - p^{n-1}_m), \mathrm{dev}\,\mathrm{sym}\, p^n_m\Big)_\Omega\\
&\quad + \frac{1}{m}\frac{1}{h}\Big(p^n_m - p^{n-1}_m, p^n_m\Big)_\Omega + C_2\frac{1}{h}\Big(\mathrm{Curl}(p^n_m - p^{n-1}_m), \mathrm{Curl}\, p^n_m\Big)_\Omega\\
&\quad + \int_\Omega \phi^*\left(\frac{p^n_m - p^{n-1}_m}{h}\right)dx + \int_\Omega \phi(\Sigma^{\mathrm{lin}}_{n,m})dx = \frac{1}{h}\left(b^n_m, u^n_m - u^{n-1}_m\right)_\Omega.
\end{aligned}$$

Multiplying by $h$ and summing the obtained relation for $n = 1, \ldots, l$ for any fixed $l \in [1, 2^m]$ we derive the inequality (here $\mathbb{B} := \mathbb{C}^{-1}$)

$$\begin{aligned}
&\frac{1}{2}\left(\|\mathbb{B}^{1/2}\sigma^l_m\|^2_2 + C_1\|\,\mathrm{dev}\,\mathrm{sym}\, p^l_m\|^2_2 + \frac{1}{m}\|p^l_m\|^2_2 + C_2\|\,\mathrm{Curl}\, p^l_m\|^2_2\right)\\
&\quad + h\sum_{n=1}^{l}\int_\Omega \phi^*\left(\frac{p^n_m - p^{n-1}_m}{h}\right)dx + h\sum_{n=1}^{l}\int_\Omega \phi\left(\Sigma^{\mathrm{lin}}_{n,m}\right)dx\\
&\le C^{(0)} + h\sum_{n=1}^{l}\left(b^n_m, \frac{u^n_m - u^{n-1}_m}{h}\right)_\Omega,
\end{aligned} \tag{38}$$

where[5]

$$2C^{(0)} := \|\mathbb{B}^{1/2}\sigma^{(0)}\|_2^2 + C_1\|\operatorname{dev}\operatorname{sym} p^{(0)}\|_2^2 + \frac{1}{m}\|p^{(0)}\|_2^2 + C_2\|\operatorname{Curl} p^{(0)}\|_2^2.$$

We estimate now the right-hand side of the previous inequality. Since $u_m^n$ is a solution of the linear elliptic problem formed by (27)–(28) and (31), it satisfies (see [7, Theorem 10.15]) the inequality

$$\|u_m^n\|_{1,q^*,\Omega} \le C\big(\|b_m^n\|_{q^*} + \|p_m^n\|_{q^*}\big), \tag{39}$$

where $C$ is a positive constant independent of $n$ and $m$. Therefore, using the linearity of the problem formed by (27)–(28) and (31), the inequality (39), and Young's inequality with $\epsilon > 0$ we get that

$$\begin{aligned}\left(b_m^n, \frac{u_m^n - u_m^{n-1}}{h}\right)_\Omega &\le \|b_m^n\|_q \|(u_m^n - u_m^{n-1})/h\|_{1,q^*,\Omega} \le CC_\epsilon \|b_m^n\|_q^q \\ &\quad + C\epsilon\|(b_m^n - b_m^{n-1})/h\|_{q^*}^{q^*} + C\epsilon\|(p_m^n - p_m^{n-1})/h\|_{q^*}^{q^*},\end{aligned} \tag{40}$$

where $C_\epsilon$ is a positive constant appearing in the Young inequality. Combining the inequalities (38) and (40), applying the condition (11), and choosing an appropriate value for $\epsilon > 0$ we obtain the estimate

