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The Structure of Almost Connected Pro-Lie Groups

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Abstract. It is shown that the compact open topology makes the automorphism group $\mathrm{Aut}\,\mathfrak{g}$ of a semisimple pro-Lie algebra $\mathfrak{g}$ a topological group in which the identity component $(\mathrm{Aut}\,\mathfrak{g})_0$ is exactly the group $\mathrm{Inn}\,\mathfrak{g}$ of inner automorphism and which has a totally disconnected supplement Δ such that $\mathrm{Aut}\,\mathfrak{g} = (\mathrm{Inn}\,\mathfrak{g})\Delta$ and $\mathrm{Aut}\,\mathfrak{g}/\,\mathrm{Inn}\,\mathfrak{g} \cong \mathrm{Inn}\,\mathfrak{g}/(\mathrm{Inn}\,\mathfrak{g}\cap\Delta)$ as topological groups. The group $\mathrm{Inn}\,\mathfrak{g}$ is a product of a family of connected simple centerfree Lie groups. These results are used to show that for a pro-Lie group G which has a compact group of connected components G/G_0, there is a compact zero-dimensional, that is, profinite, subgroup D such that $G = G_0D$. There are sets I, J, a compact connected semisimple group S, and a compact connected abelian group A such that G and $\mathbb{R}^I \times (\mathbb{Z}/2\mathbb{Z})^J \times S \times A$ are homeomorphic.

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1. Introduction

A pro-Lie group is a topological group which is a projective limit of a projective system of finite dimensional real Lie groups. Equivalently, it is a topological group that is isomorphic to a closed subgroup of a cartesian product of Lie groups. The class of *connected* pro-Lie groups and its Lie theory and structure theory was extensively discussed in [5]. The Lie theory of pro-Lie groups is an infinite dimensional one; the way in which it overlaps differentiable Lie group theory based on manifolds modelled on locally convex vector spaces was clarified in [6]. The upshot of the structure theory of pro-Lie groups as published so far is that it is largely determined by the structure of pro-Lie algebras, that is, projective limits of finite dimensional Lie algebras on the one hand, and compact groups on the other. The Lie theory of the connection between pro-Lie groups and Pro-Lie algebras can only reach as far as the identity component. On the other hand, compact groups reach out to compact extensions of connected pro-Lie groups, namely, those pro-Lie groups G which are almost connected in the sense that G/G_0 is compact. In many instances, the structure theory of *connected* pro-Lie groups in [5] included almost connected pro-Lie groups. However, so far it failed to produce one general result which one might expect if one is guided by locally compact groups, namely, the proposition that

an almost connected pro-Lie group G has a maximal compact subgroup M, all maximal compact subgroups are conjugate, and $G = G_0M$.

These results we discuss and prove below. Our detailed knowledge of compact groups then lets us demonstrate that M contains a profinite subgroup D such that $G = G_0D$ and that the two groups G and $G_0 \times G/G_0$ are homeomorphic. From [5] we know that G_0 is homeomorphic to $\mathbb{R}^I \times C$ for a set I and a compact connected group C. From [4] we know that every profinite group is homeomorphic to a product $(\mathbb{Z}/2Z)^J$ for a set J, and that a compact connected group is a semidirect product of a compact connected semisimple group S and a compact connected abelian group A. Therefore we will be finally able to say that

an almost connected pro-Lie group is homeomorphic to a group of the form

$$\mathbb{R}^I \times S \times A \times (\mathbb{Z}/2\mathbb{Z})^J.$$

The topological structure of an almost connected pro-Lie group is therefore clarified as explicitly as one could desire, notably in the light of the known structure of the compact factors S and A (see [4]).

The crucial access to this new chapter of pro-Lie group theory is provided by a careful analysis of the structure of the automorphism group of a semisimple pro-Lie algebra which, of course, is also of independent interest. The relevance of this information becomes clear at once if one recalls from [5] that a connected pro-Lie group G possess a unique largest normal connected pro-solvable subgroup $R(G)$, where a pro-Lie group is *prosolvable* if all Lie group homomorphic images are solvable. The factor group $S \stackrel{\text{def}}{=} G_0/R(G)$ is a connected semisimple pro-Lie group, whose Lie algebra $\mathfrak{s}$ is semisimple. The group G acts via inner autimorphisms on S and then via adjoint representation on $\mathfrak{s}$ and that, after careful inspection the details, gives us a representation $f: G \to \operatorname{Aut}\mathfrak{s}$. That $\operatorname{Aut}\mathfrak{s}$ is in need of a natural topological group topology and that f is a morphism of topological groups is established in the processe, but hardly any of the details is obvious. The structure of $\operatorname{Aut}\mathfrak{s}$ turns out to emulate that of the automorphism group of a finite dimensional semisimple Lie group, best known in the case of a compact semisimple Lie algebra (see [4], Lemma 6.57ff. and Lemma 9.80ff.). The group $\operatorname{Inn}\mathfrak{s} = \langle e^{\operatorname{ad}\mathfrak{s}} \rangle \subseteq \operatorname{Aut}\mathfrak{s}$ of *inner automorphisms* is a product $\prod_{j\in J} S_j$ of centerfree simple real connected Lie groups, and it is the identity component of $\operatorname{Aut}\mathfrak{s}$. An essential ingredient is the result that it is nearly a semidirect factor; indeed there is a totally disconnected cofactor Δ such that $\operatorname{Aut}\mathfrak{s} = (\operatorname{Inn}\mathfrak{s})\cdot\Delta$, but for all we know there may be a nontrivial intersection $(\operatorname{Inn}\mathfrak{s}) \cap \Delta$. Here the theory is somewhat hampered by a deficit in our knownledge of the classical theory of finite dimensional real simple Lie algebras: If a simplie real Lie algebra is either compact or the underlying real Lie algebra of a complex simple Lie algebra, then its automorphism group is a semidirect product of the inner automorphism group and a suitable finite (usually very small) cofactor. While for a number of special classes of real Lie algebras beyond the ones mentioned this conclusion persists and one has no counterexamples, it seems to be unknown whether it is true in general. At any rate, in the cofactor Δ is carefully analyzed, and we understand its structure well. Unfortunately, as a topological group, it fails to be a prodiscrete group in general. However, we do understand its compact subgroups and we see that *any almost connected closed subgroup* $G \subseteq \operatorname{Aut}\mathfrak{s}$ *containing* $\operatorname{Inn}\mathfrak{s}$ *does have the form* $G = G_0M$ *with a maximal compact subgroup* M *of* G.

Our knowledge of the structure of $\operatorname{Aut}\mathfrak{s}$ then permits us to reduce our program concerning the structure theory of almost connected pro-Lie groups G to the case that G_0 is prosolvable. This is a case that was already dealt with in [5] and in this fashion our investigation is completed.

The general architecture of our text is as follows:

2. The automorphism group of semisimple pro-Lie algebras

Recall from [5], Theorem 10.29 p. 435 that a connected pro-Lie group G is semisimple iff its Lie algebra $\mathfrak{g}$ is semisimple iff (by [5], Corollary 7.29, p. 283) there is a family $\{\mathfrak{s}_j : j \in J\}$ of finite dimensional simple real Lie algebras such that $\mathfrak{g} \cong \prod_{j\in J} \mathfrak{s}_j$. Any automorphism $\alpha\colon G \to G$ yields an automorphism $\mathfrak{L}(\alpha)\colon \mathfrak{g} \to \mathfrak{g}$, and

$$\alpha \mapsto \mathfrak{L}(\alpha) : \operatorname{Aut} G \to \operatorname{Aut}\mathfrak{g}$$

is an injective morphism of groups. If for $g \in G$ the function $I_g\colon G \to G$ is the inner automorphism defined by $I_g(x) = gxg^{-1}$, then $g \mapsto I_g : G \to \operatorname{Aut}(G)$ is a morphism of groups whose kernel is the center $Z(G)$ of G, and the composition $G \to \operatorname{Aut}(G) \to \operatorname{Aut}(\mathfrak{g})$ given by $g \mapsto \mathfrak{L}(I_g)$ is none other than the adjoint representation $\operatorname{Ad}\colon G \to \operatorname{Aut}(\mathfrak{g})$ discussed in [5], pp. 131f.
So if one wants to talk about automorphisms of connected semisimple pro-Lie groups one must necessarily focus on the automorphism group of semisimple pro-Lie algebras.
In the following we assume that
$j \mapsto \mathfrak{s}_j$ is a function from a set J into the class of simple finite dimensional real Lie algebras and that

$$\mathfrak{g} = \prod_{j\in J} \mathfrak{s}_j.$$

and that $\operatorname{Aut}\mathfrak{g}$ is the group of all automorphisms of the topological Lie algebra $\mathfrak{g}$.

THE ALGEBRAIC THEORY OF $\operatorname{Aut}\mathfrak{g}$

For each $k \in J$ define a function $\operatorname{inc}_k\colon \mathfrak{s}_k \to G$ by $\operatorname{inc}_k(X) = (Y_{jk})_{j\in J}$, where

$$Y_{jk} = \begin{cases} X & \text{if } j = k, \\ 0_j & \text{if } j \neq k, \end{cases}$$

and set

$$\mathfrak{m}_k = \mathrm{inc}_k(\mathfrak{s}_k).$$

As usual, let $\mathrm{pr}_k\colon \mathfrak{g} \to \mathfrak{s}_k$ be the projection, given by $\mathrm{pr}_k((X_j)_{j\in J}) = X_k$. Then $\mathrm{pr}_k \circ \mathrm{inc}_k = \mathrm{id}_{\mathfrak{s}_k}$ and $\mathrm{inc}_k \circ \mathrm{pr}_k\colon \mathfrak{g} \to \mathfrak{g}$ is an idempotent endomorphism of $\mathfrak{g}$ with image $\mathfrak{m}_k$. For each $k \in J$, the subset $\mathfrak{m}_k$ is a minimal ideal of the Lie algebra $\mathfrak{g}$.

Lemma 2.1. *Let $\mathfrak{m}$ be a minimal ideal of $\mathfrak{g}$. Then there is a $k \in J$ such that $\mathfrak{m} = \mathfrak{m}_k$.*

Proof. If $\mathfrak{g} = \mathfrak{m}$, then $\mathfrak{g}$ is simple and there is nothing to prove. Thus we may assume that J has more than one element. Let $0 \neq X \in \mathfrak{m}$ and denote with $\mathcal{F}$ the set of finite subsets of J. For a subset $I \subseteq J$ we identify the partial product $\prod_{j\in J} \mathfrak{s}_j$ in an obvious way with an ideal of $\mathfrak{g}$. Since $\bigcap_{F\in\mathcal{F}} \prod_{j\in J\setminus F} \mathfrak{s}_j = \{0\}$, we find an ideal (indeed partial product) $\mathfrak{n} = \prod_{j\in J\setminus F} \mathfrak{s}_j$ for some $F \in \mathcal{F}$ such that $X \notin \mathfrak{n}$. Since $\mathfrak{m}$ is a minimal ideal, $\mathfrak{m} \cap \mathfrak{n} = \{0\}$, and so $\mathfrak{m} + \mathfrak{n}$ is algebraically a direct sum. (In fact, by [5], Theorem A2.12(c), $\mathfrak{m} \oplus \mathfrak{n}$ is also a topological direct sum.) Now $\mathfrak{g}/\mathfrak{n} \cong \prod_{j\in F} \mathfrak{s}_j$ is a finite direct sum of ideals $(\mathfrak{m}_j + \mathfrak{n})/\mathfrak{n}$, $j \in F$, and $(\mathfrak{m} + \mathfrak{n})/\mathfrak{n}$ is an isomorphic copy of $\mathfrak{m}$ and an ideal in $\mathfrak{g}/\mathfrak{n}$. Every vector subspace containing $\mathfrak{n}$ is of the form $\mathfrak{m}' \oplus \mathfrak{n}$ for some vector subspace $\mathfrak{m}'$ of $\mathfrak{m}$. Now let $(\mathfrak{m}' + \mathfrak{n})/\mathfrak{n}$, $\mathfrak{m}' \subseteq \mathfrak{m}$, $\mathfrak{m}' \neq \mathfrak{m}$, be a proper ideal of $\mathfrak{g}/\mathfrak{n}$ contained in $(\mathfrak{m} + \mathfrak{n})/\mathfrak{n}$. Let $X \in \mathfrak{g}$ and $Y \in \mathfrak{m}' \subseteq \mathfrak{m}$, then $[X, Y] \in (\mathfrak{m}' + \mathfrak{n}) \cap \mathfrak{m} = \mathfrak{m}'$ by the modular law. Thus $\mathfrak{m}' \neq \mathfrak{m}$ is an ideal of $\mathfrak{g}$ and then $\mathfrak{m}' = \{0\}$ by the minimality of $\mathfrak{m}$. Thus $(\mathfrak{m} + \mathfrak{n})/\mathfrak{n}$ is a minimal ideal of $\mathfrak{g}/\mathfrak{n}$. The minimal ideals of the finite dimensional semisimple Lie algebra $\mathfrak{g}/\mathfrak{n}$ are exactly the ideals $(\mathfrak{m}_j + \mathfrak{n})/\mathfrak{n}$, $j \in F$. Hence there is a unique $k \in F$, depending on $\mathfrak{n}$, such that $\mathfrak{m} \oplus \mathfrak{n} = \mathfrak{m}_k \oplus \mathfrak{n}$. It follows that $\mathfrak{m} \cong \mathfrak{m}_k \cong \mathfrak{s}_k$. If $\mathfrak{n}'$, the canonical image of $\prod_{j\in J\setminus F'} \mathfrak{s}_j$ with $F \subseteq F' \in \mathcal{F}$ is an ideal contained in $\mathfrak{n}$, then there is a unique $k' \in F'$ such that $\mathfrak{m} \oplus \mathfrak{n}' = \mathfrak{m}_{k'} \oplus \mathfrak{n}'$. This equation implies $\mathfrak{m} \oplus \mathfrak{n} = \mathfrak{m}_{k'} \oplus \mathfrak{n}$, and thus $\mathfrak{m}_{k'} \oplus \mathfrak{n} = \mathfrak{m}_k \oplus \mathfrak{n}$ and this equality entails $k' = k$. Thus we have $\mathfrak{m} \subseteq \mathfrak{m} \oplus \mathfrak{n}' = \mathfrak{m}_k \oplus \mathfrak{n}'$ for all cofinite dimensional ideals $\mathfrak{n}'$ contained in $\mathfrak{n}$ and thus $\mathfrak{m} \subseteq \bigcap_{n'\subseteq\mathfrak{n}} (\mathfrak{m}_k \oplus \mathfrak{n}') = \mathfrak{m}_k$. Since the finite dimensional ideals $\mathfrak{m}$ and $\mathfrak{m}_k$ are isomorphic, $\mathfrak{m} = \mathfrak{m}_k$ follows which proves the Lemma. □

Note that in the abelian pro-Lie algebra $\mathbb{R}^{\mathbb{N}}$ every one-dimensional vector subspace is a minimal ideal and not only the subspaces

$$\mathfrak{m}_k \stackrel{\text{def}}{=} \{(r_{kn})_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : r_{kn} = 0 \text{ if } n \neq k\}.$$

In fact the abelian Lie algebra $\mathbb{R}^2$ has more minimal ideals than just $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$.

An automorphism $\alpha \in \mathrm{Aut}\,\mathfrak{g}$ must permute the minimal ideals. Let $P(J)$ denote the group of all permutations of J. So there is an element $\sigma(\alpha) \in P(J)$ such that

$$(\forall j \in J)\, \alpha(\mathfrak{m}_j) = \mathfrak{m}_{\sigma(\alpha)^{-1}(j)}.$$

Let $\mathcal{S}$ be a set containing exactly one specimen of each isomorphism class of finite dimensional simple real Lie algebras. Then each semisimple pro-Lie algebra $\mathfrak{g}$ is determined uniquely up to isomorphism by a partition of the index set J:

$$J = \bigcup_{\mathfrak{s}\in\mathcal{S}} J(\mathfrak{s})$$

into subsets $J(\mathfrak{s}) = J_{\mathfrak{g}}(\mathfrak{s})$ such that we may (and will) write

$$\mathfrak{g} = \prod_{\mathfrak{s}\in\mathcal{S}} \mathfrak{s}^{J(\mathfrak{s})}$$

where we call the powers $F_{\mathfrak{s}} \overset{\text{def}}{=} \mathfrak{s}^{J(\mathfrak{s})}$ the *isotypic factors.*
Set $P_{\mathcal{S}}(J) = \{\sigma \in P(J) : \sigma(J(\mathfrak{s})) = J(\mathfrak{s})\}$, a subgroup of $P(S)$. Notice that

$$P_{\mathcal{S}}(J) \cong \prod_{\mathfrak{s}\in\mathcal{S}} P(J(\mathfrak{s})). \tag{1}$$

Each automorphism α must preserve the isotypic factor $F_{\mathfrak{s}}$ and thus $\sigma(\alpha) \in P_{\mathcal{S}}(J)$.

