Polynomial Approximation on Domains Bounded by Diffeomorphic Images of Graphs

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Abstract

We investigate the approximation of functions on domains $\Omega \subset \mathbb{R}^d$ by polynomials of a given coordinate order $\nu \in \mathbb{N}^d$ and also of a given total order $n \in \mathbb{N}$. In the first case, results in the spirit of the Bramble-Hilbert Lemma are obtained for connected domains which are bounded by a finite number of axis-aligned graphs of continuous functions. In the second case, the boundary of the domains may be formed by diffeomorphic images of such graphs.

1 Introduction

In its original form [2, 3], the famous Bramble-Hilbert Lemma guarantees that, for any function $f \in W_p^n(\Omega)$, there exists a polynomial π of order n with

$$\|\partial^\beta(f-\pi)\|_{L^p(\Omega)} \le c \sum_{|\alpha|=n} \|\partial^\alpha f\|_{L^p(\Omega)}, \quad |\beta| \le n,$$

for some constant c, provided that the domain $\Omega \subset \mathbb{R}^d$ satisfies a cone condition. This result, together with many variants and generalizations, is of fundamental importance when studying approximation properties of piecewise polynomial functions on domains.

The approach of Bramble and Hilbert, which in turn relies on work by Morrey [9], is non-constructive. In particular, the dependence of the constant c on the domain Ω or the parameters n, d, p, β is not specified. Subsequent work on the problem aimed at filling this gap, and at weakening the assumntions on Ω. In particular, Dupont and Scott [7] define π as an averaged

Taylor polynomial for the case that Ω is the finite union of sub-domains which are star-shaped with respect to an open ball, thus relaxing the cone property. While Dupont and Scott reveal the principle dependence of the constant on the given parameters, Durán [8] comes up with an explicit formula for c. Amongst others, further results are due to Dechevski and Quak [5], who assume star-shapedness with respect to a point, and to Dekel and Leviathan [6], who prove existence of a uniform constant for convex domains.

However, the existing literature is leaving some questions unanswered: For instance, consider the domain $\Omega_\varepsilon := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 2 + \sin(x/\varepsilon)\}$. Following [7], the number of star-shaped subdomains required to cover Ω_ε is increasing as $\varepsilon \to 0$, thus causing c to grow unboundedly. Is this fact just a consequence of the assumptions required for the known proofs, or does it indicate large approximation errors on domains with highly oscillating boundary? Or consider a domain as simple as $\Omega_* := \{(x, y) \in [0, 1]^2 : x^2 \leq y \leq 2x^2\}$. It neither satisfies a cone condition, nor is it star-shaped in a vicinity of the origin. Thus, available theory does not apply. Is polynomial approximation still possible in this case, or is it inhibited by the horn-like cusp? The results to be derived in this paper imply that there exists a uniform constant c for Ω_ε as $\varepsilon \to 0$, and that also functions on Ω_* can be approximated well by polynomials.

In the next section, we study approximation properties of *tensor product spaces* of polynomials. This case has been addressed in [4] for domains with Lipschitz boundary, and in [7] for star-shaped domains. The interesting point is that the L^p-norm of the error can be estimated by certain *pure* partial derivatives alone. Here, results will be derived for *aligned graph domains*. Such domains are assumed to be bounded by a finite number of axis-aligned graphs of continuous functions. This means that the graphs are defined by one coordinate x_i being a function of the remaining ones. The approximating polynomial π is defined by a standard least squares fit with respect to the L^2-norm on a subdomain of Ω.

In Section 3, we study approximation by polynomials of a given *total order*. The *generalized graph domains* to be considered are assumed to be bounded by a finite number of diffeomorphic images of graphs of continuous functions. This class of domains significantly extends the range of applicability of the Bramble-Hilbert Lemma, for instance to the domain Ω_* defined above. When seeking domains still not covered by this approach, one has to resort to fractal objects, like Koch's snow flake. The approximating polynomials are defined by requiring that the mean values of all derivatives up to the given order are to be interpolated. As a corollary, we establish the Poincaré inequality for generalized graph domains.

We use standard multi-index notation, i.e., for $\alpha = [\alpha_1, \ldots, \alpha_d] \in \mathbb{N}_0^d$, let

$$|\alpha| := \alpha_1 + \cdots + \alpha_d, \quad x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \quad \partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}.$$

The monomials of degree α in d variables are denoted by $m_\alpha : x \mapsto x^\alpha$. Further, let $p, q \in [1, \infty]$ be conjugate exponents, related by $1/p + 1/q = 1$. As usual, we set $1/p = 0$ for $p = \infty$. All proofs will be detailed for the case $p < \infty$, while the much simpler case $p = \infty$ is left as an exercise. The space dimension $d \geq 2$ as well as p, q are assumed to be fixed throughout the paper. We take care that all constants are explicit and easily computable, however without striving for minimality. All dependencies of constants, except for those on d and p, will be declared.

