Asymptotic profile of steady Stokes flow around a rotating obstacle

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Dedicated to our colleague Jiří Neustupa on the occasion of his 60th birthday

Abstract

We analyze the spatial anisotropic profile at infinity of steady Stokes flow around a rotating obstacle. It is shown that the flow is largely concentrated along the axis of rotation in the leading term and that a rotating profile can be found in the second term. The proof relies upon a detailed analysis of the associated fundamental solution tensor.

Key words: Asymptotic profile; steady Stokes flow; rotating obstacle
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1 Introduction and main result

This paper studies the asymptotic anisotropic profile near infinity of steady Stokes flow around a rotating obstacle immersed in a viscous incompressible fluid. Considering the motion of a viscous incompressible fluid in an exterior domain $D \subset \mathbb{R}^3$ with smooth boundary ∂D, particular interest is focussed on the case where the rigid body ($\equiv \mathbb{R}^3 \setminus D$) is rotating about the y_3-axis with constant angular velocity $\omega = ae_3$; here $a \in \mathbb{R} \setminus \{0\}$ and $e_3 = (0,0,1)^T$. The unknown velocity $v(y,t) = (v_1, v_2, v_3)^T$ and pressure $q(y,t)$ obey the Navier-Stokes equation

$$\partial_t v + v \cdot \nabla v = \Delta v - \nabla q, \qquad \text{div } v = 0$$

for $y \in D(t)$ subject to the boundary condition

$$v|_{\partial D(t)} = \omega \times y \equiv a(-y_2, y_1, 0)^T, \qquad v \to 0 \quad \text{as } |y| \to \infty.$$

Here, the domain $D(t)$ occupied by the fluid at time t and its boundary $\partial D(t)$ are given by

$$D(t) = \{y = O(at)x;\, x \in D\}, \qquad \partial D(t) = \{y = O(at)x;\, x \in \partial D\},$$

where

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.1}$$

We take the reference frame attached to the body to reduce the problem above to the system

$$\partial_t u + u \cdot \nabla u = \Delta u + (\omega \times x) \cdot \nabla u - \omega \times u - \nabla p, \qquad \text{div } u = 0 \tag{1.2}$$

in D subject to

$$u|_{\partial D} = \omega \times x, \qquad u \to 0 \quad \text{as } |x| \to \infty, \tag{1.3}$$

where $x = O(at)^T y$ and

$$u(x,t) = (u_1, u_2, u_3)^T = O(at)^T v(O(at)x, t), \quad p(x,t) = q(O(at)x, t), \tag{1.4}$$

see [1], [12], [20]. A difficulty is the hyperbolic operator $(\omega \times x) \cdot \nabla$, which is no longer a minor perturbation of the Laplace operator even though $|\omega|$ is small. A typical hyperbolic effect was found by Farwig and Neustupa [8] in the study of the spectrum; the essential spectrum consists of an infinite set of equally spaced half lines in the left half of the complex plane. Nevertheless, the semigroup generated by the operator in the right-hand side of (1.2) possesses a certain smoothing effect ([17], [19], [20]) and also enjoys a typical large time behavior of parabolic type ([22]).

In [13] Galdi first proved that the problem (1.2)–(1.3) has a unique steady solution which satisfies

$$|u(x)| \le \frac{C}{|x|}, \qquad |\nabla u(x)| + |p(x)| \le \frac{C}{|x|^2} \tag{1.5}$$

for large $|x|$ provided $|\omega|$ is small enough. Later on, the present authors [6] gave another outlook on (1.5) in terms of weak-L^q spaces: $u \in L^{3,\infty}$, $(\nabla u, p) \in L^{3/2,\infty}$. This class or the pointwise decay (1.5) of the steady flow is important to deduce its stability, which has been established by [14] and, later on, by

[22]. The rate of decay (1.5) is the same as that of usual Navier-Stokes flow in 3D exterior domains in which the rotation of the body is absent ($\omega = 0$). But some effects of rotation should be found in the profile of the flow, which might be anisotropic since the axis of rotation singles out a special direction. Up to now, however, we have no such information.

Toward a better understanding of the effect of rotation on the profile, in this paper, we concentrate ourselves on the linear steady problem

$$-\Delta u - (\omega \times x) \cdot \nabla u + \omega \times u + \nabla p = f, \qquad \text{div } u = 0 \qquad \text{in } D. \tag{1.6}$$

Our purpose is to find the asymptotic representation as $|x| \to \infty$ of the solution to (1.6) subject to (1.3). By [6] and [21] we already know that the optimal rate of decay of the solution to (1.6) is $1/|x|$ in general even if the external force has good properties such as, for instance, $f = \text{div } F$ with $F \in C_0^\infty(D)^{3\times 3}$. Our asymptotic representation provides its rigorous explanation when we look at the first two leading terms. Roughly speaking, the main result of this paper tells us that the profile of the leading term which decays like $1/|x|$ is the third column vector of the usual Stokes fundamental solution tensor and that the second term which decays like $1/|x|^2$ includes the rotating profile $e_3 \times x$. Thus the direction of the axis of rotation, the x_3-axis, is actually preferred in the sense that the flow can be observed mainly along that axis.

For the sake of simplicity to catch the profile, the external force is of the form $f = \text{div } F$ with $F \in C_0^\infty(\overline{D})^{3\times 3}$, the restriction of $F \in C_0^\infty(\mathbb{R}^3)^{3\times 3}$ to $\overline{D}$ (although divergence form is not needed, see Theorem 4.1 below). It is proved by the present authors [6] that, for any $F \in L^{3/2,\infty}(D)^{3\times 3}$, the problem (1.6) with $f = \text{div } F$ subject to the homogeneous boundary condition $u|_{\partial D} = 0$ possesses a unique solution (u, p) with estimate

$$\|u\|_{L^{3,\infty}} + \|(\nabla u, p)\|_{L^{3/2,\infty}} \le C\|F\|_{L^{3/2,\infty}}. \tag{1.7}$$

For the problem (1.6) subject to (1.3), we have only to subtract an auxiliary function given by (5.1) below; then, we obtain a solution (u, p) that satisfies (1.7) with $\|F\|_{L^{3/2,\infty}}$ replaced by $\|F\|_{L^{3/2,\infty}} + |\omega|$ in the right-hand side.

To describe our main result we will use the usual Stokes fundamental solution, i.e., the pair

$$E_{St}(x) = \frac{1}{8\pi}\left(\frac{1}{|x|}\mathbb{I} + \frac{x \otimes x}{|x|^3}\right), \qquad Q_{St}(x) = \nabla\left(\frac{-1}{4\pi|x|}\right) = \frac{x}{4\pi|x|^3}, \tag{1.8}$$

where $\mathbb{I}$ is the 3×3-identity matrix and $x \otimes x = (x_i x_j)_{1\le i,j\le 3}$. Corresponding to the geometry of the problem, vectors in $\mathbb{R}^3$ are often written in the form

$$x = (x', x_3)^T \quad \text{where} \quad x' = (x_1, x_2)^T.$$

Moreover, we need the Cauchy stress tensor

$$T = T(u,p) = \nabla u + (\nabla u)^T - p\mathbb{I}\,. \tag{1.9}$$

Let $\nu \in \mathbb{R}^3$ denote the exterior unit normal to the boundary ∂D. Then $\nu\cdot(T+F) = \big((\nu\cdot(T+F))', (\nu\cdot(T+F))_3\big)^T$ stands for the vector $\big(\sum_j (T_{ij} + F_{ij})\nu_j\big)_{1\le i\le 3}$.