Definition of σ, ψ, and ρ

Therefore we have defined a function

$$\sigma\colon \operatorname{Aut}\mathfrak{g} \to P_{\mathcal{S}}(J) \tag{$*$}$$

which is easily verified to be a morphism of groups by its definition. If $\tau \in P_{\mathcal{S}}(J)$, define $\psi(\tau) \in \operatorname{Aut}\mathfrak{g}$ by $\psi(\tau)((X_j)_{j\in J}) = (X_{\tau^{-1}(j)})_{j\in J}$. Since $\mathfrak{s}_{\tau^{-1}(j)} = \mathfrak{s}_j$ by the definition of $P_{\mathcal{S}}(J)$, this is a well defined automorphism satisfying $\psi(\tau)(\mathfrak{m}_j) = \mathfrak{m}_{\tau^{-1}(j)}$. Hence $\sigma(\psi(\tau)) = \tau$. Thus $\sigma\colon \operatorname{Aut}\mathfrak{g} \to P_{\mathfrak{s}}(J)$ is a homomorphic retraction and is, in particular, surjective; the function

$$\psi\colon P_{\mathcal{S}}(J) \to \operatorname{Aut}\mathfrak{g} \tag{$**$}$$

is its right inverse.

Quite generally, let $\sigma\colon A \to H$ be a homomorphic retraction of groups and set $N = \ker\sigma$. Define $\psi\colon H \to A$ to be the coretraction satisfying $\sigma \circ \psi = \mathrm{id}_H$. Then there is a morphism $\gamma\colon H \to \operatorname{Aut} A$ defined by $\gamma(h)(a) = \psi(h)a\psi(h)^{-1}$, and the function $\rho\colon N \rtimes_\gamma H \to A$, $\rho(n,h) = n\psi(h)$ is an isomorphism with inverse given by $\rho^{-1}(a) = (a\psi\sigma(a)^{-1}, \sigma(a))$. This is verified directly; homomorphic retractions are, in this sense, an alternative manifestation of semidirect products.
We want to apply these arguments to the homomorphic retraction $\sigma\colon \operatorname{Aut}\mathfrak{g} \to P_{\mathcal{S}}(J)$. Firstly, an automorphism α of $\mathfrak{g}$ is in the kernel of σ iff $\alpha(\mathfrak{m}_j) = \mathfrak{m}_j$, that is, iff there is an element $(\alpha_j)_{j\in J} \in \prod_{j\in J} \operatorname{Aut}\mathfrak{s}_j$ such that $\alpha((X_j)_{j\in J}) = (\alpha_j(X_j))_{j\in J}$.
(†) We shall henceforth identify $\ker\sigma$ with $\prod_{j\in J} \operatorname{Aut}\mathfrak{s}_j$,
and conclude

Proposition 2.2. (Algebraic Gross Structure) *The function*

$$\rho : \prod_{j\in J} \operatorname{Aut}\mathfrak{s}_j \rtimes_\gamma P_{\mathcal{S}}(J) \to \operatorname{Aut}\mathfrak{g}, \tag{$***$}$$

defined by

$$\rho\big((\alpha_j)_{j\in J},\tau\big)\big((X_j)_{j\in J}\big) = \big(\alpha_j(X_{\tau^{-1}(j)})\big)_{j\in J},$$

is an isomorphism of groups with inverse function

$$\alpha \mapsto (\alpha{\cdot}\psi\sigma(\alpha)^{-1}, \sigma(\alpha)). \qquad \square$$

Topologizing $\operatorname{Aut}\mathfrak{g}$

We use the isomorphism in the Gross Structure Theorem to introduce a group topology on $\operatorname{Aut}\mathfrak{g}$. It is easy to consider the individual constituents $\prod_{j\in J}\operatorname{Aut}\mathfrak{g}_j$ and $P_{\mathcal{S}}(J)$ of the formula for $\operatorname{Aut}\mathfrak{g}$.

Firstly, for a simple finite dimensional Lie algebra $\mathfrak{s}$, the group $\operatorname{Aut}\mathfrak{s}$, as a closed subgroup of the full linear group $\operatorname{Gl}(\mathfrak{s})$ is a Lie group0. Accordingly, $\operatorname{Aut}\mathfrak{g} = \prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j$ is a product of a family of finite dimensional Lie groups and is, therefore a pro-Lie group.

We record for later reference:

Lemma 2.3a. *The groups $\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j$ and all its closed subgroups are pro-Lie groups.* $\square$

Finally, we consider on the index set J the discrete topology. Then the function semigroup J^J has a product topology which is nondiscrete if J is infinite. It induces on the subgroup $P(J)$ the topology of pointwise convergence which agrees here with the compact open topology, making $P(J)$ into a topological group. Since the reader might prefer to have an independent recourse to this claim, we insert a proof of this claim:

Lemma 2.3b. *Let J a set and let $P(J) \subseteq J^J$ the set of all permutations of J with the topology of pointwise convergence, that is, the topology induced from the product topology of J^J where J has the discrete topology. Then $P(J)$ is a topological group.*

Proof. The function

$$(f,g) \mapsto f\circ g : P(J)\times P(J) \to P(J)$$

is continuous, since for fixed $f,g\in P(J)$ and $j\in J$, the set

$$\{(f',g')\in P(J)\times P(J) : f'\big(g'(j)\big) = f(g(j))\}$$

contains the open set

$$\{f'\in P(J) : f'\big(g(j)\big) = f\big(g(j)\big)\} \times \{g'\in P(J) : g'(j) = g(j)\}.$$

Next we show that $f\mapsto f^{-1}: P(J)\to P(J)$ is continuous by showing that $f\mapsto f^{-1}(j) : P(J)\to J$. But for fixed $f\in P(J)$ and $j\in J$ the set

$$\{F\in P(J) : F^{-1}(j) = f^{-1}(j)\} = \{F\in P(J) : F\big(f^{-1}(j)\big) = j\}$$

is open in $P(J)$. This proves the Lemma. $\square$

If J is finite, $P(J)$ is finite and has the alternating group as normal subgroup of index 2; the alternating group is simple if $\operatorname{card} J \geq 5$. If J is infinite, in the normal subgroup of all permutations leaving a cofinite subset of J elementwise fixed is simple every element, like any permutation on finitely many elements, has a signature "even or odd"; the subgroup of even permutations is an infinite simple subgroup. The topological group $P(J)$ is totally disconnected.

Remark 2.4. *If J is infinite, then $P(J)$ is not closed in J^J and fails to be complete.*

Proof. Since an infinite set contains a copy of $\mathbb{N}$, it suffices to show that the subgroup $P(\mathbb{N})$ of $\mathbb{N}^{\mathbb{N}}$ is not closed in $\mathbb{N}^{\mathbb{N}}$. We depict a function $f \in \mathbb{N}^{\mathbb{N}}$ by

$$f = \begin{pmatrix} 1 & 2 & \cdots & n & \cdots \\ f(1) & f(2) & \cdots & f(n) & \cdots \end{pmatrix}.$$

Then the sequence

$$f_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n & \cdots \\ 2 & 3 & \cdots & n & 1 & \cdots \end{pmatrix}, \quad n = 2, 3, \ldots$$

converges to

$$f = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n & \cdots \\ 2 & 3 & \cdots & n & n+1 & \cdots \end{pmatrix}.$$

Moreover, $f_n \in P(\mathbb{N})$ while $f(m) \neq 1$ for all $m \in \mathbb{N}$ and thus fails to be in $P(\mathbb{N})$. The sequence $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the sense that for every identity neighborhood U in $P(\mathbb{N})$ there is an $N \in \mathbb{N}$ such that $m, n \in \mathbb{N}$ and $m, n > N$ implies $f_m f_n^{-1} \in U$. Since it does not converge in $P(\mathbb{N})$, the group $P(\mathbb{N})$ fails to be complete. $\square$

In particular, for infinite J, the group $P(J)$ is not prodiscrete.

Let us discuss the compact subgroups $K \subseteq P(J)$.

Proposition 2.5. (a) *Let R be any equivalence relation on J, giving us a partition J/R of J into finite cosets. Then the subgroup*

$$C_R \stackrel{\text{def}}{=} \{\tau \in P(J) : (\forall j \in J)\, \tau(R(j)) \subseteq R(j)\}$$

is a compact subgroup of $P(J)$ and is isomorphic to $\prod_{I\in J/R} P(I)$.

(b) *For a subgroup $K \subseteq P(J)$ the following statements are equivalent:*

(i) *K is compact.*

(ii) *K is closed and has finite orbits $K(j)$.*

(iii) *K is closed in J^J and there is an equivalence relation R of J such that the cosets $R(j)$ are finite and invariant under the action of K for all $j \in J$.*

(c) *Let K be a compact subgroup of $P(J)$ and let R be the equivalence relation with the orbits $K(j)$, $j \in J$ as cosets. Then $K \subseteq C_R$.*

(d) *If R and R' are equivalence relations with finite cosets such that $R \subseteq R'$, then $C_R \subseteq C_{R'}$.*

Proof. (a) Clearly, C_R is a closed subgroup contained in $\prod_{j\in J} R(j) \subseteq J^J$ and is therefore compact. If $\tau \in C_R$, then for each $I \in J/R$, the permutation τ induces a permutation $\tau|I : I \to I$. Then function

$$\tau \mapsto (\tau|I)_{I\in J/R} : C_R \to \prod_{I\in J/R} P(I)$$

is readily seen to be an isomorphism of topological groups.

Next we prove (b).

(i)⇒(ii): Assume (i). Then K is a compact subset of J^J and is therefore contained in a product $\prod_{j\in J} J_j$ with finite subsets $J_j \subseteq J$. This implies that the orbits $K(j) \subseteq J_j$ are finite for all $j \in J$. Since K is compact, K is closed in $P(J)$.

(ii) ⇒(iii): Let R be the equivalence relation whose cosets are the orbits of K.

(iii)⇒(i): We observe that $K = \overline{K} \subseteq C_R$ which proves that K by the compactness of C_R according to (a).

(c) and (d) are straightforward. □

The subgroup of $P(\mathbb{N})$ consisting of all $f \in P(\mathbb{N})$ satisfying $f(2n-1) = 2n$, $f(2n) = 2n-1$ for finitely many n and fixing all other elements has finite orbits but is not compact.

Remark. *Every compact totally disconnected group G has a faithful continuous representation $\pi\colon G \to P(J)$ for a suitable set J.*

Proof. Every totally disconnected compact group G is profinite. (See [4], Theorem 1.34). In other words, if $\mathcal{N}$ is the filter basis of open normal subgroups N, then the natural morphism $f\colon G \to \prod_{N\in\mathcal{N}} G/N$ is faithful. Let $J = \bigcup_{N\in\mathcal{N}} G/N$ and define $\pi\colon G \to P(J)$ by $\pi(g)(hN) = ghN$ for $h \in G$, $N \in \mathcal{N}$ then π is the required faithful representation. □

Recall the morphisms of groups

$$\begin{aligned} \sigma &: \operatorname{Aut}\mathfrak{g} \to P_{\mathcal{S}}(J), & \sigma(\mathfrak{m}_j) &= \mathfrak{m}_{\sigma(\alpha)^{-1}(j)}, \\ \psi &: P_{\mathcal{S}}(J) \to \operatorname{Aut}\mathfrak{g}, & \psi(\tau)((X_j)_{j\in J}) &= (X_{\tau^{-1}(j)})_{j\in J}, \end{aligned}$$

satisfying $\sigma \circ \psi = \mathrm{id}$.

Lemma 2.6. (i) *The group $G \stackrel{\text{def}}{=} \prod_{j\in J} \operatorname{Aut}\mathfrak{s}_j \rtimes P_{\mathcal{S}}(J)$ is a topological group.*

(ii) *If all sets* $J(\mathfrak{s})$ *are finite,then* $P_{\mathcal{S}}(J)$ *is compact and*

$$G = \prod_{\mathfrak{s}\in\mathcal{S}} \big(\operatorname{Aut}\mathfrak{s}^{J(\mathfrak{s})} \rtimes P(J(\mathfrak{s}))\big)$$

is a pro-Lie group.
(iii) *If there is an* $\mathfrak{s}\in\mathcal{S}$ *such that* $J(\mathfrak{s})$ *is infinite, then* $P(J(\mathfrak{s}))$ *and* G *are incomplete topological groups.*

Proof. (i) In order to show that G is a topological group we have to show that the automorphic action of $P_{\mathcal{S}}(J)$ on $\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j = \prod_{\mathfrak{s}\in\mathcal{S}}(\operatorname{Aut}\mathfrak{s})^{J(\mathfrak{s})}$ is jointly continuous. We write $p = (\alpha_j)_{j\in J} \in \prod j \in J \operatorname{Aut}\mathfrak{s}_j$ and we have to verify that $(\pi, p) \mapsto \psi(\pi)(p) = (\alpha_{\tau^{-1}(j)})_{j\in J}$ is a continuous function

$$P_{\mathcal{S}}(J) \times \prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j to \prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j.$$

We recall that a function $f\colon A \to \prod_{j\in J} B_j$ from a topological space to a product of topological spaces is continuous iff $\operatorname{pr}_j \circ f\colon A \to B_j$ is continuous for each j where pr_j is the projection onto the factor B_j. Thus, if we fix $k \in J$ we note that

$$(\pi, (\alpha_j)_{j\in J}) \mapsto \alpha_{\tau^{-1}(k)} : P_{\mathcal{S}}(J) \times \prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j \to \operatorname{Aut}\mathfrak{s}_{\tau^{-1}(k)}$$

is indeed continuous in view of the topology of pointwise convergence on $P(J)$. Thus $G = (\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j) \rtimes P_{\mathcal{E}}(J)$ is a topological group.
(ii) The compactness of $P_{\mathcal{S}}(J) \cong \prod_{\mathfrak{s}\in\mathcal{S}} P(J(is))$ an immediate consequence of the Remark we observed above preceding Proposition 2.5.
The group G is a pro-Lie group iff $(\operatorname{Aut}\mathfrak{s})^{J(\mathfrak{s})} \rtimes P(J(\mathfrak{s}))$ is a pro-Lie group for all $\mathfrak{s}\in\mathcal{S}$ which is certainly the case if $J(\mathfrak{s})$ if finite for all $\mathfrak{s}$.
(iii) If $J(\mathfrak{s})$ is infinite for some $\mathfrak{s}$, then $P(J(\mathfrak{s}))$ and thus $\operatorname{Aut}\mathfrak{s}^{J(\mathfrak{s})} \rtimes P(J(\mathfrak{s}))$ and incomplete by our Remark preceding Proposition 2.5. Hence neither of these group is a pro-Lie group. ☐

Let $|\mathfrak{g}|$ be the underlying weakly complete topological vector space of the semisimple pro-Lie algebra $\mathfrak{g}$.
Let $\mathbf{1}$ be the identity automorphism of $\mathfrak{g}$. For a subset $K \subseteq \mathfrak{g}$, and an open 0-neighborhood $U \subseteq \mathfrak{g}$ we set
$W(K, U) = \{\alpha \in \operatorname{Aut}\mathfrak{g} : (\forall k \in K)\, \alpha(k) - k \in U\}$.
The compact-open topology of $\operatorname{Aut}\mathfrak{g}$ has a basis of sets $W(K, U)\cdot\gamma$ as K ranges through the compact subsets of $|\mathfrak{g}|$, U through the zero neighborhoods of $|\mathfrak{g}|$, and γ through $\operatorname{Aut}\mathfrak{g}$).
It will be useful to focus on certain basic identity neighborhoods $W(K', U')$ as follows: let $W(K, U)$ be a basic identity neighborhood of $\operatorname{Aut}\mathfrak{g}$. Define $K' = \prod_{j\in J} C_j$ with a compact subset $C_j = \operatorname{pr}_j(K)$ of $\mathfrak{s}_j$; then $K \subseteq K'$ and K' is compact. Let $U' = \prod_{j\in J} V_j$ where all V_j are zero neighborhoods of $\mathfrak{s}_j$ and where,

for a suitable finite subset F of J, we have $U_j = \mathfrak{s}_j$ for $j \in J \setminus F$ so that $U' \subseteq U$. Then $W(K',U') \subseteq W(K,U)$ and we shall say that a basic identity neighborhood $W(K,U)$ is special with respect to a finite subset $F \subseteq J$ if K and U are products as K' and U' above, respectively; we shall then write $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$, where

$$\begin{aligned} \mathfrak{g}_1 &= \prod_{j\in F} \mathfrak{s}_j, \\ \mathfrak{g}_2 &= \prod_{j\in J\setminus F} \mathfrak{s}_j. \end{aligned} \tag{2}$$

Then the definition of K' implies $K = K_1 \times K_2$, where

$$\begin{aligned} K_1 &= \prod_{j\in F} C_j, \\ K_2 &= \prod_{j\in J\setminus F} C_j. \end{aligned} \tag{3}$$

We also note $U = U_1 \times \mathfrak{g}_2$, where

$$U_1 = \prod_{j\in F} V_j. \tag{4}$$

We have observed that every identity neighborhood of $\operatorname{Aut}\mathfrak{g}$ contains a special one.