2 Tensor product spaces

For $\nu \in \mathbb{N}^d$, let $\mathbb{P}^\nu := \operatorname{span}\{m_\alpha : \alpha < \nu\}$ denote the tensor product space of d-variate polynomials of coordinate order ν. We start with considering the approximation of an integrable function f by a polynomial in $\mathbb{P}^\nu$ on the unit cube $U := [0,1]^d$. More concretely, the *approximation operator* $P^\nu : L^1(U) \to \mathbb{P}^\nu$ is defined as follows: Let $\ell_\alpha(x) = \ell_{\alpha_1}(x_1) \cdots \ell_{\alpha_d}(x_d)$ denote the normalized tensor product Legendre polynomial of degree $\alpha \in \mathbb{N}_0^d$ on U, i.e.,

$$\langle \ell_\alpha, \ell_\beta \rangle := \int_U \ell_\alpha \ell_\beta = \delta_{\alpha,\beta}, \quad \alpha, \beta \in \mathbb{N}_0^d.$$

Then

$$P^\nu f := \sum_{\alpha<\nu} \langle f, \ell_\alpha \rangle \ell_\alpha$$

is the orthogonal projection of f onto $\mathbb{P}^\nu$ in the L^2-sense. The corresponding *error operator* E^ν is given by $E^\nu f := f - P^\nu f$. For $n \in \mathbb{N}$, let

$$c_0(n) := \sum_{j=0}^{n-1} \|\ell_j\|_{L^p(u)} \|\ell_j\|_{L^q(u)}, \quad u := [0,1],$$

be a constant associated with the univariate Legendre polynomials up to order n. It holds $1 = c_0(1) \leq c_0(n)$. The corresponding approximation operator $P^n : L^1(u) \to \mathbb{P}^n$ is bounded via Hölder's inequality by

$$\|P^n f\|_{L^p(u)} \le c_0(n)\|f\|_{L^p(u)}. \tag{1}$$

For $\Omega \subset \mathbb{R}^d$, the *anisotropic Sobolev space* $W_p^\nu(\Omega)$ is defined as the closure of the set of smooth functions on the interior of Ω with respect to the norm

$$\|f\|_{W_p^\nu(\Omega)} := \|f\|_{L^p(\Omega)} + |f|_{W_p^\nu(\Omega)}, \quad |f|_{W_p^\nu(\Omega)} := \sum_{i=1}^d \|\partial_i^{\nu_i} f\|_{L^p(\Omega)},$$

see [1] for an introduction. That is, the semi-norm $|\cdot|_{W_p^\nu(\Omega)}$ involves only pure partial derivatives of orders $\nu_1, \ldots, \nu_d$ in the respective coordinate directions. The following result provides an estimate for the approximation error of P^ν on the unit cube U. Existence of some constant $c_1(\nu)$ is clear, but the specific value provided below requires an explanation.

Lemma 2.1 *For any $\nu \in \mathbb{N}^d$ and $f \in W_p^\nu(U)$, the error of the approximation $P^\nu f \in \mathbb{P}^\nu$ satisfies*

$$\|E^\nu f\|_{L^p(U)} \le c_1(\nu)\, |f|_{W_p^\nu(U)}, \tag{2}$$

where $c_1(\nu) := c_0(\nu_1) \cdots c_0(\nu_d) d$.

Proof: By density, it suffices to verify the theorem for smooth f. The extension of the univariate operator P^{ν_i} to the multivariate setting is defined in the natural way by

$$(P_i f)(x) := \sum_{j=0}^{\nu_i - 1} \ell_j(x_i) \int_0^1 \ell_j(x_i) f(x)\, dx_i, \quad i = 1, \ldots, d.$$

In other words, P_i is acting on $f(x) = f(x_1, \ldots, x_d)$ like P^{ν_i}, where only x_i is treated as a variable, and all other components as constants. With this notation, the fact that $P^\nu = P^{\nu_1} \otimes \cdots \otimes P^{\nu_d}$ is the tensor product of univariate operators is equivalent to $P^\nu = P_1 \cdots P_d$. The estimate (1) immediately implies

$$\|P_i f\|_{L^p(U)} \le c_0(\nu_i)\|f\|_{L^p(U)}. \tag{3}$$

Now, we consider the operator $E_i := \mathrm{Id} - P_i$. Since f is assumed to be smooth, all partial derivatives $\partial_i^k E_i f$ up to order $k = \nu_i - 1$ have at least one zero on any line in U parallel to the ith coordinate axis. That is, by

