A sequence algebra of finite sections, convolution and multiplication operators on $L^p(\mathbb{R})$

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Abstract

This paper is concerned with the applicability of the finite sections method to operators belonging to the closed subalgebra of $\mathcal{L}(L^p(\mathbb{R}))$, $1 < p < \infty$, generated by operators of multiplication by piecewise continuous functions in $\dot{\mathbb{R}}$ and operators of convolution by piecewise continuous Fourier multipliers. For this, we introduce a larger algebra of sequences, which contains the special sequences we are interested and the usual operator algebra generated by the operators of multiplication and convolution. There is a direct relationship between the applicability of the finite section method for a given operator and invertibility of the corresponding sequence in this algebra. Exploring this relationship and using local principles, we construct locally equivalent representations that allow to derive invertibility criteria.

1 Introduction

From a formal point of view, to solve an operator equation $Au = v$ numerically by a direct method, one specifies a sequence of (in a certain sense) simple operators A_τ which converge strongly to A, and replaces the equation $Au = v$ by the sequence of the (simpler) equations $A_\tau u_\tau = v$. The crucial question is if this method *applies*, *i.e.* if the equations $A_\tau u_\tau = v$ possess unique solutions for every right-hand side v and for every sufficiently large τ, say for $\tau \geq \tau_0$, and if the sequence $(u_\tau)_{\tau \geq \tau_0}$ converges to the solution u of the original equation $Au = v$. The applicability of the method is equivalent to the stability of the sequence (A_τ), *i.e.* to the invertibility of the operators A_τ for τ being large enough and to the uniform boundedness of the norms of their inverses.

In the present paper, we are interested in operators which are constituted by operators of multiplication by piecewise continuous functions and operators of convolution by piecewise continuous Fourier multipliers. These operators are considered on L^p-spaces over the real line $\mathbb{R}$, and *simpler* means in that context that we replace the operator A by its compressions to the compact intervals $[-\tau, \tau]$ with $\tau \in (0, \infty)$. These compressions are also called the finite sections of A, whence the name *finite sections method* for this kind of approximate solution.

The stability of a sequence of operators is equivalent to an invertibility problem in the algebra of all bounded sequences of operators, factored by the ideal of all sequences converging to zero in the norm, as observed by Kozak [6] in 1973. Kozak's observation led to a whole new field of research, by inserting algebraic methods into numerical analysis. But there was still an obstacle: usually, the algebraic tools one wants to employ require to work in a much smaller algebra than the algebra of all bounded sequences factored by zero sequences. This problem was solved by one of the authors of the present paper in [11] where he examined

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the applicability of the finite sections method to one-dimensional Toeplitz operators with piecewise continuous coefficients by employing a local principle. It was a crucial step in [11] to introduce an algebra of sequences (smaller than the algebra of all bounded sequences) and an ideal of that algebra (larger than the ideal of all zero sequences) such that the resulting quotient algebra is, on the one hand, small enough to apply algebraic tools (such as local principles) and, on the other hand, large enough to contain all (cosets of) sequences one is actually interested in. This approach has proved extremely useful since.

As we have tried to indicate, the algebraic approach to study stability typically involves algebras of extremely different sizes: a very large algebra (where invertibility in that algebra is equivalent to stability) and a relatively small one (where the algebraic tools work). The passage from the large algebra to the small one rises the so-called inverse closedness problem: Does the invertibility of a sequence in the large algebra (i.e. stability) imply its invertibility in the small one (which hopefully can be studied by algebraic tools)? Note that this passage is *not* a problem if one only deals with operators on Hilbert spaces and C^*-algebras of sequences: C^*-subalgebras of C^*-algebras are always inverse closed.

In case $p = 2$, the results of the present paper extend the results from [7, 8] only slightly: the sequence algebra considered here is somewhat larger, since we include here all constant sequences of operators and the sequence (P_τ) of finite sections projections separately. But in case $p \neq 2$, a serious inverse closedness problems arises. This situation was not fully recognized until recently, and it is a main goal of the paper to offer a way to deal with the inverse closedness problem by constructing suitable intermediate algebras.

The paper is organized as follows. In Section 2 we present technical background material. In particular we introduce some families of strong limits which are later used to identify local algebras, and which also appear in the formulation of our main (stability) result. In Section 3 we recall some basic facts from [9] where Allan's local principle is applied to study the Fredholm property in the algebra generated by operators of multiplication and convolution by piecewise continuous functions. This repetition is for three reasons: first, the operator algebra examined in [9] is a subalgebra of our sequence algebra, and the approach from [9] strongly motivates and supports the approach of the present paper, second, we were able to simplify some proofs, and third, we are aware of the fact that the booklet [9] not well accessible. The main part of the present paper is Section 4, where the criteria for the applicability of the finite section method are derived. The concluding section is devoted to some examples.

2 Notation and basic results

Throughout this paper, $1 < p < \infty$. We will exclusively work on the Lebesgue space $L^p(\mathbb{R})$. Given a subinterval Γ of the real axis, we consider $L^p(\Gamma)$ as a closed subspace of $L^p(\mathbb{R})$ in the natural way. In particular, we identify the identity operator on $L^p(\Gamma)$ with the operator $\chi_\Gamma I$ of multiplication by the characteristic function χ_Γ of the interval Γ, acting on $L^p(\mathbb{R})$. More general, each bounded linear operator A on $L^p(\Gamma)$ is identified with the operator $\chi_\Gamma A \chi_\Gamma I$ acting on $L^p(\mathbb{R})$. These identifications will be used without further comment.

We write the *Fourier transform* F on the Schwartz space of rapidly decreasing infinite differentiable functions as

$$(Fu)(y) = \int_{-\infty}^{+\infty} e^{-2\pi \mathbf{i} yx} u(x)\, dx, \qquad y \in \mathbb{R}. \tag{1}$$

Then its inverse is given by

$$(F^{-1}v)(x) = \int_{-\infty}^{+\infty} e^{2\pi \mathbf{i} xy} v(y)\, dy, \qquad x \in \mathbb{R}. \tag{2}$$

It is well known that the operators F and F^{-1} can be extended continuously to bounded and unitary operators on the Hilbert space $L^2(\mathbb{R})$ and that F extends continuously to a bounded operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ where $q := p/(p-1)$ if $1 < p \le 2$ (see, for instance, [12, Theorem 74]).

Let $\mathcal{M}_p$ denote the set of all *Fourier multipliers*, *i.e.*, the set of all functions $a \in L^\infty(\mathbb{R})$ with the following property: if $u \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$, then $F^{-1}aFu \in L^p(\mathbb{R})$, and there is a constant c_p independent of u such that $\|F^{-1}aFu\|_p \le c_p \|u\|_p$. If $a \in \mathcal{M}_p$, then the operator $F^{-1}aF : (L^2(\mathbb{R}) \cap L^p(\mathbb{R})) \to L^p(\mathbb{R})$ extends continuously to a bounded operator on $L^p(\mathbb{R})$. This extension is called a *(Fourier) convolution operator* , and we denote it by $W^0(a)$. The function a is also called the *generating function* (or the symbol or presymbol) of $W^0(a)$. In particular, the convolution operator $W^0(\mathrm{sgn})$ can be identified the singular integral operator of Cauchy type,

$$(S_{\mathbb{R}}u)(t) := \frac{1}{\pi i} \int_{\mathbb{R}} \frac{u(s)}{s-t}\, ds, \qquad t \in \mathbb{R}.$$

We denote the associated projections by $P_{\mathbb{R}} := (I + S_{\mathbb{R}})/2$ and $Q_{\mathbb{R}} := I - P_{\mathbb{R}}$. We denote the characteristic functions of the positive and negative half axis by χ_+ and χ_-, respectively. Given $a \in \mathcal{M}_p$, the restriction of the operator $\chi_+ W^0(a) \chi_+ I$ onto $L^p(\mathbb{R}^+)$ is a *Wiener-Hopf operator* and will be denoted by $W(a)$.

The set $\mathcal{M}_p$ of all Fourier multipliers forms a Banach algebra when equipped with the operations inherited from $L^\infty(\mathbb{R})$ and the norm

$$\|a\|_{\mathcal{M}_p} := \|W^0(a)\|_{\mathcal{L}(L^p(\mathbb{R}))}. \tag{3}$$

We call a function $a \in L^\infty(\mathbb{R})$ *piecewise constant* (resp. *piecewise linear*) if there is a partition $-\infty = t_0 < t_1 < \ldots < t_n = +\infty$ of the real line such that a is constant (resp. linear) on each interval $[t_k, t_{k+1}]$. Stetchkin's inequality (see for instance [2]) entails that the multiplier algebra $\mathcal{M}_p$ contains the (non-closed) algebras C_0 of all continuous and piecewise linear functions on $\mathbb{R}$ and PC_0 of all piecewise constant functions on $\mathbb{R}$. Let C_p and PC_p denote the closures of C_0 and PC_0 in $\mathcal{M}_p$, respectively.

In the remainder of this section, we are going to define several types of "shift" operators and to introduce several strong limits associated to these shifts. These strong limits will be our main tool to identify local algebras in the following sections.

For $s, t \in \mathbb{R}$ and $\tau \in (0, \infty)$, consider the operators

$$U_s : L^p(\mathbb{R}) \mapsto L^p(\mathbb{R}), \quad (U_s u)(x) = e^{-2\pi \mathbf{i} xs} u(x), \tag{4}$$

$$V_t : L^p(\mathbb{R}) \mapsto L^p(\mathbb{R}), \quad (V_t u)(x) = u(x - t), \tag{5}$$

$$Z_\tau : L^p(\mathbb{R}) \mapsto L^p(\mathbb{R}), \quad (Z_\tau u)(x) := \tau^{-1/p} u(x/\tau). \tag{6}$$

Clearly, $U_s^{-1} = U_{-s}$, $V_t^{-1} = V_{-t}$, and $Z_\tau^{-1} = Z_{\tau^{-1}}$, and these operators have norm 1. The following lemma is easy to check.

Lemma 2.1. *If $a \in \mathcal{M}_p$ and $s \in \mathbb{R}$, then $U_{-s} W^0(a) U_s = W^0(V_s a V_{-s})$ and $V_s W^0(a) V_{-s} = W^0(a)$. Moreover, if $p = 2$ then*

$$U_s F^{-1} = F^{-1} V_{-s}, \quad F U_s = V_{-s} F, \quad V_s F^{-1} = F^{-1} U_s, \quad F V_s u = U_s F.$$

Lemma 2.2. (a) *The operators* V_t *converge weakly to zero as* $t \to \pm\infty$.
(b) *The operators* $Z_\tau^{\pm 1}$ *converge weakly to zero as* $\tau \to \infty$.

Proof. We will prove assertion (b) only. Let $u := \chi_{[a,b]}$ and $v := \chi_{[c,d]}$ be characteristic functions of intervals in $\mathbb{R}$. Then

$$\langle v, Z_\tau u\rangle = \int_c^d \tau^{-1/p}\chi_{[a\tau,b\tau]}(x)\,dx \le \tau^{-1/p}(d-c) \to 0$$

as $\tau \to \infty$. This implies that $\langle \tilde{v}, Z_\tau \tilde{u}\rangle \to 0$ for arbitrary piecewise constant functions $\tilde{u} \in L^p(\mathbb{R})$ and $\tilde{v} \in L^q(\mathbb{R}) = (L^p(\mathbb{R}))^*$ with $q = (p-1)/p$. As these functions are dense in $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$, the assertion for Z_τ follows. For Z_τ^{-1} and characteristic functions u and v as above one has

$$\langle v, Z_\tau^{-1} u\rangle = \int_c^d \tau^{1/p}\chi_{[\frac{a}{\tau},\frac{b}{\tau}]}(x)\,dx \le \tau^{1/p}(\frac{b}{\tau} - \frac{a}{\tau}) \to 0,$$

and one can argue as before. ■

Let $A \in \mathcal{L}(L^p(\mathbb{R}))$. If the strong limit

$$\operatorname*{s-lim}_{\tau\to+\infty} Z_\tau V_{-s} A V_s Z_\tau^{-1} \tag{7}$$

exists for some $s \in \mathbb{R}$, we denote it by $\mathsf{H}_{s,\infty}(A)$. Analogously, if the strong limit

$$\operatorname*{s-lim}_{\tau\to+\infty} Z_\tau^{-1} U_{-t} A U_t Z_\tau \tag{8}$$

exists for some $t \in \mathbb{R}$, we denote it by $\mathsf{H}_{\infty,t}(A)$. It is easy to see the set of all operators for which the strong limit $\mathsf{H}_{s,\infty}(A)$ (resp. $\mathsf{H}_{\infty,t}(A)$) exists forms a Banach algebra, that

$$\|\mathsf{H}_{s,\infty}(A)\|_{\mathcal{L}(L^p(\mathbb{R}))} \le \|A\|_{\mathcal{L}(L^p(\mathbb{R}))} \tag{9}$$

resp.

$$\|\mathsf{H}_{\infty,t}(A)\|_{\mathcal{L}(L^p(\mathbb{R}))} \le \|A\|_{\mathcal{L}(L^p(\mathbb{R}))} \tag{10}$$

for all operators in this algebra and that, hence, the operators $\mathsf{H}_{s,\infty}$ and $\mathsf{H}_{\infty,t}$ act as bounded algebra homomorphisms.

For $x \in \mathbb{R}$, let $b(x^\pm)$ denote the right/left one-sided limit of the piecewise continuous function b at x. The following results appeared for the first time in [9] (Propositions 13.1 and 13.2).

Proposition 2.3. *Let* $t \in \mathbb{R}$, *and let* $a \in PC(\mathbb{R})$, $b \in PC_p$ *and* K *a compact operator. Then*

(i) $\mathsf{H}_{\infty,t}(aI) = a(-\infty)\chi_- + a(+\infty)\chi_+$,

(ii) $\mathsf{H}_{\infty,t}(W^0(b)) = b(t^-)Q_\mathbb{R} + b(t^+)P_\mathbb{R}$,

(iii) $\mathsf{H}_{\infty,t}(K) = 0$.