The right translations $\alpha \mapsto \alpha\gamma : \operatorname{Aut}\mathfrak{g} \to \operatorname{Aut}\mathfrak{g}$ are certainly homeomorphisms of $\operatorname{Aut}\mathfrak{g}$. If $f\colon A \to B$ is a morphism of groups between groups carrying a topology such that all right translations of A, respectively, B are homeomorphisms, then f is continuous iff f is continuous at the identity of A.

Lemma 2.7. *If* $\operatorname{Aut}\mathfrak{g}$ *is given the compact open topology, then both*

$$\sigma\colon \prod_{j\in J} \operatorname{Aut}\mathfrak{s}_j \to P_{\mathcal{S}}(J) \tag{$*$}$$

and

$$\psi\colon P_{\mathcal{S}}(J) \to \prod_{j\in J} \operatorname{Aut}\mathfrak{s}_j \tag{$**$}$$

are continuous.

Proof. Regarding σ: we need to establish continuity at $\mathbf{1}$. By the definition of the topology of pointwise convergence of $P_{\mathcal{S}}(J)$ a basic open neighborhood V_F of the identity in this group is given by a finite subset $F \subseteq J$ so that

$$V_F = \{\tau \in P_{\mathcal{S}}(J) : (\forall j \in F)\, \tau(j) = j\}. \tag{5}$$

We have to show that $\sigma^{-1}(V)$ is an identity neighborhood. If we let $V_j = \{\tau \in P_{\mathcal{S}}(J) : \tau(j) = j\}$, then $V_k = \bigcap_{j\in F} V_j$, and it suffices to show that $\sigma^{-1}(V_j)$ is an identity neighborhood for each $j \in F$. So assume now that $F = \{k\}$ and write $V = V_k$. Then $\alpha \in \operatorname{Aut}\mathfrak{g}$ is in $\sigma^{-1}(V)$ iff $\alpha(\mathfrak{m}_k) = \mathfrak{m}_k$. Let $X \neq 0$ in $\mathfrak{m}_k$ and $\mathfrak{h}$ the closed vector subspace $\{0_k\} \times \prod_{j\in J\setminus\{k\}} \mathfrak{s}_j$. Then $\mathfrak{g} = \mathfrak{m}_k \oplus \mathfrak{h}$ and $X \notin \mathfrak{h}$. Now we find a 0-neighborhood U of $\mathfrak{g}$ so small that

$$(X + U) \cap \mathfrak{h} = \emptyset.$$

Now $W(\{X\}, U)$ is an identity neighborhood of $\operatorname{Aut}\mathfrak{g}$. If $\alpha \in W(\{X\}, U)$, then $\alpha(X) - X \in U$, i.e. $\alpha(X) \in X + U$. Thus $(X + U) \cap \alpha(\mathfrak{m}_k) \neq \emptyset$. Now either $\alpha(\mathfrak{m}_k) = \mathfrak{m}_k$ or $\alpha(\mathfrak{m}_k) \subseteq \mathfrak{h}$. The latter case would entail $(X+U)\cap\mathfrak{h} \neq \emptyset$, contrary to $(X + U) \cap \mathfrak{h} = \emptyset$. Thus $\alpha(\mathfrak{m}_k) = \mathfrak{m}_k$. Hence $\alpha \in \sigma^{-1}(V)$. We have shown $W(\{X\}, U) \subseteq \sigma^{-1}(V)$ and thus $\sigma^{-1}(V)$ is an identity neighborhood.
Regarding ψ: let $W(K, U)$ be a special basic identity–neighborhood of $\operatorname{Aut}\mathfrak{g}$ with respect to $F \subseteq J$. We let $W = \{\tau \in P_{\mathcal{S}}(J) : (\forall j \in F)\, \tau(j) = j\}$. Then W is an identity neighborhood of $P_{\mathcal{S}}(J)$. If $\tau \in W$, then by the definition of ψ the automorphism $\psi(\tau)$ of $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ leaves $\mathfrak{g}_1$ and $\mathfrak{g}_2$ invariant and indeed $\mathfrak{g}_1$ elementwise so. Now let $X \in K$, that is $X = (X_1, X_2)$ with $X_i \in K_i$, $i = 1, 2$. Then $\psi(\tau)(X) = (X_1, X_2')$ for some $X_2' \in \mathfrak{g}_2$. Thus $\psi(\tau)(X) - X = (0, X_2 - X_2') \in U_1 \times \mathfrak{g}_2 = U$. So $W \subseteq \psi^{-1}(W(K, U))$, and this proves our claim, thereby concluding the proof of the Lemma. □

Lemmas 2.6 and 2.7 imply that the groups $P_{\mathcal{S}}(J)$ and $\psi(P_{\mathcal{S}}(J)) \subseteq \operatorname{Aut}\mathfrak{g}$ are isomorphic topological groups and prove the following

Lemma 2.8. *The algebraic isomorphism*

$$\rho^{-1}\colon \operatorname{Aut}\mathfrak{g} \to \prod_{j\in J} \operatorname{Aut}\mathfrak{s}_j \rtimes_\gamma P_{\mathcal{S}}(J), \quad \rho^{-1}(\alpha) = (\alpha \circ \psi\sigma(\alpha), \sigma(\alpha))$$

is continuous. □

Lemma 2.9. *The morphism*

$$\rho\colon \prod_{j\in J} \operatorname{Aut}\mathfrak{s}_j \rtimes_\gamma P_{\mathcal{S}}(J) \to \operatorname{Aut}\mathfrak{g}, \quad \rho(\alpha, \tau) = \alpha \circ \gamma(\tau)$$

is continuous.

Proof. It suffices again to prove continuity at the identity. Let $W(K, U)$ be a special identity neighborhood of $\operatorname{Aut}\mathfrak{g}$ with respect to $F \subseteq J$. Recall from (7) and (8) the compact subset K_1 of $\mathfrak{g}_1$ and the zero neighborhood U_1 of $\mathfrak{g}_1$. Also recall the zero neighborhood V_F of $P_{\mathcal{S}}(J)$ from(9). Then

$$W \stackrel{\text{def}}{=} (W(K_1, U_1) \times \prod_{j\in J\setminus F} \operatorname{Aut}\mathfrak{s}_j) \times V_F$$

is an identity neighborhood of $\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j \times P_{\mathcal{S}}(J)$. Now let

$$(\alpha,\tau) = (((\alpha_j)_{j\in F}, (\alpha_j)_{j\in J\setminus F}), \tau) \in W.$$

Then $\rho(\alpha,\tau) = \alpha\circ\psi(\tau)$. Let $(X_j)_{j\in J} = ((X_j)_{j\in F}, (X_j)_{j\in J\setminus F}) \in K = K_1 \times K_2$. Then $\rho(\alpha,\tau)((X_j)_{j\in J}) - (X_j)_{j\in J}$

$$= ((\alpha_j(X_j) - X_j)_{j\in F}, (\alpha_j(X_{\tau^{-1}(j)}) - X_j)_{j\in J\setminus F}) \in U_1 \times \prod_{j\in J\setminus F}\mathfrak{s}_j = U$$

since $(\alpha,\tau)\in W$. This shows that $\rho(W)\subseteq W(K,U)$, establishing finally the continuity of ρ. □

THE TOPOLOGICAL GROUP $\operatorname{Aut}\mathfrak{g}$.

For a pro-Lie algebra $\mathfrak{g}$, the *inner automorphism group* of $\mathfrak{g}$ is the group

$$\operatorname{Inn}\mathfrak{g} \overset{\text{def}}{=} \langle e^{\operatorname{ad}\mathfrak{g}}\rangle,$$

algebrically generated by all automorphisms of the form $e^{\operatorname{ad}X}$ for $X\in\mathfrak{g}$. The group $\operatorname{Inn}\mathfrak{g}$ is normal in $\operatorname{Aut}\mathfrak{g}$; the factor group $\frac{\operatorname{Aut}\mathfrak{g}}{\operatorname{Inn}\mathfrak{g}}$ is written $\operatorname{Out}\mathfrak{g}$ and is called the outer automorphism group (somewhat of a misnomer!).
We are now ready for the first principal result of this section, recalling that a topological group G is almost connected iff G/G_0 is compact.

Main Theorem 2.10. *Let $\mathfrak{g} = \prod_{j\in J}\mathfrak{s}_j$ be a semisimple pro-Lie algebra and $\operatorname{Aut}\mathfrak{g}$ the group of all automorpisms of the topological Lie algebra $\mathfrak{g}$. Then the compact open topology on the group $\operatorname{Aut}\mathfrak{g}$ makes it a topological group, and, in the sense of topological groups, there is an isomorphism*

$$\rho\colon \operatorname{Aut}\mathfrak{g} \to \prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j \rtimes_\gamma P_{\mathcal{S}}(J), \quad \rho((\alpha_j)_{j\in J},\tau)((X_j)_{j\in J}) = (\alpha_j(X_{\tau^{-1}(j)}))_{j\in J}.$$

Firstly, the group

$$P_{\mathcal{S}}(J) \cong \prod_{\mathfrak{s}\in\mathcal{S}} P\big(J(\mathfrak{s})\big)$$

as defined in (1) *is a permutation group endowed with the group topology of pointwise convergence.*
Secondly, the group $\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j$ is an almost connected pro-Lie group and the identity component $(\operatorname{Aut}\mathfrak{g})_0$ of $\operatorname{Aut}\mathfrak{g}$ is the group $\operatorname{Inn}\mathfrak{g}$ of inner automophisms corresponding via ρ to $\prod_{j\in J}\operatorname{Inn}\mathfrak{s}_j \cong \prod_{\mathfrak{s}\in\mathcal{S}}(\operatorname{Inn}\mathfrak{s})^{J(\mathfrak{s})}$.

Proof. By Proposition 2.2, ρ is an algebraic isomorphism, and by Lemmas 2.8 and 2.9, ρ is a homeomorphism. By Lemma 2.6, the image of ρ is a topological group. Thus the domain $\operatorname{Aut}\mathfrak{g}$ of ρ is a topological group as well, and ρ is an isomorphism of topological groups. It remains to show that $(\operatorname{Aut}\mathfrak{g})_0 = \operatorname{Inn}\mathfrak{g}$ and that $\prod_{j\in J}\mathfrak{s}_j$ is almost connected. We shall do this by considering the topological group $(\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j)\rtimes P_{\mathcal{S}}(J)$. The group $P_{\mathcal{S}}(J)$ is totally disconnected. Hence we may concentrate on $\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j$. It is well established that the identity component $(\operatorname{Aut}\mathfrak{s}_j)_0$ is the subgroup $\operatorname{Inn}(\mathfrak{s}_j)$ of inner automorphisms and that $\operatorname{Out}\mathfrak{s}_j \stackrel{\rm def}{=} \operatorname{Aut}\mathfrak{s}_j/\operatorname{Inn}\mathfrak{s}_j$ is finite. (See e.g. Murakami [10] or [2].) For $\mathfrak{g}=\prod_{j\in J}\mathfrak{s}_j$ it follows that $(\operatorname{Aut}\mathfrak{g})_0=(\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j)_0=\prod_{j\in J}(\operatorname{Aut}\mathfrak{s}_j)_0=\prod_{j\in J}\operatorname{Inn}\mathfrak{s}_j=\operatorname{Inn}\mathfrak{g}$, since the equation $\operatorname{Inn}\mathfrak{g}=\prod_{j\in J}\operatorname{Inn}\mathfrak{s}_j$ is immediate from the definitions. We conclude that

$$\frac{\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j}{(\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j)_0}\cong\prod_{j\in J}\operatorname{Out}\mathfrak{s}_j$$

as a product of finite sets is compact totally disconnected. Thus $\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j$ is an almost connected pro-Lie group in view of Lemma 2.3a. □

The Ingredient of Compactness in $\operatorname{Aut}\mathfrak{g}$

We have seen
Now we recall some facts from the theory of automorphisms of a finite dimensional simple Lie algebra $\mathfrak{s}$. Dong Hoon Lee's Supplement Theorem ([9], Lemma 2.11) shows that for each $\mathfrak{s}\in\mathcal{S}$ there is a finite subgroup $E_\mathfrak{s}$ of $\operatorname{Aut}\mathfrak{s}$ such that $\mu\colon\operatorname{Inn}\mathfrak{s}\rtimes E_\mathfrak{s}\to\operatorname{Aut}\mathfrak{s}$, $\mu(\alpha,\epsilon)=\alpha\circ\epsilon$ is a surjective morphism whose kernel is isomorphic to $E_\mathfrak{s}\cap\operatorname{Inn}\mathfrak{s}$.
For many simple real Lie algebras $\mathfrak{s}$, the morphism μ is known to be an isomorphism, but surprisingly this appears to be unknown in general. Nevertheless, we shall get by with the weaker part of the information afforded by Lee's Theorem.
Accordingly, the subgroup $\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j$ of $\operatorname{Aut}\mathfrak{g}$ according to $(*)$ has the property that, due to our convention on the isotypic factors $F_\mathfrak{s}=\mathfrak{s}^{J(\mathfrak{s})}$ for each $\mathfrak{s}\in\mathcal{S}$ we have $\operatorname{Aut}\mathfrak{s}_j=\operatorname{Aut}\mathfrak{s}_k$ for all $j,k\in J(\mathfrak{s})$, $\mathfrak{s}\in\mathcal{S}$ so that, in fact we can write

$$\begin{aligned}\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j&=\prod_{\mathfrak{s}\in\mathcal{S}}(\operatorname{Aut}\mathfrak{s})^{J(\mathfrak{s})}\\&=\prod_{\mathfrak{s}\in\mathcal{S}}(\operatorname{Inn}\mathfrak{s}\cdot E_\mathfrak{s})^{J(\mathfrak{s})}\\&=\Big(\prod_{\mathfrak{s}\in\mathcal{S}}(\operatorname{Inn}\mathfrak{s})^{J(\mathfrak{s})}\Big)\cdot\Big(\prod_{\mathfrak{s}\in\mathcal{S}}E_\mathfrak{s}^{J(\mathfrak{s})}\Big)\end{aligned}\qquad(6)$$

It follows from this representation that the automorphic action of the group $P_{\mathcal{S}}(J)$ on $\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j$ respects both the factors $\prod_{\mathfrak{s}\in\mathcal{S}}(\operatorname{Inn}\mathfrak{s})^{J(\mathfrak{s})}$ and $\prod_{\mathfrak{s}\in\mathcal{S}}E_\mathfrak{s}^{J(\mathfrak{s})}$. Let us write $E_j=E_\mathfrak{s}$ for $j\in J(\mathfrak{s})$; then we may define

$$E\stackrel{\rm def}{=}\prod_{j\in J}E_j=\prod_{\mathfrak{s}\in\mathcal{S}}E_\mathfrak{s}^{J(\mathfrak{s})}.\qquad(7)$$

Since the group $P_{\mathcal{S}}(J)$ acts on E as a group of automorphisms, the semidirect product

$$\Delta \stackrel{\text{def}}{=} E \rtimes P_{\mathcal{S}}(J) \tag{8}$$

is a well-defined subgroup of $(\prod_{j\in J} \operatorname{Aut} \mathfrak{s}_{)}) \rtimes P_{\mathcal{S}}(J)$. Also we may write

$$\operatorname{Inn} \mathfrak{g} = \prod_{j\in J} \operatorname{Inn} \mathfrak{s}_j = \prod_{\mathfrak{s}\in\mathcal{S}} (\operatorname{Inn} \mathfrak{s})^{J(\mathfrak{s})}. \tag{9}$$

It is helpful to use the following convention on topological groups: Let G be a topological group, N a closed normal subgroup and H a closed subgroup such that $G = NH$. Now H acts automorphically on N via inner automorphisms, the semidirect product $N \rtimes H$ and the natural surjective morphism $\mu\colon N \rtimes H \to G$, $\mu(n, h) = nh$ are well-defined; its kernel is $K \stackrel{\text{def}}{=} \{(h^{-1}, h) \in N \rtimes H : h \in N \cap H\}$ and the function $h \mapsto (h^{-1}, h) : H \to K$ is an isomorphism of topological groups. Denote the natural bijective morphism $(h(N \cap H) \mapsto hN : H/(N \cap H) \to G$ by α and the the quotient maps $g \mapsto gN : G \to G/N$ and $h \mapsto h(N \cap H) : H \to H/(N \cap H)$ by p, respectively, q. Note that α is an isomorphism of topological groups if $H/(N \cap H)$ is compact, and thus in particular if H is compact.
We say that a morphism $f\colon A \to B$ of topological groups has a *cross-section* $s\colon B \to A$ if s is a continuous function such that $f \circ s = \mathrm{id}_B$ and $s(1) = 1$. If f has a cross-section, then f is open, and $a \mapsto (as(f(a))^{-1}, f(a)) : A \to \ker f \times B$ is a homeomorphism with inverse map $(k, b) \mapsto ks(b)$.

