Stability of Tensor Product B-Splines on Domains

Bernhard Mößner and Ulrich Reif

Abstract

We construct a uniformly stable family of bases for tensor product spline approximation on domains in $\mathbb{R}^d$. These bases are derived from the standard B-spline basis by normalization with respect to the L^p-norm and a selection process relying on refined estimates for the de Boor-Fix functionals.

1 Introduction

Uniform stability of tensor product B-spline bases in $\mathbb{R}^d$ is a well known fact [2] and one of the many favorable properties of this class of functions. However, when approximating functions on a domain $\Omega \subset \mathbb{R}^d$, stability is typically lost because of B-splines with only small parts of their support lying inside the domain. This problem was observed in [6], and probably also much earlier, and taken for granted ever since.

In [9, 8, 7], an extension procedure is suggested to stabilize B-spline bases. There, outer B-splines supported near the boundary of the domain are suitably coupled with inner ones, and it can be shown that the resulting basis combines stability with full approximation power, despite its reduced cardinality.

In this paper, we revisit the stability problem and suggest a "skip-and-scale" strategy to form uniformly stable subsequences of normalized B-splines, which are still useful for approximation. In the next section, we briefly recall the standard estimates for stability on $\mathbb{R}^d$ and explain possible sources of instability on bounded domains. In Section 3, we define *proper B-splines* in terms of geometric conditions on the intersection of the domain and the support, and show that the sequence of all such B-splines is uniformly stable. Further, we provide a sufficient analytic condition for properness. In Section 4, we consider approximation properties of spline spaces spanned by proper B-splines. For bounded domains with sufficiently smooth boundary, we can show that the L^p-approximation order is optimal for $p \leq 2$, and modestly reduced for larger exponents. Finally, in Section 5, we discuss generalizations of the concept and define even larger classes of B-splines leading to uniformly stable sequences.

2 Preliminaries

For an open set $\Omega \subset \mathbb{R}^d$, an index set $I \subset \mathbb{Z}^d$, and $1 \leq p \leq \infty$, let $B = (b_i)_{i \in I}$ be a sequence of functions $b_i \in L^p(\Omega)$. Linear combinations with real coefficients $F = (f_i)_{i \in I}$ in $\ell^p(I)$ are denoted by

$$f := BF := \sum_{i \in I} f_i b_i,$$

whenever the sum is convergent in $L^p(\Omega)$. B is a *Riesz sequence* if the constants

$$c := \inf\{\|BF\|_{p,\Omega} : \|F\|_p = 1\}, \qquad C := \sup\{\|BF\|_{p,\Omega} : \|F\|_p = 1\} \quad (1)$$

are positive and finite. In this case, the ratio $\operatorname{cond}_p B := C/c$ is called the *condition number* of B with respect to the p-norm.

In the univariate setting, let $T = (\tau_i)_{i \in \mathbb{Z}}$ be a monotone increasing bi-infinite knot sequence for a spline space of degree n. The corresponding order is denoted by $\overline{n} := n + 1$. The B-splines $(b_{i,n})_{i \in \mathbb{Z}}$ have supports

$$S_{i,n} := \operatorname{supp} b_{i,n} = [\tau_i, \tau_{i+\overline{n}}], \quad |S_{i,n}| := \tau_{i+\overline{n}} - \tau_i,$$

and satisfy *Marsden's identity*

$$(t - \tau)^n = \sum_{i \in \mathbb{Z}} b_{i,n}(t) \psi_{i,n}(\tau), \quad \psi_{i,n}(\tau) := \prod_{j=1}^{n} (\tau_{i+j} - \tau) \quad (2)$$

for all $\tau \in \mathbb{R}$ and $t \in \bigcup_{i \in \mathbb{Z}} [\tau_i, \tau_{i+1})$. The function $\psi_{i,n}$ is a polynomial of degree n with zeros located at the inner knots $\tau_{i+1}, \ldots, \tau_{i+n}$ of the B-spline

$b_{i,n}$. For $0 \le \nu \le n$, the νth derivative of $\psi_{i,n}$ can be written as

$$D^\nu \psi_{i,n}(\tau) = (-1)^\nu \frac{n!}{(n-\nu)!} \prod_{j=1}^{n-\nu} (\tau_{i,j}^\nu - \tau). \tag{3}$$

By Rolle's theorem, the zeros $\tau_{i,j}^\nu$ are all real and lie in the interval $S_{i,n}$. Hence, for $\tau \in S_{i,n}$, we have

$$|D^\nu \psi_{i,n}(\tau)| \le n! \, |S_{i,n}|^{n-\nu}, \quad 0 \le \nu \le n. \tag{4}$$

To prove stability properties of B-splines, we use the *de Boor-Fix functionals* [3] given by

$$\lambda_{i,n}(u) := \frac{1}{n!} \sum_{\nu=0}^{n} (-1)^{n-\nu} D^{n-\nu} \psi_{i,n}(\xi_i) D^\nu u(\xi_i),$$

where ξ_i is an arbitrary point with $b_{i,n}(\xi_i) > 0$. The basic duality property is

$$\lambda_{i,n}(b_{j,n}) = \delta_{ij}, \quad i, j \in \mathbb{Z}. \tag{5}$$

In the multivariate setting, points $p \in \mathbb{R}^d$ are understood as row-vectors, and their components are indexed by superscripts, $p = (p^1, \dots, p^d)$. The component-wise product of two points $p, q \in \mathbb{R}^d$ is denoted by

$$p * q := (p^1 q^1, \dots, p^d q^d) \in \mathbb{R}^d.$$

If $p \le q$ component-wise, then the two points define the closed rectangular *box*

$$P := [p, q] := [p^1, q^1] \times \cdots \times [p^d, q^d] \subset \mathbb{R}^d.$$

The vector of edge lengths, also called the *size* of P, is denoted by

$$|P| := q - p \in \mathbb{R}^d.$$

The univariate knot sequences $T^1, \dots, T^d$ define a multivariate *knot grid* $T := T^1 \times \cdots \times T^d$ with knots $\tau_i := (\tau_{i^1}^1, \dots, \tau_{i^d}^d)$ and *grid cells* $T_i := [\tau_i, \tau_{i+1}]$, $i \in \mathbb{Z}^d$. The basis functions of the tensor product spline space of coordinate degree $n = (n^1, \dots, n^d) \in \mathbb{N}_0^d$ with knots T are products of univariate B-splines, i.e.,

$$b_{i,n}(x) := b_{i^1,n^1}(x^1) \cdots b_{i^d,n^d}(x^d), \quad i \in \mathbb{Z}^d.$$

Denoting the order of the spline space again by $\overline{n} := n + (1, \dots, 1)$, their supports are the boxes

$$S_{i,n} := \operatorname{supp} b_{i,n} = [\tau_i, \tau_{i+\overline{n}}].$$

With the usual multi-index notation, we denote partial differentiation by $\partial^n := \partial_1^{n^1} \cdots \partial_d^{n^d}$, and factorials by $n! = n^1! \cdots n^d!$. The multivariate de Boor-Fix functionals are given by

$$\lambda_{i,n}(u) = \frac{1}{n!} \sum_{\nu \le n} (-1)^{n-\nu} \partial^{n-\nu} \psi_{i,n}(\xi_i) \partial^\nu u(\xi_i), \tag{6}$$

where $\psi_{i,n}(\xi) := \psi_{i^1,n^1}(\xi^1) \cdots \psi_{i^d,n^d}(\xi^d)$, and ξ_i is chosen such that $b_{i,n}(\xi_i) > 0$. To simplify notation throughout, we fix the degree $n \in \mathbb{N}_0^d$ and drop the corresponding subscript.