Proof. (i) Since $U_{-t}aU_t = aI$, it is sufficient to check the assertion for $t = 0$. Taking into account that $(Z_\tau^{-1} a Z_\tau u)(x) = a(\tau x)u(x)$, we have

$$\|(a(+\infty)\chi_+ - Z_\tau^{-1} a Z_\tau)u\|^p = \int_0^{+\infty} |(a(+\infty) - a(\tau x))u(x)|^p\,dx.$$

Given $\epsilon > 0$, choose $x_\epsilon \in \mathbb{R}^+$ such that $|a(+\infty) - a(x)|^p \|u\|^p < \frac{\epsilon^p}{2}$ for $x \geq x_\epsilon$. For some $\tau > 1$, we write the above integral as the sum

$$\int_0^{\frac{x_\epsilon}{\tau}} (|a(+\infty) - a(\tau x)|\,|u(x)|)^p \, dx + \int_{\frac{x_\epsilon}{\tau}}^{+\infty} (|a(+\infty) - a(\tau x)|\,|u(x)|)^p \, dx$$

which is not greater than

$$\max_{0<x<x_\epsilon} |a(+\infty) - a(x)| \int_0^{\frac{x_\epsilon}{\tau}} |u(x)|^p \, dx + \frac{\epsilon^p}{2}.$$

For sufficiently large τ, the first term of this sum becomes as small as desired. Thus, $\mathsf{H}_{\infty,0}(a) = a(\infty)\chi_+$.

(ii) Since $U_{-s} W^0(b) U_s = W^0(V_s b V_{-s})$ and the shifted function $V_s b V_{-s}$ belongs to PC_p, it is again sufficient to prove the assertion for $s = 0$. Write b as $b(0^-)\chi_- + b(t^+)\chi_+ + b_0$ where the function $b_0 \in PC_p$ is continuous at 0 and takes the value 0 there. Since

$$W^0 \left(b(0^-)\chi_- + b(t^+)\chi_+ \right) = b(0^-) Q_\mathbb{R} + b(0^+) P_\mathbb{R}$$

and the operators $P_\mathbb{R}$ and $Q_\mathbb{R}$ commute with Z_τ, it remains to show that $Z_\tau^{-1} W^0(b_0) Z_\tau \to 0$ strongly as $\tau \to \infty$. Since PC_p is continuously embedded into L^∞ and thus into PC, we can approximate the function b_0 in the multiplier norm as closely as desired by a piecewise constant function b_{00} which is zero in an open neighborhood U of 0. It is thus sufficient to show that $Z_\tau^{-1} W^0(b_{00}) Z_\tau \to 0$ strongly as $\tau \to \infty$. Since the operators on the left hand side are uniformly bounded with respect to τ, it is finally sufficient to show that

$$Z_\tau^{-1} W^0(b_{00}) Z_\tau u \to 0$$

for all functions u in a certain dense subset of $L^p(\mathbb{R})$. For consider the set of all functions u in the Schwartz space $\mathcal{S}$ of the rapidly decreasing infinitely differentiable function the Fourier transform Fu of which has a compact support. This space is indeed dense in $L^p(\mathbb{R})$ since the space $\mathcal{D}$ of the compactly supported infinitely differentiable functions is dense in $\mathcal{S}$, since F is a continuous bijection on $\mathcal{S}$, and since $\mathcal{S}$ is dense in $L^p(\mathbb{R})$ (see [10], Theorem 7.10). If u is a function with these properties, then

$$Z_\tau^{-1} W^0(b_{00}) Z_\tau u = F^{-1} Z_\tau b_{00} Z_\tau^{-1} F u. \tag{11}$$

If τ is sufficiently large, then the support of Fu is contained in U; hence, the function on the right hand side of (11) is the zero function.

(iii) If K is compact, then $U_{-t} K U_t$ is compact, and the assertion follows immediately from the weak convergence of Z_τ to zero. ■

Let Q_t denote the characteristic function of the interval $\mathbb{R} \setminus [-t, t]$. We let $\bar{\mathcal{M}}_p$ refer to the set of all multipliers $a \in \mathcal{M}_p$ for which there are numbers $a(-\infty)$ and $a(+\infty)$ such that

$$\lim_{t \to \infty} \|Q_t(a - a(-\infty)\chi_- - a(+\infty)\chi_+)\|_{\mathcal{M}_p} = 0. \tag{12}$$

Notice that this definition makes sense since, by the Stetchkin inequality, the characteristic functions Q_t, χ_+ and χ_- of $\mathbb{R} \setminus [-t, t]$, $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively, belong to $\mathcal{M}_p$. Also notice that the numbers $a(-\infty)$ and $a(+\infty)$ are uniquely determined by a. Further, let $\dot{\mathcal{M}}_p$ denote the class of all multipliers $a \in \bar{\mathcal{M}}_p$ such that $a(-\infty) = a(+\infty)$. Via [9, Proposition 12.2] one easily gets that

$$PC_p \subseteq \bar{\mathcal{M}}_p \text{ and } C_p \subseteq \dot{\mathcal{M}}_p.$$

The assertions of the following lemma are either taken directly from the preceding proposition or they follow by repeating some arguments of its proof.

Lemma 2.4. (i) *If $a \in PC$, then $V_{-\tau} a V_\tau \to a(\pm\infty) I$ as $\tau \to \pm\infty$.*

(ii) *If $b \in \bar{\mathcal{M}}_p$, then $U_{-\tau} W^0(b) U_\tau \to b(\pm\infty) I$ as $\tau \to \pm\infty$.*

The following is the analogue of Propositon 2.3 for the other family of strong limits. Its proof follows along the same lines, taking into account that $(V_{-s} a V_s)(t) = a(t+s)$ and $V_{-s} W^0(b) V_s = W^0(b)$.

Proposition 2.5. *Let $s \in \mathbb{R}$, and let $a \in PC$, $b \in \bar{\mathcal{M}}_p$, and K a compact operator. Then*

(i) $\mathsf{H}_{s,\infty}(aI) = a(s^-)\chi_- + a(s^+)\chi_+$,

(ii) $\mathsf{H}_{s,\infty}(W^0(b)) = b(-\infty) Q_{\mathbb{R}} + b(+\infty) P_{\mathbb{R}}$,

(iii) $\mathsf{H}_{s,\infty}(K) = 0$.

Finally, for $A \in \mathcal{L}(L^p(\mathbb{R}))$, we consider the strong limits

$$\mathsf{H}^{\pm\pm}(A) := \operatorname*{s-lim}_{t \to \pm\infty} \operatorname*{s-lim}_{s \to \pm\infty} U_{-t} V_{-s} A V_s U_t. \tag{13}$$

Here, by convention, the first superscript in $\mathsf{H}^{\pm\pm}$ refers to the strong limit with respect to $s \to \pm\infty$ and the second one to $t \to \pm\infty$. It is again easy to see that the mappings $\mathsf{H}^{\pm\pm}$ act as bounded algebra homomorphisms on the Banach algebra of all operators A for which the strong limits (13) exist.

Proposition 2.6. *Let $a \in PC$, $b \in \bar{\mathcal{M}}_p$, and K a compact operator. Then*

(i) $\mathsf{H}^{+\pm}(aI) = a(+\infty) I, \qquad \mathsf{H}^{-\pm}(aI) = a(-\infty) I$,

(ii) $\mathsf{H}^{\pm+}(W^0(b)) = b(-\infty) I, \qquad \mathsf{H}^{\pm-}(W^0(b)) = b(+\infty) I$,

(iii) $\mathsf{H}^{\pm\pm}(K) = 0$.

3 The Fredholm property

In this section we recall some results on the Fredholm property of operators in the smallest closed subalgebra $\mathcal{A}(PC(\mathbb{R}), PC_p)$ of $\mathcal{L}(L^p(\mathbb{R}))$ which contains all operators aI of multiplication by a function $a \in PC$ and all operators of convolution $W^0(b)$ where $b \in PC_p$. These results appeared for the first time in their present form in [9]. We write $\mathcal{A}^{\mathcal{K}}(PC(\mathbb{R}), PC_p)$ for the image of the algebra $\mathcal{A}(PC(\mathbb{R}), PC_p)$ in the Calkin algebra and Φ for the canonical homomorphism

$$\mathcal{A}(PC(\mathbb{R}), PC_p) \to \mathcal{A}^{\mathcal{K}}(PC(\mathbb{R}), PC_p).$$

Analogously, we define the algebras $\mathcal{A}(C(\dot{\mathbb{R}}), C_p)$ and $\mathcal{A}^{\mathcal{K}}(C(\dot{\mathbb{R}}), C_p)$. The following is proved in [9, Proposition 12.6].

Lemma 3.1. *The algebra* $\mathcal{A}^{\mathcal{K}}\big(C(\dot{\mathbb{R}}), C_p\big)$ *lies in the center of the algebra* $\mathcal{A}^{\mathcal{K}}\big(PC(\mathbb{R}), PC_p\big)$*, and its maximal ideal space is homeomorphic to the subset* $(\dot{\mathbb{R}} \times \{\infty\}) \cup (\{\infty\} \times \dot{\mathbb{R}})$ *of the torus* $\dot{\mathbb{R}} \times \dot{\mathbb{R}}$*, with the point* $(s,\, t) \in (\dot{\mathbb{R}} \times \{\infty\}) \cup (\{\infty\} \times \dot{\mathbb{R}})$ *corresponding to the maximal ideal* $\big\{\Phi\big(fW^0(g)\big) \colon f \in C(\dot{\mathbb{R}}),\ g \in C_p \text{ and } f(s) = g(t) = 0\big\}$.

This lemma offers the way to employ Allan's local principle (see for instance [5, Theorem 4.8]) to localize the algebra $\mathcal{A}^{\mathcal{K}}\big(PC(\mathbb{R}), PC_p\big)$ with respect to its central subalgebra $\mathcal{A}^{\mathcal{K}}\big(C(\dot{\mathbb{R}}), C_p\big)$.

Given $(s, t) \in (\dot{\mathbb{R}} \times \{\infty\}) \cup (\{\infty\} \times \dot{\mathbb{R}})$, let $\mathcal{I}_{s,t}$ denote the smallest closed ideal of the Banach algebra $\mathcal{A}^{\mathcal{K}}\big(PC(\mathbb{R}), PC_p\big)$ which contains the point (s, t), and let $\Phi^{\mathcal{K}}_{s,t}$ refer to the canonical homomorphism from $\mathcal{A}\big(PC(\mathbb{R}), PC_p\big)$ to the local quotient algebra

$$\mathcal{A}^{\mathcal{K}}_{s,t} := \mathcal{A}^{\mathcal{K}}\big(PC(\mathbb{R}), PC_p\big)/\mathcal{I}_{s,t}.$$

Theorem 3.2 ([9], Theorem 15.1)**.** *Let* $A \in \mathcal{A}\big(PC(\mathbb{R}), PC_p\big)$.

(i) *The coset* $A + \mathcal{K}(L^p(\mathbb{R}))$ *is invertible in* $\mathcal{A}^{\mathcal{K}}\big(PC(\mathbb{R}), PC_p\big)$ *if and only if the coset* $\Phi^{\mathcal{K}}_{s,t}(A)$ *is invertible in* $\mathcal{A}^{\mathcal{K}}_{s,t}$ *for each* $(s, t) \in (\dot{\mathbb{R}} \times \{\infty\}) \cup (\{\infty\} \times \dot{\mathbb{R}})$.

(ii) *For* $s \in \mathbb{R}$*, the local algebra* $\mathcal{A}^{\mathcal{K}}_{s,\infty}$ *is isometrically isomorphic to the closed subalgebra* $\mathrm{alg}\{I, \chi_+, S_{\mathbb{R}}\}$ *of* $\mathcal{L}(L^p(\mathbb{R}))$*, and the isomorphism is given by*

$$\Phi^{\mathcal{K}}_{s,\infty}(A) \mapsto \mathsf{H}_{s,\infty}(A) \tag{14}$$

for each operator $A \in \mathcal{A}\big(PC(\mathbb{R}), PC_p\big)$.

(iii) *For* $t \in \mathbb{R}$*, the local algebra* $\mathcal{A}^{\mathcal{K}}_{\infty,t}$ *is isometrically isomorphic to the closed subalgebra* $\mathrm{alg}\{I, \chi_+, S_{\mathbb{R}}\}$ *of* $\mathcal{L}(L^p(\mathbb{R}))$*, and the isomorphism is given by*

$$\Phi^{\mathcal{K}}_{\infty,t}(A) \mapsto \mathsf{H}_{\infty,t}(A) \tag{15}$$

for each operator $A \in \mathcal{A}\big(PC(\mathbb{R}), PC_p\big)$.

(iv) *The local algebra* $\mathcal{A}^{\mathcal{K}}_{\infty,\infty}$ *is generated by the four idempotent elements*

$$\Phi^{\mathcal{K}}_{\infty,\infty}(W(\chi_-)\chi_-),\ \Phi^{\mathcal{K}}_{\infty,\infty}(W(\chi_-)\chi_+),\ \Phi^{\mathcal{K}}_{\infty,\infty}(W(\chi_+)\chi_-),\ \Phi^{\mathcal{K}}_{\infty,\infty}(W(\chi_+)\chi_+),$$

and the coset $\Phi^{\mathcal{K}}_{\infty,\infty}(A)$ *is invertible if and only if the four operators*

$$\mathsf{H}^{\pm\pm}(A),$$

which are complex multiples of the identity operator, are invertible.