Lemma FIT. (On the First Isomorphism Theorem) *The following conditions are equivalent:*

(i) *q has a cross-section $s\colon H/(N \cap H) \to H$, and α is an isomorphism of topological groups.*
(ii) *p has a cross section $S\colon G/N \to G$ with values in H.*
(iii) There is an idempotent self-map $P\colon G \to G$ with image N and $P^{-1}(n) = nP^{-1}(1) \subseteq nH$ for $n \in N$.

These conditions imply

(iv) *μ has a cross-section $\nu\colon G \to N \times H$.*

Proof. (i)⇒(ii): Let $j\colon H \to G$ be the inclusion map and define S by $S = j \circ s \circ \alpha^{-1}$. This is a continuous function. Every element of G/N is of the form Nh with a suitable $h \in H$ and so $S(Nh) = s(N \cap H)h) \in (N \cap H)h \in H \cap Nh$.
(ii)⇒(i): We let $S'\colon G \to H$ denote the corestriction of S. Then $\alpha^{-1} = q \circ S'$: indeed $(N \cap H)S(Nh) = \alpha^{-1}(Nh)$. Thus α^{-1} is continuous. Further define $s\colon H/(N \cap H) to H$ by $s((N \cap H)h) = S(Nh)$, i.e., $s = S' \circ \alpha$. Then $q \circ s = q \circ S' \circ \alpha = \alpha^{-1} \circ \alpha$.
(ii)⇒(iii): Set $P(g) = gS(p(g))^{-1}$. Write $g = nS(p(g))$ for a suitable element $n \in N$; then $P(g) = n$. If $g = n \in N$, then $S(p(g)) = 1$ and so $P(g) = g$. Assume $P(g) = 1$, that is $g = S(p(g))$. Then $g \in H$ by (ii). If $P(g) = n$, then $g = nS(p(g)) \in nP^{-1}(1) \subseteq nH$.

(iii)⇒(ii): Define $\eta(g) = P(g)^{-1}g$. Since $P(g) \in N$ and $g \in P(g)H$ we can write $g = P(g)h$ for some $h \in H$ and thus $\eta(g) = h \in H$ and therefore $Ng = N\eta(g)$ for all $g \in G$. Consequently,
(i)⇒(iv): Assume s exists, then define $\phi: G \to H$ by $\phi = s \circ \alpha^{-1} \circ p$. Let $g = nh \in G$ with $n \in N$ and $h \in H$. Then we may write $\alpha^{-1}(p(g)) = h(N \cap H)$ and $\phi(g)$ is an element $h' \in h(N \cap H)$, whence $g\phi(g)^{-1} = nhh'^{-1} \in N(N \cap H) = N$, $\phi(1) = 1$. Define $\nu: G \to N \rtimes H$ by $\nu(g) = (g\phi(g)^{-1}, \phi(g))$. Then $\mu \circ \nu = \mathrm{id}_G$, $\nu(1) = (1, 1)$. Since α is assumed to be an isomorphism of topological groups, then ϕ and therefore ν are continuous. □

If the condition (i) of this lemma is satisfied, then we shall write $G = N \bullet H$. This then indicates, in particular that G is homeomorphic to $N \times (H/(N \cap H)$.
In our present situation, since E_j is finite for each $j \in J$, the conditions of Lemma FIT(i) are satisfied for $G = \prod_{j\in J} \mathrm{Aut}\,\mathfrak{s}_j$, $N = \prod_{j\in J} \mathrm{Inn}\,\mathfrak{s}_j$, $H = \prod_{j\in J} E_j$.
With this notation we have the following refinement of the results obtained so far:

Theorem 2.11. (Refined Structure Theorem) *For a semisimple pro-Lie algebra $\mathfrak{g}$ there are isomorphisms of topological groups*

$$\mathrm{Inn}\,\mathfrak{g} \bullet \Delta \cong (\mathrm{Inn}\,\mathfrak{g} \bullet E) \rtimes P_{\mathcal{S}}(J) \cong \mathrm{Aut}\,\mathfrak{g}.$$

Proof. By Theorem 2.10 we have an isomorphism

$$\rho: \prod_{j\in J} \mathrm{Aut}\,\mathfrak{s}_j \rtimes_\gamma P_{\mathcal{S}}(J) \to \mathrm{Aut}\,\mathfrak{g}.$$

By (6), (7), (9) and the Lemma above we have $\prod_{j\in J} \mathrm{Aut}\,\mathfrak{s}_j = \mathrm{Inn}_{j\in J} \bullet E$ and therefor an isomorphism

$$(\mathrm{Inn}\,\mathfrak{g} \bullet E) \rtimes P_{\mathcal{S}}(J) \to \mathrm{Aut}\,\mathfrak{g}.$$

Since E is left invariant under the action of $P_{\mathcal{S}}(J)$, we have the identity

$$(\mathrm{Inn}\,\mathfrak{g} \bullet E) \rtimes P_{\mathcal{S}}(J) \cong \mathrm{Inn}\,\mathfrak{g} \bullet (E \rtimes P_{\mathcal{S}}(J) = \mathrm{Inn}\,\mathfrak{g} \rtimes \Delta$$

in view of (8). This proves the theorem. □

The closed subgroup $P_{\mathcal{S}}(J)$ is isomorphic to $\prod_{\mathfrak{s}\in\mathcal{S}} P(J(\mathfrak{s}))$ and is therefore prodiscrete if and only if all $J(\mathfrak{s})$ are finite.
The group $\Delta = E \rtimes P_{\mathcal{S}}(J)$ has a unique maximal compact normal subgroup, namely, $M \stackrel{\mathrm{def}}{=} E \times \prod_{\mathrm{card}(J(\mathfrak{s}))<\infty} P(\mathfrak{s})$.
From the developments of Theorems 2.10 and 2.11. we obtain

Corollary 2.12. *Let $\mathfrak{g} = \prod_{j\in J} \mathfrak{s}_j$ be a semisimple pro-Lie algebra. Then*

(a) *The automorphism group* $\operatorname{Aut}\mathfrak{g}$ *of is isomorphic to* $\operatorname{Inn}\mathfrak{g}\bullet\Delta$*, where* $(\operatorname{Aut}\mathfrak{g})_0 = \operatorname{Inn}\mathfrak{g} = \prod_{j\in J}\operatorname{Inn}\mathfrak{s}_j$ *is a product of connected simple centerfree Lie groups and is, accordingly, a connected pro-Lie group, and where* Δ *is totally disconnected.*
(b) $\Delta = E\rtimes P_{\mathcal{S}}(J)$ *is itself a semidirect product; here* E *is a product of finite groups, and* $P_{\mathcal{S}}(J)$ *is a closed subgroup of the totally disconnected full permutation group* $P(J)$ *of the index set* J*.*
(c) *The intersection* $\Delta\cap\operatorname{Inn}\mathfrak{g}$ *is a compact totally disconnected subgroup.*
(d) *Suppose that* H *is a closed almost connected subgroup of* $\operatorname{Aut}\mathfrak{g}$ *containing* $(\operatorname{Aut}\mathfrak{g})_0$*. Then there is a compact totally disconnected subgroup* C *of* Δ *such that* $H = H_0C$*.*

Proof. The Statements (a), (b), (c) are summaries of what was shown above. In order to prove (d) we let H be a closed subgroup with $H_0 = \operatorname{Inn}\mathfrak{g}$ and H/H_0 compact. Let $C = \Delta\cap H$. Since Δ is totally disconnected, C is totally disconnected. By the Modular Law, $H = (\operatorname{Inn}\mathfrak{g})\Delta = H_0(H\cap\Delta) = H_0C$. By Theorem 2.11, the morphism

$$a(H_0\cap C)\mapsto aH_0 : H/(H_0\cap C)\to H/H_0$$

is an isomorphism of topological groups. Hence $C/(H_0\cap C)$ is compact. Since $H_0\cap C = (\operatorname{Inn}\mathfrak{g})\cap\Delta$ is compact by (c), the subgroup C is compact. □

Corollary 2.13. *The automorphism group* $\operatorname{Aut}\mathfrak{g}$ *is topologically a product of its identity component* $(\operatorname{Aut}\mathfrak{g})_0 = \operatorname{Inn}\mathfrak{g}$ *and the totally disconnected group* $\frac{E}{E\cap\operatorname{Inn}\mathfrak{g}}\rtimes P_{\mathcal{S}}(J)$*. If all automorphism groups* $\operatorname{Aut}\mathfrak{s}_j$ *split over* $\operatorname{Inn}\mathfrak{s}_j$*, then the product is a semidirect product of topological groups.* □

Automorphic Action of a Compact Group

First let us return to $\mathfrak{s}^J$ for a set J and a simple Lie algebra $\mathfrak{s}$ and assume that Ω is a compact group acting automorphically on $\mathfrak{s}^J$, that is, there is a continuous action

$$(\omega, g)\mapsto\omega{\cdot}g : \Omega\times\mathfrak{s}^J\to\mathfrak{s}^J$$

such that $g\mapsto\omega{\cdot}g$ is in $\operatorname{Aut}(\mathfrak{s}^J)$ for all $\omega\in\Omega$.
By Proposition 2.5, every compact subgroup of $P(J)$ is contained in a subgroup of the form $C_R = \{\tau\in P(J) : (\forall j\in J)\,\tau(R(j))\subseteq R(j)\}$ for an equivalence relation R on J with finite cosets. Consequently, if $P(J)$ acts on $\mathfrak{s}^J$ by $\tau{\cdot}(s_j)_{j\in J} = (s_{t^{-1}(j)})_{j\in J}$, then

$$\mathfrak{s}^J = \prod_{\xi\in J/R}\mathfrak{s}^\xi$$

is a decomposition of $\mathfrak{s}^J$ into a product of finite dimensional semisimple Lie algebras which are invariant under the action of C_R. Accordingly, if Ω is a compact group acting automorphically on $\mathfrak{s}^J$, then there is a morphism $\omega \mapsto (\omega_j)_{j\in J} : \Omega \to (\mathrm{Aut}\,\mathfrak{s})^J$ of topological groups and an action $(\omega, j) \mapsto \omega{\cdot}j : \Omega \times J \to J$ such that

$$\omega{\cdot}(s_j)_{j\in J} = (\omega_j(s_{\omega^{-1}\cdot j}))_{j\in J}$$

As a consequence, $\mathfrak{s}^J = \prod_{\xi\in J/\Omega} \mathfrak{s}^\xi$ is a decomposition of $\mathfrak{s}^J$ into a product of finite dimensional semisimple Lie algebras which are invariant under the action of Ω. Continuing this notation, we shall obtain

Lemma 2.13. *There is a maximal compactly embedded subalgebra $\mathfrak{k}$ of $\mathfrak{s}^J$ which is invariant under Ω.*

Proof. Let $\xi \in J/R$. Then there is an automorphic action of Ω on the finite dimensional Lie algebra $\mathfrak{s}^\xi$. That is, there is a morphism of topological groups $\pi\colon \Omega \to \mathrm{Aut}\,\mathfrak{s}^\xi$. Since Ω is compact, $L \overset{\mathrm{def}}{=} \pi(\Omega)$ is a compact Lie subgroup of the Lie group $\mathrm{Aut}\,\mathfrak{s}^\xi$. We set

$$S \overset{\mathrm{def}}{=} \Gamma(\mathfrak{s}).$$

Since there is natural isomorphism $\alpha \mapsto \mathfrak{L}(\alpha) : \mathrm{Aut}\,\mathfrak{s} \to \mathrm{Aut}\,S$ we have an automorphic action of the compact Lie group L on the Lie group $\Gamma(\mathfrak{s}^\xi) \cong S^\xi$. We can form the almost connected Lie group $S^\xi \rtimes L$ and its quotient

$$G \overset{\mathrm{def}}{=} (S/Z(S))^\xi \rtimes L \cong \frac{S^\xi \rtimes L}{Z(S)^\xi \times \{1\}}.$$

We abbreviate $P \overset{\mathrm{def}}{=} \big(S/Z(S)\big)^\xi \times \{1\}$. Let K be a maximal compact subgroup of the simple centerfree (adjoint) connected Lie group $S/Z(S)$ Then K is connected and $N(K, S/Z(S)) = K$ (see [12], Lemma 1.1.3.7. on p. 28.). Set $K_1 \overset{\mathrm{def}}{=} K^\xi \times \{1\}$. It follows that $N(K_1, P) = K_1$, where $N(K_1, P)$ as usual denotes the normalizer of K_1 in P.
Let us briefly pause for a recollection of the

Frattini Argument. *Let Γ be a group acting on a set X and $\Sigma \subseteq \Gamma$ a subgroup acting transitively, then $\Gamma = \Sigma{\cdot}\Gamma_x$ where Γ_x is the isotropy group $\{\gamma : \gamma{\cdot}x = x\}$ at any $x \in X$.* □

(See e.g. [4], Lemma preceding Corollary 6.35, p. 216.)

The inner automorphisms of G act transitively on the maximal compact subgroups of P and so the Frattini Argument yields $G = PN(K_1, G)$ and $P \cap N(K_1, G) = K_1$. We note that $N(K_1, G)/K_1 = N(K_1, G)/\big(P \cap N(K_1, G)\big) \cong G/P = L$ is compact, and so $N(K_1, G)$ is compact. We claim that $N(K_1, G)$ is maximal compact in G. We isolate this claim in the following

Lemma Max. *Let M be a compact subgroup of G and P a normal subgroup of G such that $G = PM$ and M contains a maximal compact subgroup Q of P. Then M is maximal compact in G.*

Proof. Let C^* be a compact subgroup of G containing M, then $Q \subseteq P \cap C^*$. By the maximality of Q in P we have $Q = P \cap C^*$. But then $C^* = (P \cap C^*)M = QM = M$, as claimed. $\square$

It is known that in almost connected Lie groups the maximal compact subgroups are conjugate (see [3], p. 380, Theorem 3.1). Consequently, there is a $g \in G$ such that $\{1\} \times L \subseteq gN(K_1, G)g^{-1} = N(gK_1g^{-1}, G)$. Let $I_g \in \mathrm{Aut}(P)$ be the automorphism induced by the inner automorphism implemented by g on P. Thus the maximal compact subgroup $I_g(K_1)$ is L-invariant. Its Lie algebra $\mathfrak{k}_\xi$ is a maximal compactly embedded Ω-invariant subalgebra of $\mathfrak{s}^\xi$. Now we set $\mathfrak{k} = \prod_{\xi \in J/\Omega} \mathfrak{k}_\xi \subseteq \prod_{\xi \in J/\Omega} \mathfrak{s}^\xi = \mathfrak{s}^J$. Then $\mathfrak{k}$ is maximally compactly embedded in $\mathfrak{s}$ and is Ω-invariant. $\square$

Proposition 2.14. *Let a compact group Ω act automorphically on a semisimple pro-Lie algebra $\mathfrak{g}$. Then there is a maximal compactly embedded subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$ which is Ω-invariant.*

Proof. We let

$$\mathfrak{g} = \prod_{\mathfrak{s} \in \mathcal{S}} F_\mathfrak{s}, \quad F_\mathfrak{s} = \mathfrak{s}^{J(\mathfrak{s})}$$

be the isotypic decomposition of $\mathfrak{g}$. Since Ω acts automorphically, it preserves the factors $\mathfrak{s}^{J(\mathfrak{s})}$. On each of these, Ω acts automorphically and thus by Lemma 4.6, there is a maximal compactly embedded subalgebra $\mathfrak{k}_\mathfrak{s} \subseteq \mathfrak{s}^{J(\mathfrak{s})}$ which is Ω-invariant. Then $\mathfrak{k} = \prod_{\mathfrak{s} \in \mathcal{S}} \mathfrak{k}_\mathfrak{s}$ is a maximal compactly embedded subalgebra that is invariant under the action of Ω. $\square$

3. The automorphism groups of semisimple pro-Lie groups

In the spirit of Lie theory, from the information on the automorphism group of a pro-Lie group $\mathfrak{g}$ we draw conclusions on the automorphism group of a connected semisimple Lie group G. Let $\mathfrak{L}$ be the Lie algebra functor as detailed in [5], Chapters 2, 3 and 4. For a pro-Lie group G, we denote by

$$L_G \colon \mathrm{Aut}\, G \to \mathrm{Aut}\, \mathfrak{g} \quad \text{the function implemented by } L_G(\alpha) = \mathfrak{L}(\alpha),$$

where $\alpha \colon G \to G$ is an automorphism of G. Assume first that G is simply connected. Then we may identify G with $\Gamma(\mathfrak{g})$ as in [5], Theorem 6.5., p. 253. By Theorem 6.6(vii) on p. 256 of [5]), $\Gamma \circ \mathfrak{L}$ is naturally isomorphic to the identity functor of the category of pro-Lie groups and $\mathfrak{L} \circ \Gamma$ is naturally isomorphic to the identity functor of the category of pro-Lie algebras. Therefore, $L_G \colon \mathrm{Aut}\, G \to \mathrm{Aut}\, \mathfrak{g}$ is inverted by $\beta \mapsto \Gamma(\beta)$, So we conclude that $L_G \colon \mathrm{Aut}\, G \to \mathrm{Aut}\, \mathfrak{g}$ is an isomorphism of topological groups.