Friedrichs' inequality, $\|E_i f\|_{L^p(U)} \le \|\partial_i^1 E_i f\|_{L^p(U)} \le \cdots \le \|\partial_i^{\nu_i} E_i f\|_{L^p(U)}$, and hence

$$\|E_i f\|_{L^p(U)} \le \|\partial_i^{\nu_i} E_i f\|_{L^p(U)} = \|\partial_i^{\nu_i} f\|_{L^p(U)} \le |f|_{W_p^\nu(U)}. \tag{4}$$

Using (3), (4), and the representation

$$E^\nu f = \sum_{i=1}^{d} P_1 \cdots P_{i-1} E_i f$$

of the error, the estimate (2) follows easily. □

Let $|a|_1 := \sum_{j=0}^{n-1} |a_j|$ denote the 1-norm or the vector $a \in \mathbb{R}^n$. Identifying a with the univariate polynomial $a(t) := \sum_{j=0}^{n-1} a_j (t-1)^j$, we define the constant

$$c_2(n) := \max_{a \ne 0} \frac{|a|_1}{\|a\|_{L^1(u)}}.$$

It holds $1 = c_2(1) \le c_2(n)$. Given numbers $M \ge 1$ and $m \in [0, M]$, we have $\|a\|_{L^p([1,1+m])} \le M^n |a|_1 \le M^n c_2(n) \|a\|_{L^1(u)}$, and hence

$$\|a\|_{L^p([1,1+m])} \le c_2(n) M^n \|a\|_{L^p(u)}. \tag{5}$$

We remark that the assumption $M \ge 1$ is made only for technical reasons. In principle, $M \ge 0$ is equally possible, but our choice avoids case distinctions when specifying constants later on. We continue with a lemma concerning one-sided extensions of the unit cube U.

Lemma 2.2 *Let $Y := [0,1]^{d-1}$ and $M \ge 1$. Given a continuous function $\varphi : Y \to [1, 1+M]$, we define the sets*

$$U^* := \{(y, z) \in \mathbb{R}^d : y \in Y,\, 1 \le z \le \varphi(y)\}$$

lying between the hyperplane $z = 1$ and the graph of φ, and $U^+ := U \cup U^*$. Then

$$\|\Delta\|_{L^p(U^*)} \le 2 c_2(\nu_d) M^{\nu_d} \left(\|\Delta\|_{L^p(U)} + \|\partial_d^{\nu_d} \Delta\|_{L^p(U^+)} \right)$$

for any function $\Delta \in W_p^\nu(U^+)$.

Proof: Again, by density, we may assume that Δ is smooth. We write $\Delta(y, z) = T(y, z) + R(y, z)$, where

$$T(y, z) = \sum_{j=0}^{\nu_d - 1} \frac{\partial_d^j \Delta(y, 1)}{j!} (z-1)^j$$

is a function built on univariate Taylor polynomials of Δ, and

$$R(y,z) = \int_1^z \frac{(z-t)^{\nu_d-1}}{(\nu_d-1)!}\,\partial_d^{\nu_d}\Delta(y,t)\,dt$$

is the corresponding remainder in integral form. Considering norms on U^* and on U, respectively, we find

$$\begin{aligned}\|\Delta\|_{L^p(U^*)} &\le \|T\|_{L^p(U^*)} + \|R\|_{L^p(U^*)}\\ \|T\|_{L^p(U)} &\le \|\Delta\|_{L^p(U)} + \|R\|_{L^p(U)}.\end{aligned}$$

By (5), we have $\|T(y,\cdot)\|_{L^p([1,\varphi(y)])} \le c_2(\nu_d)M^{\nu_d}\|T(y,\cdot)\|_{L^p([0,1])}$, and integration over Y yields

$$\|T\|_{L^p(U^*)} \le c_2(\nu_d)M^{\nu_d}\|T\|_{L^p(U)}.$$

Hence,

$$\begin{aligned}\|\Delta\|_{L^p(U^*)} &\le c_2(\nu_d)M^{\nu_d}\|T\|_{L^p(U)} + \|R\|_{L^p(U^*)}\\ &\le c_2(\nu_d)M^{\nu_d}\big(\|\Delta\|_{L^p(U)} + \|R\|_{L^p(U)}\big) + \|R\|_{L^p(U^*)}.\end{aligned} \tag{6}$$

The remainder can be estimated on U^* by

$$|R(y,z)| \le \frac{M^{\nu_d-1}}{(\nu_d-1)!}\int_1^{\varphi(y)} |\partial_d^{\nu_d}\Delta(y,t)|\,dt, \quad z \ge 1.$$

By Hölder's inequality,

$$|R(y,z)|^p \le \frac{M^{p(\nu_d-1+1/q)}}{((\nu_d-1)!)^p}\int_1^{\varphi(y)} |\partial_d^{\nu_d}\Delta(y,t)|^p\,dt.$$