Theorem 1.1 *Let $\omega = ae_3$ with $a \in \mathbb{R}\setminus\{0\}$. Given $f = \operatorname{div} F$ with $F \in C_0^\infty(\overline{D})^{3\times 3}$, let (u,p) be the solution to (1.6) subject to (1.3). Then it has the representation*

$$u(x) = U_{1st}(x) + U_{2nd}(x) + \left(1 + \frac{1}{|a|}\right) O\left(\frac{1}{|x|^3}\right), \tag{1.10}$$

$$p(x) = P_{1st}(x) + O\left(\frac{1}{|x|^3}\right) \tag{1.11}$$

for $|x| \to \infty$ with

$$\begin{aligned} U_{1st}(x) &= \frac{1}{8\pi}\int_{\partial D} (\nu\cdot(T+F))_3\, d\sigma_y \left(\frac{e_3}{|x|} + \frac{x_3 x}{|x|^3}\right) \\ &= E_{St}(x)\begin{pmatrix} 0 \\ 0 \\ \int_{\partial D}(\nu\cdot(T+F))_3\, d\sigma_y \end{pmatrix}, \end{aligned} \tag{1.12}$$

$$\begin{aligned} U_{2nd}(x) &= \frac{1}{8\pi|x|^3}\begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \frac{3\,(x\otimes x)}{8\pi|x|^5}\begin{pmatrix} \frac{\alpha'}{2}x_1 \\ \frac{\alpha'}{2}x_2 \\ \alpha_3 x_3 \end{pmatrix} \\ &= \frac{\beta(e_3\times x)}{8\pi|x|^3} + \big(\alpha_3 - \frac{\alpha'}{2}\big)\,\frac{|x'|^2 - 2x_3^2}{|x|^2}\,\frac{x}{8\pi|x|^3}\,, \end{aligned} \tag{1.13}$$

and

$$P_{1st}(x) = \int_{\partial D} \{(\nu\cdot(\Delta u))\,y - p\nu + \nu\cdot F\}\, d\sigma_y \cdot Q_{St}(x). \tag{1.14}$$

Here, the quantities α, α', α_3 and β are defined as

$$\begin{aligned} \alpha &= -\int_{\partial D} y\cdot(\nu\cdot(T+F))\, d\sigma_y + \int_D \operatorname{tr} F\, dy = \alpha' + \alpha_3, \\ \alpha' &= -\int_{\partial D} y'\cdot(\nu\cdot(T+F))'\, d\sigma_y + \int_D (F_{11}+F_{22})\, dy, \\ \alpha_3 &= -\int_{\partial D} y_3(\nu\cdot(T+F))_3\, d\sigma_y + \int_D F_{33}\, dy, \\ \beta &= e_3\cdot\int_{\partial D} y\times(\nu\cdot(T+F))\, d\sigma_y + \int_D (F_{12}-F_{21})\, dy\,. \end{aligned}$$

Remark 1.1 *In view of* (1.12)*, the profile of the leading term is* $E_{St}(x)e_3$*, that is, the third column vector of the Stokes fundamental solution* (1.8)*. Since* $E_{St}(x)e_3$ *is symmetric about the axis of rotation (*x_3*-axis), we have*

$$(\omega \times x) \cdot \nabla E_{St}(x)e_3 - \omega \times E_{St}(x)e_3 = 0$$

in $\mathbb{R}^3 \setminus \{0\}$ *and, furthermore, this equality holds in* $\mathcal{D}'(\mathbb{R}^3)^3$*; thus, we find*

$$-\Delta E_{St}(x)e_3 - (\omega \times x) \cdot \nabla E_{St}(x)e_3 + \omega \times E_{St}(x)e_3 + \nabla\left(e_3 \cdot Q_{St}(x)\right) = \delta e_3$$

together with $\operatorname{div} E_{St}(x)e_3 = 0$ *in the sense of distributions, where* δ *denotes the Dirac measure at* 0*.*

Remark 1.2 *When* $\omega = 0$*, it is easy to see that the usual Stokes flow satisfies the asymptotic representation*

$$u(x) = \int_{\partial D} \nu \cdot (T + F)\, d\sigma_y \cdot E_{St}(x) + O\left(\frac{1}{|x|^2}\right)$$

for $|x| \to \infty$*, where* $\int_{\partial D} \nu \cdot (T + F)\, d\sigma$ *stands for the total net force exerted on the boundary* ∂D *by the fluid and the external force term* F*. Compared to this case, our leading term* $U_{1st}(x)$ *given by* (1.12) *shows that the third component of the net force is sufficient to control the rate of decay of* $u(x)$*. For example, if the obstacle, the external force and the solution are symmetric with respect to the* x_1x_2*-plane, we find* $e_3 \cdot \int_{\partial D} \nu \cdot (T + F)\, d\sigma = 0$ *to conclude that* $u(x) = O(1/|x|^2)$*.*

Remark 1.3 *In* (1.10) *the remaining term which decays like* $1/|x|^3$ *must be singular as* $a \to 0$ *in view of Remark 1.2; in fact, the last term of* (1.10) *means that*

$$|u(x) - \{U_{1st}(x) + U_{2nd}(x)\}| \leq \left(1 + \frac{1}{|a|}\right) \frac{C}{|x|^3}$$

for large $|x|$*. On the other hand, the leading term* $P_{1st}(x)$ *of the pressure in* (1.11) *takes the same form as in the case* $\omega = 0$*. This is because*

$$\operatorname{div}\, [(\omega \times x) \cdot \nabla u - \omega \times u] = (\omega \times x) \cdot \nabla \operatorname{div}\, u = 0 \qquad (1.15)$$

and because

$$\nu \cdot [(\omega \times x) \cdot \nabla u - \omega \times u]\Big|_{\partial D} = 0. \qquad (1.16)$$

Remark 1.4 *When the direction $\omega/|\omega|$ of the rotating axis is generally prescribed (this can be reduced to the case of $\omega/|\omega| = e_3$ discussed here), the leading term is given by*

$$\left(\frac{\omega}{|\omega|} \cdot \int_{\partial D} \nu \cdot (T + F)\, d\sigma_y\right) E_{St}(x) \frac{\omega}{|\omega|}.$$

Remark 1.5 *In the second term $U_{2nd}(x)$, see* (1.13)*, the first part of the coefficient β of the rotating profile $e_3 \times x$ is the third component of*

$$\int_{\partial D} y \times (\nu \cdot (T + F))\, d\sigma_y,$$

which stands for the total torque exerted on the boundary ∂D.

The proof of Theorem 1.1 is based on the potential representation formula (3.1) of the solution in terms of the fundamental solution $\Gamma(x, y)$ of the equation (2.1). The crucial step is to divide $\Gamma(x, y)$ exactly into three parts $\Phi_1(x) + \Phi_2(x, y) + O(1/|x|^3)$ with $\Phi_k \sim 1/|x|^k$ $(k = 1, 2)$ for $|x| \geq 2R \geq 2|y|$, where $R > 0$ is fixed. The principle that oscillations may imply rapid decay plays an important role; so, we have to find which part does not oscillate to derive $\Phi_1(x)$ and $\Phi_2(x, y)$.

For the problem in which the body is translating with constant velocity, the anisotropic decay structure yielding a wake region behind the body is well known; see Finn [9], [10], Farwig [3], [4], Galdi [11], Shibata [28] and the references therein. One may take into account translation of the body along the axis of rotation together with rotation. In this case as well, one can still find a wake region, existence of which was proved by [15] for the Navier-Stokes problem and also by [25] for the Stokes problem in terms of weighted-L^2 spaces. In both papers, however, no effect of rotation was found; it seems that such an effect is hidden behind the drastic effect of translation and the corresponding wake.

We should note an advantage arising from the translation of the body. For the purely translating problem without rotation, the leading profile of the Navier-Stokes flow is the Oseen fundamental solution because the nonlinear term decays faster due to a better decay structure outside the wake region, see for instance [4]. If we take both translation and rotation into account, very probably, the same reasoning implies that the leading profile of the Stokes flow becomes that of the Navier-Stokes flow as well. However, that is not the case when the translation of the body is absent. In fact, Deuring and Galdi [2] proved that the leading profile of the Navier-Stokes flow (with zero velocity at infinity) is no longer the Stokes fundamental solution; in other words, the effect of nonlinearity must be involved. On the other hand, Kozono, Sohr and Yamazaki [24] suggested that the leading term should be related to the net force

$$N = \int_{\partial D} \nu \cdot (T(u,p) + F - u \otimes u)\; d\sigma.$$

In fact, they showed that $N = 0$ if the Navier-Stokes flow u (as weak solution with finite Dirichlet integral) belongs to L^3. The asymptotic representation has been studied by Nazarov and Pileckas [27], and recently, by Korolev and Šverák [23]. In particular, the latter paper shows that the leading term is the Landau solution which is a (-1)-homogeneous solution of the Navier-Stokes equation in $\mathbb{R}^3$ with the forcing term $N\delta$, where δ denotes the Dirac measure at 0 and N is the net force above of the given Navier-Stokes flow. This fact combined with Theorem 1.1 suggests that the leading term of the steady solution to (1.2)–(1.3) is the Landau solution for the forcing term $(e_3 \cdot N)e_3\delta$; this will be discussed in a forthcoming paper.