In what follows, we study stability properties of sequences $B = (b_i)_{i \in I}$ of tensor product B-splines on a domain $\Omega \subset \mathbb{R}^d$. For a given knot grid T, a natural choice of B is to select all *relevant* B-splines, i.e., all B-splines that do not vanish on Ω, i.e.,

$$B = (b_{i|\Omega})_{i \in I}, \quad I := \{i \in \mathbb{Z}^d : S_i \cap \Omega \ne \emptyset\}. \tag{7}$$

The *restricted support* of a relevant B-spline b_i is defined by

$$S_i^\Omega := \operatorname{supp} b_{i|\Omega} = S_i \cap \overline{\Omega}.$$

Typically, the results to be derived later are invariant with respect to *axis-aligned affine maps*, or briefly *a^3-maps* in $\mathbb{R}^d$. Such maps have the form

$$\mathcal{A}\colon\ \mathbb{R}^d \ni x \mapsto (a_\mathrm{s} * x + a_\mathrm{t}) A_\mathrm{p} \in \mathbb{R}^d,$$

where $A_\mathrm{p} \in \{0,1\}^{d \times d}$ is a permutation matrix, $a_\mathrm{s} \in (\mathbb{R}_{\ne 0})^d$ is a scaling vector, and $a_\mathrm{t} \in \mathbb{R}^d$ defines a translation. $\mathcal{A}$ is called *isometric* if all components of the scaling vector have modulus 1. The set of all a^3-maps is a subgroup of the affine group. The relevance of this subgroup in the context of tensor product splines is due to the fact that it operates on the set of knot grids. Other affine transformations, like shearings, destroy the special structure. Transformed objects in $\mathbb{R}^d$ are denoted by

$$\tilde{x} := \mathcal{A}(x), \quad \tilde{\Omega} := \mathcal{A}(\Omega), \quad \tilde{T} := \mathcal{A}(T), \quad \tilde{n} := n A_\mathrm{p},$$

and so on. Equally, for functions $u\colon \mathbb{R}^d \to \mathbb{R}$ we write

$$\tilde{u} := u \circ \mathcal{A}^{-1}.$$

With this notation, we observe that the transformed B-spline $\tilde{b}_i$ equals the B-spline of degree $\tilde{n}$ with knots $\tilde{T}$ and support $\tilde{S}_i$, since for example the corresponding de Boor-Fix functional satisfies $\tilde{\lambda}_i(\tilde{u}) = \lambda_i(u)$. Formally speaking, the group of all a^3-maps is generating equivalence classes of spline spaces. We say that a property of objects related to a spline space is *a*^{3}*-invariant* if it is shared by the corresponding objects for all members of the equivalence class. For instance, duality of B-splines and de Boor-Fix functionals is a^3-invariant,

$$\lambda_j(b_i) = \tilde{\lambda}_j(\tilde{b}_i) = \delta_{i,j}.$$

Also the condition number $\operatorname{cond}_p(B) = C/c$ of B is a^3-invariant, while the values of c and C are not, unless $p = \infty$. When proving a statement which is a^3-invariant, it is sufficient to consider only a single representative whose characteristics can be geared to simplify the argument. For example, to prove the a^3-invariant statement $c|f_i| \le \|BF\|_{\infty,\mathbb{R}^d}$, we may assume $S_i = [0,1]^d$.

For $\Omega = \mathbb{R}^d$, the classical result on the uniform stability of B-splines [2] states that $\operatorname{cond}_\infty B$ is bounded by a constant M depending only on the degree n and the dimension d, but not on the choice of knots. A similar result holds for p-norms, $1 \le p < \infty$, if the B-splines are normalized in a suitable way. Notably, $\operatorname{cond}_\infty B$ can be arbitrarily large for general Ω.

Let us illustrate this phenomenon by a simple univariate example. For

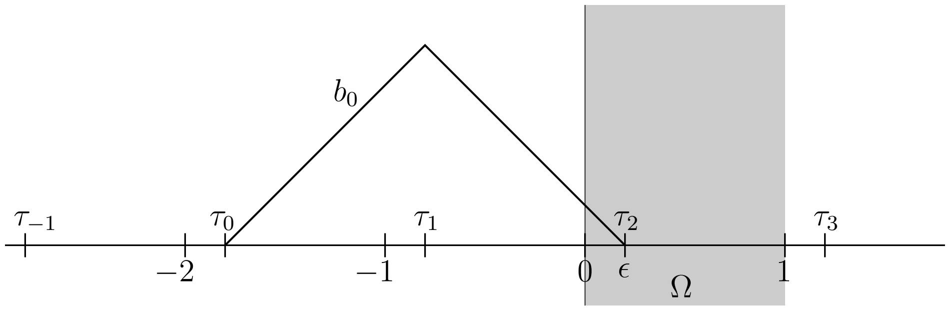

Figure 1: B-spline basis with large condition number .

$0 < \epsilon < 1$, we consider splines of degree $n = 1$ with knots $\tau_i = i - 2 + \epsilon$ on the domain $\Omega = (0,1)$. Figure 1 shows the spline $BF = b_0$ corresponding to the coefficients $f_i = \delta_{i,0}$. From $\|b_0\|_{\infty,\Omega} = \epsilon$, we conclude that the lower bound in (1) is $c \leq \epsilon$, while the upper bound is obviously $C = 1$. Hence, $\operatorname{cond}_\infty B \geq 1/\epsilon$ is not uniformly bounded.

To understand the differences between this case and the well known uniform stability on $\mathbb{R}$, let us briefly review the classical proof. For $\Omega = \mathbb{R}$ and $B = (b_i)_{i\in\mathbb{Z}}$, consider the estimate

$$c\|F\|_\infty \leq \|BF\|_{\infty,\mathbb{R}} \leq C\|F\|_\infty, \quad F \in \ell^\infty(\mathbb{Z}).$$

By partition of unity, the upper estimate holds with the optimal constant $C = 1$. For the lower estimate, we note, as mentioned above, that the statement $c|f_i| \leq \|f\|_{\infty,\mathbb{R}}$ is a^3-invariant. Hence we can assume $S_i = [0,1]$. By (5),

$$\begin{aligned}|f_i| = |\lambda_i(f)| &= \frac{1}{n!}\left|\sum_{\nu=0}^{n}(-1)^{n-\nu}D^{n-\nu}\psi_i(\xi_i)D^\nu f(\xi_i)\right| \\ &\leq \frac{1}{n!}\max_{\nu\leq n}\bigl|D^\nu\psi_i(\xi_i)\bigr|\sum_{\nu=0}^{n}\bigl|D^\nu f(\xi_i)\bigr|.\end{aligned}$$

With $S_i = [0,1]$, (4) yields $|D^\nu\psi_i(\xi)| \leq n!$ for all ν. Further, there exists an interval $Q_i \subset S_i = [0,1]$ of length $1/\overline{n}$ which does not contain a knot. Hence, $f_{|Q_i} \in \mathbb{P}_n(Q_i)$, i.e., the restriction of f to Q_i is a polynomial of degree $\leq n$. With ξ_i the center of Q_i, the sum in the above estimate is a norm on the space $\mathbb{P}_n(Q_i)$. By equivalence of norms on finite-dimensional vector spaces, this norm is bounded from above by the L^∞-norm on Q_i with a constant C_n depending only on n. We obtain

$$|f_i| \leq \sum_{\nu=0}^{n}|D^\nu f(\xi_i)| \leq C_n\|f_{|Q_i}\|_{\infty,Q_i} \leq C_n\|f\|_{\infty,\mathbb{R}}$$

for all $i \in \mathbb{Z}$ showing that $C_n^{-1}\|F\|_\infty \leq \|BF\|_{\infty,\mathbb{R}}$ for all $F \in \ell^\infty(\mathbb{Z})$. Hence, $\operatorname{cond}_\infty B \leq M := C_n$ is bounded independently of the knot sequence. It should be noted that, typically, the value of the constant M obtained that way largely over-estimates the actual condition number, which may be hard to determine [4].

For arbitrary domains $\Omega \subset \mathbb{R}$, the above argument can fail since it might be impossible to find an interval Q_i of length $1/\overline{n}$ in $S_i \cap \Omega$, if this set is small. The counterexample given above is based exactly on this observation. Of course, that problem is readily removed by adapting the knot sequence appropriately, for instance by setting $T = [0, 0, \epsilon, 1, 1]$. In this way, the instability is removed without changing the spline space on Ω. Unfortunately, this method does not work in general for domains in higher dimensions.