Proposition 3.1. *Let G be a simply connected semisimple pro-Lie group. Then*

(i) $\operatorname{Aut} G$ *is the semidirect product of the normal subgroup* $\prod_{j\in J} S_j$ *for a family of simply connected simple real Lie groups* S_j *with simple Lie algebras, and a subgroup isomorphic to* $P_{\mathcal{S}}(J)$.

(ii) $\operatorname{Aut} G$ *is a topological group with respect to the compact open topology.*

(iii) *The morphism* $L_G\colon \operatorname{Aut} G \to \operatorname{Aut} \mathfrak{g}$ *is an isomorphism of topological groups.*

Proof. In order to prove (i), we determine the structure of $\operatorname{Aut} G$ in complete analogy to that of $\operatorname{Aut} \mathfrak{g}$; in the process we shall prove (ii) as well. By Theorem 10.29 on p. 435 of [5], G may be written as $\prod_{j\in J} S_j$ with a family of simply connected simple Lie groups S_j. Just as in Section 2, we may identify $\prod_{j\in J} \operatorname{Aut} S_j$ with a subgroup of $\operatorname{Aut} G$. For each $\mathfrak{s} \in P_{\mathcal{S}}(J)$ there is a simply connected simple Lie group $S_{\mathfrak{s}}$ such that $\mathfrak{L}(S_{\mathfrak{s}}) = \mathfrak{s}$. (In the terminology of [5], $S_{\mathfrak{s}} = \Gamma(\mathfrak{s})$).) Again as in Section 2 we may assume that two factors S_j and S_k which are isomorphic, are in fact equal. Then the permutation group $P_{\mathcal{S}}(J)$ acts on $\prod_{j\in J} S_j$ via a morphism $\underline{\gamma}\colon P_{\mathcal{S}}(J) \to \operatorname{Aut}(\prod_{j\in J} S_j)$ given by

$$\underline{\gamma}(\tau)((\alpha_{j\in J}))((s_j)_{j\in J}) = (\alpha_j(s_{\tau^{-1}(j)})_{j\in J}).$$

As in the case of pro-Lie algebras, this morphism is continuous as is the function $(\tau, \alpha) \mapsto \Psi(\tau)(\alpha)$ for α abbreviating $(\alpha_j)_{j\in J}$. Thus $(\prod_{j\in J} \operatorname{Aut} S_j) \rtimes_{\underline{\gamma}} P_{\mathcal{S}}(J)$ is a topological group. As in the proof of 2.10, the morphism

$$\underline{\rho}\colon \prod_{j\in J} \operatorname{Aut} S_j \rtimes_{\underline{\gamma}} P_{\mathcal{S}}(J) \to \operatorname{Aut} G, \quad \underline{\rho}(\alpha, \tau) = \alpha \circ \underline{\gamma}(\tau)$$

is an isomorphism of groups and a homeomorphism if $\operatorname{Aut} G$ is given the compact open topology. This means that $\operatorname{Aut} G$ is in fact a topological group. This establishes (ii).

Proof of (iii). We notice that by construction, we have a commutative diagram

$$\begin{array}{ccc}
\prod_{j\in J} \operatorname{Aut} \mathfrak{s}_j \rtimes_{\gamma} P_{\mathcal{S}}(J) & \xrightarrow{\ \rho\ } & \operatorname{Aut} \mathfrak{g} \\
\Big\downarrow {\scriptstyle \prod_{j\in J} L_{S_j} \times \mathrm{id}_{P_{\mathcal{S}}(J)}} & & \Big\downarrow {\scriptstyle L_G} \\
\prod_{j\in J} S_j \rtimes_{\underline{\gamma}} P_{\mathcal{S}}(J) & \xrightarrow{\ \underline{\rho}\ } & \operatorname{Aut} G
\end{array}$$

From the theory of finite dimensional Lie groups we know that all maps

$$L_{S_j}\colon \operatorname{Aut} S_j \to \operatorname{Aut} \mathfrak{s}_j$$

are isomorphisms of topological groups. Thus the left vertical map is a homeomorphism. The two horizontal maps are isomorphisms. It follows that the right vertical map L_G is an isomorphism as well. □

Corollary 3.2. *Let G be a connected centerfree semisimple pro-Lie group. Then*

(i) $\operatorname{Aut} G$ *is the semidirect product of the normal subgroup* $\prod_{j\in J} S_j$ *for a family of simply connected simple (and centerfree) real Lie groups* S_j, *and a subgroup isomorphic to* $P_{\mathcal{S}}(J)$.

(ii) $\operatorname{Aut} G$ *is a topological group with respect to the compact open topology.*

(iii) *The morphism* $L_G\colon \operatorname{Aut} G \to \operatorname{Aut} \mathfrak{g}$ *is an isomorphism of topological groups.*

Proof. Let $\widetilde{G}$ be the universal group of G (see [5], paragraph following the end of the proof of Theorem 6.6 on p. 259)which happens to be the *universal covering group* (see [4], Definitions A.2.19 on p. 701 ($2^{\rm nd}$ ed.)) of G as well. (Note in passing that every pro-Lie group has a universal group, but not every pro-Lie group, not even every compact one, has a universal covering group.) We invoke Theorem 10.29 on p. 435 of [5] and observe that we have natural isomorphisms

$$G \cong \widetilde{G}/Z(\widetilde{G}) = \frac{\prod_{j\in J}\widetilde{S_j}}{\prod_{j\in J} Z(\widetilde{S_j})} \cong \prod_{j\in J} S_j,$$

for a famliy of centerfree adjoint simple connected Lie groups $S_j \cong \widetilde{S_j}/Z(\widetilde{S_j})$. Every morphism α of $\widetilde{G}$ satisfies $\alpha(Z(\widetilde{G})) \subseteq Z(\widetilde{G})$ and thus induces a morphism $\zeta(\alpha)$ of G and this yields a morphism of groups $\zeta\colon \operatorname{Aut}\widetilde{G} \to \operatorname{Aut} G$. The morphism

$$t\colon \operatorname{Aut} G \to \operatorname{Aut}\widetilde{G}, \quad t(\alpha) = \widetilde{\alpha}$$

obtained from the functoriality of $G \mapsto \widetilde{G}$ inverts ζ, and thus ζ is an isomorphism of groups with $\zeta^{-1} = t$. The isomorphism ζ maps $\prod_{j\in J}\operatorname{Aut}\widetilde{S_j}$ onto $\prod_{j\in J}\operatorname{Aut} S_j$ and preserves the copies of $P_{\mathcal{S}}(J)$ in $\operatorname{Aut}\widetilde{G}$, respectively, $\operatorname{Aut} G$, giving us a commutative diagram

$$\begin{array}{ccc}
\prod_{j\in J}\operatorname{Aut}\widetilde{S_j} \rtimes_{\underline{\gamma}} P_{\mathcal{S}}(J) & \xrightarrow{\ \underline{\rho}\ } & \operatorname{Aut}\widetilde{G} \\
{\scriptstyle \prod_{j\in J}\zeta_j\times\mathrm{id}}\Big\downarrow & & \Big\downarrow{\scriptstyle\zeta} \\
\prod_{j\in J}\operatorname{Aut} S_j \rtimes_{\underline{\gamma}'} P_{\mathcal{S}}(J) & \xrightarrow{\ \underline{\rho}'\ } & \operatorname{Aut} G.
\end{array}$$

The left downmap is an isomorphism of topological groups; the morphism $\underline{\rho}$ we know to be an isomorphism of topological groups. That $\underline{\rho}'$ is an isomorphism of groups is verified as in 3.1 (respectively, 2.10), and likewise that it is a homeomorphism, if $\operatorname{Aut} G$ is given the compact open topology. Thus we know that $\operatorname{Aut} G$ is a topological group with respect to the compact open topology and that $\underline{\rho}'$ is an isomorphism. It follows from the commutativity of the diagram that the right downmap ζ is an isomorphism as well. □

We recall from [5], Chapter 2ff. that each pro-Lie algebra $\mathfrak{g}$ determines functorially a simply connected pro-Lie group $\Gamma(\mathfrak{g})$ such that $\mathfrak{g} \cong \mathfrak{L}\big(\Gamma(\mathfrak{g})\big)$. If $\mathfrak{g}$ is semisimple, so are $\Gamma(\mathfrak{g})$ and $G \stackrel{\rm def}{=} \Gamma(\mathfrak{g})/Z\big(\Gamma(\mathfrak{g})\big) \cong \operatorname{Ad}\big(\Gamma(\mathfrak{g})\big)$. The upshot of these supplementary results is that all three of $\mathfrak{g}$, $\Gamma(\mathfrak{g})$ and G have "the same" topological group as automorphism group. Theorem 2.10 gives additional details on the fine structure of this automorphism group, notably on the fact that it splits over its identity component.

We now wish to extend the structure theory to connected semisimple pro-Lie groups in general. We endow the automorphism group $\mathrm{Aut}\,G$ of a pro-Lie group with the compact-open topology without claiming that, in general, this will make it a topological group. The subgroup of all inner automorphisms $I_G(g)$, $I_G(g)(x) = gxg^{-1}$ will be denoted $\mathrm{Inn}\,G \subseteq \mathrm{Aut}\,G$. We shall denote the composition

$$G \xrightarrow{I_G} \mathrm{Aut}\,G \xrightarrow{L_G} \mathrm{Aut}\,\mathfrak{g}$$

by $\mathrm{Ad}: G \to \mathrm{Aut}\,\mathfrak{g}$, as is customary.

Lemma 3.3. *Let G be a topological group then*

(i) *the function $I_G: G \to \mathrm{Aut}\,G$ is continuous if $\mathrm{Aut}\,G$ is given the compact open topology.*

(ii) *Let N be a characteristic closed subgroup of G and $\mathrm{res}_N: \mathrm{Aut}\,G \to \mathrm{Aut}\,N$ the function defined by $\mathrm{res}_N(f)(n) = f(n)$, $n \in N$. Then res_N is a continuous function and a morphism of groups.*

(iii) *Let M and $N \subseteq M$ be closed characteristic subgroups of G. Then the function $f: G \to \mathrm{Aut}\,M/N$, $f(g)(mN) = gmg^{-1}N$ is a continuous morphism of groups.*

Proof. (i) Since translations are continuous in $\mathrm{Aut}\,G$, it suffices to observe continuity of I at the origin. Thus let K be a compact subset of G and V and identity neighborhood of G. Then $W(K,V) = \{\alpha \in \mathrm{Aut}\,G : (\forall k \in K)\alpha(k) \subseteq Vk\}$ is a subbasic and indeed a basic identity neighborhood of $\mathrm{Aut}\,G$. For each $k \in K$ we now find an identity neighborhood U_k of G and a neighborhood C_k of k such that $[u,c] = ucu^{-1}c^{-1} \in V$ for all $u \in U$ and $c \in C_k$. By compactness of K, there is a finite subset $F \subseteq K$ such that $K \subseteq \bigcup_{k\in F} C_k$. Let $U = \bigcap_{k\in F} U_k$. Then U is an identity neighborhood satisfying $[U,K] \subseteq V$, and thus $uku^{-1} \in Vk$ for all $u \in U$ and $k \in K$. Thus $I_G(U) \subseteq W(K,V)$, proving the continuity of I_G.
(ii) Restriction of automorphisms to a characteristic subgroup implements a morphisms of groups. We must show continuity at the origin. A basic identity neighborhood $W_N(C,V)$ of $\mathrm{Aut}\,N$ is given by a compact subset $C \subseteq N$ and an open identity neighborhood V of N so that $\beta \in W_N(C,V)$ is given by $\beta(c)c^{-1} \in V$ for all $c \in C$. If we pick an open identity neighborhood U of G such that $U \cap N \subseteq V$, then $W_G(C,U)$ is an identity neighborhood of $\mathrm{Aut}\,G$ and $\mathrm{res}\big(W_G(C,U)\big) \subseteq W_N(C,V)$.
(iii) Let $q: G \to G/N$ be the quotient morphism. Then the function f is the composition

$$G \xrightarrow{q} \frac{G}{N} \xrightarrow{I_{G/N}} \mathrm{Aut}\,\frac{G}{N} \xrightarrow{\mathrm{res}_{M/N}} \mathrm{Aut}\,\frac{M}{N}.$$

Clearly all maps in sight are group morphisms, q is continuous, $I_{G/N}$ is continuous by (i) and $\mathrm{res}_{M/N}$ is continuous by (ii). □

For the following summary recall the notation $N \bullet H$ for a topological group with a normal subgroup N and subgroup H: it means the existens of a continu map $\nu: G \to N \rtimes H$ such that $\mu \circ \nu = \mathrm{id}_G$, $\mu(n,h) = nh$.