This paper consists of five sections. In the next section we look for the fundamental solution of our linear equation in $\mathbb{R}^3$. In terms of it, Section 3 is devoted to the deduction of a potential representation formula of the solution. Section 4 is the main part in which a detailed analysis of the fundamental solution is carried out. With its help we derive the leading and second terms of the solution u to (1.6) subject to the homogeneous boundary condition $u|_{\partial D} = 0$. In the final section we complete the proof of Theorem 1.1 by reducing the problem with the boundary condition (1.3) to the case discussed in section 4. The asymptotic representation of the pressure is also considered finally.

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2 Fundamental solution

In this section we find an explicit representation of the fundamental solution, closely related to the profile of the flow and thus playing a crucial role.

We say that the pair of a 3×3-matrix $\Gamma(x,y)$ and a column vector $Q(x,y)$ is the fundamental solution of the equation

$$-\Delta u - (\omega \times x) \cdot \nabla u + \omega \times u + \nabla p = f, \qquad \operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^3 \tag{2.1}$$

if the volume potentials

$$u(x) = \int_{\mathbb{R}^3} \Gamma(x,y) f(y)\, dy, \qquad p(x) = \int_{\mathbb{R}^3} Q(x,y) \cdot f(y)\, dy$$

solve (2.1) for all $f = (f_1, f_2, f_3)^T \in C_0^\infty(\mathbb{R}^3)^3$. It follows from (2.1) that $\Delta p = \text{div}\ f$ on account of (1.15); consequently, we find

$$Q(x,y) = \nabla_y \frac{1}{4\pi|x-y|} = Q_{St}(x-y),$$

see (1.8). We thus look for $\Gamma(x,y)$ in the form

$$\Gamma(x,y) = \Gamma^0(x,y) + \Gamma^1(x,y) = \big(\Gamma^0_{ij}(x,y) + \Gamma^1_{ij}(x,y)\big)_{1\le i,j\le 3}$$

in such a way that

$$u^m(x) = \int_{\mathbb{R}^3} \Gamma^m(x,y) f(y)\, dy \qquad (m = 0, 1)$$

solves

$$\begin{aligned} &-\Delta u^0 - (\omega \times x)\cdot \nabla u^0 + (\omega \times u^0) = f, \\ &-\Delta u^1 - (\omega \times x)\cdot \nabla u^1 + (\omega \times u^1) = -\nabla p\,, \end{aligned} \tag{2.2}$$

respectively. Therefore, once we have $\Gamma^0(x,y)$ which is called the fundamental solution of the operator

$$L = -\Delta - (\omega \times x)\cdot \nabla + \omega \times, \tag{2.3}$$

we get from (1.8) and (2.2) that

$$\Gamma^1(x,y) = \int_{\mathbb{R}^3} \text{div}_z \Gamma^0(x,z) \otimes \nabla_y \frac{1}{4\pi|z-y|}\, dz, \tag{2.4}$$

or more exactly,

$$\Gamma^1_{ij}(x,y) = \int_{\mathbb{R}^3} \sum_k \partial_{z_k} \Gamma^0_{ik}(x,z) \frac{z_j - y_j}{4\pi|z-y|^3}\, dz.$$

Setting $u = u^0 + u^1$, we easily see that (2.2) implies

$$-\Delta \text{div}\ u - (\omega \times x)\cdot \nabla \text{div}\ u = 0 \qquad \text{in } \mathbb{R}^3$$

since $\Delta p = \text{div}\ f$. Consequently, $v = \text{div}\, u \in \mathcal{S}'(\mathbb{R}^3)$, a tempered distribution, satisfies $-\Delta v - (\omega \times x)\cdot \nabla v = 0$, and by [7] (and also [21, Lemma 4.2]) we conclude that its Fourier transform $\widehat{v}$ satisfies Supp $\widehat{v} \subset \{0\}$. From this it follows that div $u = 0$, since we know $|\nabla u(x)| \le C/|x|^2$ for large $|x|$ (see [6, Proposition 3.3]). Set

$$G(x,t) = (4\pi t)^{-3/2} e^{-|x|^2/(4t)}$$

which is the heat kernel in $\mathbb{R}^3$ and satisfies

$$\int_0^\infty G(x,t)\,dt = \frac{1}{4\pi|x|}\,. \tag{2.5}$$

In view of the derivation of the equation (1.2) from (1.4), the function

$$U(x,t) = O(at)^T \int_{\mathbb{R}^3} G(O(at)x - y, t) f(y)\,dy$$

solves the initial value problem

$$\partial_t U = \Delta U + (\omega \times x)\cdot \nabla U - \omega \times U, \qquad U(x,0) = f(x)$$

in $\mathbb{R}^3$, where $O(t)$ is given by (1.1) and $\omega = ae_3$. We thus find that

$$\begin{aligned}\Gamma^0(x,y) &= \int_0^\infty O(at)^T G(O(at)x - y, t)\,dt \\ &\equiv \int_0^\infty O(at)^T (4\pi t)^{-3/2} e^{-|O(at)x-y|^2/(4t)}\,dt\end{aligned} \tag{2.6}$$

is a fundamental solution of L defined by (2.3). In fact, one can justify

$$\begin{aligned}L\int_{\mathbb{R}^3} \Gamma^0(x,y) f(y)\,dy &= L\int_0^\infty U(x,t)\,dt \\ &= -\int_0^\infty \partial_t U(x,t)\,dt = U(x,0) = f(x)\end{aligned}$$

in, say, $L^r(\mathbb{R}^3)^3$ provided that $f \in C_0^\infty(\mathbb{R}^3)^3$. As for another deduction of (2.6) by use of the Fourier transform, see [7]. By (2.4), (2.5) and (2.6) we employ the semigroup property of $G(x,t)$ to obtain

$$\begin{aligned}&\Gamma^1_{ij}(x,y) \\ &= \int_{\mathbb{R}^3} \left(\sum_k \partial_{z_k} \int_0^\infty O(at)^T_{ik} G\left(O(at)x - z, t\right)\,dt \right) \partial_{y_j} \int_0^\infty G(z-y,\tau)\,d\tau dz \\ &= -\int_0^\infty \int_0^\infty \int_{\mathbb{R}^3} \partial_{x_i} G\left(O(at)x - z, t\right) \partial_{y_j} G(z-y,\tau)\,dz dt d\tau \\ &= -\int_0^\infty \int_0^\infty \partial_{x_i}\partial_{y_j} G\left(O(at)x - y, t+\tau\right)\,d\tau dt\end{aligned}$$

which implies

$$
\begin{aligned}
\Gamma^1(x,y) &= -\int_0^\infty \int_0^s \nabla_x \nabla_y G(O(at)x-y,s)\,dtds \\
&= \int_0^\infty (4\pi s)^{-3/2} \int_0^s e^{-|O(at)x-y|^2/(4s)} \\
&\quad \cdot \left\{ \frac{\left(x-O(at)^T y\right)\otimes (O(at)x-y)}{4s^2} - \frac{1}{2s}\,O(at)^T \right\} dtds,
\end{aligned}
\tag{2.7}
$$

where $z\otimes w = (z_i w_j)_{1\le i,j\le 3}$.

Proposition 2.1 *Let $\omega = ae_3$. Then the pair $\{\Gamma(x,y), Q_{St}(x-y)\}$ with $\Gamma(x,y)=\Gamma^0(x,y)+\Gamma^1(x,y)$ is a fundamental solution of the equation* (2.1)*, where $\Gamma^0(x,y), \Gamma^1(x,y)$ and $Q_{St}(x)$ are given by* (2.6)*,* (2.7) *and* (1.8)*, respectively.*

Set

$$
H(x,t) = \int_t^\infty \nabla^2 G(x,s)\,ds = \int_t^\infty G(x,s)\left(\frac{x\otimes x}{4s^2} - \frac{\mathbb{I}}{2s}\right) ds.
$$