With $\varphi(y)-1 \le M$, integration yields

$$\begin{aligned}\|R\|^p_{L^p(U^*)} &\le \frac{M^{p\nu_d-1}}{((\nu_d-1)!)^p}\int_Y\int_1^{\varphi(y)}\int_1^{\varphi(y)} |\partial_d^{\nu_d}\Delta(y,t)|^p\,dtdzdy\\ &\le \frac{M^{p\nu_d}}{((\nu_d-1)!)^p}\|\partial_d^{\nu_d}\Delta\|^p_{L^p(U^*)}.\end{aligned}$$

Hence, we find the estimate

$$\|R\|_{L^p(U^*)} \le M^{\nu_d}\|\partial_d^{\nu_d}\Delta\|_{L^p(U^*)}.$$

Equally, using $|R(y,z)| \le \int_0^1 |\partial_d^{\nu_d}\Delta(y,t)|\,dt$ for $z \le 1$, the remainder on U is bounded by

$$\|R\|_{L^p(U)} \le \|\partial_d^{\nu_d}\Delta\|_{L^p(U)}.$$

Inserting the last two displays into (6), and using $c_2(\nu_d) \ge 1$, we obtain the desired result. □

Now, we consider domains which are the finite union of images of sets of type U^+ under *aligned isometries* in $\mathbb{R}^d$. Such maps are given by a permutation of coordinates followed by a translation.

Definition 2.3 *Let* $M = [M_1, \ldots, M_J] \in [1,\infty)^J$. *The set* $\Omega \subset \mathbb{R}^d$ *is called an* aligned graph domain *with parameter* M *if there is a nested sequence of subsets*

$$U =: \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_J = \Omega$$

with the following property: For every $j \in \{1, \ldots, J\}$, *there exists a continuous function* $\varphi_j : Y \to [1, 1+M_j]$ *and an aligned isometry* $\mathcal{T}_j$ *such that*

$$\mathcal{T}_j(U) \subset \Omega_{j-1} \quad \text{and} \quad \Omega_j = \Omega_{j-1} \cup \mathcal{T}_j(U_j^*),$$

where U_j^* *denotes the set of points lying between* $z = 1$ *and the graph of* φ_j.

By definition, the aligned isometry $\mathcal{T}_j$ can be written in the form $\mathcal{T}_j(x) = \Pi_j x + v_j$, where Π_j is a $(d \times d)$-permutation matrix and $v_j \in \mathbb{R}^d$. Let

$$V_j := \mathcal{T}_j(U), \quad V_j^* := \mathcal{T}_j(U_j^*), \quad V_j^+ := V_j \cup V_j^*, \quad U_j^+ := U \cup U_j^*.$$

Then $\Delta \in W_p^\nu(V_j^+)$ implies $\Delta' := \Delta \circ \mathcal{T}_j \in W_p^{\nu'}(U_j^+)$, where $\nu' := \Pi_j^{-1}\nu$ is the inversely permuted order vector. Denoting the last component of ν' by $n_j := \nu'_d$, Lemma 2.2 yields

$$\begin{aligned}\|\Delta'\|_{L^p(U_j^*)} &\le 2c_2(n_j)M_j^{n_j}\big(\|\Delta'\|_{L^p(U)} + \|\partial_d^{n_j}\Delta'\|_{L^p(U_j^+)}\big)\\ &\le 2c_2(n_j)M_j^{n_j}\big(\|\Delta'\|_{L^p(U)} + |\Delta'|_{W_p^{\nu'}(U_j^+)}\big)\end{aligned}$$

for any function $\Delta \in W_p^\nu(V_j^+)$. Hence,

$$\|\Delta\|_{L^p(V_j^*)} \le 2c_2(n_j)M_j^{n_j}\big(\|\Delta\|_{L^p(V_j)} + |\Delta|_{W_p^\nu(V_j^+)}\big). \tag{7}$$

Since Ω contains U, we can use the approximation operator P^ν also for approximating functions on Ω.