$$\begin{aligned}&\frac{1}{2}\left(\|\mathbb{B}^{1/2}\sigma_m^l\|_2^2 + C_1\|\operatorname{dev}\operatorname{sym} p_m^l\|_2^2 + \frac{1}{m}\|p_m^l\|_2^2 + C_2\|\operatorname{Curl} p_m^l\|_2^2\right) \\ &\quad + h\hat{C}_\epsilon \sum_{n=1}^{l} \int_\Omega \left|\frac{p_m^n - p_m^{n-1}}{h}\right|^{q^*} dx + h\tilde{C}\sum_{n=1}^{l}\int_\Omega |\Sigma_{n,m}^{\text{lin}}|^q dx \\ &\le C^{(0)} + h\tilde{C}_\epsilon \sum_{n=1}^{l}\left(\|b_m^n\|_q^q + \|(b_m^n - b_m^{n-1})/h\|_{q^*}^{q^*}\right),\end{aligned} \tag{41}$$

where $\tilde{C}, \tilde{C}_\epsilon$, and $\hat{C}_\epsilon$ are some positive constants. Now, taking Remark 8.15 in [22] and the definition of Rothe's approximation functions into account we rewrite (41) as follows:

$$\begin{aligned}&\|\mathbb{B}^{1/2}\bar{\sigma}_m(t)\|_2^2 + C_1\|\operatorname{dev}\operatorname{sym}\bar{p}_m(t)\|_2^2 + \frac{1}{m}\|\bar{p}_m(t)\|_2^2 + C_2\|\operatorname{Curl}\bar{p}_m(t)\|_2^2 \\ &\quad + 2\hat{C}_\epsilon \int_0^{T_e}\int_\Omega |\partial_t p_m(x,t)|^{q^*} dxdt + 2\tilde{C}\int_0^{T_e}\int_\Omega |\bar{\Sigma}_m^{\text{lin}}(x,t)|^q dxdt \\ &\le 2C^{(0)} + 2\tilde{C}_\epsilon\|b\|_{W^{1,q}(0,T_e;L^q(\Omega,\mathbb{R}^3))}^q.\end{aligned} \tag{42}$$

[5] Here we use the inequality

$$\begin{aligned}\sum_{n=1}^{l}(\phi_m^n - \phi_m^{n-1}, \phi_m^n)_\Omega &= \frac{1}{2}\sum_{n=1}^{l}\left(\|\phi_m^n\|_2^2 - \|\phi_m^{n-1}\|_2^2\right) \\ &\quad + \frac{1}{2}\sum_{n=1}^{l}\|\phi_m^n - \phi_m^{n-1}\|_2^2 \ge \frac{1}{2}\|\phi_m^l\|_2^2 - \frac{1}{2}\|\phi_m^0\|_2^2\end{aligned}$$

for any family of functions $\phi_m^0, \phi_m^1, \ldots, \phi_m^m$.

Altogether, from estimate (42) we then get that

$$\{p_m\}_m \text{ is uniformly bounded in } W^{1,q^*}(0,T_e;L^{q^*}(\Omega,\mathcal{M}^3)), \tag{43}$$

$$\{\operatorname{dev}\operatorname{sym}\bar{p}_m\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{M}^3)), \tag{44}$$

$$\{\bar{\sigma}_m\}_m, \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{S}^3)), \tag{45}$$

$$\{\operatorname{Curl}\bar{p}_m\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{M}^3)), \tag{46}$$

$$\left\{\frac{1}{\sqrt{m}}\bar{p}_m\right\}_m \text{ is uniformly bounded in } L^\infty(0,T_e;L^2(\Omega,\mathcal{M}^3)), \tag{47}$$

$$\left\{\bar{\Sigma}_m^{\text{lin}}\right\}_m \text{ is uniformly bounded in } L^q(0,T_e;L^q(\Omega,\mathcal{M}^3)). \tag{48}$$

In particular, the uniform boundedness of the sequences in (43)–(48) yields

$$\{u_m\}_m \text{ is uniformly bounded in } W^{1,q^*}(0,T_e;W_0^{1,q^*}(\Omega,\mathbb{R}^3)), \tag{49}$$

$$\{\operatorname{Curl}\operatorname{Curl}\bar{p}_m\}_m \text{ is uniformly bounded in } L^2(0,T_e;L^2(\Omega,\mathcal{M}^3)). \tag{50}$$