It is well known that $G(x,t)\mathbb{I}+H(x,t)$ is the fundamental solution of the usual unsteady Stokes equation ($\omega=0$), and it is worth noting the relation

$$
\Gamma^1(x,y) = \int_0^\infty O(at)^T H(O(at)x-y,t)\,dt.
$$

When we have to take care of the dependence of the fundamental solution $\Gamma(x,y)$ on the angular velocity $\omega = ae_3$ (the direction e_3 of the rotating axis is fixed), we write

$$
\Gamma_a(x,y) = \left(\Gamma_{a,ij}(x,y)\right)_{1\le i,j\le 3}.
$$

Note that Proposition 2.1 covers the case $\omega=0$; in fact,

$$
\Gamma_0(x,y) = \int_0^\infty \{G(x-y,t)\mathbb{I} + H(x-y,t)\}\,dt = E_{St}(x-y),
$$

see (1.8). Since the operator $(\omega\times x)\cdot\nabla - \omega\times$ is skew-symmetric, $\Gamma_{-a}(x,y)$ is (the velocity part of) the fundamental solution of the adjoint equation

$$
-\Delta v + (\omega\times x)\cdot\nabla v - \omega\times v - \nabla q = g, \qquad \mathrm{div}\ v = 0 \qquad \text{in } \mathbb{R}^3 \tag{2.8}
$$

and it is easy to verify the relation

$$
\Gamma_{-a}(x,y)^T = \Gamma_a(y,x). \tag{2.9}
$$

Pointwise estimates of $\Gamma(x,y)$ for $|x|, |y| \to \infty$ are very complicated. Even $\Gamma^0(x,y)$ given by (2.6), does not satisfy an estimate from above by $C/|x-y|$ when we take, for instance, $x = \rho e_1, y = \rho e_2$ with $\rho \to \infty$, see [7]. However, when y is fixed and $|x| \to \infty$, it is easy to estimate $\Gamma(x,y)$. Let $|x| \geq 2|y|$. Then we have $e^{-|O(at)x-y|^2/(4s)} \leq e^{-|x|^2/(16s)}$ in (2.7). From this together with the simple equality (where $\gamma(\cdot)$ denotes the gamma function)

$$\int_0^\infty s^{-m/2} e^{-c|x|^2/s}\, ds = \frac{c^{-m/2+1}\gamma(m/2-1)}{|x|^{m-2}}, \qquad m > 2,\ c > 0, \tag{2.10}$$

it follows that $|\Gamma^1(x,y)| \leq C/|x|$ and $|\nabla_x \Gamma^1(x,y)| \leq C/|x|^2$. Since $\Gamma^0(x,y)$ can be estimated similarly, we obtain

$$|\Gamma(x,y)| \leq \frac{C}{|x|}, \qquad |\nabla_x \Gamma(x,y)| \leq \frac{C}{|x|^2} \tag{2.11}$$

for $|x| \geq 2|y|$, which will be used to show Proposition 3.1.

3 Potential representation formula

This section is devoted to the deduction of the following potential representation formula (3.1) of the solution $u(x)$ to (1.6) in terms of the fundamental solution $\Gamma(x,y)$ given by Proposition 2.1.

Proposition 3.1 *Let $f \in C_0^\infty(\overline{D})^3$, the restriction of $f \in C_0^\infty(\mathbb{R}^3)^3$ to $\overline{D}$. Then the solution (u,p) to (1.6) with $u|_{\partial D} = 0$ can be represented as*

$$u(x) = \int_D \Gamma(x,y) f(y)\, dy + \int_{\partial D} \Gamma(x,y)\, (\nu \cdot T(u,p))\,(y)\, d\sigma_y, \tag{3.1}$$

$$\begin{aligned} p(x) = &\int_D Q_{St}(x-y)\cdot f(y)\, dy + \int_{\partial D} \frac{\nu \cdot (\nabla p - f)(y)}{4\pi|x-y|}\, d\sigma_y \\ &- \int_{\partial D} \nu \cdot Q_{St}(x-y) p(y)\, d\sigma_y. \end{aligned} \tag{3.2}$$

Here, $Q_{St}(x)$ is given by (1.8) and the formula (3.2) for the pressure holds true even for the boundary condition $u|_{\partial D} = \omega \times x$.

For the proof, we derive the Green formula (3.5) below associated with the Stokes equation with rotation effect. In what follows we always assume that div $u =$ div $v = 0$. Let $W \subset \mathbb{R}^3$ be a bounded domain with smooth boundary ∂W. We start with the classical hydrodynamical Green formula, see e.g. [26, Chap. 2., Sec. 3, (10)-(11)],

$$\begin{aligned}\int_W \{v \cdot (\Delta u - \nabla p) - u \cdot (\Delta v + \nabla q)\}\ dx \\ = \int_{\partial W} \nu \cdot \{v \cdot T(u,p) - u \cdot T(v,-q)\}\ d\sigma,\end{aligned} \tag{3.3}$$

where ν is the exterior unit normal to ∂W and $T(u,p)$ is the Cauchy stress tensor (1.9). Moreover, we need that

$$\begin{aligned}&\int_W [v \cdot \{(\omega \times x) \cdot \nabla u\} + u \cdot \{(\omega \times x) \cdot \nabla v\}]\, dx \\ &= \int_W \sum_{i,j} \partial_j \{(\omega \times x)_j u_i v_i\}\ dx = \int_{\partial W} \nu \cdot (\omega \times x)(u \cdot v)\, d\sigma.\end{aligned} \tag{3.4}$$

We collect (3.3), (3.4) and the obvious equality $v \cdot (\omega \times u) + u \cdot (\omega \times v) = 0$ to obtain that

$$\begin{aligned}&\int_W [v \cdot \{\Delta u + (\omega \times x) \cdot \nabla u - \omega \times u - \nabla p\} \\ &\qquad - u \cdot \{\Delta v - (\omega \times x) \cdot \nabla v + \omega \times v + \nabla q\}]\, dx \\ &= \int_{\partial W} [\nu \cdot \{v \cdot T(u,p) - u \cdot T(v,-q)\} + \nu \cdot (\omega \times x)(u \cdot v)]\ d\sigma.\end{aligned} \tag{3.5}$$

Proof of Proposition 3.1. Suppose that (u,p) is the solution to (1.6) with $f \in C_0^\infty(\overline{D})^3$ subject to (1.3) or $u|_{\partial D} = 0$. Then we know that it is smooth and satisfies

$$|u(x)| \le \frac{C}{|x|}, \qquad |\nabla u(x)| + |p(x)| \le \frac{C}{|x|^2}, \qquad |\nabla p(x)| \le \frac{C}{|x|^3} \tag{3.6}$$

for large $|x|$, see Galdi [13, Theorem 4.1]. Let $y \in D$ be a given point and let $R \ge 2|y|$. We use (3.5) in the domain $W = D_R = D \cap B_R$, where $B_R = \{x \in \mathbb{R}^3; |x| < R\}$. Then we get

$$\begin{aligned}&-\int_{D_R} [v \cdot f + u \cdot \{\Delta v - (\omega \times x) \cdot \nabla v + \omega \times v + \nabla q\}]\ dx \\ &= \int_{\partial D_R} \nu \cdot \{v \cdot T(u,p) - u \cdot T(v,-q)\}\ d\sigma + \int_{\partial D} \nu \cdot (\omega \times x)(u \cdot v)\, d\sigma.\end{aligned} \tag{3.7}$$

Note that the integral of $\nu \cdot (\omega \times x)(u \cdot v)$ on the sphere ∂B_R vanishes since $x \cdot (\omega \times x) = 0$. Let us recall the fundamental solution $\{\Gamma_{-a}(x,y), -Q_{St}(x-y)\}$ of the adjoint equation (2.8). For each $k \in \{1,2,3\}$, we denote by $\Gamma_{-a,k}(x,y)$ the k-th column vector of $\Gamma_{-a}(x,y)$ and set $Q_{St,k}(x) = x_k / (4\pi|x|^3)$; then, we have

$$L_x^* \Gamma_{-a,k}(x,y) + \nabla_x Q_{St,k}(x-y) = \delta(x-y) e_k, \qquad \mathrm{div}_x \Gamma_{-a,k}(x,y) = 0$$

in $\mathcal{D}'(\mathbb{R}^3)^3$, where

$$L_x^* = -\Delta_x + (\omega \times x) \cdot \nabla_x - \omega \times,$$

and $\delta(\cdot - y)$ is the Dirac measure at y. In (3.7) we take

$$v(x) = \Gamma_{-a,k}(x,y), \qquad q(x) = -Q_{St,k}(x-y)$$

and restrict ourselves to the case $u|_{\partial D} = 0$ to find

$$\begin{aligned} u_k(y) = &\int_{D_R} \sum_i \Gamma_{-a,ik}(x,y) f_i(x)\, dx \\ &+ \int_{\partial D} \sum_{i,j} \nu_j \Gamma_{-a,ik}(x,y) T_{ij}(u,p)(x)\, d\sigma_x + I_{\partial B_R} \end{aligned} \tag{3.8}$$

with

$$\begin{aligned} I_{\partial B_R} = \int_{\partial B_R} \sum_{i,j} \frac{x_j}{R} \Big\{ &\Gamma_{-a,ik}(x,y) T_{ij}(u,p)(x) \\ &- u_i(x) T_{ij}\Big(\Gamma_{-a,k}(\cdot,y), Q_{St,k}(\cdot - y)\Big)(x) \Big\} d\sigma_x. \end{aligned}$$

By (2.11) together with (3.6) we see that $|I_{\partial B_R}| \to 0$ as $R \to \infty$, so that

$$u(y) = \int_D \Gamma_{-a}(x,y)^T f(x)\, dx + \int_{\partial D} \Gamma_{-a}(x,y)^T \left(\nu \cdot T(u,p)\right)(x)\, d\sigma_x,$$

which implies (3.1) because of (2.9).