For the proof of (ii), given in [9], one shows that the operators $\mathsf{H}_{s,\infty}(A)$ depend on the coset $\Phi^{\mathcal{K}}_{s,\infty}(A)$ only and that the kernel of the homomorphism (14) is just the local ideal $\mathcal{I}_{s,\infty}$. Assertions (iii) and (iv) follow similarly, and (i) is then a consequence of (ii) – (iv) and Allan's local principle.

Corollary 3.3. *The algebra* $\mathcal{A}^{\mathcal{K}}\big(PC(\mathbb{R}), PC_p\big)$ *is inverse-closed in the Calkin algebra of* $L^p(\mathbb{R})$.

Proof. Consider the smallest non-closed subalgebra $\mathcal{A}_0$ of $\mathcal{A}\big(PC(\mathbb{R}), PC_p\big)$, which contains all operators of multiplication and convolution by piecewise constant functions. Applying Theorem 3.2 to an operator $A \in \mathcal{A}_0$, we find that the spectrum of the coset $A + \mathcal{K}(L^p(\mathbb{R}))$ in $\mathcal{A}^{\mathcal{K}}\big(PC(\mathbb{R}), PC_p,\big)$ is a thin subset (i.e., a set with empty interior) of the complex plane. Since $\mathcal{A}_0$ is dense in $\mathcal{A}\big(PC(\mathbb{R}), PC_p,\big)$, the assertion follows via [3, Remark 2]. ■

Corollary 3.4. *An operator $A \in \mathcal{A}\big(PC(\mathbb{R}), PC_p\big)$ is Fredholm if and only if the operators $\mathsf{H}_{s,\infty}(A)$, $\mathsf{H}_{\infty,t}(A)$ and $\mathsf{H}^{\pm\pm}(A)$ are invertible for all $s, t \in \mathbb{R}$.*

To illustrate the previous results, we consider paired convolution operators. These are operators of the form

$$A = a_1 W^0(b_1) + a_2 W^0(b_2) \tag{16}$$

with $a_1, a_2 \in PC(\dot{\mathbb{R}})$ and $b_1, b_2 \in PC_p$. The following result is an immediate consequence of Corollary 3.4.

Theorem 3.5. *The operator A in (16) is Fredholm on $L^p(\mathbb{R})$ if and only if*

(i) *the operator $c_+ P_\mathbb{R} + c_- Q_\mathbb{R}$ with*

$$\begin{aligned} c_\pm(s) \;&:=\; \big(a_1(s^-)b_1(\pm\infty) + a_2(s^-)b_2(\pm\infty)\big)\,\chi_- \\ &\qquad + \big(a_1(s^+)b_1(\pm\infty) + a_2(s^+)b_2(\pm\infty)\big)\,\chi_+ \end{aligned}$$

is invertible on $L^p(\mathbb{R})$ for each $s \in \mathbb{R}$,

(ii) *the operator $d_+ P_\mathbb{R} + d_- Q_\mathbb{R}$ with*

$$\begin{aligned} d_\pm(t) \;&:=\; \big(a_1(-\infty)b_1(t^\pm) + a_2(-\infty)b_2(t^\pm)\big)\,\chi_- \\ &\qquad + \big(a_1(+\infty)b_1(t^\pm) + a_2(+\infty)b_2(t^\pm)\big)\,\chi_+ \end{aligned}$$

is invertible on $L^p(\mathbb{R})$ for each $t \in \mathbb{R}$,

(iii) *none of the numbers*

$$a_1(+\infty)b_1(\pm\infty) + a_2(+\infty)b_2(\pm\infty), \qquad a_1(-\infty)b_1(\pm\infty) + a_2(-\infty)b_2(\pm\infty)$$

is zero.

Of particular interest are paired operators of the form

$$A = a_1 W^0(\chi_+) + a_2 W^0(\chi_-) = a_1 P_\mathbb{R} + a_2 Q_\mathbb{R} \tag{17}$$

with $a_1, a_2 \in PC(\dot{\mathbb{R}})$, which can also be written as the singular integral operator

$$\frac{a_1 + a_2}{2} I + \frac{a_1 - a_2}{2} S_\mathbb{R}.$$

For these operators, Corollary 3.4 implies the following.

Corollary 3.6. *Let $a_1, a_2 \in PC(\dot{\mathbb{R}})$. The singular integral operator $a_1 P_\mathbb{R} + a_2 Q_\mathbb{R}$ is Fredholm on $L^p(\mathbb{R})$ if and only if*

(i) *the operator $(a_1(s^-)\chi_- + a_1(s^+)\chi_+)P_\mathbb{R} + (a_2(s^-)\chi_- + a_2(s^+)\chi_+)Q_\mathbb{R}$ is invertible on $L^p(\mathbb{R})$ for each $s \in \mathbb{R}$ and*

(ii) *the operator $(a_1(-\infty)\chi_- + a_1(+\infty)\chi_+)P_\mathbb{R} + (a_2(-\infty)\chi_- + a_2(+\infty)\chi_+)Q_\mathbb{R}$ is invertible on $L^p(\mathbb{R})$.*

4 Stability of the finite sections method

We consider finite sections $P_\tau A P_\tau$ of operators $A \in \mathcal{L}(L^p(\mathbb{R}))$ with respect to the projections P_τ, $\tau > 0$, given by

$$P_\tau : L^p(\mathbb{R}) \to L^p(\mathbb{R}), \quad (P_\tau u)(t) := \begin{cases} u(t) & \text{if} \quad |t| \le \tau, \\ 0 & \text{if} \quad |t| > \tau. \end{cases}$$

It will prove to be convenient to consider the *extended* finite sections $P_\tau A P_\tau + Q_\tau$ with $Q_\tau := I - P_\tau$ instead of the usual $P_\tau A P_\tau$. The passage from finite sections to extended finite sections does not involve any complications since, of course, both sequences $(P_\tau A P_\tau + Q_\tau)_{\tau>0}$ and $(P_\tau A P_\tau)_{\tau>0}$ are simultaneously stable or not. One technical advantage of using extended finite sections is that the operator A and its extended finite sections act on the same space.

Our approach to analyze the stability of the finite sections method will follow a general scheme to treat approximation problems, which can be summarized as follows. Suppose we are interested in the stability of sequences in a set $\mathcal{A}$.

1. **Algebraization:** Find a unital Banach algebra $\mathcal{E}$ which contains $\mathcal{A}$ and a closed ideal $\mathcal{G} \subset \mathcal{E}$ such that the original stability problem becomes equivalent to an invertibility problem in the quotient algebra $\mathcal{E}/\mathcal{G}$.

2. **Essentialization:** Find a unital subalgebra $\mathcal{F}$ of $\mathcal{E}$ which contains $\mathcal{A}$ and a closed ideal $\mathcal{J}$ of $\mathcal{F}$ which contains $\mathcal{G}$, such that $\mathcal{J}$ can be lifted. The latter means that one has full control about the difference between the invertibility of a coset of a sequence $(A_\tau) \in \mathcal{F}$ in the algebra $\mathcal{E}/\mathcal{G}$ and the invertibility of the coset of the same sequence in $\mathcal{F}/\mathcal{J}$. This control is usually guaranteed by a lifting theorem; see below.

3. **Localization:** Find a unital subalgebra $\mathcal{F}_0$ of $\mathcal{F}$ which contains $\mathcal{A}$ and $\mathcal{J}$ such that $\mathcal{F}_0/\mathcal{F}$ is inverse closed in $\mathcal{F}/\mathcal{J}$ and such that the quotient algebra $\mathcal{F}_0/\mathcal{J}$ has a large center. Use a local principle to translate the invertibility problem in the algebra $\mathcal{F}_0/\mathcal{J}$ to a family of simpler invertibility problems in local algebras.

4. **Identification:** Find necessary and sufficient conditions for the invertibility of the cosets of sequences in $\mathcal{A}$ in the local algebras.

The first (algebraization) step is simple. We let $\mathcal{E}$ stand for the Banach algebra of all bounded sequences $(A_\tau)_{\tau>0}$ of operators $A_\tau \in \mathcal{L}(L^p(\mathbb{R}))$ and write $\mathcal{G}$ for the closed ideal of $\mathcal{E}$ which consists of all sequences tending to zero in the norm. A standard Neumann series argument shows that then a sequence in $\mathcal{E}$ is stable if and only if its coset modulo $\mathcal{G}$ is invertible in the quotient algebra $\mathcal{E}/\mathcal{G}$. The sequences we are interested in belong to the smallest closed subalgebra $\mathcal{A} = \mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$ of $\mathcal{E}$ which contains all constant sequences (aI) of operators of multiplication by a function $a \in PC(\dot{\mathbb{R}})$, all constant sequences $(W^0(b))$ of operators of convolution by a multiplier $b \in PC_p$, the sequence $(P_\tau)_{\tau>0}$, and the ideal $\mathcal{G}$. This algebra can be seen as an extension of the algebra $\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p\big)$, studied in the previous section, with the addition of the non-constant sequence (P_τ).

4.1 Essentialization

Let $\mathcal{F}$ denote the set of all sequences $\mathbf{A} := (A_\tau) \in \mathcal{E}$ which have the following properties (all limits are considered with respect to the strong convergence as $\tau \to \infty$):

- there is an operator $\mathsf{W}_0(\mathbf{A})$ such that $A_\tau \to \mathsf{W}_0(\mathbf{A})$ and $A_\tau^* \to \mathsf{W}_0(\mathbf{A})^*$;
- there are operators $\mathsf{W}_{\pm 1}(\mathbf{A})$ such that

$$V_{-\tau} A_\tau V_\tau \to \mathsf{W}_1(\mathbf{A}) \quad \text{and} \quad (V_{-\tau} A_\tau V_\tau)^* \to \mathsf{W}_1(\mathbf{A})^*$$

  and

$$V_\tau A_\tau V_{-\tau} \to \mathsf{W}_{-1}(\mathbf{A}) \quad \text{and} \quad (V_\tau A_\tau V_{-\tau})^* \to \mathsf{W}_{-1}(\mathbf{A})^*;$$

- for each $y \in \mathbb{R}$, there is an operator $\mathsf{H}_{\infty,y}(\mathbf{A})$ such that

$$Z_\tau^{-1} U_y A_\tau U_{-y} Z_\tau \to \mathsf{H}_{\infty,y}(\mathbf{A}) \quad \text{and} \quad (Z_\tau^{-1} U_y A_\tau U_{-y} Z_\tau)^* \to \mathsf{H}_{\infty,y}(\mathbf{A})^*;$$

- for each $x \in \mathbb{R}$, there is an operator $\mathsf{H}_{x,\infty}(\mathbf{A})$ such that

$$Z_\tau V_{-x} A_\tau V_x Z_\tau^{-1} \to \mathsf{H}_{x,\infty}(\mathbf{A}) \quad \text{and} \quad (Z_\tau V_{-x} A_\tau V_x Z_\tau^{-1})^* \to \mathsf{H}_{x,\infty}(\mathbf{A})^*.$$

Proposition 4.1. *(i) The set $\mathcal{F}$ is a unital closed subalgebra of $\mathcal{E}$. The mappings W_i with $i \in \{-1, 0, 1\}$, $\mathsf{H}_{\infty,y}$ with $y \in \mathbb{R}$, and $\mathsf{H}_{x,\infty}$ with $x \in \mathbb{R}$ act as bounded homomorphisms on $\mathcal{F}$, and the ideal $\mathcal{G}$ of $\mathcal{F}$ lies in the kernel of each these homomorphisms.*

(ii) The algebra $\mathcal{F}$ is inverse-closed in $\mathcal{E}$, and the algebra $\mathcal{F}/\mathcal{G}$ is inverse-closed in $\mathcal{E}/\mathcal{G}$.

Proof. *(i)* The only assertion which is not completely trivial is the closedness of $\mathcal{F}$ in $\mathcal{E}$. To prove it, let $(\mathbf{A}_k)_{k\in\mathbb{N}}$ with $\mathbf{A}_k := (A_\tau^{(k)})_{\tau>0}$ be a sequence in $\mathcal{F}$ which converges to a sequence $\mathbf{A} := (A_\tau)_{\tau>0}$ in $\mathcal{E}$. Since $(\mathbf{A}_k)_{k\in\mathbb{N}}$ is a Cauchy sequence and $\|\mathsf{W}_0(\mathbf{B})\| \le \|\mathbf{B}\|$ for every sequence $\mathbf{B} \in \mathcal{F}$, we conclude that $(\mathsf{W}_0(\mathbf{A}_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}(L^p(\mathbb{R}))$. Let A^0 denote its limit. We show that A_0 is the strong limit of the sequence $\mathbf{A}$. For let $u \in L^p(\mathbb{R})$. For every $\epsilon > 0$, there exist a $\tau_0 > 0$ and a $k_0 \in \mathbb{N}$ such that, for $\tau > \tau_0$,

$$\begin{aligned} \|(A^0 - A_\tau)u\| &\le \|(A^0 - A_\tau^{(k_0)})u\| + \|A_\tau^{(k_0)} - A_\tau\|\|u\| \\ &\le \|(A^0 - A_\tau^{(k_0)})u\| + \|\mathbf{A} - \mathbf{A}_{k_0}\|\|u\| < \epsilon, \end{aligned}$$

which establishes the existence of the strong limit $\mathsf{W}_0(\mathbf{A})$. In a similar way, the existence of the strong limits $\mathsf{W}_{\pm 1}(\mathbf{A})$, $\mathsf{H}_{x,\infty}(\mathbf{A})$ and $\mathsf{H}_{\infty,y}(\mathbf{A})$ follows. Thus, $\mathbf{A} \in \mathcal{F}$, whence the closedness of $\mathcal{F}$.