Main Theorem 3.4. *Let G be a connected semisimple pro-Lie group. Then*

(i) *the function* $\mathrm{Ad} \stackrel{\mathrm{def}}{=} L_G \circ I_G : G \to \mathrm{Aut}\,\mathfrak{g}$ *implements an open morphism from* G *onto* $\mathrm{Inn}\,\mathfrak{g}$ *and an isomorphism of topological groups* $G/Z(G) \to \mathrm{Inn}\,\mathfrak{g}$. *In particular,* $G/Z(G)$ *is a pro-Lie group.*

(ii) *The morphism* $L_G\colon \mathrm{Aut}\,G \to \mathrm{Aut}\,\mathfrak{g}$ *is an embedding of topological groups.*

(iii) $\mathrm{Aut}\,G$ *is a topological group with respect to the compact open topology.*

(iv) $\mathrm{Aut}\,G$ *is the product* $(\mathrm{Aut}\,G)_0 \bullet D$ *of a normal subgroup* $(\mathrm{Aut}\,G)_0$ *isomorphic to* $\mathrm{Inn}\,\mathfrak{g} \cong \prod_{j\in J} S_j$ *for a family of simply connected simple (and centerfree) real Lie groups* S_j*, and a totally disconnected topological group* D *isomorphic to a subgroup of* $\Delta = E \rtimes P_{\mathcal{S}}(J)$. *The intersection* $(\mathrm{Aut}\,G)_0 \cap D$ *is compact and totally disconnected.*

Proof. (i) The groups G and $\mathrm{Inn}\,\mathfrak{g}$ are connected pro-Lie groups by hypothesis, respectively, Corollary 2.11. Thus the Open Mapping Theorem for Pro-Lie Groups [5], 9.60 on p. 409 applies to the morphism $\mathrm{Ad}\colon G \to \mathrm{Inn}\,\mathfrak{g}$ and shows that it is open. For $g \in G$ we have $(\forall x \in \mathfrak{g})\ \mathrm{Ad}(g)(x) = x$ and thus $g(\exp x)g^{-1} = \exp \mathrm{Ad}(g)(x) = \exp x$; since $\langle \exp \mathfrak{g}\rangle$ is dense in G (see [5], Corollary 4.22(i)), this is the case if g commutes with all $y \in G$, that is, $y \in Z(G)$. Thus $\ker \mathrm{Ad} = Z(G)$. Therefore the corestriction $\mathrm{Ad}\colon G \to \mathrm{Inn}\,\mathfrak{g}$ of Ad induces an embedding $G/Z(G) \to \mathrm{Inn}\,\mathfrak{g}$. Now let $\alpha \in \mathrm{Inn}\,\mathfrak{g}$ be an inner automorphism of the pro-Lie algebra $\mathfrak{g}$. We know that we may write $\mathfrak{g} = \prod_{j\in J} \mathfrak{g}_j$ for simple real (finite dimensional) Lie algebras $\mathfrak{g}_j$. Accordingly, we may identitify $\mathrm{Inn}\,\mathfrak{g}$ with $\prod_{j\in J} \mathrm{Inn}\,\mathfrak{g}_j$. In that sense we can write $\alpha = (\alpha_j)_{j\in J}$ with $\alpha_j \in \mathrm{Inn}\,\mathfrak{g}_j$. Let $\widetilde{G} = \Gamma(\mathfrak{g})$ be the universal group of G with the universal morphism $\pi_G\colon \widetilde{G} \to G$ ([5], p. 259). We know $\widetilde{G} = \prod_{j\in J} \widetilde{G}_j$ with simply connected real simple Lie groups $\widetilde{G}_j$ and that the exponential function decomposes accordingly

$$\exp_{\widetilde{G}} = \prod_{j\in J} \exp_{\widetilde{G}_j}\colon \mathfrak{g} = \prod_{j\in J} \mathfrak{g}_j \to \prod_{j\in J} \widetilde{G}_j = \widetilde{G}.$$

From the theory of finite dimensional Lie groups one knows that for each j there is an element $g_j \in \widetilde{G}_j$ such that $\mathrm{Ad}_{G_j}(g_j) = \alpha_j$. Set $\widetilde{g} = (g_j)_{j\in J} \in \widetilde{G}$. Then $\mathrm{Ad}_{\widetilde{G}}(\widetilde{g}) = \alpha$. Now set $g \stackrel{\mathrm{def}}{=} \pi_G(\widetilde{g}) \in G$. Then $\mathrm{Ad}_G(g) = \alpha$, as we deduce from our identifying $\mathfrak{L}(\widetilde{G}) = \mathfrak{g} = \mathfrak{L}(G)$ and the commuting of the diagram

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\ \mathrm{Ad}_{\widetilde{G}}\ } & \mathrm{Inn}\,\mathfrak{g} \\ {\scriptstyle \pi_G}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}_G} \\ G & \xrightarrow[\ \mathrm{Ad}_G\ ]{} & \mathrm{Inn}\,\mathfrak{g}. \end{array}$$

Therefore $\mathrm{Inn}\,\mathfrak{g} = \mathrm{Ad}_G(G)$, and

$$gZ(G) \mapsto \mathrm{Ad}(g)\colon G/Z(G) \to \mathrm{Inn}\,\mathfrak{g}$$

is an isomorphism of topological groups. Since $\mathrm{Inn}\,\mathfrak{g}$ is a pro-Lie group by Lemma 2.5, the factor group $G/Z(G)$ is a pro-Lie group.

(ii) We abbreviate the center $Z(G)$ of G by Z. Set $\underline{G} = G/Z$. Then $\underline{G}$ is isomorphic to the topological group $\mathrm{Inn}\,\mathfrak{g}$ by (i), a centerfree semisimple connected pro-Lie group and we know from 3.2(iii) that $L_{\underline{G}}\colon \mathrm{Aut}\,\underline{G} \to \mathrm{Aut}\,\mathfrak{g}$ is an isomorphism of topological groups. It therefore suffices to guarantee that the morphism $\theta\,\mathrm{Aut}\,G \to \mathrm{Aut}\,\underline{G}$, $\theta(\alpha)(gZ) = \alpha(g)Z$ is an embedding. Firstly, it is injective: $\theta(\alpha) = \mathrm{id}_{\underline{G}}$ means $\forall g \in G)\,\alpha(g)Z = gZ$, that is, $(\forall g \in G)\,g^{-1}\alpha(g) \in Z$. But G is connected, Z is totally disconnected and $g \mapsto g^{-1}\alpha(g) : G \to Z$ is continuous, hence constant. It follows that $\alpha = \mathrm{id}_G$. This proves that θ is injective. Secondly, we claim that θ is continuous. Let $q\colon G \to \underline{G}$, $q(g) = gZ$ denote the quotient morphism.

Lemma. (a) *For each compact set K in $\underline{G}$ there is a compact set C in G such that $K \subseteq q(C)$.*
(b) *For any compact subset C of G there is a compact* connected *subset C^* of G containing C.*

Proof of the Lemma. (a) We may write $\underline{G} = \prod_{j\in J} S_j$ with a family of connected, centerfree simple Lie groups S_j. Let K_j be the projection of K into $\S_j$. Then $K \subseteq \prod_{j\in J} K_j$ and the right hand side is compact. Now let $\widetilde{S}_j$ be the simply connected covering group of S_j and let $\widetilde{K}_j$ be a compact subset of $\widetilde{S}_j$ such that its image in S_j contains K_j. Let $\pi_G\colon \widetilde{G} = \prod_{j\in J} \widetilde{S}_j \to G$ denote the universal morphism of G. Set $C = \pi_G(\prod_{j\in J} \widetilde{K}_j)$. Then

$$q(C) = \pi_{\underline{G}}(\textstyle\prod_{j\in J} \widetilde{K}_j) \supseteq \prod_{j\in J} K_j \supseteq K.$$

(b) (This assertion does not depend on semisimplicity!) By [5], Theorem 12.81, p.551, G is homeomorphic to $\mathbb{R}^J \times M$ where J is a set and M a (maximal) compact connected subgroup of G. If $C \subseteq \mathbb{R}^J \times M$ is compact, and C_j is the projction of C into the j-th factor $\mathbb{R}$ of $\mathbb{R}^J$, $j \in J$, then we find a connected compact subset C_j^* in that factor containing C_j. Then $C^* \stackrel{\mathrm{def}}{=} (\prod_{j\in J} C_j^*) \times M$ is compact, connected and contains C. This finishes the proof of the Lemma.

Now the continuity of θ! Let $W(K, V)$ for a compact subset K of $\underline{G}$ and an open identity neighborhood V of $\underline{G}$ be a basic open identity neighborhood of $\underline{G}$ for the compact-open topology. By part (a) of the Lemma, we find a compact subset C of G with $q(C) \supseteq K$ and an open identity neighborhood U of G such that $q(U) \subseteq V$. Now let $\alpha \in W(C, U)$ implies $\alpha(c)c^{-1} \in U$ for all $c \in C$; if $k \in K$ then by the choice of C there is $c \in C$ such that $k = q(c) = cZ$, and thus $\theta(\alpha)(k)k^{-1} = \alpha(c)c^{-1}Z \in UZ/Z \subseteq V$.

Thirdly, we show that the morphism θ is open onto its image. Let C be a compact set in G, and U an open identity neighborhood in G. We must find a compact subset K of $\underline{G}$ and an open identity neighborhood V of $\underline{G}$, such that $W(K, V) \cap \mathrm{im}\,\theta \subseteq W(C, V)$, that is, such that for every $\alpha \in \mathrm{Aut}\,G$ with $\theta(\alpha)(k)k^{-1} \in V$ for all $k \in K$ we have $\alpha(c)c^{-1} \in U$ for all $c \in C$. By Part (b) of the Lemma we pick a compact connected subset C^* of G containing C and choose $K = q(C^*) = C^*Z/Z$ and we propose to find a suitable open identity neighborhood U' of G and choose $V = q(U') = U'Z/Z$, so that whenever we assume that $\theta(\alpha)(k)k^{-1} \in V$ for all $k \in K$ we shall be able to conclude $\alpha(c)c^{-1} \in U$. Now let $c \in C^*$. Then $k \stackrel{\mathrm{def}}{=} cZ \in K$ and thus $\alpha(c)c^{-1}Z \in V = U'Z/Z$, that is, $\alpha(c)c^{-1} \in U'Z$; since the set $\{\alpha(c)c^{-1} : c \in C^*\}$ is connected we actually have $\alpha(c)c^{-1} \in (U'Z)_0$, the connected component of $U'Z$ containing the identity. Our task is to find U' in such a fashion that this will imply $\alpha(c)c^{-1} \in U$ for all $c \in C^*$. This will be accomplished whenever $(U'Z)_0 \subseteq U$.

Since G is a pro-Lie group, we find a closed normal subgroup $N \subseteq U \subseteq G$ such that G/N is a Lie group which, by the assumptions on G, is connected and semisimple. By making U smaller, if necessary, we may assume that $UN = U$ (see [5], Proposition 3.26(iii), p. 150). We then let $U' \subseteq U$ be an open subset of G containing 1 such that $NU' = U'$ and that $U'N/N$ is an open cell contained in U/N. The quotient morphism $G \to G/N$ maps Z onto the finitely generated discrete center $ZN/N \cong Z/(Z \cap N)$. By making U' smaller, if necessary, we may assume that $U'z_1/N \cap U'z_2/N \neq \emptyset$ in G/N for $z_1, z_2 \in Z$ implies $z_2z_1^{-1} \in N \cap Z$; that is, $U'z_1 \cap U'z_2 \neq \emptyset$ in G for $z_1, z_2 \in Z$ implies $z_2z_1^{-1} \in N \cap Z$. Thus $U'Z = U' \cup (U'Z \setminus U')$ is a disjoint union of open subsetes of $U'Z$. In other words, U' is open closed in $U'Z$. Therefore $(U'Z)_0 \subseteq U' \subseteq U$, and this is what we had to accomplish.

(iii) By (ii) above, $\operatorname{Aut} G$ is algebraically and topologically isomorphic to a subgroup of $\operatorname{Aut}\mathfrak{g}$. By Theorem 2.10, $\operatorname{Aut}\mathfrak{g}$ is a topological group, and thus $\operatorname{Aut} G$ is a topological group with respect to the compact open topology.

(iv) From (ii) we know that $\operatorname{Aut} G$ is isomorphic as a topological group to a subgroup A of $\operatorname{Aut}\mathfrak{g}$ which contains $\operatorname{Inn}\mathfrak{g}$. From Theorem 2.11 and Corollary 2.12 we know that $\operatorname{Aut}\mathfrak{g} = (\operatorname{Inn}\mathfrak{g}) \bullet \Delta$ with a totally disconnected group $\Delta \cong E \rtimes P_{\mathcal{S}}(J)$ (see Section 2) such that $(\alpha, \beta) \mapsto \alpha\beta : \operatorname{Inn}\mathfrak{g} \rtimes \Delta \to \operatorname{Aut}\mathfrak{g}$ is a quotient morphism of topological groups. Now let $D \overset{\text{def}}{=} \Delta \cap A$. Then $A = (\operatorname{Inn}\mathfrak{g})D$ and $D \cap \operatorname{Inn}\mathfrak{g}$ is compact totally disconnected. The group A is a quotient of $\operatorname{Inn}\mathfrak{g} \rtimes D$ modulo a compact totally disconnected normal subgroup. □

We notice that $D = E \rtimes P_{\mathcal{S}}(J)$ if G is simply connected or centerfree by 3.1 and 3.2.

If G is a pro-Lie group and N a closed normal subgroup, then G/N sometimes fails to be a pro-Lie group due to the possible lack of completeness ([5], Corollary 4.11, p. 179). If G is connected, then [5], Theorem 4.28 on p. 202 gives sufficient conditions for G/N to be a pro-Lie group: such as N being almost connected, or first countable, or locally compact. None of these apply to $Z(G)$ for a semisimple connected pro-Lie groups G. It is therefore not a priori obvious that $G/Z(G)$ is a pro-Lie group. It does however satisfy condition (iv) of Theorem 4.28 = Corollary 9.57 of [5] by our Main Theorem 3.4(i).

For examples of connected semisimple pro-Lie groups we refer to [5], p. 608ff.

Compact Automorphic Actions on Semisimple Pro-Lie Groups

Recall that we say that a topological group Ω *acts automorphically* on a topological group G if there is a continuous action $(\omega, g) \mapsto \omega{\cdot}g : \Omega \times G \to G$ such that $g \mapsto \omega{\cdot}g$ is in $\operatorname{Aut}(G)$ for all $\omega \in \Omega$.

Corollary 3.5. *Let G be a connected semisimple pro-Lie group and Ω a compact group acting automorphically on G. Then there is a maximal compactly embedded connected subgroup K of G with $\Omega{\cdot}K \subseteq K$.*

Proof The compact group Ω acts linearly and automorphically on $\mathfrak{g}$ via

$$\exp_G(\omega{\cdot}X) = \omega{\cdot} \exp_G X \quad \text{for all} \quad X \in \mathfrak{g}.$$

Then, by Proposition 2.14, there is a maximal compactly embedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ that is Ω-invariant. Then $K \stackrel{\rm def}{=} \overline{\exp \mathfrak{k}}$ is a maximal compactly embedded closed subgroup, and it is Ω-invariant. (See [5], Proposition 12.52, p. 524.) □

4. Topological Groups with Pro-Lie Identity Component

In this and the next section we apply our results about the structure of the automorphism groups of semisimple pro-Lie algebras and pro-Lie groups to the structure theory of pro-Lie groups in general.
First we consider an arbitrary topological group G about we assume nothing except that the identity component G_0 is a pro-Lie group. We recall from [5], Definition 10.23ff., notably, Theorem 10.25, that we have a unique largest connected prosolvable characteristic subgroup $R(G)$. Let G_0 denote the identity component of G, also a characteristic subgroup of G. We know that $G_0/R(G)$ is a connected semisimple pro-Lie group (See [5], Theorem 10.28ff.) Let $\mathbf{R}(G)$ be that subgroup of G_0 containing $R(G)$ for which $\mathbf{R}(G)/R(G)$ is the prodiscrete center of $G_0/R(G)$. That is, $\mathbf{R}(G)$ is the largest prosolvable normal subgroup of G_0, another characteristic subgroup of G_0. We define $S \stackrel{\rm def}{=} G_0/\mathbf{R}(G)$ and denote the Lie algebra of S by $\mathfrak{s}$. Then S is a centerfree semisimple pro-Lie group.

The group G acts on S via inner automorphisms:

$$\gamma\colon G \to \operatorname{Aut} S, \quad \gamma(g)(g_0\mathbf{R}(G)) = gg_0g^{-1}\mathbf{R}(G).$$

By Lemma 3.3(iii), γ is a morphism of topological groups. An element $g \in G$ is in the kernel of γ iff $\gamma(g) = \mathrm{id}$ iff $[g,x] = gxg^{-1}x^{-1} \in \mathbf{R}(G)$ for all $x \in G_0$, and so

$$\ker\gamma = \{g \in G : [g,x] \in \mathbf{R}(G) \text{ for all } x \in G_0\},$$

the centralizer of G_0 mod $\mathbf{R}(G)$. Obviously, $\mathbf{R}(G) \subseteq \ker\gamma$.
The key link to the previous discussion is now provided by the morphism

$$L_S\colon \operatorname{Aut} S \to \operatorname{Aut} \mathfrak{s}$$

of topological groups implemented by the Lie algebra functor $\mathfrak{L}$ (see Proposition 3.1), which is an embedding of topological groups by Theorem 3.4(ii).

Definition 4.1. Let G be a topological group whose identity component G_0 is a pro-Lie group. The composition $L_S \circ \gamma$ is called the *standard representation* $\Phi_G\colon G \to \operatorname{Aut} \mathfrak{s}$.

The standard representation is a morphism of topological groups.

Proposition 4.2. *The standard representation Φ maps the identity component G_0 openly onto* $\operatorname{Inn}\mathfrak{g}$*, implementing an isomorphism of topological groups* $S \stackrel{\text{def}}{=} G_0/\mathbf{R}(G) \to \operatorname{Inn}\mathfrak{s}$*. The kernel of Φ satisfies* $G_0 \cap \ker\Phi = \mathbf{R}(G)$

Proof By Theorem 2.20, the group $\operatorname{Inn}\mathfrak{s}$ is the identity component of $\operatorname{Aut}\mathfrak{s}$. By Theorem 3.4(i) it is an isomorphic image of S implemented by Φ since S is centerfree.
It follows, among other things, that $G_0 \cap \ker\Phi \subseteq \mathbf{R}(G)$; on the other hand we noticed above that $\mathbf{R}(G) \subseteq \ker\gamma \subseteq ker\Phi$. Hence $G_0 \cap \ker\Phi = \mathbf{R}(G)$. $\square$

Almost Connectedly Prosolvable Groups

Recall that a topological group G is called *almost connected* if G/G_0 is compact. Examples we have encountered here include the groups
$\prod_{j\in J}\operatorname{Aut}\mathfrak{s}_j$. In the end, we want to show that in any almost connected pro-Lie group G there are maximal compact subgroups C, that two of them are conjugate and, notably, that $G = G_0C$; these results hold for connected pro-Lie groups according to [5]. Let us first record that in [5] we have also provided these pieces of information for a special classes of almost connected pro-Lie groups which we introduce now:

Definition 4.3. A topological group G is called *almost connectedly prosolvable* if $G/R(G)$ is compact for the radical $R(G)$, the largest connected pro-solvable normal subgroup (see [5], Proposition 7.45 and Definition 7.46 on p. 291 and Theorem 7.53 on p. 295).