It remains to show (3.2). By (1.6) together with (1.15) we have

$$\Delta p = \mathrm{div}\ f \qquad \text{in } D. \tag{3.9}$$

It thus follows from the usual Green formula that

$$\begin{aligned} p(y) = &-\int_{D_R} \frac{\mathrm{div}\ f(x)}{4\pi|x-y|}\, dx + \int_{\partial D_R} \frac{(\nu \cdot \nabla p)(x)}{4\pi|x-y|}\, d\sigma_x \\ &- \int_{\partial D_R} \nu \cdot \nabla_x \left(\frac{1}{4\pi|x-y|}\right) p(x)\, d\sigma_x. \end{aligned}$$

By integration by parts in the first integral and by letting $R \to \infty$, we obtain (3.2) since the integrals on the sphere ∂B_R go to zero on account of (3.6). This completes the proof. □

4 Case $u = 0$ on ∂D

In this section we will prove (1.10) with (1.12) and (1.13) for the case of the homogeneous boundary condition $u|_{\partial D} = 0$. The original boundary condition (1.3) will be reduced to this case in the next section, in which we will see that we have the same coefficients in the leading and second terms for both cases: $u|_{\partial D} = 0$ and $u|_{\partial D} = \omega \times x$.

Since we prefer the representation of the net force $\int_{\partial D} \nu \cdot (T + F)\, d\sigma$ which involves F, we have taken the external force of the form $f = \text{div } F$ in Theorem 1.1. But, actually, divergence form is not needed to catch the profile. In fact, we show the following.

Theorem 4.1 *Let $\omega = ae_3$ with $a \in \mathbb{R}\backslash\{0\}$. Given $f = (f', f_3)^T \in C_0^\infty(\overline{D})^3$, let u be the solution to* (1.6) *subject to $u|_{\partial D} = 0$. Then it satisfies the asymptotic representation* (1.10) *for $|x| \to \infty$ with*

$$U_{1st}(x) = \frac{1}{8\pi}\left(\int_{\partial D} (\nu \cdot T)_3\, d\sigma_y + \int_D f_3\, dy\right)\left(\frac{e_3}{|x|} + \frac{x_3 x}{|x|^3}\right)$$

and $U_{2nd}(x)$ of the same form as in (1.13), *where the coefficients are given by*

$$\begin{aligned}
\alpha &= -\int_{\partial D} y \cdot (\nu \cdot T)\, d\sigma_y - \int_D y \cdot f\, dy = \alpha' + \alpha_3, \\
\alpha' &= -\int_{\partial D} y' \cdot (\nu \cdot T)'\, d\sigma_y - \int_D y' \cdot f'\, dy, \\
\alpha_3 &= -\int_{\partial D} y_3 (\nu \cdot T)_3\, d\sigma_y - \int_D y_3 f_3\, dy, \\
\beta &= \int_{\partial D} (e_3 \times y) \cdot (\nu \cdot T)\, d\sigma_y + \int_D (e_3 \times y) \cdot f\, dy.
\end{aligned}$$

Corollary 4.1 *Assume $f = \text{div } F$ with $F \in C_0^\infty(\overline{D})^{3\times 3}$ in Theorem 4.1. Then we have* (1.10) *with* (1.12) *and* (1.13), *where the coefficients are exactly the same as in Theorem 1.1.*

Once Theorem 4.1 is obtained, Corollary 4.1 is obvious since

$$\int_D y_i \sum_k \partial_k F_{jk}\, dy = \int_{\partial D} y_i \sum_k \nu_k F_{jk}\, d\sigma_y - \int_D F_{ji}\, dy$$

for $1 \le i, j \le 3$. For instance, we have

$$\int_D (e_3 \times y) \cdot \text{div } F\, dy = \int_{\partial D} (e_3 \times y) \cdot (\nu \cdot F)\, d\sigma_y + \int_D (F_{12} - F_{21})\, dy,$$

which combined with the identity $(b \times c) \cdot d = b \cdot (c \times d)$ for arbitrary vectors $b, c, d \in \mathbb{R}^3$ implies the representation of β given in Theorem 1.1.

We fix $R > 0$ such that $f(y) = 0$ for $|y| \geq R$. The proof of Theorem 4.1 is based on the solution formula (3.1), by which we may assume $|y| \leq R$ and $|x| \geq 2R$ in what follows. Our task is now to find $\Phi_1(x) \sim 1/|x|$ and $\Phi_2(x, y) \sim 1/|x|^2$ so that the fundamental solution is represented as

$$\Gamma(x, y) = \Phi_1(x) + \Phi_2(x, y) + \left(1 + \frac{1}{|a|}\right) O\left(\frac{1}{|x|^3}\right) \tag{4.1}$$

for $|x| \to \infty$. The last term means that

$$|\Gamma(x, y) - \{\Phi_1(x) + \Phi_2(x, y)\}| \leq \left(1 + \frac{1}{|a|}\right) \frac{C_R}{|x|^3}$$

for $|x| \geq 2R \geq 2|y|$, where $C_R > 0$ is independent of $a \in \mathbb{R} \setminus \{0\}$. Let $\Gamma^0(x, y)$ and $\Gamma^1(x, y)$ be as in (2.6) and (2.7), respectively.

Proposition 4.1 *For $|y| \leq R$ and $|x| \to \infty$ we have*

$$\Gamma^0(x, y) = \Phi_1^0(x) + \Phi_2^0(x, y) + \left(1 + \frac{1}{|a|}\right) O\left(\frac{1}{|x|^3}\right)$$

where

$$\Phi_1^0(x) = \frac{1}{4\pi|x|} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.2}$$

$$\Phi_2^0(x, y) = \frac{1}{8\pi|x|^3} \begin{pmatrix} x' \cdot y' & (e_3 \times x) \cdot y & 0 \\ -(e_3 \times x) \cdot y & x' \cdot y' & 0 \\ 0 & 0 & 2x_3 y_3 \end{pmatrix}. \tag{4.3}$$

Proposition 4.2 *For $|y| \leq R$ and $|x| \to \infty$ one has*

$$\Gamma^1(x, y) = \Phi_1^1(x) + \Phi_2^1(x, y) + \left(1 + \frac{1}{|a|}\right) O\left(\frac{1}{|x|^3}\right)$$

with

$$\Phi_1^1(x) = \frac{1}{8\pi|x|^3} \begin{pmatrix} 0 & 0 & x_1 x_3 \\ 0 & 0 & x_2 x_3 \\ 0 & 0 & -|x'|^2 \end{pmatrix}, \tag{4.4}$$

$$\Phi_2^1(x,y) = \frac{-1}{8\pi|x|^3}\left\{x\otimes y + \begin{pmatrix} x_1y_1 & x_1y_2 & 0 \\ x_2y_1 & x_2y_2 & 0 \\ 0 & 0 & 2x_3y_3 \end{pmatrix}\right\} + \Psi(x,y), \tag{4.5}$$

where

$$\Psi(x,y) = \frac{3x}{8\pi|x|^5}\otimes\left\{\frac{|x'|^2}{2}(y',0)^T + x_3^2(0,0,y_3)^T\right\}. \tag{4.6}$$

We postpone the proof of these propositions to that of Theorem 4.1.

Proof of Theorem 4.1. It follows from Propositions 4.1 and 4.2 that (4.1) holds with

$$\Phi_1(x) = \Phi_1^0(x) + \Phi_1^1(x) = \frac{1}{8\pi|x|^3}\begin{pmatrix} 0 & 0 & x_1x_3 \\ 0 & 0 & x_2x_3 \\ 0 & 0 & |x|^2 + x_3^2 \end{pmatrix},$$

$$\begin{aligned}\Phi_2(x,y) &= \Phi_2^0(x,y) + \Phi_2^1(x,y) \\ &= \frac{-1}{8\pi|x|^3}\{x\otimes y - (e_3\times x)\otimes(e_3\times y)\} + \Psi(x,y),\end{aligned}$$

where $\Psi(x,y)$ is given by (4.6). This combined with (3.1) provides the desired asymptotic representation of $u(x)$. □

For the proof of Proposition 4.1, we use the following elementary decay property due to oscillation, which should be compared to (2.10). For our purpose, the case $n=1$ is enough.