For assertion (ii), let $\mathbf{A} := (A_\tau) \in \mathcal{F}$, and suppose that $\mathbf{A} + \mathcal{G} \in \mathcal{F}/\mathcal{G}$ is invertible in $\mathcal{E}/\mathcal{G}$. Then there exist a sequence $\mathbf{B} := (B_\tau) \in \mathcal{E}$ and a sequence $(G_\tau) \in \mathcal{G}$ such that $B_\tau A_\tau = I + G_\tau$ for every $\tau > 0$. Let $u \in L^p(\mathbb{R})$. Then

$$\|u\| = \|(B_\tau A_\tau - G_\tau)u\| \le c\|A_\tau u\| + \|G_\tau u\|$$

with a constant $c := \|\mathbf{B}\| > 0$. Taking the limit as $\tau \to \infty$ we obtain

$$\|u\| \le c\|\mathsf{W}_0(\mathbf{A})u\|,$$

which implies that the kernel of $\mathsf{W}_0(\mathbf{A})$ is $\{0\}$ and the range of $\mathsf{W}_0(\mathbf{A})$ is closed. Applying the same argument to the adjoint sequence we find that the kernel of $\mathsf{W}_0(\mathbf{A}^*) = \mathsf{W}_0(\mathbf{A})^*$ is $\{0\}$, too. Hence, $\mathsf{W}_0(\mathbf{A})$ is invertible. Further, for $u \in L^p(\mathbb{R})$, we have

$$\begin{aligned} &\|B_\tau u - \mathsf{W}_0(\mathbf{A})^{-1}u\| \\ &\quad = \|B_\tau u - (B_\tau A_\tau - G_\tau + Q_\tau)\mathsf{W}_0(\mathbf{A})^{-1}u\| \\ &\quad \le \|B_\tau\|\|u - A_\tau \mathsf{W}_0(\mathbf{A})^{-1}u\| + \|(-G_\tau + Q_\tau)\mathsf{W}_0(\mathbf{A})^{-1}u\| \\ &\quad = \|B_\tau\|\|\mathsf{W}_0(\mathbf{A})v - A_\tau v\| + \|(-G_\tau + Q_\tau)\mathsf{W}_0(\mathbf{A})^{-1}u\| \end{aligned}$$

with $v = \mathsf{W}_0(\mathbf{A})^{-1}u$. Since the right-hand side of this estimate tends 0 zero as $\tau \to \infty$, the inverse sequence $\mathbf{B}$ is strongly convergent, too. Similarly, one shows the strong convergence of the adjoint sequence $\mathbf{B}^*$. Further, from

$$V_{-\tau}B_\tau V_\tau \, V_{-\tau}A_\tau V_\tau = V_{-\tau}B_\tau A_\tau V_\tau = V_{-\tau}IV_\tau + V_{-\tau}G_\tau V_\tau = I + G'_\tau$$

with $(G'_\tau) \in \mathcal{G}$ one concludes that the sequence $(V_{-\tau}A_\tau V_\tau)$ is invertible in $\mathcal{E}/\mathcal{G}$. As above, one shows that the strong limit $\mathsf{W}_1(\mathbf{A})$ is invertible and that the sequences $(V_{-\tau}B_\tau V_\tau)$ and $(V_{-\tau}B_\tau V_\tau)^*$ are strongly convergent. A similar reasoning yields the strong convergence of the sequences $(V_\tau B_\tau V_{-\tau})$, $(Z_\tau^{-1}U_y B_\tau U_{-y} Z_\tau)$, $(Z_\tau V_{-x} B_\tau V_x Z_\tau^{-1})$ and of their adjoints. Thus, the sequence $\mathbf{B}$ belongs to $\mathcal{F}$, whence the inverse-closedness of $\mathcal{F}/\mathcal{G}$. The inverse-closedness of $\mathcal{F}$ in $\mathcal{E}$ is an obvious consequence of the previous result. ■

The W-homomorphisms and the H-homomorphisms will play different roles in what follows. Whereas the W-homomorphisms are needed to define an ideal of $\mathcal{F}$ which is subject to the lifting theorem and for which the quotient algebra has a center which is useful for applying Allan's local principle, the family of the H-homomorphisms will be used to identify the corresponding local algebras (of a suitable subalgebra of $\mathcal{F}$).

Let us turn to the lifting theorem. Let $\mathcal{K}$ denote the ideal of the compact operators on $L^p(\mathbb{R})$, and set

$$\mathcal{J} := \{(V_\tau K_1 V_{-\tau}) + (K_0) + (V_{-\tau}K_{-1}V_\tau) + (G_\tau) : K_{-1}, K_0, K_1 \in \mathcal{K},\ (G_\tau) \in \mathcal{G}\}.$$

Proposition 4.2. *$\mathcal{J}$ is a closed ideal of $\mathcal{F}$.*

Proof. We will only prove that $\mathcal{J}$ is a subset of $\mathcal{F}$. Once this is clear, the remainder of the proof will run completely parallel to the proof of Proposition 1.7 in [5].

In order to show that $\mathcal{J} \subset \mathcal{F}$, we have to show that all strong limits required in the definition of $\mathcal{F}$ exist for the sequences (K_0), $(V_{-\tau}K_{-1}V_\tau)$ and $(V_\tau K_1 V_{-\tau})$ with compact operators K_i. This is evident for the W-homomorphisms: One clearly has

$$\mathsf{W}_{-1}(V_{-\tau}K_{-1}V_\tau) = K_{-1}, \quad \mathsf{W}_0(K_0) = K_0, \quad \mathsf{W}_1(V_\tau K_1 V_{-\tau}) = K_1, \tag{18}$$

whereas all other W-homomorphisms give 0 when applied to these sequences since the sequences $(V_{-\tau})$ and (V_τ) are uniformly bounded and tend weakly to zero as τ tends to $+\infty$.

We claim that the H-homomorphisms applied to a sequence in $\mathcal{J}$ also give zero. This will follow once we have checked that the sequences

$$(U_{-y}Z_\tau),\ (V_\tau U_{-y}Z_\tau),\ (V_{-\tau}U_{-y}Z_\tau) \quad \text{and} \quad (V_x Z_\tau^{-1}),\ (V_\tau V_x Z_\tau^{-1}),\ (V_{-\tau}V_x Z_\tau^{-1})$$

are uniformly bounded and converge weakly to zero as $\tau \to +\infty$ for every choice of $x, y \in \mathbb{R}$. Since the operators U_{-y} and V_x are independent of τ and since V_x commutes with V_τ, it is sufficient to check these assertions for the sequences

$$(Z_\tau),\ (V_\tau U_{-y}Z_\tau),\ (V_{-\tau}U_{-y}Z_\tau) \quad \text{and} \quad (Z_\tau^{-1}),\ (V_\tau Z_\tau^{-1}),\ (V_{-\tau}Z_\tau^{-1}).$$

For the sequences $(Z_\tau^{\pm 1})$, this is the assertion of Lemma 2.2. For the other sequences, the uniform boundedness is evident, and for their weak convergence to zero we can argue similarly as in the proof of that lemma. For let B_τ denote any of the operators $V_\tau U_{-y}$ and $V_{-\tau}U_{-y}$ with $y \in \mathbb{R}$. Then

$$|(B_\tau \chi_{[\tau a, \tau b]})(x)| \le 1 \quad \text{and} \quad |(B_\tau^* \chi_{[c,d]})(x)| \le 1$$

for every possible choice of a, b, c, d, τ, s and x. Hence,

$$\begin{aligned} |\langle \chi_{[c,d]}, B_\tau Z_\tau \chi_{[a,b]} \rangle| &= \left| \frac{1}{\tau^{1/p}} \langle \chi_{[c,d]}, B_\tau \chi_{[\tau a, \tau b]} \rangle \right| \\ &= \frac{1}{\tau^{1/p}} \left| \int_c^d (B_\tau \chi_{[\tau a, \tau b]})(x)\, dx \right| \le \frac{1}{\tau^{1/p}} (d-c) \to 0 \end{aligned}$$

and

$$\begin{aligned} |\langle \chi_{[c,d]}, B_\tau Z_\tau^{-1} \chi_{[a,b]} \rangle| &= |\tau^{1/p} \langle B_\tau^* \chi_{[c,d]}, \chi_{[\frac{a}{\tau}, \frac{b}{\tau}]} \rangle| \\ &= \tau^{1/p} \left| \int_{\frac{a}{\tau}}^{\frac{b}{\tau}} (B_\tau^* \chi_{[c,d]})(x)\, dx \right| \le \tau^{1/p} (\frac{b}{\tau} - \frac{a}{\tau}) \to 0. \end{aligned}$$

The claimed weak convergence follows since the linear combinations of functions of the form $\chi_{[c,d]}$ lie dense in $L^p(\mathbb{R})$ and in its dual space. ■

Now one can apply the Lifting Theorem [5, Theorem 1.8] to obtain the following. Note that the stability of a sequence (A_τ) in $\mathcal{F}$ is equivalent to the invertibility of the coset $(A_\tau)+\mathcal{G}$ in the quotient algebra $\mathcal{F}/\mathcal{G}$ due to the inverse-closedness of $\mathcal{F}/\mathcal{G}$ in $\mathcal{E}/\mathcal{G}$ by Proposition 4.1.

Theorem 4.3. *Let $\mathbf{A} := (A_\tau) \in \mathcal{F}$. The sequence $\mathbf{A}$ is stable if and only if the operators $\mathsf{W}_{-1}(\mathbf{A})$, $\mathsf{W}_0(\mathbf{A})$ and $\mathsf{W}_1(\mathbf{A})$ are invertible in $\mathcal{L}(L^p(\mathbb{R}))$ and if the coset $\mathbf{A} + \mathcal{J}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{J}$.*

In that sense, the operators $\mathsf{W}_{-1}(\mathbf{A})$, $\mathsf{W}_0(\mathbf{A})$ and $\mathsf{W}_1(\mathbf{A})$ control the difference between the invertibility in $\mathcal{E}/\mathcal{G}$ and $\mathcal{F}/\mathcal{J}$.

The goal of the following lemmas is to show that all strong limits required in the definition of the algebra $\mathcal{F}$ exist for the generating sequences of the algebra $\mathcal{A} = \mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$, which implies that $\mathcal{A}$ is a closed subalgebra of $\mathcal{F}$.

Lemma 4.4. *Let $a \in PC(\dot{\mathbb{R}})$ and $b \in PC_p$. Then the strong limit, $\mathsf{W}_0(A_\tau)$, [1] $\mathsf{W}_{-1}(A_\tau)$ and $\mathsf{W}_1(A_\tau)$ exist for the following sequences in $\mathcal{A}$:*

(i) $\mathsf{W}_0(P_\tau) = I, \quad \mathsf{W}_0(aI) = aI, \quad \mathsf{W}_0(W^0(b)) = W^0(b)$;

(ii) $\mathsf{W}_{-1}(P_\tau) = \chi_+ I, \quad \mathsf{W}_{-1}(aI) = a(-\infty)I, \quad \mathsf{W}_{-1}(W^0(b)) = W^0(b)$;

(iii) $\mathsf{W}_1(P_\tau) = \chi_- I, \quad \mathsf{W}_1(aI) = a(+\infty)I, \quad \mathsf{W}_1(W^0(b)) = W^0(b)$.

For the constant sequences (aI), these assertions are shown in Lemma 2.4, and the remaining assertions are evident. For the H-homomorphisms one has the following.

Lemma 4.5. *Let $y \in \mathbb{R}$. The strong limit $\mathsf{H}_{\infty,y}(A_\tau)$ exists for the following sequences in $\mathcal{A}$:*

(i) $\mathsf{H}_{\infty,y}(P_\tau) = P_1$;

(ii) $\mathsf{H}_{\infty,y}(aI) = a(-\infty)\chi_- I + a(+\infty)\chi_+ I$ *for* $a \in PC(\dot{\mathbb{R}})$;

(iii) $\mathsf{H}_{\infty,y}(W^0(b)) = b(y^-)W^0(\chi_-) + b(y^+)W^0(\chi_+)$ *for* $b \in PC_p$.

[1] We write $W_0(A_\tau)$ and not $W_0((A_\tau))$ to make the notation less heavy, but remember that all homomorphisms act on sequences, and not on particular operators.

Lemma 4.6. *Let $x \in \mathbb{R}$. The strong limit $\mathsf{H}_{x,\infty}(A_\tau)$ exists for the following sequences in $\mathcal{A}$:*

(i) $\mathsf{H}_{x,\infty}(P_\tau) = I$;

(ii) $\mathsf{H}_{x,\infty}(aI) = a(x^-)\chi_- I + a(x^+)\chi_+ I$ *for* $a \in PC(\dot{\mathbb{R}})$;

(iii) $\mathsf{H}_{x,\infty}(W^0(b)) = b(-\infty)W^0(\chi_-) + b(+\infty)W^0(\chi_+)$ *for* $b \in PC_p$.

Indeed, these assertions are easy to check for the sequence (P_τ), and they were shown in Propositions 2.3 and 2.5 for the other sequences.

Corollary 4.7. $\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$ *is a closed subalgebra of the algebra* $\mathcal{F}$.

Let us emphasize that for a pure finite sections sequence $(A_\tau) = (P_\tau A P_\tau + Q_\tau)$ with (A) a constant sequence in $\mathcal{F}$, the strong limits are given by

$$\mathsf{W}_{-1}(A_\tau) = \chi_+ \mathsf{W}_{-1}(A)\chi_+ I + \chi_- I, \quad \mathsf{W}_0(A_\tau) = A, \quad \mathsf{W}_1(A_\tau) = \chi_- \mathsf{W}_1(A)\chi_- I + \chi_+ I$$

and

$$\mathsf{H}_{\infty,y}(A_\tau) = P_1 \mathsf{H}_{\infty,y}(A) P_1 + Q_1, \quad \mathsf{H}_{x,\infty}(A_\tau) = \mathsf{H}_{x,\infty}(A)$$

for all $x, y \in \mathbb{R}$.