Theorem 2.4 *Let Ω be an aligned graph domain with parameter $M \in [1,\infty)^J$. For any $\nu \in \mathbb{N}^d$ and $f \in W_p^\nu(\Omega)$, the error of the approximation $P^\nu f \in \mathbb{P}^\nu$ is bounded by*

$$\|E^\nu f\|_{L^p(\Omega)} \leq c_3(\nu, M)\, |f|_{W_p^\nu(\Omega)},$$

where the constant is given by

$$c_3(\nu, M) := c_1(\nu) \prod_{j=1}^{J} \bigl(1 + 4c_2(n_j) M_j^{n_j}\bigr).$$

Proof: The proof is by induction on J. The base case $J = 0$ yields $\Omega = U$, and thus reduces to Theorem 2.1. We consider the domain $\Omega = \Omega_{J-1} \cup V_J^*$ with parameter $M = [\tilde{M}, M_J]$ and assume that the assertion is correct for $J - 1$. That is, we know that the error $\Delta := E^\nu f$ satisfies

$$\|\Delta\|_{L^p(\Omega_{J-1})} \leq c_3(\nu, \tilde{M}) |f|_{W_p^\nu(\Omega_{J-1})} \leq c_3(\nu, \tilde{M}) |f|_{W_p^\nu(\Omega)}.$$

By (7) and $|\Delta|_{W_p^\nu(\Omega)} = |f|_{W_p^\nu(\Omega)}$, we obtain

$$\begin{aligned}
\|\Delta\|_{L^p(\Omega)} &\leq \|\Delta\|_{L^p(\Omega_{J-1})} + \|\Delta\|_{L^p(V_J^*)} \\
&\leq \|\Delta\|_{L^p(\Omega_{J-1})} + 2c_2(n_J) M_J^{n_J} \bigl(\|\Delta\|_{L^p(V_J)} + |\Delta|_{W_p^\nu(V_J^+)}\bigr) \\
&\leq \bigl(1 + 2c_2(n_J) M_J^{n_J}\bigr) \|\Delta\|_{L^p(\Omega_{J-1})} + 2c_2(n_J) M_J^{n_J} |\Delta|_{W_p^\nu(\Omega)} \\
&\leq \Bigl(\bigl(1 + 2c_2(n_J) M_J^{n_J}\bigr) c_3(\nu, \tilde{M}) + 2c_2(n_J) M_J^{n_J}\Bigr) |f|_{W_p^\nu(\Omega)}.
\end{aligned}$$

Since $c_3(\nu, \tilde{M}) \geq 1$, it follows

$$\|\Delta\|_{L^p(\Omega)} \leq \bigl(1 + 4c_2(n_J) M_J^{n_J}\bigr) c_3(\nu, \tilde{M}) |f|_{W_p^\nu(\Omega)} = c_3(\nu, M) |f|_{W_p^\nu(\Omega)},$$

and the proof is complete. □

Let us briefly discuss some variants on the results derived so far without going into details:

- The result of Theorem 2.4 can be generalized by scaling and shifting the domain. Given an invertible diagonal matrix $H := \mathrm{diag}(h_1, \ldots, h_d)$ and some shift vector $s \in \mathbb{R}^d$, let $\mathcal{H}(x) := Hx + s$. Let $\Omega \subset \mathbb{R}^d$ be an aligned graph domain with parameter M, and $\Omega_{\mathcal{H}} := \mathcal{H}(\Omega)$ a non-degenerate scaled copy of it. Then, defining the approximation operator $P^\nu_{\mathcal{H}} : L^1(\Omega_{\mathcal{H}}) \to \mathbb{P}^\nu$ on $\Omega_{\mathcal{H}}$ by $P^\nu_{\mathcal{H}} f := P^\nu(f \circ \mathcal{H}) \circ \mathcal{H}^{-1}$, we obtain the estimate

$$\|f - P^\nu_{\mathcal{H}} f\|_{L^p(\Omega_{\mathcal{H}})} \leq c_3(\nu, M) \sum_{i=1}^{d} |h_i|^{\nu_i} \|\partial_i^{\nu_i} f\|_{L^p(\Omega_{\mathcal{H}})}$$

for any function $f \in W^\nu_p(\Omega_{\mathcal{H}})$.

- The aligned isometries $\mathcal{T}_j$ used in Definition 2.3 can be generalized by permitting to scale down the image by some factor r_j,

  $$\mathcal{T}_j(x) := r_j \Pi_j x + v_j, \quad r_j \in (0, 1].$$

  Using a scaling argument as before, one can easily show that the crucial estimate (7), and hence also Theorem 2.4, remains valid in this case. This means may be useful to resolve small features of domains without unnecessarily inflating the number J of subdomains.