3 Stability

As shown above, the sequence B considered in (7) is not necessarily stable. A trivial way to circumvent this problem is to discard those B-splines for which a suitable box Q_i does not exist. For instance, if the knot sequence is uniform, these are exactly those B-splines which do not have a complete grid cell of their support in Ω. Although in this way only relatively few B-splines near the boundary of Ω are ruled out, it is easily shown that the resulting spline space reveals a substantial loss of approximation power. A much more appropriate solution is based on the concept of *extension* as introduced in [9]. There, the outer B-splines causing instability were suitably attached to inner ones so that a uniformly stable basis with full approximation power was obtained. In what follows, we suggest an even simpler approach to the problem which is based on normalization of B-splines with respect to the L^p-norm. More precisely, for a given domain $\Omega \subset \mathbb{R}^d$ and $1 \le p \le \infty$, we define the *normalized B-splines*

$$b_i^p := \frac{b_i}{\|b_i\|_{p,\Omega}}, \quad i \in I,$$

where I is the index set of relevant B-splines introduced in (7) so that the denominator is positive. In Definition 3.2 we introduce a subsequence of normalized B-splines which, in Theorem 3.3, is shown to be uniformly stable with respect to the knot grid. This result is essentially based on the following estimate for univariate B-splines:

Lemma 3.1 *Let b_i be a univariate B-spline of degree n with $\tau_i = 0$, and let $P := [0, 1]$. Then*

$$\|b_i\|_{\infty,P} \, \|D^\nu \psi_i\|_{\infty,P} \le n!$$

for all $\nu = 0, \ldots, n$.

Proof: The proof is by induction on ν, proceeding backwards from $\nu = n$. For $\nu = n$, the estimate follows immediately from $\|b_i\|_{\infty,P} \le 1$ and (4). Now, we assume that the estimate holds for $\nu + 1$. If $D^\nu \psi_i$ has a zero in $[0, 1]$, then, by the mean value theorem, $\|D^\nu \psi_i\|_{\infty,P} \le \|D^{\nu+1}\psi_i\|_{\infty,P}$, implying that the estimate holds also for ν. Otherwise, all zeros of $D^\nu \psi_i$ are greater than 1. In this case, $|D^\nu \psi_i|$ is monotone decreasing on $P = [0, 1]$. Hence,

$$\|b_i\|_{\infty,P} \, \|D^\nu \psi_i\|_{\infty,P} = \|b_i\|_{\infty,P} \, |D^\nu \psi_i(0)|.$$

Further, $\tau_{i+\overline{n}} > 1$ so that Marsden's identity is valid for $t \in P$. Since b_i and ψ_i depend only on the knots $\tau_i, \ldots, \tau_{i+\overline{n}}$, we can assume $\tau_j = \tau_i = 0$ for all $j \le i$ without loss of generality. Differentiating (2) with respect to τ, we obtain for $\tau = 0$

$$\sum_{j\in\mathbb{Z}} b_j(t) D^\nu \psi_j(0) = (-1)^\nu \frac{n!}{(n-\nu)!} t^{n-\nu}, \quad t \in P.$$

By the special choice of knots, all zeros $\tau_{j,l}^\nu$ of $D^\nu \psi_j$ are non-negative. Thus, (3) yields $D^\nu \psi_j(0) = (-1)^\nu |D^\nu \psi_j(0)|$, and we obtain

$$b_i(t)|D^\nu \psi_i(0)| \le \sum_{j\in\mathbb{Z}} b_j(t)|D^\nu \psi_j(0)| = \frac{n!}{(n-\nu)!} t^{n-\nu}, \quad t \in P.$$

Finally,

$$\|b_i\|_{\infty,P} \, |D^\nu \psi_i(0)| \le \frac{n!}{(n-\nu)!} \le n!,$$

and the proof is complete. □

When establishing stability properties, it is evident that the degenerate case of vanishing B-splines has to be exluded, i.e., we require

$$|S_i| = \tau_{i+\overline{n}} - \tau_i > 0, \quad i \in \mathbb{Z}^d.$$

Further, to facilitate approximation on the whole domain Ω, it must be covered by the union of grid cells $T_i = [\tau_i, \tau_{i+1}]$, i.e.,

$$\Omega \subset \bigcup_{i\in\mathbb{Z}^d} T_i.$$

In the following, these two properties are taken for granted without further notice. The next definition introduces a subset of B-splines which can be stabilized by normalization.

Definition 3.2 *Let $\Omega \subset \mathbb{R}^d$ be an open domain, and I the set of relevant indices of the given spline space according to (7). A B-spline $b_i, i \in I$, with support S_i and restricted support $S_i^\Omega = S_i \cap \overline{\Omega}$ is called* proper*, if there exist boxes P_i, R_i with the following properties:*

a) $R_i \subset S_i^\Omega \subset P_i$

b) The sizes of R_i and P_i are related by $2d\,|R_i| = |P_i|$.

c) The boxes P_i and S_i have one corner in common.

For $p \in [1, \infty]$, the sequence $B_\bullet^p$ of normalized proper B-splines is defined by

$$B_\bullet^p := (b_i^p)_{i \in I_\bullet}, \quad I_\bullet := \{i \in I : b_i \text{ is proper }\}.$$

Let us briefly discuss the concept. The property of being proper is a^3-*invariant* since so are all three conditions. That is, if b_i is proper with respect to T, n, Ω, then $\tilde{b}_i$ is proper with respect to $\tilde{T}, \tilde{n}, \tilde{\Omega}$ for any a^3-map. Another important fact is that the sizes of the boxes P_i, R_i are *not* related to the size of the support S_i. Hence, even B-splines with an arbitrarily small restricted support may be proper.

When searching for a pair of boxes R_i, P_i to establish properness of a B-spline b_i, one observes the following: If $S_i^\Omega \subset P_i \subset P_i'$ for some boxes P_i, P_i' having one corner in common with S_i, it is advantageous to choose the smaller one, P_i, because this alleviates the problem of finding a corresponding box R_i inside S_i^Ω. If S_i^Ω contains a corner of S_i, this implies that the optimal choice for P_i is the bounding box of S_i^Ω. Otherwise, one may restrict the search for P_i to the finitely many bounding boxes of sets of the form $S_i^\Omega \cup c_i$, where c_i is a corner of S_i. In any case, we may assume $P_i \subset S_i$ without loss of generality.

Obviously, all B-splines with $S_i \subset \Omega$ are proper. This guarantees that the set of proper B-splines is not empty if the knot sequence is sufficiently fine. Throughout, we exclude the degenerate case $I_\bullet = \emptyset$ without further notice. Now, we focus on *boundary B-splines* which are characterized by $i \in I$ and $S_i \not\subset \Omega$. Figure 2 shows a few typical cases. On the left hand side, the domain Ω is locally bounded by a hyperplane. We choose $P_i =: p_i + [0, h_i]$ as the bounding box of S_i^Ω, and note that the dashed box $p_i + [0, h_i/d]$ is contained in S_i^Ω. Hence, we can choose the even smaller box $R_i := p_i + [0, h_i/(2d)]$ to show that b_i is proper. Due to the margin provided by R_i in the linear case, this construction typically remains valid for small perturbations of the boundary, as they arise for a smooth boundary and a relatively fine knot grid. This situation is depicted in the middle of the figure. The argument can fail in the following situation: if the boundary is a small perturbation of a hyperplane which is close to a face of S_i, then the maximal permitted size of R_i may be arbitrarily small, while P_i is still required to be relatively large. Such a situation is depicted on the right hand side of the figure. However, we see that in this case the measure of S_i^Ω is *very* small so that discarding the corresponding B-spline seems to be reasonable. The heuristics suggested here will be made precise in Theorems 3.5 and 4.1.

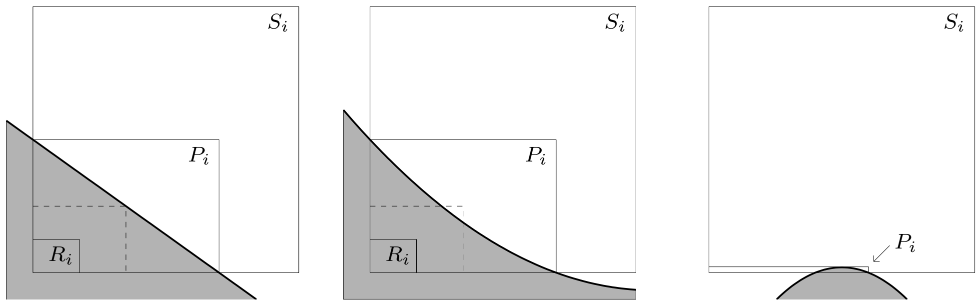

Figure 2: Proper *(left, middle)* and non-proper *(right)* B-splines.

Theorem 3.3 *The condition number of the sequence $B^p_\bullet$ of normalized proper B-splines is bounded by*

$$\operatorname{cond}_p B^p_\bullet \leq M,$$

where the constant M depends on n, d, p, but neither on T nor on Ω.