Clearly, every almost connectedly prosolvable pro-Lie group is almost connected.

We have rather good structural information on almost pro-slovable groups through [5], Theorem 11.28.B on p. 486 and Remark 11.33 on p. 490 (holding for *almost* prosolvable pro-Lie groups). We summarize the information we have in the following

Proposition 4.4. *Let G be an almost connectedly prosolvable pro-Lie group. Then G has a maximal compact subgroup C and all compact subgroups have a conjugate inside C. Moreover, $G = G_0C$.* $\square$

Now the following result is an outgrowth of our preceding discussions.

Theorem 4.5. *Let G be a topological group whose identity component G_0 is a pro-Lie group, and let C be any compact subgroup of G. Then G_0C is an almost connected subgroup of G and C is contained in a closed subgroup A with the following properties:*

(i) *$A = A_0C$ and A_0 contains $\mathbf{R}(G)$.*
(ii) *A is almost connectedly prosolvable.*

(iii) *The factor group* $A/\mathbf{R}(G)$ *is compact and* $A_0/\mathbf{R}(G)$ *is maximal compact in* $G_0/\mathbf{R}(G)$*, and*

(iv) A_0 *contains a maximal compact subgroup of* G_0*.*

(v) $A/\mathbf{R}(G)$ *is a maximal compact subgroup of* $G_0C/\mathbf{R}(G)$*.*

Proof. The group C it acts automorphically on G under inner automorphisms and hence it acts automorphically on on the semisimple pro-Lie group $G_0/R(G)$, and so by Corollary 3.5, there is a subgroup K of G_0 containing $R(G)$ such that $K/R(G)$ is a maximal compactly embedded connected subgroup of $G_0/R(G)$ so that K is normalized by C. Hence $A \stackrel{\text{def}}{=} KC$ is a closed subgroup of G.

(i) The group $K/R(G)$, being maximally compactly embedded in $G_0/R(G)$ is connected, and since $R(G)$ is likewise connected, K is connected and so $K \subseteq A_0$, whence $A = KC \subseteq A_0C$. Moreover, $K/R(G)$ contains the center $Z\big(G/Z(G)\big)$, and so $\mathbf{R}(G) \subseteq K \subseteq A_0$ by the definition of $\mathbf{R}(G)$.

(ii) Next we show that A is almost connectedly prosolvable. A compactly embedded subgroup like $K/R(G)$ is potentially compact (see [5], Definition 12.46 on p. 521). Let B be that closed subgroup of K containing $R(G)$ for which $B/R(G) = R\big(K/R(G)\big) = Z\big(K/R(G)\big)_0$ (see [5], Theorem 12.48 on p. 522). Then, firstly, B is connected and prosolvable, for instance by [5], Theorem 10.18, pp. 427, 428, since $\mathfrak{L}(B)$ does not contain a finite dimensional simple Lie algebra by its definition. Hence $B \subseteq R(K)$. Moreover, $K/B \cong \big(K/R(G)\big)/R\big(K/R(G)\big)$ is compact by Theorem 12.48 of [5]. Then $A/B = KC/B$ is compact. This shows that A is almost connectedly prosolvable.

(iii) Since the group $K/R(G)$ is a maximal compactly embedded connected subgroup of $G_0/R(G)$, the factor group $K/\mathbf{R}(G) = \big(K/R(G)\big)/Z\big(K/R(G)\big)$ is compact. Since $A/K = KC/K$ is compact, the compactness of $A/\mathbf{R}(G)$ follows.

(iv) Let M be a maximal compact subgroup of G_0. Then $K'R(G)/R(G)$ is compact and thus is contained in a compact subgroup of $G_0/R(G)$ and, in particular, a compactly embedded connected subgroup of $G_0/R(G)$. It follows that a conjugate is contained in the maximal compactly embedded connected subgroup $K/R(G)$ of $G_0/R(G)$ (see [5], Theorem 12.53, p. 525). Therefore a conjugate of M is contained in A_0.

(v) Note $G_0C = G_0A$ and so $G_0C/\mathbf{R}(G) = (G_0/\mathbf{R}(G))(A/\mathbf{R}(G))$.
Then by Lemma Max in the proof of Lemma 2.13, $A/\mathbf{R}(G)$ is maximal compact in $G/\mathbf{R}(G)$. □

Before we secure the existence of a compact subgroup C such that $G = G_0C$ we temporarily return to the general case of a topological group G where we merely assume that G_0 is a pro-Lie group.

We consider the standard representation $\Phi_G\colon G \to \operatorname{Aut}\mathfrak{s}$. From Theorem 3.4 and Proposition 4.2 we know that Φ implements an isomorphism of topological groups $G_0/\mathbf{R}(G) \to \operatorname{Inn}\mathfrak{s}$ and that $\operatorname{Aut}\mathfrak{s} = \operatorname{Inn}\mathfrak{s} \bullet D$ for a subgroup D of the totally disconnected subgroup $E \rtimes P_{\mathcal{S}}(J)$.

We consider the following hypothesis on $\operatorname{Aut}\mathfrak{s}$:

(H) *There is a subgroup* H *of* $\operatorname{Aut}\mathfrak{s}$ *such that* $\operatorname{Aut}\mathfrak{s} = \operatorname{Inn}\mathfrak{s} \bullet H$*.*

This means that $\operatorname{Aut}\mathfrak{s}$ is a product to which Lemma FIT preceding 2.11 applies. Specifically, this says that
there is an idempotent self-map $P\colon \Phi_G(G) \to \Phi_G(G)$ *with* image $\operatorname{Inn}\mathfrak{s}$ and

$$P^{-1}(\theta) = \theta P^{-1}(1) \subseteq \theta D \quad \text{for}\theta \in \operatorname{Inn}\mathfrak{s}.$$

One example for the subgroup H is the group $H = D$ of Theorem 3.4. Another important example will arise in the next section where H is a compact subgroup.

We define $G_1 = \Phi_G^{-1}(D) = \Phi_G^{-1}(E \rtimes P_{\mathcal{S}}(J))$.

Theorem 4.6. *Let G be a topological group whose identity component G_0 is a pro-Lie group and for which* $\operatorname{Aut}\mathfrak{s}$ *satisfies Hypothesis* (H)*. Then there is a closed subgroup G_1 containing $\mathbf{R}(G)$ with the following properties:*

(i) $G_0 \cap G_1 \supseteq \mathbf{R}(G)$*, and* $\frac{G_0 \cap G_1}{\mathbf{R}(G)} \cong (\operatorname{Inn}\mathfrak{s}) \cap H$*.*
(ii) $G = G_0 \bullet G_1$*.*
(iii) $G_1/(G_0 \cap G_1) \cong G/G_0$
(iv) *If* $\operatorname{Inn}\mathfrak{s} \cap H$ *is totally disconnected, then* $(G_1)_0 = R(G)$*.*

Proof.
(i) By 4.2, $\mathbf{R}(G) = G_0 \cap \ker \Phi_G \subseteq G_0 \cap G_1$. Since $G_0/\mathbf{R}(G)$ is mapped isomorphically onto $\operatorname{Inn}\mathfrak{s}$ and $G\_1 = \Phi_G^{-1}(H)$, assertion (i) follows.
(ii) Let $g \in G$. Then $\Phi_G(g) = \theta\delta$ for an inner automorphism θ and an element $\delta \in H$. Hence we find a $g_0 \in G_0$ mapped onto θ and an element $g_1 \in G_1$ mapped onto δ. Then $k \stackrel{\text{def}}{=} g_1^{-1} g_0^{-1} g$ is in the kernel of Φ_G which is contained in $G_1 = \Phi_G^{-1}(H)$. Hence $g = g_0(g_1 k) \in G_0 G_1$. Thus $G = G_0 G_1$. Let $\phi\colon G/\mathbf{R}(G) \to \operatorname{Aut}\mathfrak{s}$ be the morphism induced by Φ_G, and let $P\colon \operatorname{Aut}\mathfrak{s} \to \operatorname{Aut}\mathfrak{s}$ be the idempotent map with image $\operatorname{Inn}\mathfrak{s}$ and $P^{=1}(\theta) = \theta P^{-1}(1)$ for $\theta \in \operatorname{Inn}(\mathfrak{s})$ which we have according to Theorem 3.4. Recall that $f \stackrel{\text{def}}{=} (\phi|G_0/\mathbf{R}(G))\colon G_0/\mathbf{R}(G) \to \operatorname{Inn}\mathfrak{s}$ is an isomorphism of topological groups. Let $j\colon G_0/\mathbf{R}(G) \to G/\mathbf{R}(G)$ be the inclusion and $P'\colon \operatorname{Aut}\mathfrak{s} \to \operatorname{Inn}\mathfrak{s}$ the corestriction of P.
Now define $Q\colon G \to G$ as the map given by $Q = j \circ f^{-1} \circ P' \circ \phi$. The image of Q is $G_0/\mathbf{R}(G)$. Let $g_0 \in G_0$ and set $\overline{g} = g_0\mathbf{R}(G)$. Then one computes readily that $Q(\overline{g}) = \overline{g}$. Thus Q is idempotent. Moreover, $Q^{-1}(\overline{g}) = \phi^{-1}(P^{-1}(\phi(\overline{g}))) = \phi^{-1}(\phi(\overline{g})P^{-1}(1)) = \overline{g}Q^{-1}(1)$. Thus Q satisfies Condition (iii) of Lemma FIT (preceding 2.11). This shows that $G = G_0 \bullet G_1$.
(iii) is a consequence of (ii) in view of Lemma FIT.
(vi) $(G_1/(G_0 \cap G_1)$ is totally disconnected by (iii), and $(G_0 \cap G_1)/R(G)$ is totally disconnected by (i) and the assumption that $\operatorname{Inn}\mathfrak{s} \cap H$ is totally disconnected. Hence $G_1/R(G)$ is totally disconnected while $R(G)$ is connected. Thus $(G_1)_0 = R(G)$. ☐

This Theorem is a by-product of the structure theory of $\operatorname{Aut}\mathfrak{s}$ and will not be used in the special case that G is almost connected below. One of the conclusions at this point is the fact that for a pro-Lie group G, the quotient G/G_0 is isomorphic to a quotient $G_1/(G_0 \cap G_1)$ where G_1 is likely to have more special properties than G: For instance if $H = D$ then G_1 is a pro-Lie group whose identity component is prosolvable. Since we do not know in general whether G/G_0 is complete and therefore is a pro-Lie group (i.e., prodiscrete in this case), this may be a useful piece of information which we do not pursue at this point.

5. Almost Connected Pro-Lie Groups

We apply the Theorem 4.5 above to the situation of Corollary 2.12.(d).

Proposition 5.1. *Assume that G is an almost connected closed subgroup of the group* $\mathrm{Aut}\,\mathfrak{s}$ *for a semisimple pro-Lie algebra $\mathfrak{s}$ and assume that $G_0 = (\mathrm{Aut}\,\mathfrak{s})_0 = \mathrm{Inn}\,\mathfrak{s}$. Then there is a compact subgroup C of G such that $G = G_0C$, and there is a maximal compact subgroup M of G containing C such that $M_0 = G_0 \cap M$ is maximal compact in G_0.*
Moreover, $\mathrm{Aut}\,\mathfrak{s} = \mathrm{Inn}\,\mathfrak{s} \bullet M$ *that is,* $\mathrm{Aut}\,\mathfrak{s}$ *satisfies Hypothesis* (H) *preceding* Theorem 4.6*.*

Proof. We obtain C from 2.12(d) and then apply Theorem 4.5, noting that $\mathbf{R}(G) = \{1\}$ in the present situation and calling here the subgroup A of 4.5 rather M, for "maximal compact."
In order to verify Hypothesis (H) we observe that condition (i) of Lemma FIT preceding 2.11 is satisfied for $N = \mathrm{Inn}\,\mathfrak{s}$ and $H = M$: Firstly $M \mapsto M/(\mathrm{Inn}\,\mathfrak{s} \cap M) = M/M_0$ has a cross-section (see [4] , Corollary 10.38) and $\alpha: M/(\mathrm{Inn}\,\mathfrak{s} \cap M) \to G$ is an isomorphism of topological groups, since M is compact. The assertion follows. □

Theorem 5.2. *Let G be an almost connected pro-Lie group. Then there is a compact subgroup C such that $G = G_0C$.*

Proof. Let again $S = G_0/\mathbf{R}(G)$ and consider the standard representation $\Phi_G: G \to \mathrm{Aut}\,\mathfrak{s}$. By Proposition 4.2 we have $S \cong \mathrm{Inn}\,\mathfrak{s} = (\mathrm{Aut}\,\mathfrak{s})_0 = \Phi_G(G_0)$, whence $\Phi_G(G)$ is an almost connected subgroup of $\mathrm{Aut}\,\mathfrak{s}$. From Proposition 5.1 we derive $\Phi_G(G) = \Phi_G(G_0) \bullet M$ for a maximal compact subgroup M of $\Phi_G(G)$ such that $M_0 = M \cap \Phi_G(G_0)$ is a maximal compact subgroup of $\Phi_G(G_0)$.
Now we set $A \stackrel{\mathrm{def}}{=} \Phi_G^{-1}(M)$ and apply of Theorem 4.6. Thus
(i) $G_0 \cap A$ contains $\mathbf{R}(G)$ and $(G_0 \cap A)/\mathbf{R}(G) \cong \mathrm{Inn}\,\mathfrak{s} \cap M = M_0$ is a compact connected group,
(ii) $G = G_0 \bullet A$,
(iii) $A/(G_0 \cap A) \cong G/G_0$ is compact totally disconnected.

If $g_0 \in G_0 \cap A$, then $\Phi_G(g_0) \in M \cap \Phi_G(G_0) = M_0$ and so $G_0 \cap A = G_0 \cap \Phi_G^{-1}(M_0)$. Now

$$\frac{(G_0 \cap A)}{R(G)} \Big/ Z\Big(\frac{G_0}{R(G)}\Big) = \frac{(G_0 \cap A)}{R(G)} \Big/ \frac{\mathbf{R}(G)}{R(G)} \cong (G_0 \cap A)/\mathbf{R}(G) \cong M_0$$

is a maximal compact subgroup of the semisimple centerfree pro-Lie group $G_0/\mathbf{R}(G) \cong \mathrm{Inn}(\mathfrak{s})$. Then $(G_0 \cap A)/R(G)$ is a maximally compactly embedded subgroup of the semisimple pro-Lie group $G_0/R(G)$.

Lemma. *Assume that H is a connected, semisimple pro-Lie group and K a closed subgroup containing the center $Z(H)$ such that $K/Z(H)$ is a maximally compact subgroup of $H/Z(H)$. Then K is connected.*

Proof. By [5], Theorem 10.29, p.439 there is a family $\{S_j : j \in J\}$ of simply connected simple real Lie groups and morphisms

$$\prod_{j\in J} S_j \xrightarrow{\ \alpha\ } H \xrightarrow{\ \beta\ } H/Z(H) \cong \prod_{j\in J} S_j/Z(S_j)$$

inducing isomorphisms on the pro-Lie algebra level such that for each $j \in J$ there is a maximally compactly embedded subgroup K_j of S_j containing $Z(S_j)$ such that α and β induce the sequence

$$\prod_{j\in J} K_j \xrightarrow{\ \alpha'\ } K \xrightarrow{\ \beta'\ } K/Z(H) \cong \prod_{j\in J} K_j/Z(S_j).$$

Now all K_j are connected and thus

$$K = \overline{\alpha(\prod_{j\in J} K_j)}$$

is connected. □

The Lemma shows that $G_0 \cap A$ is connected, and thus

$$A_0 = G_0 \cap A.$$

Now $A_0/R(G)$ is compactly embedded into $G_0/R(G)$ and is therefore potentially compact and thus almost connectedly prosolvable. The group $R(G)$ is prosolvable, and so A_0 is connectedly almost prosolvable. Since $G = G_0 \bullet A$ by (ii) above, the continuous bijection $A/A_0 = A/(A \cap G_0) \to G_0A/G_0 = G/G_0$ is an isomorphism. Therefore A/A_0 is compact. Hence the group A is connectedly almost prosolvable, and thus there is a compact group such that $A = A_0C$ by Theorem 4.5. Then $G = G_0A = G_0A_0C = G_0C$. □

6. Maximal compact subgroups

We have touched maximal compact subgroups in preceding articles and in [5] they play an important role. In this section we collect some general and systematic remarks.