In the following sections, it will be convenient to be able to take the strong limits $\mathsf{W}_{\pm 1}$ and $\mathsf{H}_{0,\infty}$ subsequently. For that reason we consider

$$\mathcal{F}' := \{\mathbf{A} \in \mathcal{F} : (\mathsf{W}_{\pm 1}(\mathbf{A})) \in \mathcal{F}\}.$$

It is not difficult to see that $\mathcal{F}'$ is a closed and inverse-closed subalgebra of $\mathcal{F}$ which contains the ideal $\mathcal{J}$ (compare the proof of Proposition 4.2), that $\mathcal{F}'/\mathcal{G}$ is an inverse-closed subalgebra of $\mathcal{F}/\mathcal{G}$ (see below), and that $\mathcal{F}'$ contains the algebra $\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$ (see the lemmas before Corollary 4.7). So, $\mathcal{F}'$ is the real outcome of the essentialization step.

We would like to add a general argument which yields the inverse-closedness of $\mathcal{F}'/\mathcal{G}$ in $\mathcal{F}/\mathcal{G}$ as a special case. Note that it is in general *not* true that if $\mathcal{B}$ is an inverse-closed subalgebra of an algebra $\mathcal{C}$ and if $\mathcal{I} \subset \mathcal{B}$ is an ideal of $\mathcal{B}$ and $\mathcal{C}$, then $\mathcal{B}/\mathcal{I}$ is inverse-closed in $\mathcal{C}/\mathcal{I}$.

Proposition 4.8. *Let $\mathcal{B}$ be a closed and inverse closed subalgebra of $\mathcal{F}$ which contains the ideal $\mathcal{G}$ of the zero sequences. Then $\mathcal{B}/\mathcal{G}$ is inverse closed in $\mathcal{F}/\mathcal{G}$.*

Proof. Let (A_n) be a sequence in $\mathcal{B}$ for which $(A_n) + \mathcal{G}$ is invertible in $\mathcal{F}/\mathcal{G}$. Thus, there are sequences (B_n) and (C_n) in $\mathcal{F}$ and (G_n) and (H_n) in $\mathcal{G}$ such that $B_n A_n = I - G_n$ and $A_n C_n = I - H_n$ for all $n \geq 1$. Choose n_0 such that $\|G_n\| < 1/2$ and $\|H_n\| < 1/2$ for all $n \geq n_0$. Set $A'_n := A_n$ if $n \geq n_0$ and $A'_n := I$ if $n < n_0$, and define B'_n and C'_n in the same way. Further, set $G'_n = G_n$ if $n \geq n_0$ and $G'_n := 0$ if $n < n_0$, and define H'_n analogously. Then

$$B'_n A'_n = I - G'_n \quad \text{and} \quad A'_n C'_n = I - H'_n \quad \text{for all } n \geq 1,$$

with $\|(G'_n)\| < 1/2$ and $\|(H'_n)\| < 1/2$. By a Neumann series argument, we conclude that the sequences $(I - G'_n)$ and $(I - H'_n)$ are invertible in $\mathcal{F}$ and that their inverses are of the form $(I - G''_n)$ and $(I - H''_n)$ with sequences (G''_n) and (H''_n) in $\mathcal{G}$, respectively. Hence,

$$(I - G''_n) B'_n A'_n = I \quad \text{and} \quad A'_n C'_n (I - H''_n) = I$$

for all $n \geq 1$. This shows that the sequence (A'_n) is invertible in $\mathcal{F}$. But the sequence (A'_n) differs from the sequence (A_n) only by a sequence in $\mathcal{G}$. Thus, (A'_n) is in $\mathcal{B}$, and the inverse-closedness of $\mathcal{B}$ in $\mathcal{F}$ entails that (A'_n) is invertible in $\mathcal{B}$. Hence, the coset $(A'_n) + \mathcal{G}$ is invertible in $\mathcal{B}/\mathcal{G}$, which implies the assertion, since $(A'_n) + \mathcal{G} = (A_n) + \mathcal{G}$. ∎

4.2 Localization

Since the algebra $\mathcal{F}'/\mathcal{J}$ has a trivial center, Allan's local principle is not helpful at this point. So next we are going to look for a subalgebra $\mathcal{F}_0$ of $\mathcal{F}'$ for which $\mathcal{F}_0/\mathcal{G}$ is still inverse-closed in $\mathcal{F}'/\mathcal{G}$, which contains the ideal $\mathcal{J}$, and which owns the property that the center of $\mathcal{F}_0/\mathcal{J}$ includes all cosets $(fI)+\mathcal{J}$ and $(W^0(g))+\mathcal{J}$ with $f \in C(\dot{\mathbb{R}})$ and $g \in C_p$. Note that the inverse-closedness of $\mathcal{F}_0/\mathcal{G}$ in $\mathcal{F}'/\mathcal{G}$ and, thus, in $\mathcal{E}/\mathcal{G}$ is needed to guarantee that the invertibility in $\mathcal{F}_0/\mathcal{G}$ is still equivalent to the stability.

The following construction will provide us with an algebra with the desired properties. Let $\mathcal{F}_0$ denote the set of all sequences in $\mathcal{F}'$ which commute with all constant sequences (fI) and $(W^0(g))$ where $f \in C(\dot{\mathbb{R}})$ and $g \in C_p$ modulo sequences in the ideal $\mathcal{J}$. The proof of the following proposition is straightforward. Note that, for each subset B of a unital algebra $\mathcal{C}$, the commutator $\{c \in \mathcal{C} : bc = cb \text{ for each } b \in B\}$ is an inverse-closed subalgebra of $\mathcal{C}$.

Proposition 4.9. (i) *The set $\mathcal{F}_0$ is a closed subalgebra of $\mathcal{F}'$ and contains the ideal $\mathcal{J}$.*

(ii) *The mappings W_i with $i \in \{-1,0,1\}$, $\mathsf{H}_{\infty,y}$ with $y \in \mathbb{R}$, and $\mathsf{H}_{x,\infty}$ with $x \in \mathbb{R}$ act as bounded homomorphisms on $\mathcal{F}_0$. The ideal $\mathcal{G}$ of $\mathcal{F}$ lies in the kernel of each these homomorphisms, and the ideal $\mathcal{J}$ lies in the kernel of each of the H-homomorphisms.*

(iii) *The algebra $\mathcal{F}_0$ is inverse-closed in $\mathcal{E}$, and the algebra $\mathcal{F}_0/\mathcal{G}$ is inverse-closed in $\mathcal{E}/\mathcal{G}$.*

By assertion (i), the lifting theorem applies to study invertibility in $\mathcal{F}_0/\mathcal{G}$.

Theorem 4.10. *Let $\mathbf{A} = (A_\tau) \in \mathcal{F}_0$. The sequence $\mathbf{A}$ is stable if and only if the operators $\mathsf{W}_{-1}(\mathbf{A})$, $\mathsf{W}_0(\mathbf{A})$ and $\mathsf{W}_1(\mathbf{A})$ are invertible in $\mathcal{L}(L^p(\mathbb{R}))$ and if the coset $\mathbf{A}+\mathcal{J}$ is invertible in the quotient algebra $\mathcal{F}_0^{\mathcal{J}} := \mathcal{F}_0/\mathcal{J}$.*

The algebra $\mathcal{F}_0$ is still large enough to contain all sequences that interest us.

Proposition 4.11. *$\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$ is a closed subalgebra of $\mathcal{F}_0$.*

Proof. We have to show that the generators (aI) with $a \in PC(\dot{\mathbb{R}})$, $(W^0(b))$ with $b \in PC_p$, and (P_τ) of $\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$ commute with the constant sequences (fI) where $f \in C(\dot{\mathbb{R}})$ and $(W^0(g))$ where $g \in C_p$ modulo sequences in $\mathcal{J}$. For the generators which are constant sequences this follows immediately from Lemma 3.1. For instance, one has

$$(fI)(W^0(b)) - (W^0(b))(fI) = (fW^0(b) - W^0(b)fI),$$

which is a constant sequence with a compact entry by the lemma. Hence, this sequence is in $\mathcal{J}$. It is further evident that (P_τ) commutes with (cI), and so it remains to verify that the commutator

$$(P_\tau)(W^0(g)) - (W^0(g))(P_\tau)$$

belongs to $\mathcal{J}$ for every multiplier $g \in C_p$. Write

$$\begin{aligned}
(P_\tau W^0(g) - W^0(g)P_\tau) &= (P_\tau W^0(g)Q_\tau - Q_\tau W^0(g)P_\tau) \\
&= (P_\tau \chi_+ W^0(g)\chi_+ Q_\tau - Q_\tau \chi_+ W^0(g)\chi_+ P_\tau) \\
&\quad + (P_\tau \chi_+ W^0(g)\chi_- Q_\tau - Q_\tau \chi_+ W^0(g)\chi_- P_\tau) \\
&\quad + (P_\tau \chi_- W^0(g)\chi_+ Q_\tau - Q_\tau \chi_- W^0(g)\chi_+ P_\tau) \\
&\quad + (P_\tau \chi_- W^0(g)\chi_- Q_\tau - Q_\tau \chi_- W^0(g)\chi_- P_\tau).
\end{aligned}$$

The sequences in the second and third line of the right-hand side of this equation belong to the ideal $\mathcal{G}$ since the operators $\chi_\pm W^0(g)\chi_\mp I$ are compact by Proposition [9, Proposition 12.6 (ii)] and since the Q_τ converge strongly to zero. The sequence in last line can be treated in a similar way as the sequence in the first line. So we are left on verifying that

$$(P_\tau \chi_+ W^0(g)\chi_+ Q_\tau - Q_\tau \chi_+ W^0(g)\chi_+ P_\tau) \in \mathcal{J}.$$

Write this sequence as

$$\begin{aligned}
&(\chi_{[0,\tau]} W^0(g)\chi_{[\tau,\infty)} I - \chi_{[\tau,\infty)} W^0(g)\chi_{[0,\tau]} I) \\
&\quad = (V_\tau (V_{-\tau}(\chi_{[0,\tau]} W^0(g)\chi_{[\tau,\infty)} I - \chi_{[\tau,\infty)} W^0(g)\chi_{[0,\tau]} I)V_\tau)V_{-\tau}) \\
&\quad = (V_\tau (\chi_{[-\tau,0]} W^0(g)\chi_{[0,\infty)} I - \chi_{[0,\infty)} W^0(g)\chi_{[-\tau,0]} I)V_{-\tau}) \\
&\quad = (V_\tau (\chi_{[-\tau,0]}\chi_- W^0(g)\chi_+ I - \chi_+ W^0(g)\chi_- \chi_{[-\tau,0]} I)V_{-\tau}).
\end{aligned}$$

Since the operators $\chi_\pm W^0(g)\chi_\mp I$ are compact and $\chi_{[-\tau,0]} I \to \chi_- I$ strongly as $\tau \to \infty$, we conclude that the sequence in the last line of this equality is of the form $(V_\tau K V_{-\tau}) + (G_\tau)$ with K compact and $(G_\tau) \in \mathcal{G}$. Hence, this sequence is in $\mathcal{J}$. ■

We proceed with localization. One can easily check that the commutative algebra generated by the cosets of constant sequences $(fI) + \mathcal{J}$ and $(W^0(g)) + \mathcal{J}$ with $f \in C(\dot{\mathbb{R}})$ and $g \in C_p$ is isomorphic to the subalgebra of the Calkin algebra which is generated by $fI + \mathcal{K}$ and $W^0(g) + \mathcal{K}$. By Lemma 3.1, the maximal ideal space of the latter algebra (and, thus, of our present central subalgebra) is homeomorphic to the subset $(\dot{\mathbb{R}} \times \{\infty\}) \cup (\{\infty\} \times \dot{\mathbb{R}})$ of the torus $\dot{\mathbb{R}} \times \dot{\mathbb{R}}$.

Given $(s,t) \in (\dot{\mathbb{R}} \times \{\infty\}) \cup (\{\infty\} \times \dot{\mathbb{R}})$, let $\mathcal{I}_{s,t}$ denote the smallest closed two-sided ideal of the quotient algebra $\mathcal{F}_0/\mathcal{J}$ which contains the maximal ideal corresponding to the point (s,t), and let $\Phi^{\mathcal{J}}_{s,t}$ refer to the canonical homomorphism from $\mathcal{F}_0/\mathcal{J}$ onto the quotient algebra $\mathcal{F}^{\mathcal{J}}_{s,t} := (\mathcal{F}_0/\mathcal{J})/\mathcal{I}_{s,t}$. In order not to burden the notation, we write $\Phi^{\mathcal{J}}_{s,t}(A_\tau)$ instead of $\Phi^{\mathcal{J}}_{s,t}((A_\tau) + \mathcal{J})$ for every sequence $(A_\tau) \in \mathcal{F}_0$.

Let $(s,t) \in (\dot{\mathbb{R}} \times \{\infty\}) \cup (\{\infty\} \times \dot{\mathbb{R}})$. One cannot expect that the local algebra $\mathcal{F}^{\mathcal{J}}_{s,t}$ can be identified completely. But we will be able to identify the smallest closed subalgebra $\mathcal{A}^{\mathcal{J}}_{s,t}$ of $\mathcal{F}^{\mathcal{J}}_{s,t}$ which contains all cosets $(P_\tau) + \mathcal{I}_{s,t}$, $(aI) + \mathcal{I}_{s,t}$ with $a \in PC(\dot{\mathbb{R}})$ and $(W^0(b)) + \mathcal{I}_{s,t}$ with $b \in PC_p$, and this identification will be sufficient for our purposes. We will identify the algebras $\mathcal{A}^{\mathcal{J}}_{s,t}$ by means of the family of the H-homomorphisms. Note that, by assertion *(ii)* of Proposition 4.9, the operators $\mathsf{H}_{\infty,y}(\mathbf{A})$ and $\mathsf{H}_{x,\infty}(\mathbf{A})$ depend only on the coset of the sequence $\mathbf{A}$ modulo $\mathcal{J}$. Thus, the quotient homomorphisms

$$\mathbf{A} + \mathcal{J} \mapsto \mathsf{H}_{\infty,y}(\mathbf{A}) \quad \text{and} \quad \mathbf{A} + \mathcal{J} \mapsto \mathsf{H}_{x,\infty}(\mathbf{A})$$

are well defined. We denote them again by $\mathsf{H}_{\infty,y}$ and $\mathsf{H}_{x,\infty}$, respectively.