- In some cases, the value of the constant $c_3(\nu, M)$ may be unduly pessimistic by assuming that every cube V_j is grabbing the whole error accumulated so far on Ω_{j-1}. Tighter bounds can be obtained by determining the smallest index $\ell = \ell(j)$ such that $V_j \subset \Omega_{\ell(j)}$. In doing so, the constant in Theorem 2.4 can be reduced to a possibly much smaller value $\tilde{c}_3(\nu, M)$, defined by the recursion

  $$\begin{aligned} \tilde{c}_0 &:= c_1(\nu) \\ \tilde{c}_j &:= \tilde{c}_{j-1} + \tilde{c}_{\ell(j)}(1 + 2c_2(\nu) M_j^{n_j}), \quad j = 1, \ldots, J, \\ \tilde{c}_3(\nu, M) &:= \tilde{c}_J. \end{aligned}$$

Finally, we remark that, in general, it is not possible to estimate derivatives of the error in terms of the pure partial derivatives of f. As an example, consider the function $f(x, y) := x^2 \ln(1 + y)$ defined on Ω, the unit disc. For $\nu = [3, 1]$, we have $f \in W^\nu_\infty(\Omega)$, and Theorem 2.4 shows that the approximation error is bounded by a constant times $|f|_{W^\nu_\infty(\Omega)} = 2$. However, $\partial_x^2 \partial_y f$ is unbounded, and so is $\partial_x^2 \partial_y E^\nu f$.

3 Total order spaces

For $n \in \mathbb{N}$, let $\mathbb{P}^n := \operatorname{span}\{m_\alpha : |\alpha| < n\}$ denote the space of polynomials in d variables of total order n. The standard Sobolev space $W^n_p(\Omega)$ is defined as the closure of the set of smooth functions on the interior of Ω with respect to the norm

$$\|f\|_{W_p^n(\Omega)} := \sum_{k=0}^{n} |f|_{W_p^k(\Omega)}, \quad |f|_{W_p^k(\Omega)} := \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(\Omega)}.$$

For $n = 1$ and $\nu = [1, \dots, 1]$, the spaces $W_p^n(\Omega)$ and $W_p^\nu(\Omega)$ coincide. Hence, Theorem 2.4 yields the Poincaré-type inequality

$$\|f - P^1 f\|_{L^p(\Omega)} \le c_3([1, \dots, 1], M)\, |f|_{W_p^1(\Omega)},$$

where $P^1 f := \fint_U f$ is the mean value of f on U. So far, Ω was assumed to have the special properties of an aligned graph domain. However, the estimate above is valid for a much larger class of domains. These domains are defined recursively, just as in Definition 2.3, but now, the sets V_j^+ and U_j^+ may be related by arbitrary diffeomorphisms.

Definition 3.1 *Let $\mathcal{T} = [\mathcal{T}_1, \dots, \mathcal{T}_J]$ be a sequence of diffeomorphisms $\mathcal{T}_j : \mathbb{R}^d \to \mathbb{R}^d$, and let $M = [M_1, \dots, M_J] \in [1, \infty)^J$. The set $\Omega \subset \mathbb{R}^d$ is called a* generalized graph domain *with parameters $\mathcal{T}$ and M if there is a nested sequence of subsets*

$$U =: \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_J = \Omega$$

with the following property: For every $j \in \{1, \dots, J\}$, there exists a continuous function $\varphi_j : Y \to [1, 1 + M_j]$ such that

$$\mathcal{T}_j(U) \subset \Omega_{j-1} \quad \text{and} \quad \Omega_j = \Omega_{j-1} \cup \mathcal{T}_j(U_j^*),$$

where U_j^* denotes the set of points lying between $z = 1$ and the graph of φ_j.

In the preceding section, the assumption that $\mathcal{T}_j$ be an aligned isometry was only used to derive the estimate (7) from Lemma 2.2. So let us adapt the argument to the special case $\nu = [1, \dots, 1]$ and the relaxed assumptions on Ω. As before, let

$$V_j := \mathcal{T}_j(U), \quad V_j^* := \mathcal{T}_j(U_j^*), \quad V_j^+ := V_j \cup V_j^*, \quad U_j^+ := U \cup U_j^*.$$

Since $\mathcal{T}_j$ is assumed to be a diffeomorphism on all of $\mathbb{R}^d$, its Jacobian $\partial \mathcal{T}_j = [\partial_1 \mathcal{T}_j, \dots, \partial_d \mathcal{T}_j]$ is bounded on the compact set U^+, and $|\det \partial \mathcal{T}_j|$ is bounded away from zero. Hence, with $t_j := |\partial_d \mathcal{T}_j|_q$ denoting the q-norm of the vector $\partial_d \mathcal{T}_j$, the constants

$$\underline{\tau}_j := \| \det \mathcal{T}_j^{-1} \|_{L^\infty(U_j^+)}^{1/p}, \quad \overline{\tau}_j := \| \det \mathcal{T}_j \|_{L^\infty(U_j^+)}^{1/p}, \quad \tau_j := \|t_j\|_{L^\infty(U_j^+)}$$

and

$$T_j := \underline{\tau}_j \overline{\tau}_j \max\{1, \tau_j\}$$

are well defined. In particular, if $\mathcal{T}_j$ is an isometric map, then $\underline{\tau}_j = \overline{\tau}_j = 1$, and $\partial_d \mathcal{T}_j$ is some constant vector with length $|\partial_d \mathcal{T}_j|_2 = 1$ so that $T_j \leq \sqrt{d}$.