Employing (37), the estimates (43)–(48) further imply that the sequences $\{\sigma_m\}_m$, $\{\operatorname{dev}\operatorname{sym}p_m\}_m$, $\{\operatorname{Curl}p_m\}_m$, $\{p_m/\sqrt{m}\}_m$, $\{\Sigma_m^{\text{lin}}\}_m$, and $\{\operatorname{Curl}\operatorname{Curl}p_m\}_m$ are also uniformly bounded in the corresponding spaces. As a result, we have

$$\{p_m\}_m \text{ is uniformly bounded in } L^{q^*}(0,T_e;V^{q^*}_{\operatorname{Curl}}(\Omega,\mathcal{M}^3)). \tag{51}$$

Moreover, due to (43) and the obvious relation

$$p_m^l = p_m^0 + h\sum_{n=1}^{l}\left(\frac{p_m^n - p_m^{n-1}}{h}\right),$$

we conclude that $\{\bar{p}_m\}_m$ is uniformly bounded in $L^{q^*}(0,T_e;L^{q^*}(\Omega,\mathcal{M}^3))$.

**Existence of weak solutions.** By estimates (43)–(51) and at the expense of extracting a subsequence, we have that the sequences in (43)–(51) converge with respect to weak and weak-star topologies in corresponding spaces, respectively. Next, we claim that weak limits of $\{\bar{p}_m\}_m$ and $\{p_m\}_m$ coincide. Indeed, using (43) this can be shown as

$$\begin{aligned}\|p_m-\bar{p}_m\|^{q^*}_{q^*,\Omega_{T_e}} &= \sum_{n=1}^{m}\int_{(n-1)h}^{nh}\left\|(p_m^n-p_m^{n-1})\frac{t-nh}{h}\right\|^{q^*}_{q^*}dt\\ &= \frac{h^{q^*+1}}{q^*+1}\sum_{n=1}^{m}\left\|\frac{p_m^n-p_m^{n-1}}{h}\right\|^{q^*}_{q^*} = \frac{h^{q^*}}{q^*+1}\left\|\frac{dp_m}{dt}\right\|^{q^*}_{q^*,\Omega_{T_e}},\end{aligned}$$

which implies that $\bar{p}_m - p_m$ converges strongly to 0 in $L^{q^*}(\Omega_{T_e},\mathcal{M}^3)$. The proof that the difference $\bar{\sigma}_m-\sigma_m$ converges weakly to 0 in $L^2(\Omega_{T_e},\mathcal{S}^3)$ can be performed as in [22, p. 210]. For the reader's convenience we reproduce here the reasoning from that work. Let us choose some appropriate number $d\in\mathbb{N}$ and then fix any integer $n_0\in[1,2^d]$. Let $h_0=T_e/2^{n_0}$. Consider functions $I_{[h_0(n_0-1),h_0n_0]}v$ with $v\in L^2(\Omega,\mathcal{S}^3)$, where $I_K$ denotes the indicator function of a set $K$. We note that according to [22, Proposition 1.36], the linear combinations of all such functions are dense in $L^2(\Omega_{T_e},\mathcal{S}^3)$. Then for any $h \le h_0$[6]

$$\begin{aligned}
&\big(\sigma_m - \bar\sigma_m, I_{[h_0(n_0-1),h_0 n_0]} v\big)_{\Omega_{T_e}} \\
&\quad = \int_{h_0(n_0-1)}^{h_0 n_0} (\sigma_m(t) - \bar\sigma_m(t), v)_\Omega \, dt \\
&\quad = \sum_{n=h_0(n_0-1)/h+1}^{h_0 n_0/h} \int_{(n-1)h}^{nh} \left( (\sigma_m^n - \sigma_m^{n-1}) \frac{t-nh}{h}, v \right)_\Omega dt \\
&\quad = -\frac{h}{2} \left( \sigma_m^{h_0 n_0/h} - \sigma_m^{h_0(n_0-1)/h}, v \right)_\Omega = -\frac{h}{2} \left( \bar\sigma_m(h_0 n_0) - \bar\sigma_m(h_0(n_0-1)), v \right)_\Omega .
\end{aligned}$$