Proof: We determine constants $c', C' > 0$ such that

$$c'\|F\|_p \leq \|B^p_\bullet F\|_{p,\Omega} \leq C'\|F\|_p$$

for all $F \in \ell^p(I_\bullet)$. The upper bound follows from locality of B-splines. On every grid cell T_j, there are $N := \overline{n}^1 \cdots \overline{n}^d$ non-vanishing basis functions. With $\|b_i^p\|_{p,\Omega} = 1$, standard arguments show that $C' := N^{1/q}$ is a valid constant, where p and q are related by $1/p + 1/q = 1$.

To prove the lower estimate, we consider a proper B-spline b_i and the boxes P_i, R_i according to Definition 3.2. Being a subset of S_i, the box R_i

is partitioned into at most $\overline{n}^1 \times \cdots \times \overline{n}^d$ boxes by the knot grid T. Hence, there exists a box $Q_i \subset R_i$ with $\overline{n} * |Q_i| = |R_i|$ which is completely contained in a grid cell. Like the conditions on P_i, Q_i, R_i, the lower estimate $c'\|F\|_p \leq \|B^p_\bullet F\|_{p,\Omega}$ is also a^3-invariant, i.e.,

$$\|B^p_\bullet F\|_{p,\Omega} = \|\tilde{B}^p_\bullet F\|_{p,\tilde{\Omega}}.$$

Hence, when estimating the coefficient f_i, we can assume $P_i = [0,1]^d$, and that the common corner of P_i and S_i is the origin.

Applying the de Boor-Fix functional (6) to the spline $f = B^p_\bullet F$, we obtain the estimate

$$\begin{aligned} \frac{|f_i|}{\|b_i\|_{p,\Omega}} = |\lambda_i(f)| &\leq \frac{1}{n!} \sum_{\nu \leq n} |\partial^{n-\nu} \psi_i(\xi_i)|\, |\partial^\nu f(\xi_i)| \\ &\leq \frac{1}{n!} \left(\max_{\nu \leq n} |\partial^\nu \psi_i(\xi_i)| \right) \sum_{\nu \leq n} |\partial^\nu f(\xi_i)|, \end{aligned} \tag{8}$$

where ξ_i is the center of Q_i. Since Q_i is contained in a grid cell, f restricted to Q_i is a polynomial of order $\overline{n}$. The sum in (8) is a norm on the space $\mathbb{P}_n(Q_i)$ of polynomials of degree $\leq n$ on Q_i. Therefore, by equivalence of norms on finite-dimensional vector spaces, this norm is bounded from above by the p-norm on Q_i times a constant $C_{n,d,p}$. This constant depends only on n, d, p because the size of Q_i is fixed by $2d\overline{n} * |Q_i| = 2d\,|R_i| = (1, \ldots, 1)$. From $S_i^\Omega \subset P_i = [0,1]^d$, there follows the estimate

$$\|b_i\|_{p,\Omega} \leq \|b_i\|_{\infty,P_i}. \tag{9}$$

Hence, using $\xi_i \in Q_i \subset P_i$, we obtain

$$|f_i| \leq \frac{C_{n,d,p}}{n!} \|f\|_{p,Q_i} \|b_i\|_{\infty,P_i} \max_{\nu \leq n} \|\partial^\nu \psi_i\|_{\infty,P_i}.$$

We apply Lemma 3.1 to all univariate factors of b_i and $\partial^\nu \psi_i$ and find

$$|f_i| \leq C_{n,d,p} \|f\|_{p,Q_i}.$$

In view of the fact that at most $N := \overline{n}^1 \cdots \overline{n}^d$ of the boxes Q_i can overlap, a standard argument finally yields

$$\|F\|_p^p = \sum_{i \in I_\bullet} |f_i|^p \leq C^p_{n,d,p} \sum_{i \in I_\bullet} \|f\|^p_{p,Q_i} \leq N C^p_{n,d,p} \|f\|^p_{p,\Omega},$$

and thus, $c' = (N^{1/p}C_{n,d,p})^{-1}$ is a valid constant for the lower estimate. Together, the condition number is bounded by

$$\operatorname{cond}_p B^p_\bullet \le M := \frac{C'}{c'} = C_{n,d,p}N^{1/p+1/q} = C_{n,d,p}N.$$

□

With the help of this result, we can easily derive a condition for the univariate case $d = 1$, saying that a B-spline can be non-proper only if its support contains the domain, and the domain is relatively small.

Theorem 3.4 *Let $\Omega \subset \mathbb{R}$ be an open interval. Then a relevant B-spline b_i with support S_i is proper if*

$$\Omega \not\subset S_i \quad \text{or} \quad 3|\Omega| > |S_i|.$$

Proof: The case $\Omega = \mathbb{R}$ is trivial. Otherwise, let us assume that the condition is not satisfied. By a^3-invariance, we can assume $S_i = [0, 1]$, and further that the center of $\Omega = (a, b)$ is not greater than the center of S_i,

$$(a + b)/2 \le 1/2.$$

Here, also the case $\Omega = (-\infty, b)$ is formally covered by permitting $a = -\infty$. If $\Omega \not\subset S_i$, then $\tau_i = 0 \in (a, b)$, and with $h_i := \min(1, b)$ we set

$$P_i := S_i^\Omega = [0, h_i], \quad R_i := [0, h_i/2].$$

Otherwise, if $\Omega \subset S_i$ and $3(b - a) > |S_i| = 1$, it is $a < b/2$, and we set

$$P_i := [0, b], \quad R_i := [b/2, b].$$

In both cases, the intervals P_i, R_i satisfy the conditions of Definition 3.2. □

As an example, we consider the uniform knot sequence $T := 2\mathbb{Z}$ and the domain $\Omega := (1, 1 + \delta)$, $\delta > 0$. Figure 3 shows L^2-condition numbers of the standard basis $(b_i)_{i\in I}$ *(left)* and the normalized basis $(b_i^2)_{i\in I}$ *(right)* as a function of δ for degrees $n = 1, \ldots, 4$. As explained already in Section 2, the standard basis is ill-conditioned if it contains B-splines with small support in Ω. Indeed, we observe large condition numbers if δ is either small or slightly larger than an odd integer. For normalized B-splines, the figure confirms uniform stability according to Theorem 3.4 for δ beyond the marked value

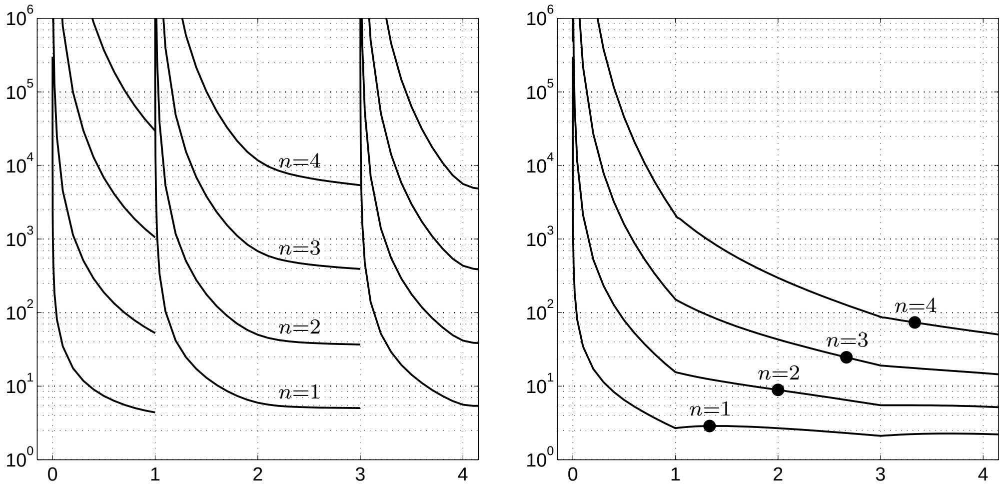

Figure 3: L^2-condition number of standard B-splines *(left)* and normalized B-splines *(right)*.

$2\overline{n}/3 = |S_i|/3$, for which all relevant B-splines are proper. It is apparent (and provable) that the condition numbers grow unboundedly as $\delta \to 0$, showing that the lower bound on the size of the domain is indispensable.

Now, we focus on the less trivial multivariate case, and assume $d \geq 2$ for the remainder of this section without further notice. While the conceptual design of proper B-splines is relatively simple, the *geometric conditions* of Definition 3.2 are inconvenient for verification, say, by a computer program. They are also not suitable for an analysis of approximation properties. Therefore, we are going to provide sufficient *analytic conditions* for proper B-splines which can be verified by estimating values and gradients of local parametrizations of the boundary of the domain.