4.3 Identification of the local algebras

We start with describing the local algebras $\mathcal{A}^{\mathcal{J}}_{s,\infty}$.

Theorem 4.12. *Let $s \in \mathbb{R}$. The algebra $\mathcal{A}^{\mathcal{J}}_{s,\infty}$ is isometrically isomorphic to the subalgebra $\mathrm{alg}\{I, \chi_+ I, W^0(\chi_+)\}$ of $\mathcal{L}(L^p(\mathbb{R}))$, and the isomorphism is given by*

$$\Phi^{\mathcal{J}}_{s,\infty}(\mathbf{A}) \mapsto \mathsf{H}_{s,\infty}(\mathbf{A}). \tag{19}$$

Proof. By definition, $\mathcal{I}_{s,\infty}$ is the smallest two-sided ideal of $\mathcal{F}_0^{\mathcal{J}}$ which contains the cosets $(fW^0(g)) + \mathcal{J}$ with $f(s) = 0$ and $g(\infty) = 0$. From Lemma 4.6 we infer that $\mathsf{H}_{s,\infty}(\mathcal{I}_{s,\infty}) = 0$. Thus, the homomorphism $\mathsf{H}_{s,\infty}$ is well defined on the quotient algebra $\mathcal{A}_{s,\infty}^{\mathcal{J}}$. The same lemma also implies that $\mathsf{H}_{s,\infty}$ maps $\mathcal{A}_{s,\infty}^{\mathcal{J}}$ to $\text{alg}\{I, \chi_+ I, W^0(\chi_+)\}$.

We claim that the homomorphism $\mathsf{H}_{s,\infty} : \mathcal{A}_{s,\infty}^{\mathcal{J}} \to \text{alg}\{I, \chi_+ I, W^0(\chi_+)\}$ is an isometry. This will follow once we have shown that the identity

$$\Phi_{s,\infty}^{\mathcal{J}}(\mathbf{A}) = \Phi_{s,\infty}^{\mathcal{J}}(V_s \mathsf{H}_{s,\infty}(\mathbf{A}) V_{-s}) \tag{20}$$

holds for all generators of the algebra $\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$ in place of the sequence $\mathbf{A}$. Note that the right-hand side of (20) makes sense since the constant sequence $(V_s \mathsf{H}_{s,\infty}(\mathbf{A}) V_{-s})$ belongs to the algebra $\mathcal{F}_0$ by Proposition 4.11.

For the generators (aI) and $(W^0(b))$ with $a \in PC(\dot{\mathbb{R}})$ and $b \in PC_p$ of the algebra $\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$, the identity (20) was already established in the proof of Theorem 14.2 in [9]. For the generating sequence (P_τ) in place of $\mathbf{A}$, the right-hand side of (20) is the identity element. So we have to show that $\Phi_{s,\infty}^{\mathcal{J}}(P_\tau)$ is the identity element of the local algebra.

Choose $y \in \mathbb{R}$ greater than $|s|$, and let f_s be a continuous function supported on the interval $(-y, y)$ such that $f_s(s) = 1$. Since $\Phi_{s,\infty}^{\mathcal{J}}(f_s I)$ is the identity in the local algebra, we have

$$\Phi_{s,\infty}^{\mathcal{J}}(Q_\tau) = \Phi_{s,\infty}^{\mathcal{J}}(f_s I) \Phi_{s,\infty}^{\mathcal{J}}(Q_\tau) = \Phi_{s,\infty}^{\mathcal{J}}(f_x Q_\tau).$$

But $f_s Q_\tau = 0$ for τ sufficiently large. Thus, the sequence $(f_s Q_\tau)_{\tau > 0}$ belongs to the ideal $\mathcal{G}$, whence $\Phi_{s,\infty}^{\mathcal{J}}(Q_\tau) = 0$. Hence, $\Phi_{s,\infty}^{\mathcal{J}}(P_\tau) = \Phi_{s,\infty}^{\mathcal{J}}(I - Q_\tau)$ is the identity element. ■

The previous theorem implies that, for sequences $\mathbf{A} \in \mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$, the coset $\Phi_{s,\infty}^{\mathcal{J}}(\mathbf{A})$ is invertible in the local algebra $\mathcal{A}_{s,\infty}^{\mathcal{J}}$ if and only if the operator $\mathsf{H}_{s,\infty}(\mathbf{A})$ is invertible in $\text{alg}\{I, \chi_+ I, W^0(\chi_+)\}$. Of course, one would prefer to check the invertiblity of the operator $\mathsf{H}_{s,\infty}(\mathbf{A})$ in $\mathcal{L}(L^p(\mathbb{R}))$, not in $\text{alg}\{I, \chi_+ I, W^0(\chi_+)\}$. In the present setting, this causes no problem since the algebra $\text{alg}\{I, \chi_+ I, W^0(\chi_+)\}$ is known to be inverse-closed in $\mathcal{L}(L^p(\mathbb{R}))$. The following proposition and its proof show how the desired invertibility condition can be derived without any a priori information on the inverse-closedness of the local algebras.

Proposition 4.13. *Let $\mathbf{A} \in \mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$. Then the coset $\Phi_{s,\infty}^{\mathcal{J}}(\mathbf{A})$ is invertible in the local algebra $\mathcal{F}_{s,\infty}^{\mathcal{J}}$ if and only if the operator $\mathsf{H}_{s,\infty}(\mathbf{A})$ is invertible in $\mathcal{L}(L^p(\mathbb{R}))$.*

Proof. For $s \in \mathbb{R}$, let $\mathcal{D}_{s,\infty}$ denote the set of all operators $A \in \mathcal{L}(L^p(\mathbb{R}))$ with the property that the constant sequence $(V_s A V_{-s})$ belongs to the algebra $\mathcal{F}_0$. One easily checks that $\mathcal{D}_{s,\infty}$ is a closed subalgebra of $\mathcal{L}(L^p(\mathbb{R}))$. Moreover, $\mathcal{D}_{s,\infty}$ is inverse-closed in $\mathcal{L}(L^p(\mathbb{R}))$, which can be seen as follows.

Let $A \in \mathcal{D}_{s,\infty}$ be invertible in $\mathcal{L}(L^p(\mathbb{R}))$. The constant sequence $(V_s A V_{-s})$ is invertible in the algebra $\mathcal{E}$ of all bounded sequences, and its inverse is the sequence $(V_s A^{-1} V_{-s})$. Since $(V_s A V_{-s}) \in \mathcal{F}_0$ by hypothesis, and since $\mathcal{F}_0$ is inverse-closed in $\mathcal{E}$ by Proposition 4.9 (*iii*), we conclude that $(V_s A^{-1} V_{-s}) \in \mathcal{F}_0$. Hence, $A^{-1} \in \mathcal{D}_{s,\infty}$.

Let now $\mathbf{A} \in \mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$. If the coset $\Phi_{s,\infty}^{\mathcal{J}}(\mathbf{A})$ is invertible in $\mathcal{F}_{s,\infty}^{\mathcal{J}}$, then $\mathsf{H}_{s,\infty}(\mathbf{A})$ is invertible in $\mathcal{L}(L^p(\mathbb{R}))$, since $\mathsf{H}_{s,\infty}$ acts as a homomorphism on that local algebra. Conversely, let $\mathsf{H}_{s,\infty}(\mathbf{A})$ be invertible in $\mathcal{L}(L^p(\mathbb{R}))$. We know already that $\mathsf{H}_{s,\infty}(\mathbf{A})$ belongs to the algebra $\text{alg}\{I, \chi_+ I, W^0(\chi_+)\}$, and one easily checks that this algebra is contained in

$\mathcal{D}_{s,\infty}$. By the inverse-closedness of $\mathcal{D}_{s,\infty}$, the operator $\mathsf{H}_{s,\infty}(\mathbf{A})$ possesses an inverse in $\mathcal{D}_{s,\infty}$. Let B denote this inverse. From $B\mathsf{H}_{s,\infty}(\mathbf{A}) = I$ we get

$$(V_s B V_{-s})(V_s \mathsf{H}_{s,\infty}(\mathbf{A}) V_{-s}) = (I). \tag{21}$$

Note that the sequences in (21) are constant. Since the operators B and $\mathsf{H}_{s,\infty}(\mathbf{A})$ belong to $\mathcal{D}_{s,\infty}$, it is also clear that the sequences in (21) belong to $\mathcal{F}_0$. Hence, one can apply the local homomorphism $\Phi^{\mathcal{J}}_{s,\infty}$ to both sides of (21), which gives

$$\Phi^{\mathcal{J}}_{s,\infty}(V_s B V_{-s})\Phi^{\mathcal{J}}_{s,\infty}(V_s \mathsf{H}_{s,\infty}(\mathbf{A}) V_{-s}) = \Phi^{\mathcal{J}}_{s,\infty}(I).$$

From (20) we conclude that $\Phi^{\mathcal{J}}_{s,\infty}(\mathbf{A})$ is invertible in $\mathcal{F}^{\mathcal{J}}_{s,\infty}$. ■

The following is an immediate consequence of Theorem 4.12, Proposition 4.13 and the well known inverse-closedness of the algebra $\text{alg}\{I, \chi_+ I, W^0(\chi_+)\}$ in $\mathcal{L}(L^p(\mathbb{R}))$.

Corollary 4.14. *The local algebra $\mathcal{A}^{\mathcal{J}}_{s,\infty}$ is inverse-closed in $\mathcal{F}^{\mathcal{J}}_{s,\infty}$.*

Next we are going to examine the local algebras $\mathcal{A}^{\mathcal{J}}_{\infty,t}$.

Theorem 4.15. *Let $t \in \mathbb{R}$. The algebra $\mathcal{A}^{\mathcal{J}}_{\infty,t}$ is isometrically isomorphic to the subalgebra $\text{alg}\{I, \chi_+ I, P_1, W^0(\chi_+)\}$ of $\mathcal{L}(L^p(\mathbb{R}))$, and the isomorphism is given by*

$$\Phi^{\mathcal{J}}_{\infty,t}(\mathbf{A}) \mapsto \mathsf{H}_{\infty,t}(\mathbf{A}). \tag{22}$$

Proof. It follows from Lemma 4.5, that the operator $\mathsf{H}_{\infty,t}(\mathbf{A})$ belongs to the algebra $\text{alg}\{I, \chi_+ I, P_1, W^0(\chi_+)\}$ and that this operator depends on the coset $\Phi^{\mathcal{J}}_{\infty,t}(\mathbf{A})$ of the sequence $\mathbf{A}$ only. Thus, there is a well defined homomorphism

$$\mathcal{A}^{\mathcal{J}}_{\infty,y} \to \text{alg}\{I, \chi_+ I, P_1, W^0(\chi_+)\}, \quad \Phi^{\mathcal{J}}_{\infty,t}(\mathbf{A}) \mapsto \mathsf{H}_{\infty,t}(\mathbf{A})$$

which we denote by $\mathsf{H}_{\infty,y}$ again. It will follow that this homomorphism is an isometry once we have verified the identity

$$\Phi^{\mathcal{J}}_{\infty,t}(\mathbf{A}) = \Phi^{\mathcal{J}}_{\infty,t}(U_t Z_\tau \mathsf{H}_{\infty,t}(\mathbf{A}) Z_\tau^{-1} U_{-t}) \tag{23}$$

for all sequences $\mathbf{A}$ in $\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$. This is again done in the proof of Theorem 14.2 in [9] for the constant generating sequences of $\mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$, and it is evident for the sequence (P_τ). ■

The previous theorem can be completed as follows.

Proposition 4.16. *Let $\mathbf{A} \in \mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$. Then the coset $\Phi^{\mathcal{J}}_{\infty,t}(\mathbf{A})$ is invertible in the local algebra $\mathcal{F}^{\mathcal{J}}_{\infty,t}$ if and only if the operator $\mathsf{H}_{\infty,t}(\mathbf{A})$ is invertible in $\mathcal{L}(L^p(\mathbb{R}))$.*

Proof. The proof proceeds as that of Proposition 4.13. For $t \in \mathbb{R}$, introduce the algebra $\mathcal{D}_{\infty,t}$ of all operators $A \in \mathcal{L}(L^p(\mathbb{R}))$ with the property that the sequence $(U_t Z_\tau A Z_\tau^{-1} U_{-t})_{\tau>0}$ belongs to the algebra $\mathcal{F}_0$. Again one checks easily checks that $\mathcal{D}_{\infty,t}$ is an inverse-closed subalgebra of $\mathcal{L}(L^p(\mathbb{R}))$ and that the algebra $\text{alg}\{I, \chi_+ I, P_1, W^0(\chi_+)\}$ is contained in $\mathcal{D}_{\infty,t}$. ■

Corollary 4.17. *The local algebra* $\mathcal{A}^{\mathcal{J}}_{\infty,t}$ *is inverse-closed in* $\mathcal{F}^{\mathcal{J}}_{\infty,t}$ *if and only if the algebra* $\mathrm{alg}\{I, \chi_+ I, P_1, W^0(\chi_+)\}$ *is inverse-closed in* $\mathcal{L}(L^p(\mathbb{R}))$.