Lemma 4.1 *Let $a\in\mathbb{R}\setminus\{0\}$, $m>2$ and $c>0$. Given $n\in\mathbb{N}$ arbitrarily, there is a constant $K=K(n,m,c)>0$ such that*

$$\left|\int_0^\infty \begin{pmatrix}\cos at \\ \sin at\end{pmatrix} t^{-m/2}e^{-c|x|^2/t}\,dt\right| \le \frac{K}{|a|^n|x|^{2n+m-2}} \tag{4.7}$$

for $x\in\mathbb{R}^3\setminus\{0\}$.

Proof. Since

$$\int_0^\infty \left|\partial_t^n\left(t^{-m/2}e^{-c/t}\right)\right|dt < \infty$$

for every $n \in \mathbb{N}$, n-times integration by parts yields

$$\begin{aligned}\int_0^\infty e^{iat} t^{-m/2} e^{-c|x|^2/t}\,dt &= \frac{1}{|x|^{m-2}} \int_0^\infty e^{ia|x|^2 t}\, t^{-m/2} e^{-c/t}\,dt \\ &= \frac{1}{|x|^{m-2}} \left(\frac{-1}{ia|x|^2} \right)^n \int_0^\infty e^{ia|x|^2 t}\, \partial_t^n \left(t^{-m/2} e^{-c/t} \right) dt,\end{aligned}$$

where $i = \sqrt{-1}$. This immediately implies the assertion. □

Proof of Proposition 4.1. We employ the Taylor formula (with respect to y) to get that

$$\begin{aligned} e^{-|O(at)x-y|^2/(4t)} &= e^{-|x|^2/(4t)} + e^{-|x|^2/(4t)}\, \frac{(O(at)x)\cdot y}{2t} \\ &+ \frac{1}{2}\, e^{-|O(at)x-\theta y|^2/(4t)}\; y^T \frac{(O(at)x-\theta y)\otimes (O(at)x-\theta y) - 2t\mathbb{I}}{4t^2}\; y \end{aligned} \tag{4.8}$$

with some $\theta = \theta(x,y,t) \in (0,1)$. We decompose $\Gamma^0(x,y)$ as

$$\Gamma^0(x,y) = \Gamma^{01}(x,y) + \Gamma^{02}(x,y) + \Gamma^{03}(x,y)$$

correspondingly to (4.8). By (4.7) ($n = 1$, $m = 3$) together with (2.5) we find

$$\Gamma^{01}(x,y) = \Phi_1^0(x) + \frac{1}{|a|}\, O\left(\frac{1}{|x|^3} \right),$$

where $\Phi_1^0(x)$ is as in (4.2). Since the last term of (4.8) is estimated from above by $Ce^{-|x|^2/(16t)}\, R^2(|x|^2+t)/t^2$ for $|x| \ge 2R \ge 2|y|$, we obtain

$$\left|\Gamma^{03}(x,y)\right| \le \frac{C}{|x|^3}$$

by (2.10) without using (4.7). Concerning

$$\Gamma^{02}(x,y) = \int_0^\infty O(at)^T (4\pi t)^{-3/2} e^{-|x|^2/(4t)}\, \frac{(O(at)x)\cdot y}{2t}\,dt,$$

we note

$$(O(at)x)\cdot y = (x'\cdot y')\cos at + ((e_3\times x)\cdot y)\sin at + x_3 y_3 \tag{4.9}$$

and

$$\begin{pmatrix} \cos^2 at \\ \sin^2 at \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+\cos 2at \\ 1-\cos 2at \end{pmatrix}$$

to find

$$\frac{(O(at)x)\cdot y}{2}\,O(at)^T = \frac{1}{4}A(x,y) + (\text{remainder}) \tag{4.10}$$

with

$$A(x,y) = \begin{pmatrix} x'\cdot y' & (e_3\times x)\cdot y & 0 \\ -(e_3\times x)\cdot y & x'\cdot y' & 0 \\ 0 & 0 & 2x_3y_3 \end{pmatrix} \tag{4.11}$$

where the remainder contains oscillating terms $\cos kat$ and $\sin kat$ $(k=1,2)$ and has degree one with respect to x. By (4.7) $(n=1,\ m=5)$ together with (2.10) $(m=5)$ we are led to

$$\Gamma^{02}(x,y) = \frac{1}{8\pi|x|^3}A(x,y) + \frac{1}{|a|}\,O\left(\frac{1}{|x|^4}\right),$$

which completes the proof. □

Proof of Proposition 4.2. Similarly to the proof of Proposition 4.1, we use

$$\begin{aligned} e^{-|O(at)x-y|^2/(4s)} &= e^{-|x|^2/(4s)} + e^{-|x|^2/(4s)}\,\frac{(O(at)x)\cdot y}{2s} \\ &+ \frac{1}{2}\,e^{-|O(at)x-\theta y|^2/(4s)}\,y^T\frac{(O(at)x-\theta y)\otimes(O(at)x-\theta y)-2s\mathbb{I}}{4s^2}\,y \end{aligned}$$

with some $\theta=\theta(x,y,t,s)\in(0,1)$ and, correspondingly to this, decompose $\Gamma^1(x,y)$, see (2.7), as

$$\Gamma^1(x,y) = \Gamma^{11}(x,y) + \Gamma^{12}(x,y) + \Gamma^{13}(x,y).$$

It is seen that $|\Gamma^{13}(x,y)| \le C/|x|^3$ for $|x|\ge 2R\ge 2|y|$. By a series of elementary calculations we will show that

$$\Gamma^{11}(x,y) = \Phi^1_1(x) + \Phi^{11}_2(x,y) + \left(1+\frac{1}{|a|}\right)O\left(\frac{1}{|x|^3}\right), \tag{4.12}$$

$$\Gamma^{12}(x,y) = \Phi^{12}_2(x,y) + O\left(\frac{1}{|x|^3}\right) + \left(1+\frac{1}{|a|}\right)O\left(\frac{1}{|x|^4}\right). \tag{4.13}$$

Here, $\Phi^1_1(x)$ is as in (4.4), and

$$\Phi^{11}_2(x,y) = \frac{-1}{8\pi|x|^3}\begin{pmatrix} \frac{3}{2}x_1y_1-\frac{1}{2}x_2y_2 & \frac{1}{2}x_2y_1+\frac{3}{2}x_1y_2 & x_1y_3 \\ \frac{3}{2}x_2y_1+\frac{1}{2}x_1y_2 & -\frac{1}{2}x_1y_1+\frac{3}{2}x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & 2x_3y_3 \end{pmatrix},$$

$$\Phi^{12}_2(x,y) = \frac{-1}{16\pi|x|^3}A(x,y) + \Psi(x,y),$$

where $A(x,y)$ and $\Psi(x,y)$ are given by (4.11) and (4.6), respectively.