To this end, we introduce the following notational convention. The first $(d-1)$ components of a point p or a box $P = [p, q]$ in $\mathbb{R}^d$ are marked with a superscript star,

$$
\begin{aligned}
p &= (p^*, p^d), \quad p^* := (p^1, \ldots, p^{d-1}), \\
P &= P^* \times P^d, \quad P^* = [p^*, q^*], \quad P^d = [p^d, q^d].
\end{aligned}
$$

Now, we consider a relevant B-spline b_i with support S_i intersecting the boundary $\partial\Omega$ of Ω. If $\partial\Omega$ is smooth and S_i is sufficiently small, then $\partial\Omega$ can be represented locally as the graph of a smooth function over a hyperplane which is perpendicular to one of the coordinate axes. More precisely, we say that $\partial\Omega$ is *projectable* on S_i if there exists a box $U_i = [0, u_i] \subset \mathbb{R}^d$, a C^1-function $\varphi_i \colon U_i^* \to \mathbb{R}$, and an isometric a^3-map $\mathcal{I}_i$ so that the restricted support of b_i is given by

$$S_i^\Omega = \mathcal{I}_i(U_i^\Omega), \quad U_i^\Omega := \{x \in U_i : x^d \leq \varphi_i(x^*)\}. \tag{10}$$

Extrema of values and gradients of φ_i are denoted by

$$v_i^d := \max_{x^* \in U_i^*} \varphi_i(x^*), \quad \underline{w}_i := \min_{x^* \in U_i^*} \nabla\varphi_i(x^*), \quad \overline{w}_i := \max_{x^* \in U_i^*} \nabla\varphi_i(x^*),$$

where min/max of $\nabla\varphi_i$ are understood component-wise, i.e., $\underline{w}_i, \overline{w}_i \in \mathbb{R}^{d-1}$. The two drawings on the left hand side of Figure 4 illustrate the setting. Of course, the representation is not unique. First, S_i^Ω can be projectable with respect to different coordinate directions. This leads to qualitatively different parametrizations of the boundary, and the condition (11) provided below might be satisfied for one of them, but not for the other. Second, even when fixing the direction of projection, the choice of $\mathcal{I}_i$ is not unique. It can be replaced by $\mathcal{I}_i \circ \hat{\mathcal{I}}$ with the isometric a^3-map $\hat{\mathcal{I}}$ defined as follows: Disregarding permutations of the components, which are trivially possible, we choose an index set $K \subset \{1, \ldots, d-1\}$ specifying directions in which the given representation is to be reflected, and set

$$\hat{\mathcal{I}}(x) := y, \quad y^k := \begin{cases} u_i^k - x^k & \text{for } k \in K \\ x^k & \text{else.} \end{cases}$$

Further, the function $\hat{\varphi}_i \colon U_i^* \to \mathbb{R}$ is defined by $\hat{\varphi}_i(x^*) := \varphi_i(y^*)$. Then $U_i = \hat{\mathcal{I}}(U_i)$ and

$$S_i^\Omega = \mathcal{I}_i(\hat{\mathcal{I}}(\hat{U}_i^\Omega)), \quad \hat{U}_i^\Omega = \{x \in U_i : x^d \leq \hat{\varphi}_i(x^*)\},$$

see the third drawing in Figure 4. Hence, any choice of $\hat{\mathcal{I}}$ yields another valid representation of the restricted support. We note that the maximal value $\hat{v}_i^d = v_i^d$ and the difference $\hat{\overline{w}}_i - \underline{\hat{w}}_i = \overline{w}_i - \underline{w}_i$ are independent of $\hat{\mathcal{I}}$, despite the fact that the components of $\nabla\varphi_i$ and $\nabla\hat{\varphi}_i$ with index $k \in K$ have opposite sign. The following theorem provides a sufficient condition for a B-spline to be proper.

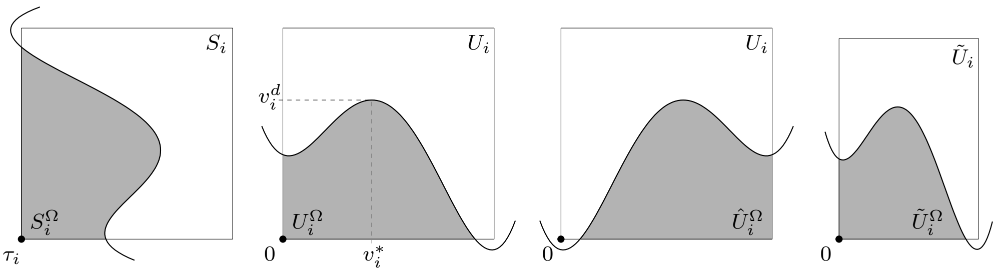

Figure 4: Variants on the representation of the restricted support S_i^Ω.

Theorem 3.5 *Let $\partial\Omega$ be projectable on the support S_i of the boundary B-spline b_i. Then, with the notation introduced above, b_i is proper if*

$$\frac{d}{d-1}\, v_i^d \geq u_i^* * (\overline{w}_i - \underline{w}_i), \tag{11}$$

where the inequality is understood component-wise.

Proof: Throughout the proof, we drop the subscript i to simplify notation, i.e., $S = S_i, S^\Omega = S_i^\Omega$, and so on. The definition of a proper B-spline as well as the conditions given in the theorem are a^3-invariant. To show this, we consider any a^3-map $\mathcal{A} : x \mapsto (a_\mathrm{s} * x + a_\mathrm{t})A_\mathrm{p}$. Let $\mathcal{I} : x \mapsto (i_\mathrm{s} * x + i_\mathrm{t})I_\mathrm{p}$ be the isometry used in (10). With $b_\mathrm{s} := |a_\mathrm{s}|I_\mathrm{p}^{-1}$, the a^3-map $\mathcal{B} : x \mapsto b_\mathrm{s} * x$ is a scaling with positive factors so that $\tilde{U} := \mathcal{B}(U)$ is a box of the form $\tilde{U} = [0, \tilde{u}]$. Further, if we define the isometric a^3-map $\tilde{\mathcal{I}} := \mathcal{A} \circ \mathcal{I} \circ \mathcal{B}^{-1}$, and the function $\tilde{\varphi} : \tilde{U}^* \to \mathbb{R}$ via the relation

$$\tilde{\varphi}\big(b_\mathrm{s}^* * x^*\big) = b_\mathrm{s}^d\, \varphi\big(x^*\big), \quad x^* \in U^*,$$

then the restricted support $\tilde{S}^\Omega = \mathcal{A}(S^\Omega)$ is given by

$$\tilde{S}^\Omega = \tilde{\mathcal{I}}(\tilde{U}^\Omega), \quad \tilde{U}^\Omega := \{\tilde{x} \in \tilde{U} : \tilde{x}^d \leq \tilde{\varphi}(\tilde{x}^*)\}.$$

The rightmost drawing in Figure 4 illustrates the effect of scaling. The transformation rules for the quantities appearing in (11) are

$$\tilde{v}^d = b_\mathrm{s}^d v^d, \quad \tilde{u}^* = b_\mathrm{s}^* * u^*, \quad b_\mathrm{s}^* * (\tilde{\overline{w}} - \tilde{\underline{w}}) = b_\mathrm{s}^d(\overline{w} - \underline{w}).$$

Hence, all scaling factors cancel out, showing that the validity of (11) is independent of the application of $\mathcal{A}$.

As a consequence of a³-invariance, we can assume that $\mathcal{I}$ is the identity, and that $S = U = [0,1]^d$, i.e., $u^* = (1,\ldots,1), u^d = 1$.

Let $v^* \in [0,1]^{d-1}$ denote a point where φ attains its maximum,

$$v^d = \varphi(v^*).$$

Of course, $v^d > 0$ since $S^\Omega \neq \emptyset$. We assume that the corner of U^* closest to v^* is the origin, i.e., $v^* \in [0,1/2]^{d-1}$. Otherwise, as described above, we replace $\mathcal{I}$ by $\mathcal{I} \circ \hat{\mathcal{I}}$, where $\hat{\mathcal{I}}$ is the isometric a³-map corresponding to the set of indices $K := \{k : (v^*)^k > 1/2\}$ not conforming to the condition. Consequently, $\nabla\varphi(v^*) \leq 0$ so that $\underline{w} \leq 0$. We define the box $P := [0,p]$ by

$$p^k := \begin{cases} 1 & \text{if } k < d \text{ and } -\overline{w}^k \leq v^d \\ -v^d/\overline{w}^k & \text{if } k < d \text{ and } -\overline{w}^k > v^d \\ \min\{1, v^d\} & \text{if } k = d, \end{cases}$$

and with $r := p/(2d)$ the box

$$R := (v^*, 0) + [0, r] = [v^*, v^* + r^*] \times [0, r^d].$$

Now, we show that $P = P_i$ and $R = R_i$ satisfy the assumptions of Definition 3.2.