Our final goal is the local algebra $\mathcal{A}^{\mathcal{J}}_{\infty,\infty}$. It is easy to see that this algebra is generated by the identity element and by the three projections $\Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ I)$, $\Phi^{\mathcal{J}}_{\infty,\infty}(W^0(\chi_+))$, and $\Phi^{\mathcal{J}}_{\infty,\infty}(P_\tau)$. The following proposition shows that this algebra has a non-trivial center.

Proposition 4.18. *The projection* $\Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ I)$ *belongs to the center of* $\mathcal{A}^{\mathcal{J}}_{\infty,\infty}$.

Proof. One only has to check the relation

$$\Phi^{\mathcal{J}}_{\infty,\infty}(W^0(\chi_+)\chi_+ I) = \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ W^0(\chi_+)). \tag{24}$$

Choose a continuous and monotonically increasing function χ'_+ which takes the values 0 at $-\infty$ and 1 at $+\infty$. Then, clearly,

$$\Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ I) = \Phi^{\mathcal{J}}_{\infty,\infty}(\chi'_+ I) \quad \text{and} \quad \Phi^{\mathcal{J}}_{\infty,\infty}(W^0(\chi_+)) = \Phi^{\mathcal{J}}_{\infty,\infty}(W^0(\chi'_+)).$$

Since the commutator $W^0(\chi'_+)\chi'_+ I - \chi'_+ W^0(\chi'_+)$ is compact by [9, Proposition 12.6 (ii)], the equality (24) follows. ∎

Proposition 4.18 implies that the local algebra $\mathcal{A}^{\mathcal{J}}_{\infty,\infty}$ splits into the direct sum

$$\mathcal{A}^{\mathcal{J}}_{\infty,\infty} = \mathcal{A}^{+}_{\infty,\infty} + \mathcal{A}^{-}_{\infty,\infty}$$

where $\mathcal{A}^{\pm}_{\infty,\infty} := \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_\pm I)\mathcal{A}^{\mathcal{J}}_{\infty,\infty}\Phi^{\mathcal{J}}_{\infty,\infty}(\chi_\pm I)$. The algebras $\mathcal{A}^{\pm}_{\infty,\infty}$ are unital, and the cosets $\Phi^{\mathcal{J}}_{\infty,\infty}(\chi_\pm I)$ can be considered as their identity elements. It is evident that the invertibility of the coset $\Phi^{\mathcal{J}}_{\infty,\infty}(\mathbf{A})$ in $\mathcal{A}^{\mathcal{J}}_{\infty,\infty}$ for $\mathbf{A} = (A_\tau) \in \mathcal{A}$ is equivalent to the invertibility of the two cosets $\Phi^{\mathcal{J}}_{\infty,\infty}(\chi_\pm A_\tau \chi_\pm I)$ in the algebras $\mathcal{A}^{\pm}_{\infty,\infty}$, respectively. Note that one obtains the same result by localizing the algebra $\mathcal{A}^{\mathcal{J}}_{\infty,\infty}$ over its central subalgebra described in Proposition 4.18.

Consider the algebra $\mathcal{A}^{+}_{\infty,\infty}$. It is another consequence of Proposition 4.18 that this algebra is generated by the two idempotent elements $p := \Phi^{\mathcal{J}}_{\infty,\infty}(P_\tau \chi_+ I)$ and $r := \Phi^{\mathcal{J}}_{\infty,\infty}(W^0(\chi_+)\chi_+ I)$ and by the identity element $e := \Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ I)$. Thus, the local algebra $\mathcal{A}^{+}_{\infty,\infty}$ is subject to the two projections theorem (see, for instance [3]). To apply this theorem, we have to determine the spectrum of the element

$$X := prp + (e-p)(e-r)(e-p) = \Phi^{\mathcal{J}}_{\infty,\infty}\left(P_\tau W^0(\chi_+)P_\tau \chi_+ I + Q_\tau W^0(\chi_-)Q_\tau \chi_+ I\right)$$

in the local algebra $\mathcal{A}^{+}_{\infty,\infty}$. We will first determine the spectrum of X in $\mathcal{F}^{\mathcal{J}}_{\infty,\infty}$. The following simple lemma will be useful. Let $\mathcal{H}$ denote the smallest closed subalgebra of $\mathcal{E}$ which contains the sequence (P_τ) and all constant sequences of homogeneous operators in $\mathcal{L}(L_p(\mathbb{R}))$.

Lemma 4.19. *Let* $(B_\tau) \in \mathcal{H}$. *Then* (B_τ) *is invertible in* $\mathcal{E}$ *if and only if* B_1 *is invertible in* $\mathcal{L}(L_p(\mathbb{R}))$.

Indeed, one has $Z_\tau^{-1} B_\tau Z_\tau = B_1$ for every $\tau > 0$.

Let I be the closed interval between p and $p/(p-1)$. For $\alpha \in I$, set

$$\mathcal{C}_\alpha := \{(1 + \coth((z + \mathbf{i}\alpha)\pi))/2 : z \in \mathbb{R}\} \cup \{0,\, 1\}$$

and $\Omega_p := \cup_{\alpha \in I} \mathcal{C}_\alpha$.

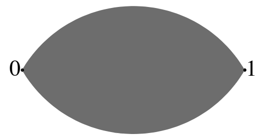


Figure 1: The lense Ω_p for $p = 3$ (or $p = 3/2$). If $p = 2$ it would be just the straight line between the points 0 and 1.

Proposition 4.20. *The spectrum of the element X in $\mathcal{F}^{\mathcal{J}}_{\infty,\infty}$ is Ω_p.*

Proof. It is elementary to check that $\sigma(X) = \Omega_p$ if and only if $\sigma(prp) = \Omega_p$. By Lemma 4.19, the spectrum of the coset

$$\left(P_\tau \chi_+ W^0(\chi_+)\chi_+ P_\tau\right) + \mathcal{G} \tag{25}$$

in $\mathcal{E}/\mathcal{G}$ is equal to the spectrum of the operator

$$P_1\chi_+ W^0(\chi_+)\chi_+ P_1 = \chi_{[0,1]} W(\chi_+)\chi_{[0,1]} I = \chi_{[0,1]} M^0(c)\chi_{[0,1]} I$$

on $L^p([0,1])$, where $M^0(c)$ stands for the operator of Mellin convolution by the function $c(y) := \coth\left((y + \mathbf{i}/p)\pi\right)$. The spectrum of the operator $\chi_{[0,1]}M^0(c)\chi_{[0,1]}I$ coincides with the spectrum of the Wiener-Hopf operator $W(c)$, and the latter can be shown to be the lentiform domain Ω_p (see [1], Prop. 9.15).

Since $\mathcal{F}_0/\mathcal{G}$ is inverse-closed in $\mathcal{E}/\mathcal{G}$ by Proposition 4.9 (*iii*), the spectrum of (25) is also Ω_p, which implies that the spectrum of prp (and, hence, that of X) in $\mathcal{F}^{\mathcal{J}}_{\infty,\infty}$ is contained in Ω_p.

To get the reverse inclusion $\Omega_p \subseteq \sigma(prp)$, let $\lambda \notin \sigma(prp)$. Then $prp - \lambda\Phi^{\mathcal{J}}_{\infty,\infty}(I)$ is invertible in $\mathcal{F}^{\mathcal{J}}_{\infty,\infty}$ and there are sequences $\mathbf{B} \in \mathcal{F}_0$ and $\mathbf{J} \in \mathcal{J}_{\infty,\infty} := \mathcal{J} + \mathcal{I}_{\infty,\infty}$ such that

$$\left(P_\tau\chi_+W^0(\chi_+)\chi_+P_\tau - \lambda I\right)\mathbf{B} = \mathbf{I} - \mathbf{J},$$

with $\mathbf{I}$ referring to the identity sequence. Without loss of generality, one can assume that $\mathbf{J}$ belongs to a dense subset of $\mathcal{J}_{\infty,\infty}$, say that there are sequences $\mathbf{C}_i, \mathbf{D}_i, \mathbf{E}_j$ and $\mathbf{F}_j$ in $\mathcal{F}_0$ and functions $f_i \in C(\dot{\mathbb{R}})$ and $g_j \in C_p$ with $f_i(\infty) = 0$ and $g_j(\infty) = 0$ such that

$$\left(P_\tau\chi_+W^0(\chi_+)\chi_+P_\tau - \lambda I\right)\mathbf{B} = \mathbf{I} - \sum_{i=1}^{M}\mathbf{C}_i f_i \mathbf{D}_i - \sum_{j=1}^{N}\mathbf{E}_j W^0(g_j)\mathbf{F}_j - \mathbf{J}',$$

with $\mathbf{J}' \in \mathcal{J}$. Applying the homomorphism W_1 to both sides of this equality we obtain the operator equality

$$\left(\chi_- W^0(\chi_+)\chi_- I - \lambda I\right)\mathsf{W}_1(\mathbf{B}) = I - \sum_{j=1}^{N}\mathsf{W}_1(\mathbf{E}_j)W^0(g_j)\mathsf{W}_1(\mathbf{F}_j) - K$$

(recall Lemma 4.4), where K is a compact operator. Applying then the homomorphism $\mathsf{H}_{0,\infty}$ to both sides of this equality (which we can consider as an equality between constant sequences; note that this is the place where the passage to the algebra $\mathcal{F}'$ becomes important), we find

$$\left(\chi_- W^0(\chi_+)\chi_- I - \lambda I\right)\mathsf{H}_{0,\infty}(\mathsf{W}_1(\mathbf{B})) = I.$$

Thus, the operator $\chi_- W^0(\chi_+)\chi_- I - \lambda I$ is invertible from the right-hand side. Analogously, the invertibility of this operator from the left-hand side follows. Thus, the operator $\chi_- W^0(\chi_+)\chi_- I - \lambda\chi_- I$ is invertible as an operator on $L^p(\mathbb{R}^-)$. Let $J : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be the flip operator defined by $(Jf)(x) := f(-x)$. Then, as one easily checks,

$$J\left(\chi_- W^0(\chi_+)\chi_- I - \lambda\chi_- I\right) J = \chi_+ W^0(\chi_-)\chi_+ I - \lambda\chi_+,$$

which implies that λ is not in the spectrum of the Wiener-Hopf operator $W(\chi_-)$ on $L^p(\mathbb{R}^+)$. Since the spectrum of this operator is the lense Ω_p (see again [1], Prop. 9.15), we conclude that $\lambda \notin \Omega_p$. Hence, $\Omega_p \subseteq \sigma(prp)$. ■

Note that the lense Ω_p is simply connected in $\mathbb{C}$. Hence, the spectrum of X in $\mathcal{F}^{\mathcal{J}}_{\infty,\infty}$ coincides with the spectrum of X in $\mathcal{A}^{\mathcal{J}}_{\infty,\infty}$ and also with its spectrum in $\mathcal{A}^{+}_{\infty,\infty}$.

The following proposition summarizes the results obtained for the case $(s,t) = (\infty,\infty)$. Note that the occurring 4×4 matrices have a 2×2-block diagonal structure, which reflects the decomposing property of the local algebra at (∞,∞). Define functions $\widehat{P}, \widehat{p}, \widehat{r} : \Omega_p \to \mathbb{C}^{4\times 4}$ by

$$\widehat{P} : x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \widehat{p} : x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\widehat{r} : x \mapsto \begin{bmatrix} x & \sqrt{x(1-x)} & 0 & 0 \\ \sqrt{x(1-x)} & 1-x & 0 & 0 \\ 0 & 0 & x & \sqrt{x(1-x)} \\ 0 & 0 & \sqrt{x(1-x)} & 1-x \end{bmatrix}.$$

Here $\sqrt{x(1-x)}$ stands for any complex number c with $c^2 = x(1-x)$.

Proposition 4.21. *(i) The mapping Ψ which sends the cosets $\Phi^{\mathcal{J}}_{\infty,\infty}(\chi_+ I)$, $\Phi^{\mathcal{J}}_{\infty,\infty}(P_\tau)$ and $\Phi^{\mathcal{J}}_{\infty,\infty}(W^0(\chi_+))$ to the functions $\widehat{P}$, $\widehat{p}$ and $\widehat{r}$ extends to a homomorphism from the algebra $\mathcal{A}^{\mathcal{J}}_{\infty,\infty}$ into the algebra of all bounded 4×4-matrix valued functions on Ω_p.*
(ii) Let $\mathbf{A} \in \mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$. Then the coset $\Phi^{\mathcal{J}}_{\infty,\infty}(\mathbf{A})$ is invertible in $\mathcal{F}^{\mathcal{J}}_{\infty,\infty}$ if and only if the associated function $\Psi(\Phi^{\mathcal{J}}_{s,\infty})$ is invertible.

Note that the intersection of each of the intervals $(-\infty,0)$ and $(1,\infty)$ with the lense Ω_p is empty. Hence, the values of the function $x \mapsto x(1-x)$ on Ω_p do not meet the negative real axis $(-\infty,0)$. One can therefore choose the square roots $\sqrt{x(1-x)}$ in such a way that $\widehat{r}$ becomes a continuous function on Ω_p, and Ψ becomes a homomorphism into $C(\Omega_p, \mathbb{C}^{4\times 4})$.

4.4 The main result

Having identified all local algebras, we can now state our main result. Write $\mathsf{H}_{\infty,\infty}(\mathbf{A})$ for the function $\Psi(\Phi^{\mathcal{J}}_{s,\infty})$. Recall also the definition of the algebra $\mathcal{A} := \mathcal{A}\big(PC(\dot{\mathbb{R}}), PC_p, (P_\tau)\big)$ as the smallest closed subalgebra of $\mathcal{E}$ which contains the constant sequences (aI) with $a \in PC(\dot{\mathbb{R}})$ and $(W^0(b))$ with $b \in PC_p$, and the sequence (P_τ).