Given a smooth function Δ on V_j^+, let $\Delta' := \Delta \circ \mathcal{T}_j$. The change of coordinates induced by $\mathcal{T}_j$ yields

$$\|\Delta'\|_{L^p(U_j^+)} \leq \underline{\tau}_j \|\Delta\|_{L^p(V_j^+)}, \quad \|\Delta\|_{L^p(V_j^+)} \leq \overline{\tau}_j \|\Delta'\|_{L^p(U_j^+)}.$$

Moreover, by the chain rule, the derivatives of Δ and Δ' are related by $\partial_d \Delta' = \langle \partial \Delta \circ \mathcal{T}_j, \partial_d \mathcal{T}_j \rangle$. Let $|a|_p$ denote the p-norm of the vector $a \in \mathbb{R}^d$. Then Hölder's inequality yields

$$\|\partial_d \Delta'\|_{L^p(U_j^+)}^p \leq \int_{U_j^+} |\partial \Delta \circ \mathcal{T}_j|_p^p \cdot |\partial_d \mathcal{T}_j|_q^p \leq (\tau_j \underline{\tau}_j)^p \int_{V_j^+} |\partial \Delta|_p^p$$

so that $\|\partial_d \Delta'\|_{L^p(U_j^+)} \leq \tau_j \underline{\tau}_j |\Delta|_{W_p^1(V_j^+)}$. By density, this estimate is valid for any function $\Delta \in W_p^1(V_j^+)$. By Lemma 2.2 and $c_2(1) = 1$,

$$\begin{aligned}
\|\Delta\|_{L^p(V_j^*)} &\leq \overline{\tau}_j \|\Delta'\|_{L^p(U_j^*)} \leq 2\overline{\tau}_j \underline{\tau}_j M_j \big(\|\Delta\|_{L^p(V_j)} + \tau_j |\Delta|_{W_p^1(V_j^+)}\big) \\
&\leq 2T_j M_j \big(\|\Delta\|_{L^p(V_j)} + |\Delta|_{W_p^1(V_j^+)}\big).
\end{aligned}$$

Replacing (7) by this estimate, a verbatim transcription of the arguments used in the proof of Theorem 2.4 for the special case $\nu = [1, \ldots, 1]$ yields

$$\|f - P^1 f\|_{L^p(\Omega)} \leq c_4(\mathcal{T}, M)\, |f|_{W_p^1(\Omega)}, \tag{8}$$

where the constant is now given by

$$c_4(\mathcal{T}, M) := d \prod_{j=1}^{J} (1 + 4T_j M_j).$$

For general $n \in \mathbb{N}$, we define the approximation operator $P^n : W_1^{n-1}(\Omega) \to \mathbb{P}^n$ and the associated error operator $E^n := \mathrm{Id} - P^n$ recursively as follows: Let $P^1 f := \int_U f$ and

$$P^n f := P^{n-1} f + \sum_{|\alpha|=n-1} \frac{P^1(\partial^\alpha f)}{\alpha!} E^{n-1} m_\alpha, \quad n \geq 2,$$

where, as before, $m_\alpha : x \mapsto x^\alpha$ are the monomials in d variables. It is easily verified by induction that $P^n f$ is the unique polynomial in $\mathbb{P}^n$ satisfying the mean value interpolation condition

$$P^1 \partial^\alpha E^n f = \int_U \partial^\alpha f - \int_U \partial^\alpha P^n f = 0, \quad |\alpha| < n. \tag{9}$$

Theorem 3.2 *Let Ω be a generalized graph domain with parameters $\mathcal{T}$ and $M \in [1, \infty)^J$. For any $n \in \mathbb{N}$ and $f \in W_p^n(\Omega)$, the error of the approximation $P^n f \in \mathbb{P}^n$ satisfies*

$$|E^n f|_{W_p^k(\Omega)} \leq c_5(\mathcal{T}, M)^{n-k} \, |f|_{W_p^n(\Omega)}, \quad 0 \leq k \leq n,$$

where the constant is given by $c_5(\mathcal{T}, M) := d c_4(\mathcal{T}, M)$.