First, we have $2d\,|R| = |P|$, and the origin is the common corner of S and P.

Second, we prove that $R \subset S^\Omega = U^\Omega$. To this end, let $x = (x^*, x^d) \in R$. The first component can be written in the form

$$x^* = v^* + \eta * r^*, \quad \eta \in [0,1]^{d-1}.$$

We have $0 \leq v^* \leq 1/2$ and $\eta * r^* \leq p^*/(2d) \leq 1/4$. Hence, $x^* \in [0,1]^{d-1}$, and it remains to show that

$$r^d \leq \varphi(v^* + \eta * r^*) \quad \text{for all} \quad \eta \in [0,1]^{d-1},$$

i.e., that the upper face of R lies below the graph of φ. Obeying $\underline{w} \leq 0$, we estimate the function value at x^* by

$$\varphi(x^*) \geq \varphi(v^*) + \sum_{k=1}^{d-1} \underline{w}^k \eta^k r^k \geq v^d + \frac{1}{2d} \sum_{k=1}^{d-1} \underline{w}^k p^k. \tag{12}$$

To estimate $\underline{w}^k p^k$ we have to distinguish the following two cases: For $-\overline{w}^k \leq v^d$ we have $p^k = 1$ and get from (11)

$$\frac{d}{d-1}\, v^d \geq \overline{w}^k - \underline{w}^k \geq -v^d - \underline{w}^k,$$

hence

$$\underline{w}^k p^k = \underline{w}^k \geq \left(-\frac{d}{d-1} - 1\right) v^d = \frac{1-2d}{d-1} v^d.$$

On the other hand, if $-\overline{w}^k > v^d$, we have $p^k = -v^d/\overline{w}^k$ and get from (11)

$$-\underline{w}^k \leq \frac{d}{d-1} v^d - \overline{w}^k \leq -\frac{d}{d-1}\overline{w}^k - \overline{w}^k = \frac{1-2d}{d-1}\overline{w}^k.$$

Finally, using $\overline{w}^k < -v^d < 0$, we find

$$\underline{w}^k p^k = -\underline{w}^k \frac{v^d}{\overline{w}^k} \geq \frac{1-2d}{d-1} v^d,$$

as in the first case. Applying this estimate to (12) we obtain

$$\varphi(x^*) \geq v^d \left(1 + \frac{(1-2d)(d-1)}{2d(d-1)}\right) = \frac{v^d}{2d} \geq \frac{p^d}{2d} = r^d.$$

Third, to prove the inclusion $S^\Omega = U^\Omega \subset P$, let $(x^*, x^d) \in U^\Omega$. Then the last coordinate satisfies $0 \leq x^d \leq \min\{1, v^d\} = p^d$, i.e.,

$$x^d \in [0, p^d].$$

It remains to show that $x^k \in [0, p^k]$ for all indices $k < d$. If $p^k = 1$, nothing has to be shown. Otherwise, if $-\overline{w}^k > v^d$, we obtain

$$0 \leq \varphi(x^*) \leq \varphi(x^* - x^k e_k) + \overline{w}^k x^k \leq v^d + \overline{w}^k x^k,$$

where e_k denotes the kth unit vector. With $\overline{w}^k < 0$, we finally get the desired estimate

$$x^k \leq -v^d/\overline{w}^k = p^k.$$

□

If the boundary $\partial\Omega$ is locally planar, then $\overline{w}_i = \underline{w}_i$ so that condition (11) is *always* satisfied. In fact, the term $u_i^* * (\overline{w}_i - \underline{w}_i)$ on the right hand side of (11) can be regarded as a measure for the local deviation of the boundary from a plane. If we assume that the size $|S_i|$ of the support is of order h, and that the boundary of Ω is $C^{1,1}$, then $\overline{w}_i - \underline{w}_i$ is also of order h. Hence, the condition is satisfied unless the maximal value v_i^d of φ_i is small of order h^2. That is, as $h \to 0$, only B-splines with a smaller and smaller fraction of their support in Ω may be non-proper. In the next section, this fact will be exploited to show that the additional approximation error related to the reduction of the basis is reasonably small.

But first, we consider a simple example to illustrate the concepts developed so far. Let $\Omega \subset \mathbb{R}^2$ be the unit circle, and $T = h\mathbb{Z} \times h\mathbb{Z}$ a uniform knot grid for bicubic B-splines. The knot spacing is given by $h = (m+\delta)^{-1}$ for some $m \in \mathbb{N}, m \geq 5$, and $\delta \in [0,1)$. The support of b_i is $S_i = ih + [0,4h]^2$, and we consider only B-splines with the center of their support lying in the first quadrant above the bisector. For the index $i = (i^1, i^2)$, this means $-2 \leq i^1 \leq i^2 \leq m$. All other cases are similar. Now, three cases of boundary B-splines b_i are distinguished:

- If $i^1 \geq 0$, then, locally, the boundary is monotone decreasing and convex. Hence, when choosing $P_i = ih + [0, h_i]$ as the bounding box of S_i^Ω, the box $R_i := ih + [0, h_i/4]$ is contained in S_i^Ω so that b_i is proper.

- If $i^1 \in \{-2,-1\}$ and $i^2 < m$, then we choose $P_i := [i^1 h, (i^1+4)h] \times [i^2 h, 1]$ and $R_i := [-h/2, h/2] \times [i^2 h, (3i^2 h + 1)/4]$. The inclusion $R_i \subset S_i^\Omega$ is verified by inspection, showing that b_i is proper, too.

- If $i^1 \in \{-2,-1\}$ and $i^2 = m$, we use the condition given in Theorem 3.5. The maximal value of the function φ_i is $v_i^d = 1 - mh = \delta/(m+\delta)$. The gradient $\varphi'(t) = -t/\sqrt{1-t^2}$ is monotone decreasing. Hence, using some elementary estimates, $\overline{w}_i - \underline{w}_i = \varphi'(i^1 h) - \varphi(i^1 h + 4h) \leq 5h$. With $u_i^* = 4h$, (11) yields the sufficient condition $\delta \geq \frac{10}{m+\delta}$, which is satisfied if $\delta \geq 10/m$.

Summarizing, there are at most $8 \times 2 = 16$ non-proper B-splines. Moreover, for a fine knot sequence (corresponding to a large value of m), only values of h for which h^{-1} is slightly larger than an integer may lead to non-proper B-splines at all. More precisely, $I = I_\bullet$ unless $h^{-1} \in (m, m + 10/m)$.

4 Approximation

Because the focus of this paper is on stability issues, the following discussion of approximation properties is not aimed at full generality. We want to estimate the additional L^p-error introduced by discarding parts of the complete basis B with the goal to show that *"skip and scale"* is a reasonable concept.

Our results rely on approximation properties of complete tensor product spline spaces, which are far from being fully understood. For instance, the classical results in [6] are based on quite restrictive conditions on the geometry of the domain, and those in [7] are geared to splines with equal degree in all coordinate directions. Both results have in common that the constants in the error estimates depend on the aspect ratio of the knot grid, and thus can grow unboundedly if, for instance, the knot sequence in one coordinate direction is repeatedly refined, while the other ones remain fixed. Presumably, this counterintuitive behavior is due to technical limitations in the proofs, and not to the actual nature of spline approximation on grids with largely differing knot spacings in the coordinate directions.