A partially ordered set $(P, \leq)$ is called *inductive*, if every totally ordered subset has an upper bound. Let G be a topological group and $\mathcal{C}(G)$ the collection of its compact subgroups. Then $\mathcal{C}(G)$ is a partially ordered set with respect to inclusion $\subseteq$. The set $(\mathcal{C}(G), \subseteq)$ may or may not be inductive.

Definition 6.1. (a) If a subgroup M of a topological group G is a maximal element of $\mathcal{C}(G)$ then it is called a *maximal compact subgroup* of G. The set of maximal subgroups of G will be written $\max\mathcal{C}(G)$.
(b) A topological group G is said to be *compactly inductive* if $\mathcal{C}(G)$ is inductive. The full subcategory of compactly inductive groups in the category $\mathbb{TOPGR}$ of topological groups is denoted $\mathbb{ICG}$. □

The group G acts on $\mathcal{C}(G)$ by conjugation:

$$(g, C) \mapsto gCg^{-1} : G \times \mathcal{C}(G) \to \mathcal{C}(G).$$

Since this action preserves partial order, it leaves $\max\mathcal{C}(G)$ invariant. In [5] it is shown that G acts transitively on $\max\mathcal{C}(G)$ if G is a connected pro-Lie group (see [5], Theorem 12.77, p. 547). If G is locally compact and almost connected then G does act transitively on $\max\mathcal{C}(G)$ (see [7]). One of our goals here is that this holds also for almost connected pro-Lie groups.

Remark 6.2. In a compactly inductive topological group G, every compact subgroup is contained in a maximal one.

Proof. Zorn's Lemma. □

The category $\mathbb{ICG}$ was investigated in [7]. If G is locally compact and almost connected, then G is compactly inductive and thus every compact subgroup is contained in a maximal one (see [7]); indeed more generally, *a locally compact group is compactly inductive if G/G_0 is compactly inductive.* In [7] a discrete (hence locally compact) example of a group D was presented, in which every finite subgroup is contained in a member of $\max\mathcal{C}(D)$ but which is not compactly inductive.
In a connected pro-Lie group G every compact subgroup is contained in a maximal compact subgroup. The group $\mathbb{Q}_p$ of p-adic rationals is locally compact but fails to have this property (see e.g. [5], Example 14.2, p. 588). In the light of the development of pro-Lie groups in the meantime, the following is relevant:

Proposition 6.3. (i) *The category $\mathbb{ICG}$ is complete.*
(ii) *An almost connected pro-Lie group is an $\mathbb{ICG}$-group.*

Proof (i) Clearly, if G is an $\mathbb{ICG}$-group, then every closed subgroup of G is an $\mathbb{ICG}$-group as well. By [5], Theorem 1.11(ii), p. 72, it suffices, therefore, to verify that $\mathbb{ICG}$ is closed under the formation of arbitrary products. Let $\{G_j : j \in J\}$ a family of compactly inductive groups. Now let $\mathcal{T}$ be a tower of compact subgroups of $P \stackrel{\rm def}{=} \prod_{j\in J} G_j$. Now $\mathrm{pr}_j(\{C : C \in \mathcal{T}\}) = \{\mathrm{pr}_j(C) : C \in \mathcal{T}\}$ is a tower of compact subgroups of G_j. Since G_j is compactly inductive, there is a compact subgroup $C_j \subseteq G_j$ containing all $\mathrm{pr}_j(C)$, $C \in \mathcal{T}$. Then $K \stackrel{\rm def}{=} \prod_{j\in J} C_j$ is compact by the Theorem of Tychonov. Now $C \subset K$ for all $C \in \mathcal{T}$. Then $\overline{\bigcup \mathcal{T}}$ is compact and this completes the proof of (i).

(ii) Let G be an almost connected pro-Lie group. As in [5], let $\mathcal{N}(G)$ denote the filterbasis of normal subgroups N of G such that G/N is a Lie group. Then $G = \lim_{N \in \mathcal{N}(G)} G/N$ and all G/N are almost connected (see [5]). Then all G/N are compactly inductive by [7], and so G is compactly inductive by (i) above. □

From 6.2 and 6.3 we have as immediate consequence:

Corollary 6.4. *A compact subgroup of an almost connected pro-Lie group is contained in a maximal compact subgroup.* □

For the purposes of our present discussion it will be convenient to have some technical notation. First we notice that for a maximal compact subgroup M of an almost connected pro-Lie group it is not clear that $G_0 \cap M$ is maximal compact in G_0.

Definition 6.5. A compact subgroup $K \in \mathcal{C}(G)$ of a topological group G is called *standard* if $K \cap G_0 \in \max \mathcal{C}(G_0)$.

Notice that $K_0 \subseteq K \cap G_0$ and that therefore $K_0 \in \max \mathcal{C}(G_0)$ implies $K_0 = K \cap G_0$.

Generalities

Lemma 6.6. (i) *If K is a compact subgroup of G then K is standard iff K_0 is maximal compact in G_0.*
(ii) *Every $\mathbb{ICG}$-group (and in particular, every almost connected pro-Lie group) has standard maximal compact subgroups.*
(iii) *If K is a standard compact subgroup such that $G = G_0 K$, then K is maximal compact in G.*
(iv) *In an $\mathbb{ICG}$-group in which all maximal compact subgroups are conjugate, all maximal compact subgroups are standard.*

Proof. (i) Let $K \in \mathcal{C}(G)$ be standard. Since $K \cap G_0 \in \max \mathcal{C}(G_0)$, the group $K \cap G_0$ is connected by [5], Theorem 12.77 on p. 547. (In the formulation of that theorem, the word "connected" is erroneously missing in the hypotheses for G.) Thus $K \cap G_0 \subseteq K_0 \subseteq K \cap G_0$. So $K \cap G_0 = K_0$ and thus $K_0 \in \max \mathcal{C}(G_0)$. Conversely, assume that $K_0 \in \max \mathcal{C}(G_0)$. We noticed already that this implies the equality $K_0 = K \cap G_0$, whence K is standard.
(ii) We consider a maximal compact subgroup C of G_0; then C is connected (see [5], Theorem 12.77, p. 547). By Corollary 6.4, C is contained in some $M \in \max \mathcal{C}(G)$. Since C is connected, $C \subseteq M_0$. Clearly M_0 is compact and contained in G_0, then by the maximality of C we conclude $C = M_0$. Since $C \subseteq M \cap G_0$ and $M \cap G_0$ is compact, by maximality of C once more, $C = M \cap G_0$.
(iii) By 6.6(i) this is Lemma Max above.
(iv) In any $\mathbb{ICG}$-group G there exist standard maximal compact subgroups by (ii) above. If all maximal compact subgroups are conjugate the assertion follows. □

Theorem 6.7. *An almost connected pro-Lie algebra has standard maximal compact subgroups satisfying $G = G_0C$.*

Proof. By Theorem 5.2 we find a maximal compact group C such that $G = G_0C$. Now by Theorem 4.5 we find an almost prosolvable closed subgroup containing C and $\mathbf{R}(G)$. We have $A = A_0C$ and if M is a maximal compact connected normal abelian subgroup of A, then $M \subseteq C$ and there is a connected closed subgroup V uch that $A/M = (V/M) \rtimes (C/M)$ and V/M is simply connected and compact free by Theorem 11.28.B of [5] on p. 486. It follows that $(C/M)_0 = C_0/M$ is maximal compact in A_0/M and therefore C_0 is maximal compact in A_0. Since A_0 contains maximal compact subgroups of G_0 by 4.10(v) and since maximal compact subgroups of A_0 are conjugate (see [5], Theorem 12.77, p. 547), C_0 is maximal compact in G_0. Thus C is standard. □

At this stage it is not obvious that, in general, an almost connected pro-Lie group cannot contain nonstandard maximal compact subgroups.

7. The conjugacy of maximal compact subgroups in almost connected pro-Lie groups

We have information on maximal compact subgroups in an almost connected Lie group G. Firstly, they exist by 6.4. Secondly, standard maximal compact subgroups exist by 6.6(ii). Most importantly, there are standard maximal subgroups C such that $G = G_0C$ by Theorem 6.7.

Theorem 7.1. *The maximal compact subgroups of an almost connected Lie group are conjugate.*

Proof. By Theorem 6.7 there exists a standard maximal compact subgroup C such that $G = G_0C$ and that C is contained in an almost prosolvable closed subgroup A which contains $\mathbf{R}(G)$ (see 4.5). Since A is a prosolvable pro-Lie group, every compact subgroup of A has a conjugate in A that is contained in C by Remark 11.33 of [5] on p. 490. (We should remark that Remark 11.33 is formulated for prosolvable pro-Lie groups, but is valid for almost prosolvable pro-Lie groups as it is based on [5], Corollary 11.32 on p. 489, dealing with almost prosolvable pro-Lie groups.)
It follows that we must show that every compact subgroup K of G has a conjugate in A. From Theorem 4.10 we know that K is contained in an almost prosolvable subgroup B containing $\mathbf{R}(G)$. In order to show that a conjugate of B is contained in A, it suffices to prove that $B/\mathbf{R}(G)$ has a conjugate in $A/\mathbf{R}(G)$. Instead of considering G we are allowed to consider $G/\mathbf{R}(G)$. It is therefore no loss of generality to assume that G_0 is centerfree semisimple, that is, we have $G_0 = \prod_{j\in J} S_j$ for a family of centerfree semisimple connected Lie groups S_j. Then $A = C$ and $C_0 \cong \prod_{j\in J} K_j$ for maximal compact subgroups K_j of S_j. Since K_j is its own normalizer as we have observed in the proof of 2.13, we have $C \subseteq N(C_0, G)$ and $N(C_0, G) \cap G_0 = N(C_0, G_0) = C_0$. and $G = G_0N(C_0, G)$. It follows that $C = N(C_0, G)$. Similarly, K_1 is a maximal compact subgroup of G_0. By 4.10(iii) the group B contains a maximal compact subgroup of G_0, that is, $B_0 \in \max\mathcal{C}(G_0)$.

Hence by [5], Theorem 12.77, p. 547, C_0 and B_0 are conjugate. We may assume that $B_0 = C_0$. But then $B \subseteq N(B_0, G) = N(C_0, G) = C$ which is what we had to show. $\square$

8. The structure of almost connected pro-Lie groups

First we summarize what we have achieved

Main Theorem 8.1. *Let G be an almost connected pro-Lie group. Then G has maximal compact subgroups and all of these are conjugate. If M is one of them, then*
(1) *$G = G_0M$.*
(2) *$M_0 = G_0 \cap M$, and this subgroup is maximal compact in G_0.* $\square$

We also record that
$N(M_0, G) = N(M_0, G_0)M$ *and* $N(M_0, G)/N(M_0, G_0) \cong G/G_0$.
We recall from [4], Theorems 9.41, p. 479 and Corollary 10.38, p. 559:

Facts 8.2. *Let G be a compact group. Then*
(1) *there is a profinite subgroup D of G such that $G = G_0D$, $G_0 \cap D$ is normal in G and central in G_0;*
(2) *there is a compact zero-dimensional subset $\Delta \subseteq G$ such that $m\colon \Delta \times G_0 \to G$, $m(\delta, g) = \delta g$, is a homeomorphism. In particular, $\delta \mapsto \delta G_0 : \Delta \to G/G_0$ is a homeomorphism, and the groups G and $G/G_0 \times G_0$ are are homeomorphic.* $\square$

From Fact 8.2(1) we obtain:

Corollary 8.3. *Let G be an almost connected pro-Lie group. Then there is a profinite subgroup D such that $G = G_0D$ and that $N(G_0 \cap D, G)$ contains at least one maximal compact subgroup.*

Proof. Let M be a maximal compact subgroup according to Theorem 8.1 and apply Fact 8.2(1) in order to obtain a profinite subgroup D such that $M = M_0D$ and $M_0 = G_0 \cap M$. Then $G = G_0M = G_0(G_0 \cap M)D = G_0D$. Further, $G_0 \cap D = G_0 \cap M \cap D = M_0 \cap D$, whence $N(G_0 \cap D, G) = N(M_0 \cap D,) \supseteq M$. $\square$

With Facts 8.2 (2) we prove the following topological splitting theorem for almost connected pro-Lie groups.

Theorem 8.4. *Let G be an almost connected pro-Lie group and let M be a maximal compact group subgroup. Then for some $p \leq 4$ there are vector subspaces $\mathfrak{v}_k \subseteq \mathfrak{g}$, $k \leq p$ such that the function*

$$(m, X_1, \dots, X_p) \mapsto m \exp_G X_1 \cdots \exp_G X_p : M \times \mathfrak{v}_1 \times \cdots \times \mathfrak{v}_p \to G$$

is a homeomorphism.

Proof. Let $C = M_0$; then C is a maximal compact subgroup of G_0 by Theorem 8.1 Thus [5], Theorem 12.81, p. 551 secures the existence of the $\mathfrak{v}_j$, $1 \leq j \leq p$ such that the function

$$(c, X_1, \ldots, X_p) \mapsto c \exp_G X_1 \cdots \exp_G X_p : C \times \mathfrak{v}_1 \times \cdots \times \mathfrak{v}_p \to G_0$$

is a homeomorphism. Now let $\Delta \subseteq M$ be as in Fact 8.3 (2). Then

$$\Delta C \exp_G \mathfrak{v}_1 \cdots \exp_G \mathfrak{v}_p = \Delta G_0 = MG_0 = G$$

and the function

$$(\delta, m, X_1, \ldots, X_p) \mapsto \delta m : \exp_G X_1 \cdots \exp_G X_p : M \times \mathfrak{v}_1 \times \cdots \times \mathfrak{v}_p \to G$$

is a homeomorphism. □

Corollary 8.5. *In an almost connected connected pro-Lie group G there is a closed subset $E \subseteq G$ which is homeomorphic to $\mathbb{R}^J$ for a set J and a maximal compact subgroup M of G such that $(m, e) \mapsto me : M \times E \to G$ is a homeomorphism.*

Proof. In Theorem 8.4 we set $E = \exp_G \mathfrak{v}_1 \cdots \exp_G \mathfrak{v}_p$. By Theorem 8.4, this set is homeomorphic to $\mathfrak{v}_1 \times \cdots \times \mathfrak{v}_p$, that is, to a weakly complete vector space and thus is homeomorphic to $\mathbb{R}^J$ (see [5], Corollary A2.9, p. 638). The assertion now follows from Theorem 8.4. □

Theorem 8.6. *Let G be an almost connected pro-Lie group. Then there is a set J such that G is homeomorphic to $\mathbb{R}^J \times C$ for a maximal compact subgroup C, and to $\mathbb{R}^J \times C_0 \times C/C_0$.*

Proof. From Theorem 8.5 we know that there is a set J and a maximal compact subgroup C such that G is homeomorphic to $C \times \mathbb{R}^J$. Since C is homeomorphic to $\Delta \times C_0$ Fact 8.1 (2)) which in turn is homeomorphic $(C/C_0) \times C_0$. □

Corollary 8.7. *Any almost connected pro-Lie group is homotopy equivalent to a compact group.*

Proof. By Theorem 8.6, an almost connected pro-Lie group is homeomorphic to $\mathbb{R}^J \times C$ for some set J and a compact group C. Since the topological vector space $\mathbb{R}^J$ is homotopy equivalent to a point, the assertion follows. □

For more detailed information we recall the following facts on compact groups

Facts 8.8. **(i)** *Let G be a compact connected group. Then G is the semidirect product of the commutator group G' which is a compact connected semisimple group and a connected compact abelian subgroup $A \cong G/G'$.*

(ii) *Let G be a compact totally disconnected group. Then there is a set J such that G and $\{0,1\}^J$ are homeomorphic.*

Proof. See [4], for (i): Theorem 9.39, and for (ii) 10.40. □

Corollary 8.9. *Let G be an almost connected pro-Lie group. Then there are sets I and J, a compact connected semisimple group S, and a connected compact abelian group A such that G and $\mathbb{R}^I \times (\mathbb{Z}/2Z)^J \times S \times A$ are homeomorphic.*

Proof. This is now an immediate consequence of Theorem 8.6 and Facts 8.8. □

Corollary 8.10. *The underlying space of an almost connected pro-Lie group is a Baire space.*

Proof. Again the assertion follows from the fact that an almost connected pro-Lie group is homeomorphic to a product of a product of lines and a compact space. (See [11], Theorem 6.) □

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