Proof: The proof is by induction on k, decrementing from the case $k = n$, which is trivial. Let us assume that the assertion is correct for some index $k > 0$. Then, for any multi-index α with $|\alpha| = k-1$, consider $\partial^\alpha \Delta := \partial^\alpha E^n f$. By (9), we have $P^1 \partial^\alpha \Delta = 0$. Hence, we conclude from (8)

$$\|\partial^\alpha \Delta\|_{L^p(\Omega)} = \|\partial^\alpha \Delta - P^1 \partial^\alpha \Delta\|_{L^p(\Omega)} \leq c_4(\mathcal{T}, M) |\partial^\alpha \Delta|_{W_p^1(\Omega)},$$

and further

$$|\Delta|_{W_p^{k-1}(\Omega)} = \sum_{|\alpha|=k-1} \|\partial^\alpha \Delta\|_{L^p(\Omega)} \leq c_4(\mathcal{T}, M) \sum_{|\alpha|=k-1} \sum_{i=1}^{d} \|\partial_i \partial^\alpha \Delta\|_{L^p(\Omega)}.$$

In the sum on the right hand side, every partial derivative $\partial^\beta \Delta$ of order $|\beta| = k$ appears at most d times. Hence, by the induction hypothesis,

$$\begin{aligned} |\Delta|_{W_p^{k-1}(\Omega)} &\leq c_5(\mathcal{T}, M) \sum_{|\beta|=k} \|\partial^\beta \Delta\|_{L^p(\Omega)} = c_5(\mathcal{T}, M) |\Delta|_{W_p^k(\Omega)} \\ &\leq c_5(\mathcal{T}, M)^{n-k+1} |f|_{W_p^n(\Omega)}, \end{aligned}$$

verifying the claim for $k-1$. $\square$

As an immediate consequence of the theorem, we can establish the Poincaré inequality for generalized graph domains:

Corollary 3.3 *Let Ω be a generalized graph domain with parameters $\mathcal{T}$ and $M \in [1,\infty)^J$, and define the mean value operator $M^1 : L^1(\Omega) \to \mathbb{P}^1$ by $M^1 f := \frac{1}{|\Omega|}\int_\Omega f$. Then, for any $f \in W_p^1(\Omega)$,*

$$\|f - M^1 f\|_{L^p(\Omega)} \le 2c_5(\mathcal{T}, M)|f|_{W_p^1(\Omega)}.$$

Proof: Since $M^1 P^1 = P^1$, we have

$$\|M^1 f - P^1 f\|_{L^p(\Omega)} = \|M^1 E^1 f\|_{L^p(\Omega)} \le |\Omega|^{1/p-1}\|E^1 f\|_{L^1(\Omega)} \le \|E^1 f\|_{L^p(\Omega)},$$

and the claim follows from

$$\|f - M^1 f\|_{L^p(\Omega)} \le \|E^1 f\|_{L^p(\Omega)} + \|P^1 f - M^1 f\|_{L^p(\Omega)} \le 2\|E^1 f\|_{L^p(\Omega)}$$

and Theorem 3.2. $\square$

Possible generalizations of the results derived in this section include the following:

- Let $\mathcal{H} : x \mapsto Hx + s$ denote an invertible affine map in $\mathbb{R}^d$ and consider the transformed domain $\Omega_{\mathcal{H}} := \mathcal{H}(\Omega)$. Then, defining the operators $P^n_{\mathcal{H}} : W_1^{n-1}(\Omega_{\mathcal{H}}) \to \mathbb{P}^n$ and $M^1_{\mathcal{H}} : L^1(\Omega_{\mathcal{H}}) \to \mathbb{P}^1$ by

  $$P^n_{\mathcal{H}} f := P^n(f \circ \mathcal{H}) \circ \mathcal{H}^{-1} \quad \text{and} \quad M^1_{\mathcal{H}} f := M^1(f \circ \mathcal{H}) \circ \mathcal{H}^{-1},$$

  respectively, one can easily show that

  $$\begin{aligned} |f - P^n_{\mathcal{H}} f|_{W_p^k(\Omega_{\mathcal{H}})} &\le h^{n-k} c_5(\mathcal{T}, M)^{n-k} |f|_{W_p^n(\Omega_{\mathcal{H}})}, \quad k \le n \\ \|f - M^1_{\mathcal{H}} f\|_{L^p(\Omega_{\mathcal{H}})} &\le 2h c_5(\mathcal{T}, M)|f|_{W_p^1(\Omega_{\mathcal{H}})}. \end{aligned}$$

  Here, with $e := [1; \ldots; 1]$ denoting the column vector of all ones, h is given by $h := |He|_q$.

- As in the discussion at the end of the preceding section, the constant in Theorem 3.2 and Corollary 3.3 can be reduced to $\tilde{c}_5(\mathcal{T}, M)$, defined recursively by

  $$\begin{aligned} \tilde{c}_0 &:= d^2 \\ \tilde{c}_j &:= \tilde{c}_{j-1} + \tilde{c}_{\ell(j)}(1 + 2T_j M_j^{n_j}), \quad j = 1, \ldots, J, \\ \tilde{c}_5(\nu, M) &:= \tilde{c}_J. \end{aligned}$$

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