To keep things as simple as possible, let us assume that the degree of the spline space is equal in all directions, i.e.,

$$\nu := n^1 = \cdots = n^d, \quad \overline{\nu} := \nu + 1.$$

Further, we define the global *fineness* and the global *mesh ratio* of the knot sequence T by

$$h := \max_{1\le j\le d} \max_{i\in\mathbb{Z}} \left(\tau^j_{i+1} - \tau^j_i\right), \quad \varrho := h^{-1} \min_{1\le j\le d} \min_{i\in\mathbb{Z}} \left(\tau^j_{i+1} - \tau^j_i\right),$$

respectively. To exclude multiple knots, we require $\varrho > 0$. Further, Ω is assumed to be a bounded $C^{1,1}$*-graph domain*. This means that there exists a finite index set J, a family of $C^{1,1}$-functions

$$\psi_j \colon V^*_j \to \mathbb{R}, \quad j \in J,$$

defined on *open* boxes $V^*_j = (0, v^*_j) \subset \mathbb{R}^{d-1}$, and isometric a^3-maps $\mathcal{J}_j$ such that the boundary $\partial\Omega$ is covered by the images of the graphs G_j of ψ_j under $\mathcal{J}_j$, i.e.,

$$\partial\Omega = \bigcup_{j\in J} \mathcal{J}_j(G_j), \quad G_j := \{(x^*, \psi_j(x^*)) \in \mathbb{R}^d : x^* \in V^*_j\}.$$

Omitting the technical details, we note that such domains have the following properties:

- Because Ω is bounded and $\partial\Omega$ is compact, there exists $h_0 > 0$ such that for any box S with size $|S| < h_0$ the intersection $S \cap \partial\Omega$ is contained in one of the images $\mathcal{J}_j(G_j)$. Hence, for a knot grid T with fineness $h < h_0$, the boundary is projectable on *every* support S_i intersecting the boundary. With $[p_i, q_i] := \mathcal{J}_j^{-1}(S_i)$, it is easily verified that

  $$U_i := [0, q_i - p_i],\ \mathcal{I}_i x := \mathcal{J}_j(x + p_i),\ \varphi_i(x^*) := \psi_j(x^* + p_i^*) - p_i^d,$$

  yields a representation of the restricted support S_i^Ω according to (10).

- By the last equation, Lipschitz constants of the gradients $\nabla\varphi_i$ are bounded by those of the finitely many $\nabla\psi_j$. Hence, there exists a constant L depending only on Ω via the functions ψ_j so that for all i in question

  $$|\nabla\varphi_i(x_1^*) - \nabla\varphi_i(x_2^*)| \le L\,|x_1^* - x_2^*|, \quad x_1^*, x_2^* \in U_i^*, \tag{13}$$

  where the absolute value is understood component-wise.

- The size of the grid cells is bounded from below by $|T_i| \ge \varrho h$. Hence, as a consequence of compactness and smoothness of $\partial\Omega$, the number of grid cells intersecting the boundary is bounded by $\#\{i : T_i \cap \partial\Omega \ne \emptyset\} \le C' h^{1-d}$, where C' depends only on Ω and ϱ. The support of a non-proper B-spline must contain one of these cells, and each cell can be shared by at most $\overline{\nu}^d$ B-splines so that the number of non-proper B-splines is bounded by

  $$\#(I \backslash I_\bullet) \le C' \overline{\nu}^d\, h^{1-d} =: C h^{1-d}. \tag{14}$$

Here and in the following,

$$C = C(\nu, d, p, \Omega, \varrho), \quad \varrho > 0,$$

denotes a generic constant depending only on the specified data, which may take different values at each occurrence. Now, we consider the approximation of a given spline BF by its reduced counterpart

$$B_\bullet F_\bullet := \sum_{i \in I_\bullet} f_i b_i.$$

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^{1,1}$-graph domain, and T a knot grid with fineness $h < h_0$ and mesh ratio $\varrho > 0$. Then there exists a constant C with*

$$\|BF - B_\bullet F_\bullet\|_{p,\Omega} \le C\|F\|_\infty h^{\nu+2/p}.$$

Proof: Using Hölder's inequality and the fact that at most $\overline{\nu}^d$ B-splines cover a given grid cell, a standard argument yields

$$\|BF - B_\bullet F_\bullet\|_{p,\Omega}^p = \Big\| \sum_{i\in I\setminus I_\bullet} f_i b_i \Big\|_{p,\Omega}^p \le \overline{\nu}^{dp/q} \|F\|_\infty^p \sum_{i\in I\setminus I_\bullet} \|b_i\|_{p,\Omega}^p,$$

where $1/p + 1/q = 1$. Further, estimating the number of summands by (14),

$$\|BF - B_\bullet F_\bullet\|_{p,\Omega} \le C\|F\|_\infty h^{(1-d)/p} \max_{i\in I\setminus I_\bullet} \|b_i\|_{p,\Omega} \tag{15}$$

so that it remains to consider the L^p-norm of any non-proper B-spline b_i. If b_i is non-proper then, by Theorem 3.5, there exists an index $j < d$ such that

$$v_i^d < (1 - 1/d)\, u_i^j (\overline{w}_i^j - \underline{w}_i^j).$$

By Lipschitz continuity according to (13), we have $\overline{w}_i^j - \underline{w}_i^j \le L u_i^j$. Further, $u_i^j \le \overline{\nu} h$ so that

$$v_i^d < (1 - 1/d) L \overline{\nu}^2\, h^2 =: Ch^2.$$

Assuming without loss of generality that the isometric a^3-map $\mathcal{I}_i$ appearing in (10) is the identity, we obtain the inclusion

$$S_i^\Omega \subset [0, \overline{\nu} h]^{d-1} \times [0, Ch^2]$$

for the restricted support of b_i. Hence,

$$\|b_i\|_{p,\Omega}^p \le Ch^{d-1} \int_0^{Ch^2} \big(b_{i^d}(t)\big)^p \, dt.$$

For $\tau = 0$, Marsden's identity (2) for the dth coordinate yields

$$\psi_{i^d}(0) b_{i^d}(t) \le \sum_{j \ge i^d} \psi_j(0) b_j(t) = t^\nu, \quad t \ge 0,$$

because $\psi_j(0) = 0$ for $i^d - n \le j < i^d$ and $\psi_j(0) \ge 0$ for $j \ge i^d$. Since the mesh ratio ϱ is assumed to be positive, we have $\psi_{i^d}(0) = \prod_{j=1}^{\nu} \tau_{i^d+j} \ge \nu!(\varrho h)^\nu$ and obtain $b_{i^d}(t) \le C\,(t/h)^\nu$. Hence,

$$\|b_i\|_{p,\Omega}^p \le C\,h^{d-1} \int_0^{Ch^2} (t/h)^{\nu p}\,dt \le C\,h^{\nu p+d+1}.$$

Substituting this estimate into (15), we obtain the desired result. □

With the help of the theorem, we can relate approximation properties of complete to reduced spline spaces. If f is a sufficiently smooth function and BF is a suitable approximating spline, we expect an error estimate of the form

$$\|f - BF\|_{p,\Omega} \le C|f|\,h^{\overline{\nu}}, \tag{16}$$

where $|f|$ is a semi-norm involving certain higher order partial derivatives. If so, the error of approximation by the reduced spline $B_\bullet F_\bullet$ is bounded by

$$\|f - B_\bullet F_\bullet\|_{p,\Omega} \le \|f - BF\|_{p,\Omega} + \|BF - B_\bullet F_\bullet\|_{p,\Omega} \le Ch^{\overline{\nu}}(|f| + h^{2/p-1}\|F\|_\infty).$$

Typically, BF is constructed by quasi-interpolation [1, 6, 10],

$$BF := \Lambda(f) = \sum_{i \in I} \lambda_i(f) b_i,$$

where the λ_i are a family of uniformly bounded functionals,

$$\|\lambda_i\|_p \le C, \quad i \in I. \tag{17}$$

For instance, such functionals can be defined as Hahn-Banach extensions of the de Boor-Fix functionals from the space of polynomials to $L^p(\Omega)$. Hence, $\|F\|_\infty \le C\|f\|_{p,\Omega}$, and

$$\|f - B_\bullet F_\bullet\|_{p,\Omega} \le Ch^{\overline{\nu}}(|f| + h^{2/p-1}\|f\|_{p,\Omega}).$$

This estimate shows that we can expect the optimal approximation order $O(h^{\overline{\nu}})$ in the reduced spline space with respect to the L^p-norm for $p \le 2$. For larger values of p, the order is diminished. The worst case appears for $p = \infty$, where the approximation order is reduced by 1 to $O(h^\nu)$.

We emphasize that the estimates (16), (17) must not be taken for granted. Known approaches to the problem include the following:

- In [6], the estimates (16) and (17) are derived for domains Ω which are *coordinate-wise convex* and satisfy other technical conditions. The important point here is that the semi-norm $|f|$ involves only *pure* partial derivatives,

  $$|f| := \sum_{k=1}^{d} \|\partial_k^{\overline{\nu}} f\|_{p,\Omega}.$$

  In general, for degree $n = (n^1, \ldots, n^d)$, the error estimates read

  $$\|f - BF\| \le C \sum_{k=1}^{d} h^{\overline{n}^k} \|\partial_k^{\overline{n}^k} f\|_{p,\Omega}$$

  and (omitting the proof)

  $$\|f - B_\bullet F_\bullet\| \le C \sum_{k=1}^{d} h^{\overline{n}^k} \big(\|\partial_k^{\overline{n}^k} f\|_{p,\Omega} + h^{2/p-1}\|f\|_{p,\Omega}\big).$$

  This type of estimate is natural for approximation in tensor product spaces, see also [5].