In fact, we further decompose $\Gamma^{11}(x,y)$ as

$$\Gamma^{11}(x,y) = \Gamma^{111}(x,y) + \Gamma^{112}(x,y)$$

with

$$\begin{aligned}
\Gamma^{111}(x,y) &= \frac{(4\pi)^{-3/2}}{4}\int_0^\infty s^{-7/2}e^{-|x|^2/(4s)}\int_0^s S(x,y,t)\,dtds, \\
\Gamma^{112}(x,y) &= \frac{-(4\pi)^{-3/2}}{2}\int_0^\infty s^{-5/2}e^{-|x|^2/(4s)}\int_0^s O(at)^T\,dtds,
\end{aligned}$$

where $S(x,y,t) = (S_{ij}(x,y,t))_{1\le i,j\le 3}$ denotes the matrix with entries

$$S_{ij}(x,y,t) = \big\{x_i - \big(O(at)^Ty\big)_i\big\}\big\{\,(O(at)x)_j - y_j\big\}. \tag{4.14}$$

Among them, we here calculate the typical cases $(i,j) = (1,1),(1,3),(3,1)$ and $(3,3)$ only (since the others are treated similarly):

$$\begin{aligned}
\int_0^s S_{11}(x,y,t)\,dt &= \left(-\frac{3}{2}x_1y_1 + \frac{1}{2}x_2y_2\right)s + |x'|^2\,\frac{\sin as}{a} \\
&\quad + (-x_1x_2 + y_1y_2)\frac{1-\cos as}{a} - \frac{1}{2}(x'\cdot y')\frac{\sin 2as}{2a} \\
&\quad - \frac{1}{2}(e_3\times x)\cdot y\,\frac{1-\cos 2as}{2a}, \\
\int_0^s S_{13}(x,y,t)\,dt &= x_1(x_3-y_3)s - (x_3-y_3)\left(y_1\frac{\sin as}{a} + y_2\frac{1-\cos as}{a}\right), \\
\int_0^s S_{31}(x,y,t)\,dt &= -(x_3-y_3)y_1s + (x_3-y_3)\left(x_1\frac{\sin as}{a} - x_2\frac{1-\cos as}{a}\right), \\
\int_0^s S_{33}(x,y,t)\,dt &= (x_3-y_3)^2 s,
\end{aligned}$$

from which together with (2.10), without using (4.7), it follows that

$$\begin{aligned}
\Gamma^{111}_{11}(x,y) &= \frac{-\frac{3}{2}x_1y_1 + \frac{1}{2}x_2y_2}{8\pi|x|^3} + \frac{1}{|a|}O\left(\frac{1}{|x|^3}\right), \\
\Gamma^{111}_{13}(x,y) &= \frac{x_1x_3}{8\pi|x|^3} + \frac{-x_1y_3}{8\pi|x|^3} + \frac{1}{|a|}O\left(\frac{1}{|x|^4}\right), \\
\Gamma^{111}_{31}(x,y) &= \frac{-x_3y_1}{8\pi|x|^3} + \left(1 + \frac{1}{|a|}\right)O\left(\frac{1}{|x|^3}\right), \\
\Gamma^{111}_{33}(x,y) &= \frac{x_3^2}{8\pi|x|^3} - \frac{2x_3y_3}{8\pi|x|^3} + O\left(\frac{1}{|x|^3}\right).
\end{aligned}$$

We eventually get

$$\Gamma^{111}(x,y) = \frac{1}{8\pi|x|^3}\begin{pmatrix} 0 & 0 & x_1x_3 \\ 0 & 0 & x_2x_3 \\ 0 & 0 & x_3^2 \end{pmatrix} + \Phi_2^{11}(x,y) + \left(1+\frac{1}{|a|}\right)O\left(\frac{1}{|x|^3}\right).$$

This combined with

$$\Gamma^{112}(x,y) = \frac{1}{8\pi|x|}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{|a|}O\left(\frac{1}{|x|^3}\right)$$

see (2.10), implies (4.12).

We proceed to $\Gamma^{12}(x,y)$, that is decomposed as

$$\Gamma^{12}(x,y) = \Gamma^{121}(x,y) + \Gamma^{122}(x,y)$$

with

$$\Gamma^{121}(x,y) = \frac{(4\pi)^{-3/2}}{4}\int_0^\infty s^{-9/2}e^{-|x|^2/(4s)}\int_0^s \frac{(O(at)x)\cdot y}{2}S(x,y,t)\,dtds,$$

$$\Gamma^{122}(x,y) = \frac{-(4\pi)^{-3/2}}{2}\int_0^\infty s^{-7/2}e^{-|x|^2/(4s)}\int_0^s \frac{(O(at)x)\cdot y}{2}O(at)^T\,dtds,$$

where $S(x,y,t)$ is as in (4.14). For each (i,j), one can write

$$\frac{(O(at)x)\cdot y}{2}S_{ij}(x,y,t) = K_{ij}(x,y) + (\text{remainder})$$

by use of (4.9), such that the remainder consists of $\cos kat$ and $\sin kat$, $k = 1,2,3$, while $K_{ij}(x,y)$ does not depend on t. It is obvious that the remainder above yields $\frac{1}{|a|}O(1/|x|^4)$ after integration in the representation of $\Gamma^{121}(x,y)$. Among all the terms in $K(x,y) = (K_{ij}(x,y))_{1\le i,j\le 3}$, what we need is the matrix

$$M(x,y) = \Big(M_1(x,y), M_2(x,y), M_3(x,y)\Big)$$

whose column vectors are given by

$$M_1(x,y) = \frac{1}{2}\left\{\frac{x'\cdot y'}{2}x_1 - \frac{(e_3\times x)\cdot y}{2}x_2\right\}x,$$

$$M_2(x,y) = \frac{1}{2}\left\{\frac{x'\cdot y'}{2}x_2 + \frac{(e_3\times x)\cdot y}{2}x_1\right\}x,$$

$$M_3(x,y) = \frac{1}{2}(x_3^2y_3)x.$$

In fact, $M(x,y)$ is the part of $K(x,y)$, whose degree with respect to x is just three, while the degree of $K(x,y)-M(x,y)$ is at most two with respect to x and, therefore, this part yields $O(1/|x|^3)$ after integration in the representation of $\Gamma^{121}(x,y)$. We set

$$\Psi(x,y)=\frac{(4\pi)^{-3/2}}{4}\int_0^\infty s^{-7/2}e^{-|x|^2/(4s)}\,ds\,M(x,y)=\frac{3}{4\pi|x|^5}M(x,y)\sim\frac{1}{|x|^2}$$

for $|x|\geq 2R\geq 2|y|$ to conclude that

$$\Gamma^{121}(x,y)=\Psi(x,y)+O\left(\frac{1}{|x|^3}\right)+\left(1+\frac{1}{|a|}\right)O\left(\frac{1}{|x|^4}\right). \tag{4.15}$$

One can rewrite $M(x,y)$ as

$$M(x,y)=\frac{x}{2}\otimes\left\{\frac{|x'|^2}{2}\,(y',0)^T+x_3^2\,(0,0,y_3)^T\right\},$$

which implies (4.6). Along the same line as above with the aid of (4.10), we obtain

$$\Gamma^{122}(x,y)=\frac{-1}{16\pi|x|^3}A(x,y)+\frac{1}{|a|}O\left(\frac{1}{|x|^4}\right),$$

where $A(x,y)$ is given by (4.11). This combined with (4.15) implies (4.13).

Finally, setting $\Phi_2^1(x,y)=\Phi_2^{11}(x,y)+\Phi_2^{12}(x,y)$ yields (4.5), which completes the proof. □

5 Case $u=\omega\times x$ on ∂D

We have two approaches to the boundary condition (1.3); one is the reduction to the homogeneous one $u|_{\partial D}=0$ and the other is the direct use of a representation formula derived from (3.7). Both lead us to the same conclusion and we here adopt the former because it is easier. Concerning the pressure, one can easily deal with (1.3) directly.

To begin with, we fix $R>0$ such that $\mathbb{R}^3\setminus D\subset B_R$. Let $\zeta:[0,\infty)\to[0,1]$ be a smooth function which satisfies $\zeta(r)=0$ for $r\geq R+1$ as well as $\zeta(r)=1$ for $r\leq R$. Since $\omega\times x=-\frac{1}{2}\mathrm{rot}\ (|x|^2\omega)$, it is reasonable to introduce the auxiliary function $b\in C_0^\infty(\mathbb{R}^3)^3$ by

$$\begin{aligned} b(x)&=-\frac{1}{2}\,\mathrm{rot}\ \{\zeta(|x|)|x|^2\omega\}=\frac{|x|^2}{2}[\omega\times\nabla\{\zeta(|x|)\}]+\zeta(|x|)(\omega\times x)\\ &=\left\{\frac{|x|}{2}\,\zeta'(|x|)+\zeta(|x|)\right\}(\omega\times x)\,, \end{aligned}\tag{5.1}$$

cf. [1]. Then it satisfies

$$\text{div } b = 0 \quad (x \in \mathbb{R}^3), \qquad b(x) = \omega \times x \quad (x \in B_R), \tag{5.2}$$

$$b_i(\omega \times x)_j - (\omega \times x)_i b_j = 0 \qquad (x \in \mathbb{R}^3,\, 1 \le i, j \le 3), \tag{5.3}$$

$$\nabla b = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \nabla b + (\nabla b)^T = O \qquad (x \in B_R), \tag{5.4}$$

and, by (5.2) and $b_3 = 0$,

$$\text{tr } \nabla b = \text{div } b = 0, \qquad \partial_1 b_1 + \partial_2 b_2 = \partial_3 b_3 = 0 \qquad (x \in \mathbb{R}^3). \tag{5.5}$$