Theorem 4.22. *A sequence $\mathbf{A} \in \mathcal{A}$ is stable if and only if the operators $\mathsf{W}_{-1}(\mathbf{A})$, $\mathsf{W}_0(\mathbf{A})$ and $\mathsf{W}_1(\mathbf{A})$ and the operators $\mathsf{H}_{s,\infty}(\mathbf{A})$ and $\mathsf{H}_{\infty,t}(\mathbf{A})$ with $s,t \in \mathbb{R}$ are invertible in $\mathcal{L}(L^p(\mathbb{R}))$ and if the matrix function $\mathsf{H}_{\infty,\infty}(\mathbf{A})$ is invertible.*

Specifying Theorem 4.22 to the case when (A_τ) is a sequence of finite sections yields the following.

Theorem 4.23. *Let A be an operator in the smallest subalgebra of $\mathcal{L}(L^p(\mathbb{R}))$ which contains the operators aI with $a \in PC(\dot{\mathbb{R}})$ and $W^0(b)$ with $b \in PC_p$. Then the finite sections method*

$$(P_\tau A P_\tau + Q_\tau)u_\tau = f \tag{26}$$

applies to the operator A if and only if the operators

$$\chi_+ \mathsf{W}_{-1}(A)\chi_+ I + \chi_- I, \quad A, \quad \text{and} \quad \chi_- \mathsf{W}_1(A)\chi_- I + \chi_+ I$$

and the operators

$$\mathsf{H}_{s,\infty}(A) \quad \text{and} \quad P_1 \mathsf{H}_{\infty,t}(A) P_1 + Q_1 \quad \text{with } s, t \in \mathbb{R}$$

are invertible on $L^p(\mathbb{R})$, and if the function $\mathsf{H}_{\infty,\infty}(P_\tau A P_\tau + Q_\tau)$ is invertible.

Formally, we proved Theorem 4.22 for the scalar case. For matrix-valued functions $a \in [PC(\dot{\mathbb{R}})]^{n\times n}$ and $b \in [PC_p]^{n\times n}$, the proof remains essentially the same. This covers, for example, systems of singular integral equations and systems of Wiener-Hopf operators. Obviously, the operators resulting from the homomorphisms then will have matrix coefficients, and it can prove difficult to study the invertibility of these operators. Note that a non-scalar version of the two-projections theorem was proved in [3].

5 Some examples

Finally, we are going to examine two simple settings where Theorem 4.23 works. Consider the singular integral operator

$$A := cW^0(\chi_+) + dW^0(\chi_-) = cP_\mathbb{R} + dQ_\mathbb{R} \tag{27}$$

with coefficients c, $d \in PC(\dot{\mathbb{R}})$. A criterion for the Fredholmness of this operator is stated in Corollary 3.6.

Theorem 5.1. *The finite sections method* (26) *applies to the singular integral operator A in* (27) *if and only if the operator A is invertible on $L^p(\mathbb{R})$ and the operator*

$$P_1\left((c(-\infty)\chi_- + c(+\infty)\chi_+)W^0(\chi_+) + (d(-\infty)\chi_- + d(+\infty)\chi_+)W^0(\chi_-)\right)P_1$$

is invertible on $L^p([-1,\, 1])$.

Proof. Let $\mathbf{A} := (P_\tau A P_\tau + (I - P_\tau))_{\tau>0}$. By Theorem 4.23, the sequence $\mathbf{A}$ is stable and, hence, the finite sections method applies to A, if and only if the following operators are invertible:

(i) $\mathsf{W}_0(\mathbf{A}) = cW^0(\chi_+) + dW^0(\chi_-)$;

(ii) $\mathsf{W}_{-1}(\mathbf{A}) = \chi_+\left(c(-\infty)W^0(\chi_+) + d(-\infty)W^0(\chi_-)\right)\chi_+ I + \chi_- I$;

(iii) $\mathsf{W}_1(\mathbf{A}) = \chi_-\left(c(+\infty)W^0(\chi_+) + d(+\infty)W^0(\chi_-)\right)\chi_- I + \chi_+ I$;

(iv) $\mathsf{H}_{s,\infty}(\mathbf{A}) = (c(s^-)\chi_- + c(s^+)\chi_+)W^0(\chi_+) + (d(s^-)\chi_- + d(s^+)\chi_+)W^0(\chi_-)$ for $s \in \mathbb{R}$;

(v) $\mathsf{H}_{\infty,0}(\mathbf{A}) = Q_1 + P_1\left((c(-\infty)\chi_- + c(+\infty)\chi_+)W^0(\chi_+) + (d(-\infty)\chi_- + d(+\infty)\chi_+)W^0(\chi_-)\right)P_1$;

(vi) $\mathsf{H}_{\infty,t}(\mathbf{A}) = P_1(c(-\infty)\chi_- + c(+\infty)\chi_+)P_1 + Q_1$ for $t > 0$;
$\mathsf{H}_{\infty,t}(\mathbf{A}) = P_1(d(-\infty)\chi_- + d(+\infty)\chi_+)P_1 + Q_1$ for $t < 0$;

(vii) $$(\mathsf{H}_{\infty,\infty}(\mathbf{A}))(z) = \begin{bmatrix} c(+\infty)z + d(+\infty)(1-z) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c(-\infty)z + d(-\infty)(1-z) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
for $z \in \Omega_p$.

Thus, the conditions stated in the theorem are necessary: the operators quoted there are $\mathsf{W}_0(\mathbf{A})$ and $\mathsf{H}_{\infty,0}(\mathbf{A})$, respectively. To prove the sufficiency, we have to show that the invertibility of $\mathsf{W}_0(\mathbf{A})$ and $\mathsf{H}_{\infty,0}(\mathbf{A})$ implies the invertibility of all other operators in (i) – (vii).

Let $s \in \mathbb{R}$. Since $\mathsf{H}_{s,\infty}(A) = \mathsf{H}_{s,\infty}(\mathbf{A})$ by Lemma 4.6, the invertibility of A implies that of $\mathsf{H}_{s,\infty}(\mathbf{A})$. Further, if $\mathsf{H}_{\infty,0}(\mathbf{A})$ is invertible then the sequence

$$\begin{aligned}\mathbf{B} \; &:= \; (P_\tau((c(-\infty)\chi_- + c(+\infty)\chi_+)W^0(\chi_+) \\ &\qquad + (d(-\infty)\chi_- + d(+\infty)\chi_+)W^0(\chi_-))P_\tau + Q_\tau)\end{aligned}$$

is stable by Lemma 4.19. Since $\mathsf{W}_{-1}(\mathbf{A}) = \mathsf{W}_{-1}(\mathbf{B})$ and $\mathsf{W}_1(\mathbf{A}) = \mathsf{W}_1(\mathbf{B})$ by Lemma 4.4, the operators $\mathsf{W}_{-1}(\mathbf{A})$ and $\mathsf{W}_1(\mathbf{A})$ are then invertible.

Similarly, if $t \in \mathbb{R} \setminus \{0\}$, then $\mathsf{H}_{\infty,t}(\mathbf{A}) = \mathsf{H}_{\infty,t}(\mathbf{B})$ by Lemma 4.5, which verifies the invertibility of the operators $\mathsf{H}_{\infty,t}(\mathbf{A})$. Finally, condition (vii) is satisfied if and only if the point 0 does not belong to the lentiform domains

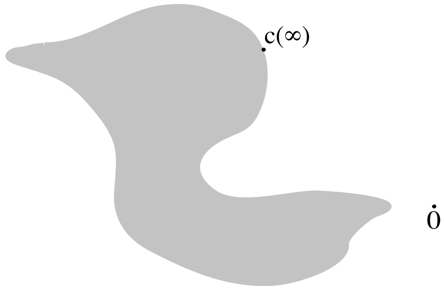
$$1 + \left(\frac{c(+\infty)}{d(+\infty)} - 1\right)\Omega_p \quad \text{and} \quad 1 + \left(\frac{c(-\infty)}{d(-\infty)} - 1\right)\Omega_p.$$

That the invertibility of $\mathsf{W}_0(\mathbf{A})$ and $\mathsf{H}_{\infty,0}(\mathbf{A})$ also implies this condition follows again by employing the invertibility criterion for singular integral operators in [4]. ■

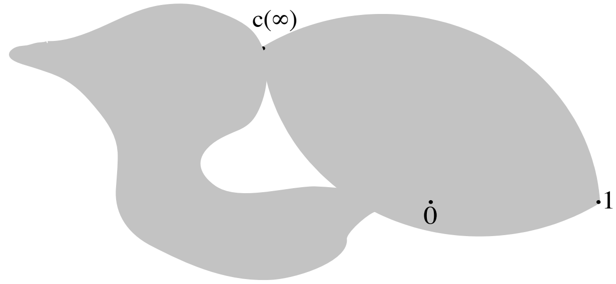
To compare the spectrum of the initial operator A with that of the finite section method (26) let $d \equiv 1$. We consider two cases. The first case is when c is continuous at ∞. In this case the spectrum of the method is the spectrum of the operator plus a lens connecting the point $c(\infty)$ with the point 1 (see Figure 5 - note that the point 1 also belongs to the spectrum of A, but it is not shown in the first image).

The case when c is discontinuous at ∞ involves the spectrum of the operator and the convex hull of three lenses, connecting the points 1, $c(-\infty)$, $c(+\infty)$. These lenses colapse to straight lines when $p = 2$. In that case, it is possible to reformulate Theorem 5.1 in the following geometric terms.

Corollary 5.2. *Let $p = 2$. The finite sections method* (26) *applies to the singular integral operator A in* (27) *if and only if the operator A is invertible on $L^2(\mathbb{R})$ and if the point* 0 *is not contained in the convex hull of the points* 1, $\frac{c(-\infty)}{d(-\infty)}$ *and* $\frac{c(+\infty)}{d(+\infty)}$.



(a) The spectrum of an operator A



(b) the spectrum of the corresponding fsm

For a second illustration of Theorem 4.23, let $a, b \in PC_p$ and consider the paired operator

$$A := W^0(a)\chi_+ I + W^0(b)\chi_- I. \tag{28}$$

Theorem 5.3. *The finite sections method* (26) *applies to the paired operator A in* (28) *if and only if the the operator A is invertible on $L^p(\mathbb{R})$, the Wiener-Hopf operators $W(b)$ and $W(\tilde{a})$ are invertible on $L^p(\mathbb{R}^+)$, the operator*

$$P_1\left((a(t^-)W^0(\chi_-) + a(t^+)W^0(\chi_+))\chi_+ I + (b(t^-)W^0(\chi_-) + b(t^+)W^0(\chi_+))\chi_- I\right)P_1$$

is invertible on $L^p([-1, 1])$ for every $t \in \mathbb{R}$, and the point 0 does not belong to the lentiform domains

$$a(-\infty) + (a(+\infty) - a(-\infty))\Omega_p \quad \textit{and} \quad b(+\infty) + (b(-\infty) - b(+\infty))\Omega_p.$$

Proof. Let again $\mathbf{A} := (P_\tau A P_\tau + (I - P_\tau))_{\tau>0}$. Theorem 4.23 implies that the finite sections method for the operator A is stable if and only if the following operators are invertible:

(i) $\mathsf{W}_0(\mathbf{A}) = A$;

(ii) $\mathsf{W}_{-1}(\mathbf{A}) = \chi_+ W^0(b)\chi_+ I + \chi_- I$;

(iii) $\mathsf{W}_1(\mathbf{A}) = \chi_- W^0(a)\chi_- I + \chi_+ I$;

(iv) $$\mathsf{H}_{0,\infty}(\mathbf{A}) = \left(a(-\infty)W^0(\chi_-) + a(+\infty)W^0(\chi_+)\right)\chi_+ I + \left(b(-\infty)W^0(\chi_-) + b(+\infty)W^0(\chi_+)\right)\chi_- I;$$

(v) $\mathsf{H}_{s,\infty}(\mathbf{A}) = a(-\infty)W^0(\chi_-) + a(+\infty)W^0(\chi_+)$ if $s > 0$;
$\mathsf{H}_{s,\infty}(\mathbf{A}) = b(-\infty)W^0(\chi_-) + b(+\infty)W^0(\chi_+)$ if $s < 0$;

(vi) $$\mathsf{H}_{\infty,t}(\mathbf{A}) = Q_1 + P_1((a(t^-)W^0(\chi_-) + a(t^+)W^0(\chi_+))\chi_+ I + (b(t^-)W^0(\chi_-) + b(t^+)W^0(\chi_+))\chi_- I)P_1 \quad \text{for } t \in \mathbb{R};$$

(vii) $$(\mathsf{H}_{\infty,\infty}(\mathbf{A}))(z) = \begin{bmatrix} a(-\infty)(1-z) + a(+\infty)z & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b(-\infty)z + b(+\infty)(1-z) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

for $z \in \Omega_p$.

The invertibility of the operators $\mathsf{W}_{-1}(\mathbf{A})$ and $\mathsf{W}_1(\mathbf{A})$ is equivalent to the invertibility of the Wiener-Hopf operators $W(b)$ and $W(\tilde{a})$, respectively. Thus, the conditions of the theorem are necessary. We show that, conversely, the invertibility of the operator A implies the invertibility of the operators in (iv) and (v). This fact follows immediately from Lemma 4.6, where it is shown that $\mathsf{H}_{s,\infty}(\mathbf{A}) = \mathsf{H}_{s,\infty}(A)$ for every $s \in \mathbb{R}$. ■

Corollary 5.4. *Let $p = 2$. The finite sections method* (26) *applies to the paired operator A in* (28) *if and only if the operator A is invertible on $L^2(\mathbb{R})$, the Wiener-Hopf operators $W(b)$ and $W(\tilde{a})$ are invertible on $L^2(\mathbb{R}^+)$ and if, for every $t \in \mathbb{R}$, the point 0 is not contained in the convex hull of the points 1, $\frac{a(t^-)}{a(t^+)}$ and $\frac{b(t^-)}{b(t^+)}$.*

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