Employing (45) we get that $\bar\sigma_m - \sigma_m$ converges weakly to 0 in $L^2(\Omega_{T_e}, \mathcal{S}^3)$. Next, by (47) the sequence $\{p_m/m\}_m$ converges strongly to 0 in $L^2(\Omega_{T_e}, \mathcal{M}^3)$. Summarizing all observations made above we may conclude that the limit functions denoted by $u, \sigma, p$, and $\Sigma^{\rm lin}$ have the properties

$$\begin{aligned}
&u \in W^{1,q^*}(0,T_e; W_0^{1,q^*}(\Omega, \mathbb{R}^3)), \quad \sigma \in L^\infty(0,T_e; L^2(\Omega, \mathcal{S}^3)), \\
&p \in L^{q^*}(0,T_e; V^{q^*}_{\rm Curl}(\Omega, \mathcal{M}^3)) \cap W^{1,q^*}(0,T_e; L^{q^*}(\Omega, \mathcal{M}^3))
\end{aligned}$$

with

$$(\operatorname{dev}\operatorname{sym} p, \operatorname{Curl} p) \in L^\infty(0,T_e; L^2(\Omega, \mathcal{M}^3 \times \mathcal{M}^3))$$

and

$$\Sigma^{\rm lin} = \sigma - C_1 \operatorname{dev}\operatorname{sym} p - C_2 \operatorname{Curl}\operatorname{Curl} p \in L^q(\Omega_{T_e}, \mathcal{M}^3).$$

Before passing to the weak limit, we note that the Rothe approximation functions satisfy the equations

$$-\operatorname{div}_x \bar\sigma_m(x,t) = \bar b_m(x,t), \tag{52}$$

$$\sigma_m(x,t) = \mathbb{C}(\operatorname{sym}(\nabla_x u_m(x,t) - p_m(x,t))), \tag{53}$$

$$\partial_t p_m(x,t) \in \partial\phi\big(\bar\Sigma_m^{\rm lin}(x,t)\big), \tag{54}$$

together with the initial and boundary conditions

$$p_m(x,0) = p^{(0)}(x), \quad x \in \Omega, \tag{55}$$

$$\operatorname{Curl} p_m(x,t) \times \tau(x) = 0, \qquad x \in \partial\Omega, \tag{56}$$

$$u_m(x,t) = 0, \qquad x \in \partial\Omega . \tag{57}$$

Repeating the arguments at the beginning of this section we arrive at the inequality

$$\begin{aligned}
&\frac{1}{2}\left( \|\mathbb{B}^{1/2}\bar\sigma_m(t)\|_2^2 + C_1 \|\operatorname{dev}\operatorname{sym}\bar p_m(t)\|_2^2 + \frac{1}{m}\|\bar p_m(t)\|_2^2 + C_2 \|\operatorname{Curl}\bar p_m(t)\|_2^2 \right) \\
&\quad + \int_0^t \int_\Omega \left( \phi^*(\partial_t p_m(x,s)) + \phi(\bar\Sigma_m^{\rm lin}(x,s)) \right) dx ds \le \big(\bar b_m, \partial_t u_m\big)_{\Omega_t} \\
&\quad + \frac{1}{2}\left( \|\mathbb{B}^{1/2}\sigma^{(0)}\|_2^2 + C_1 \|\operatorname{dev}\operatorname{sym} p^{(0)}\|_2^2 + \frac{1}{m}\|p^{(0)}\|_2^2 + C_2 \|\operatorname{Curl} p^{(0)}\|_2^2 \right)
\end{aligned} \tag{58}$$

[6] We recall that $h$ is chosen to be equal to $T_e/2^m$ for some $m \in \mathbb{N}$.

which is satisfied for a.e. $t \in (0, T_e]$.[7] Now, passing to the weak limit in (52), (53) and to the weak limit inferior in (58), respectively, we obtain the desired inequality (13) and that $(u, \sigma, p)$ is a weak solution of (1)–(6).