From (5.3) it follows that

$$\begin{aligned} &(\omega \times x) \cdot \nabla b_i - (\omega \times b)_i \\ = &\sum_j \partial_j \left\{ b_i(\omega \times x)_j - (\omega \times x)_i b_j \right\} = 0 \qquad (x \in \mathbb{R}^3,\, 1 \le i \le 3). \end{aligned} \tag{5.6}$$

By (5.4) we have

$$\nu \cdot \nabla b = \omega \times \nu, \quad y \cdot (\nu \cdot \nabla b) = y \cdot (\omega \times \nu) = -\nu \cdot (\omega \times y) \qquad (y \in \partial D). \tag{5.7}$$

Obviously, by Gauss' Theorem and (5.7),

$$\int_{\partial D} \nu \cdot \nabla b \, d\sigma_y = 0\,, \tag{5.8}$$

$$\int_{\partial D} y \cdot (\nu \cdot \nabla b)\, d\sigma_y = -\int_{\partial D} \nu \cdot (\omega \times y)\, d\sigma_y = 0\,, \tag{5.9}$$

$$\int_{\partial D} y' \cdot (\nu \cdot \nabla b)'\, d\sigma_y = -\int_{\partial D} y_3 (\nu \cdot \nabla b)_3 \, d\sigma_y = 0. \tag{5.10}$$

since by (5.4) ∇b is constant on B_R, div $(\omega \times y) = 0$ and $b_3 = 0$.

Proof of Theorem 1.1. Let (u, p) be the solution to (1.6) with $f = \text{div } F$, $F \in C_0^\infty(\overline{D})^{3\times 3}$, subject to (1.3). Set $\widetilde{u}(x) = u(x) - b(x)$. By (5.2) and (5.6) it should obey

$$-\Delta \widetilde{u} - (\omega \times x) \cdot \nabla \widetilde{u} + \omega \times \widetilde{u} + \nabla p = \text{div } (F + \nabla b), \quad \text{div } \widetilde{u} = 0 \quad \text{in } D$$

together with $\widetilde{u}|_{\partial D} = 0$. By Corollary 4.1 we know the asymptotic representation (1.10) of $\widetilde{u}(x)$, which is the same as that of $u(x)$ since $b \in C_0^\infty(\mathbb{R}^3)^3$. Both have the same stress tensor on ∂D, i.e.,

$$T(\widetilde{u}, p)(y) = T(u, p)(y) \qquad (y \in \partial D),$$

see (1.9), thanks to (5.4); thus, in what follows, we may simply write T. The leading term is

$$\frac{1}{8\pi} \int_{\partial D} (\nu \cdot (T + F + \nabla b))_3 \, d\sigma_y \left(\frac{e_3}{|x|} + \frac{x_3 x}{|x|^3} \right),$$

which immediately implies (1.12) by (5.8), or more simply, by $\nabla b_3 = 0$. We next have a look at the coefficients of the second term:

$$\begin{aligned}
\alpha &= - \int_{\partial D} y \cdot (\nu \cdot (T + F + \nabla b)) \, d\sigma_y + \int_D \mathrm{tr}\, (F + \nabla b) \, dy = \alpha' + \alpha_3, \\
\alpha' &= - \int_{\partial D} y' \cdot (\nu \cdot (T + F + \nabla b))' \, d\sigma_y + \int_D (F_{11} + F_{22} + \partial_1 b_1 + \partial_2 b_2) \, dy, \\
\alpha_3 &= - \int_{\partial D} y_3 (\nu \cdot (T + F + \nabla b))_3 \, d\sigma_y + \int_D (F_{33} + \partial_3 b_3) \, dy, \\
\beta &= e_3 \cdot \int_{\partial D} y \times (\nu \cdot (T + F + \nabla b)) \, d\sigma_y + \int_D (F_{12} - F_{21} + \partial_2 b_1 - \partial_1 b_2) \, dy.
\end{aligned}$$

From (5.5), (5.9) and (5.10) it follows that α, α' and α_3 are exactly the same as those given in Theorem 1.1. Concerning β, we find

$$\begin{aligned}
\int_D (\partial_2 b_1 - \partial_1 b_2) \, dy &= \int_{D_{R+1}} (\partial_2 b_1 - \partial_1 b_2) \, dy \\
&= \int_{\partial D} \{\nu_2 (\omega \times y)_1 - \nu_1 (\omega \times y)_2\} \, d\sigma_y \\
&= - \int_{\partial D} (e_3 \times \nu) \cdot (\omega \times y) \, d\sigma_y \\
&= -e_3 \cdot \int_{\partial D} y \times (\nu \cdot \nabla b) \, d\sigma_y
\end{aligned}$$

with the help of (5.7). We have thus shown that each of the coefficients $\alpha, \alpha', \alpha_3$ and β does not change in the leading and second terms even when we take the boundary condition (1.3).

It remains to prove (1.11). To do so, we first observe

$$\int_{\partial D} \nu \cdot (\nabla p - f) \, d\sigma_y = 0 \tag{5.11}$$

as well as (1.16). In fact, from (3.9) it follows that, for any $\rho \geq R$,

$$\int_{\partial D} \nu \cdot (\nabla p - f)\, d\sigma_y = -\int_{\partial B_\rho} \frac{y}{\rho} \cdot (\nabla p - f)\, d\sigma_y,$$

which goes to zero as $\rho \to \infty$ since we have (3.6) and since the support of f is bounded. We thus obtain (5.11). Although (1.16) was essentially found by [29, Lemma 2.1] and [14, Lemma 3], see also [8, Lemma 2.3], we give a brief proof. By (5.6) together with $(\omega \times \widetilde{u})|_{\partial D} = 0$, it is sufficient to show

$$\nu \cdot [(\omega \times x) \cdot \nabla \widetilde{u}]\Big|_{\partial D} = 0. \tag{5.12}$$

Since u satisfies (3.6), so does $\widetilde{u} = u - b$; thus, we see that $\widetilde{u} \in \widehat{H}^1_0(D)^3$ with $\operatorname{div} \widetilde{u} = 0$, where

$$\widehat{H}^1_0(D) = \{w \in L^6(D);\, \nabla w \in L^2(D)^3,\, w|_{\partial D} = 0\},$$

which coincides with the completion of $C_0^\infty(D)$ with respect to $\|\nabla(\cdot)\|_{L^2}$. Due to Heywood [18, Theorem 8], there is a sequence $\{v^k\} \subset C^\infty_{0,\sigma}(D)$ such that $\nabla v^k \to \nabla \widetilde{u}$ in $L^2(D)^{3\times 3}$ as $k \to \infty$, where $C^\infty_{0,\sigma}(D)$ denotes the class of solenoidal vector fields whose components are in $C_0^\infty(D)$. Since $\partial_j v^k \in C^\infty_{0,\sigma}(D)$ for each $j = 1, 2, 3$, we obtain $\nu \cdot (\partial_j \widetilde{u})|_{\partial D} = 0$, which implies (5.12) and, thus, (1.16).

We now denote the terms of the right-hand side of (3.2) by $p_1(x), p_2(x)$ and $p_3(x)$, respectively: $p = p_1 + p_2 + p_3$. By (5.11), (1.16) and

$$\frac{1}{4\pi|x-y|} = \frac{1}{4\pi|x|} + \frac{y\cdot x}{4\pi|x|^3} + \frac{1}{8\pi} y^T\, \frac{3(x-\theta y)\otimes(x-\theta y) - |x-\theta y|^2\mathbb{I}}{|x-\theta y|^5}\, y$$

with some $\theta \in (0,1)$, we see that

$$\begin{aligned} p_2(x) &= \int_{\partial D} \nu \cdot (\nabla p - f)\left(\frac{1}{4\pi|x-y|} - \frac{1}{4\pi|x|}\right) d\sigma_y \\ &= \int_{\partial D} (\nu \cdot (\Delta u))\, y\, d\sigma_y \cdot Q_{St}(x) + O\left(\frac{1}{|x|^3}\right) \end{aligned}$$

for $|x| \to \infty$. The terms $p_1(x)$ and $p_3(x)$ are treated similarly, so that we obtain (1.11) with

$$P_{1st}(x) = \Big(\int_{\partial D} \{(\nu \cdot (\Delta u))\, y - p\nu\}\, d\sigma_y + \int_D f\, dy\Big) \cdot Q_{St}(x),$$

which yields (1.14). We have completed the proof. □

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