- In [7], the estimates (16) and (17) are derived for web-spline approximation on domains Ω with *Lipschitz boundary.* Here an extension operator due to Stein, see [11], is used to map f to a function with comparable norm defined on an enclosing box. The semi-norm $|f|$ involves *all* partial derivatives of order $\overline{\nu}$,

  $$|f| := \sum_{|\alpha|=\overline{\nu}} \|\partial^\alpha f\|_{p,\Omega}.$$

  Unlike above, there is no natural generalization to the case of unequal degrees $n = (n^1, \ldots, n^d)$, and one has to resort to setting $\nu := \max_k n^k$.

As mentioned above, in all known estimates the constant C in (16) depends on the mesh ratio ϱ.

5 Generalizations

The theory developed so far can be generalized in several ways. Obviously, the constant $2d$ in Definition 3.2 is fixed more or less arbitrarily. Given any $\alpha \ge 1$, we can replace condition *b)* by *b') The sizes of R_i and P_i are related by $\alpha|R_i| = |P_i|$.*

B-splines satisfying this condition are called *α-proper*. The central Theorem 3.3 remains valid with the constant M now depending also on α. The changes in the proof are marginal: the size of the box Q_i satisfies $\alpha\overline{n} * |Q_i| = (1, \ldots, 1)$ so that the constant $C_{n,d,p}$ depends also on α.

When choosing $\alpha = d$, B-splines are d-proper if the boundary is locally planar. However, slight perturbations may always result in a violation of the condition. The relevance of this critical value becomes also apparent if we consider the generalization of Theorem 3.5: a sufficient condition for a B-spline to be α-proper is obtained if (11) is replaced by

$$\frac{\alpha - d}{d - 1}\, v_i^d \geq u_i^* * (\overline{w}_i - \underline{w}_i), \tag{11'}$$

which makes sense only when requiring $\alpha > d$. In the proof, the definition $r := p/(2d)$ has to be replaced by $r := p/\alpha$. Then the subsequent arguments can be carried over almost verbatim. Concerning approximation properties, Theorem 4.1 remains valid when assuming $\alpha > d$ and taking into account that the constant C now depends also on α. In the proof, only the inequality on v^d has to be adapted, $v^d < (d-1)/(\alpha - d)Lh^2 =: Ch^2$.

The choice of α can be used to trade stability for approximation properties: A larger value of α yields a larger set of proper B-splines. This results in a larger bound on the condition number, but fewer B-splines are lost for approximation. For $\alpha \leq d$, potential improvements of the condition number might be thwarted by a substantial loss of approximation power.

A careful analysis of the proof of Theorem 3.3 reveals more sources for generalization: First, the box R_i is only used to find another box Q_i with $\alpha\overline{n} * |Q_i| = |P_i|$ which is contained in a grid cell. Second, the property $S_i^\Omega \subset P_i$ of the box P_i is only used to derive the estimate (9), which can be satisfied also if $S_i^\Omega \not\subset P_i$. Further, this inequality can be generalized by introducing another constant β. If P_i is no longer required to contain S_i^Ω, it has to be ensured that there exists a point $\xi_i \in Q_i \cap P_i$ for evaluation of the de Boor-Fix functionals. This leads us to the following definition:

Definition 5.1 *Let $\Omega \subset \mathbb{R}^d$ be an open domain, and I the set of relevant indices of the given spline space according to* (7)*. A B-spline $b_i, i \in I$, with support S_i is called* (α, β, p)-proper*, if there exist boxes P_i, Q_i with the following properties:*

a') Q_i is contained in an interior grid cell, i.e., $Q_i \subset T_j \cap S_i^\Omega$ for some $j \in \mathbb{Z}^d$. Further, $Q_i \cap P_i \neq \emptyset$, and $\|b_i\|_{p,\Omega} \leq \beta \, \|b_i\|_{p,P_i}$.

b") The sizes of Q_i and P_i are related by $\alpha \overline{n} * |Q_i| = |P_i|$.

c) The boxes P_i and S_i have one corner in common.

For $p \in [1, \infty]$, the sequence of normalized (α, β, p)-proper B-splines is defined by

$$B^p_{\alpha,\beta} := (b^p_i)_{i \in I^p_{\alpha,\beta}}, \quad I^p_{\alpha,\beta} := \{i \in I : b_i \text{ is } (\alpha, \beta, p)\text{-proper}\}.$$

Also this generalized notion of properness yields uniform stability:

Theorem 5.2 *The condition number of the sequence $B^p_{\alpha,\beta}$ of normalized (α, β, p)-proper B-splines is bounded by*

$$\operatorname{cond}_p B^p_{\alpha,\beta} \leq M,$$

where the constant M depends on n, d, p, α, β, but neither on T nor on Ω.

The transcription of the proof is straightforward. Increasing the value of α or β weakens the conditions in Definition 5.1 and thus enlarges the set of (α, β, p)-proper B-splines at the cost of increasing the bound on the condition number.

To illustrate the generalized definition, we consider a non-convex $C^{1,1}$-graph domain $\Omega \subset \mathbb{R}^2$. Even for an arbitrarily fine knot grid, there may exist a B-spline b_i with disconnected restricted support S_i^Ω, see Figure 5. Given α, the B-spline b_i is *not* α-proper if the two components $S^\Omega_{i,1}$ and $S^\Omega_{i,2}$ of S^Ω are small and the gap between them is sufficiently large since P_i is required to contain both components, while R_i must be contained in one of them. On the other hand, for fixed p, consider, e.g., the case $\|b_i\|_{p,S^\Omega_{i,1}} \geq \|b_i\|_{p,S^\Omega_{i,2}}$. We choose P_i as the bounding box of $S^\Omega_{i,1}$ so that $\|b_i\|_{p,\Omega} \leq 2^{1/p} \, \|b_i\|_{p,P_i}$. If the boundary is smooth and $|S_i|$ is sufficiently small, then there exists a box $R_i \subset S^\Omega_{i,1}$ with $4|R_i| = |P_i|$. Since $R_i \subset S_i$, and S_i is subdivided into $n^1 \times n^2$ grid cells, there exists a box $Q_i \subset R_i$ with $\overline{n} * |Q_i| = |R_i|$ which is contained in a grid cell. Hence, b_i is $(4, 2^{1/p}, p)$-proper, independently of the size of the the two components or the gap between them.

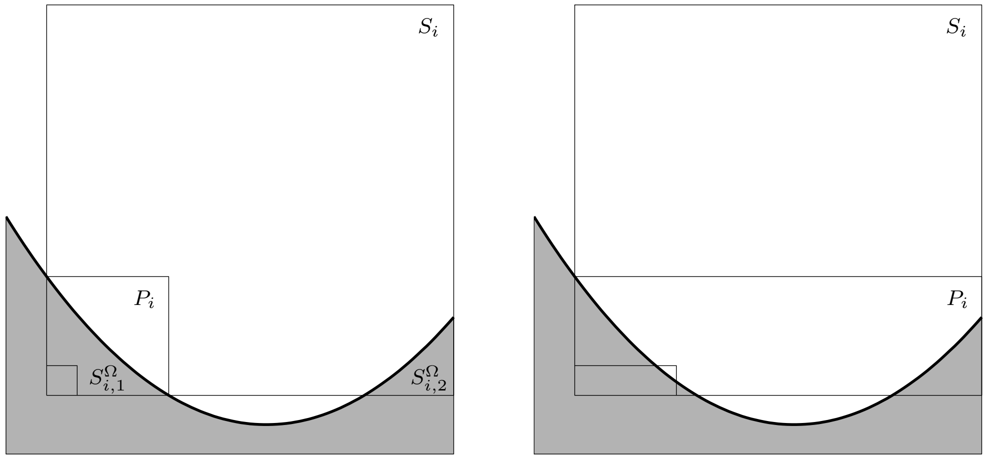

Figure 5: B-spline which is $(4, 2^{1/p}, p)$-proper *(left)* but not proper *(right)*.

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