This completes the proof of Theorem 1.2. ∎

**6. Appendix.** In this section we make some comments on the inequality (10). Suppose that $\Omega$ is a bounded domain with $C^\infty$-boundary $\partial\Omega$.

We define a space $V^q_\sigma(\Omega, \mathbb{R}^3)$ by

$$V^q_\sigma(\Omega, \mathbb{R}^3) := \{v \in V^q(\Omega, \mathbb{R}^3) \mid \operatorname{div} v = 0\}$$

and denote by $V^q_{har}(\Omega, \mathbb{R}^3)$ the $L^q$-space of harmonic functions on $\Omega$ as

$$V^q_{har}(\Omega, \mathbb{R}^3) := \{v \in V^q_\sigma(\Omega, \mathbb{R}^3) \mid \operatorname{Curl} v = 0\}.$$

Then, as shown in [12, Theorem 1], the space $V^q_{har}(\Omega, \mathbb{R}^3)$ for every fixed $q$, $1 < q < \infty$, coincides with the space $V_{har}(\Omega, \mathbb{R}^3)$ given by

$$V_{har}(\Omega, \mathbb{R}^3) = \{v \in C^\infty(\bar{\Omega}, \mathbb{R}^3) \mid \operatorname{div} v = 0, \operatorname{Curl} v = 0 \text{ with } v \times \tau = 0 \text{ on } \partial\Omega\}.$$

The space $V_{har}(\Omega, \mathbb{R}^3)$ is a finite dimensional vector space [12, Theorem 1].

If now $L$ denotes the dimension of $V_{har}(\Omega, \mathbb{R}^3)$, i.e., $\dim V_{har}(\Omega, \mathbb{R}^3) = L$, and $\{\phi_1, \ldots, \phi_L\}$ is a basis of $V_{har}(\Omega, \mathbb{R}^3)$, then it holds that $V^q(\Omega, \mathbb{R}^3) \subset W^{1,q}(\Omega, \mathbb{R}^3)$ with the estimate

$$\|v\|_q + \|\nabla v\|_q \le C \left( \|\operatorname{Curl} v\|_q + \|\operatorname{div} v\|_q + \sum_{i=1}^{L} |(v, \phi_i)| \right) \tag{59}$$

for all $v \in V^q(\Omega, \mathbb{R}^3)$, where $C = C(\Omega, q)$ [12, Theorem 2]. The proof of the inequality (59) with $\sum_{i=1}^{L} |(v, \phi_i)|$ replaced by $\|v\|_q$ is performed in [12, Lemma 4.5]. (For $q = 2$ it can be found in [5, Theorem VII.6.1].)

If we assume that the boundary $\partial\Omega$ has $L + 1$ smooth connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_L$ such that $\Gamma_1, \ldots, \Gamma_L$ lie inside $\Gamma_0$ with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$ and

$$\partial\Omega = \cup_{i=0}^{L} \Gamma_i,$$

then it holds that [12, Appendix]

$$\dim V_{har}(\Omega, \mathbb{R}^3) = L.$$

Therefore, if we suppose that the boundary $\partial\Omega$ has only one smooth connected component, i.e., $L = 0$, we obtain from (59) the inequality (10).

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[7] We have used again the inequality

$$\sum_{n=1}^{l} (\phi^n_m - \phi^{n-1}_m, \phi^n_m)_\Omega \ge \frac{1}{2}\|\phi^l_m\|_2^2 - \frac{1}{2}\|\phi^0_m\|_2^2,$$

where $\phi^0_m, \phi^1_m, \ldots, \phi^m_m$ is any family of functions.

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